Last time: Started the proof of the Jantzen - Kac - Kazhdan - Shapovalov formula

$$\boxed{\det(\langle\, ,\, \rangle^\eta) = (\text{nonzero constant}) \cdot \prod_{\alpha\in\Delta^+}\prod_{n\geq 1}\left(h_\alpha + \rho(h_\alpha) - \frac{n(\alpha,\alpha)}{2}\right)^{P(\eta-n\alpha)}}$$

in particular, we proved that $\det(\langle\, ,\, \rangle)^\eta$ indeed equals, up to a nonzero constant, a product of $\{h_\beta + \rho(h_\beta) - \frac{1}{2}(\beta,\beta) \mid \beta\text{-quasiroot}\}$. Note that if $\beta = \tau\alpha$ $(\tau\in\mathbb{Q})$, then $h_\beta + \rho(h_\beta) - \frac{1}{2}(\beta,\beta) = \tau\cdot\left(h_\alpha + \rho(h_\alpha) - \frac{\tau}{2}(\alpha,\alpha)\right)$

- Case 1: $(\alpha,\alpha)=0$

In this case, all $\beta=\tau\alpha$ will be contributing the same $h_\alpha+\rho(h_\alpha)$, hence its overall power matches the power of h_α in the leading term, i.e. $\sum_{n\geq 1} P(\eta-n\alpha)$

So: For roots $\alpha\in\Delta^+$ s.t. $(\alpha,\alpha)=0$, we get the claimed $\prod_{n\geq 1}\left(h_\alpha+\rho(h_\alpha)-\frac{n(\alpha,\alpha)}{2}\right)^{P(\eta-n\alpha)}$

- Case 2: $(\alpha,\alpha)\neq 0$.

In this case, while we do know from the leading term computation last time that the overall sum of multiplicities is $\sum_{n\geq 1} P(\eta-n\alpha)$, we want to:

i) show that multiplicity of $h_\beta+\rho(h_\beta)-\frac{1}{2}(\beta,\beta)$ with $\beta=n\alpha$, $n\in\mathbb{Z}_{>0}$, is $\dim(\mathfrak{g}_\alpha)\cdot P(\eta-\beta)$

ii) — // — with $\beta=n\alpha$, $n\notin\mathbb{Z}_{>0}$, is 0

To this end, we shall now introduce the important notion: Jantzen filtration.

[Idea: While we know that $\mathrm{Ker}(M_\lambda \twoheadrightarrow L_\lambda)$ is precisely the kernel of $\langle\, ,\, \rangle_\lambda$, we would like to count the degree of degeneracy of $\langle\, ,\, \rangle_\mu$ as $\mu\to\lambda$.

To this end, we shall deform our original setup $\mathbb{C} \rightsquigarrow R=\mathbb{C}[t]$.

Thus, we define $\mathfrak{g}_R := \mathfrak{g}\otimes_{\mathbb{C}} R$, $\mathfrak{h}_R := \mathfrak{h}\otimes_{\mathbb{C}} R$, $\mathfrak{h}^*_R := \mathfrak{h}^*\otimes_{\mathbb{C}} R = \mathrm{Hom}_R(\mathfrak{h}_R, R)$, so that all Lie algebras are now viewed over R (free modules). Also note that $U(\mathfrak{g})\otimes_{\mathbb{C}} R = U_R(\mathfrak{g}_R)$ with right-hand side being univ. enveloping/R.

For any $\tilde{\lambda}\in\mathfrak{h}^*_R$, we likewise have Verma $\mathfrak{g}_R$-module $\tilde{M}_{\tilde{\lambda}}$. Moreover, we still have the involutive anti-automorphism $\sigma: \underset{\simeq\, U(\mathfrak{g})_R}{U_R(\mathfrak{g}_R)} \to U_R(\mathfrak{g}_R)$ given by $\sigma(a\otimes F(t)) = \sigma(a)\otimes F(t)$ $\forall a\in U(\mathfrak{g})$, $F(t)\in R$

Hence, we can still introduce the form $\widetilde{\langle,\rangle}$ on $U(\mathfrak{g})_R$ via

$$\boxed{\widetilde{\langle,\rangle} : U(\mathfrak{g})_R \times U(\mathfrak{g})_R \longrightarrow U(\mathfrak{h})_R \quad \text{given by } \widetilde{\langle x,y\rangle} = \pi(\sigma(x)y)}$$

and as a result, for any $\tilde{\lambda} \in \mathfrak{h}^*_R$ we get:

$$\boxed{\widetilde{\langle,\rangle}_{\tilde{\lambda}} : \widetilde{M}_{\tilde{\lambda}} \times \widetilde{M}_{\tilde{\lambda}} \longrightarrow R = \mathbb{C}[t]}$$

Viewing $\widetilde{M}_{\tilde{\lambda}}$ as a free module $/R$, one can still compute $\det(\langle , \rangle^{\eta}_{\tilde{\lambda}}) \in R$ which are determinants of the same-sized square matrices of size $P(\eta)$.

Note: We can view $\mathbb{C}$ as R-module via specializing $t \mapsto 0$; in particular, if $\phi : \underset{F(t)}{\overset{R}{\psi}} \longrightarrow \underset{F(0)}{\overset{\mathbb{C}}{\psi}}$, then we can apply ϕ to all the constructions above.

In particular, if $\tilde{\lambda} \in \mathfrak{h}^*_R = \mathfrak{h}^* \underset{\mathbb{C}}{\otimes} \mathbb{C}[t]$ has the form $\underset{\in \mathfrak{h}^*}{\lambda} \otimes 1 + t(\dots)$, then $\phi(\tilde{\lambda}) = \lambda$.
In this case, we get $U(\mathfrak{g})_R$-module morphism $\widetilde{M}_{\tilde{\lambda}} \longrightarrow M_\lambda$, where M_λ is viewed as $U(\mathfrak{g})_R$-module also through ϕ-induced map $U(\mathfrak{g})_R \longrightarrow U(\mathfrak{g})$.

Key: $\boxed{\phi\left(\det(\langle , \rangle^{\eta}_{\tilde{\lambda}})\right) = \det \langle , \rangle^{\eta}_{\lambda}}$ in other words the order of vanishing of $\langle , \rangle^{\eta}_{\lambda}$ is controlled by the max order of t dividing $\langle , \rangle^{\eta}_{\tilde{\lambda}}$.

Def 1: For any $F(t) \in R$, we use $\nu_t(F(t))$ to denote the max power of t dividing $F(t)$

example: $\nu_t(t^2 + 7t^3) = 2$.

We shall now consider the following filtration of the Verma module

$$\boxed{\widetilde{M}_{\tilde{\lambda}} = \widetilde{M}^{(0)} \supseteq \widetilde{M}^{(1)} \supseteq \widetilde{M}^{(2)} \supseteq \dots \quad \text{with } \widetilde{M}^{(k)} := \{v \in \widetilde{M}_{\tilde{\lambda}} \mid \nu_t(\widetilde{\langle v,w\rangle}_{\tilde{\lambda}}) \geq k \ \forall w \in \widetilde{M}_{\tilde{\lambda}}\}}$$

Applying the specialization ϕ which takes $\widetilde{M}_{\tilde{\lambda}}$ to M_λ (= usual Verma module/$\mathfrak{g}$)

Def 2: The Jantzen's filtration of M_λ is obtained as ϕ-image of above:

$$\boxed{M_\lambda = M^{(0)} \supseteq M^{(1)} \supseteq M^{(2)} \supseteq \dots \quad \text{with } M^{(k)} = \phi(\widetilde{M}^{(k)})}$$

Easy properties: 1) $M^{(1)} = \mathrm{Ker}(\langle , \rangle_\lambda) = \mathrm{Ker}(M_\lambda \twoheadrightarrow L_\lambda)$

2) $\forall \eta \in Q_+ \ \exists i \gg 0$ s.t. $M_\lambda(\lambda - \eta) \cap M^{(i)} = 0$.

Lecture #29

Let's now explain how this allows to finish the proof of determinant f-la. We start with the following simple

Exercise: If $\alpha\in\Delta^+$ s.t. $(\alpha,\alpha)\neq 0$ and $\beta\in(\mathbb{Q}_{>0}\alpha)\cap Q$, then $\exists\lambda\in\mathfrak{h}^*$ s.t.
$$(\lambda+\rho)(h_\beta)=\tfrac{1}{2}(\beta,\beta),\quad (\lambda+\rho)(h_\gamma)\neq\tfrac{1}{2}(\gamma,\gamma)\ \forall\gamma\in Q_+\setminus\{0,\beta\}$$
$$(\lambda+\rho-\beta)(h_\gamma)\neq\tfrac{1}{2}(\gamma,\gamma)\ \forall\gamma\in Q_+\setminus\{0\}$$

Lemma 1: In the setup of exercise above, $\mathrm{Ker}(M_\lambda\twoheadrightarrow L_\lambda)$ is a direct sum of several copies of $M_{\lambda-\beta}$.

▷ The last condition guarantees that $M_{\lambda-\beta}$ – irreducible, i.e. $M_{\lambda-\beta}\simeq L_{\lambda-\beta}$.
Let $V:=(M_\lambda[\lambda-\beta])^{\mathrm{sing}}$ – space of singular vectors in degree $\lambda-\beta$ of Verma M_λ.
Pick a basis $\{v_1,\dots,v_N\}$ of V and use it to construct $M_{\lambda-\beta}^{\oplus N}\xhookrightarrow[\mathfrak{g}\text{-homom}]{}M_\lambda$.
Claim: $L:=M_\lambda/M_{\lambda-\beta}^{\oplus N}$ is irreducible (hence, $\simeq L_\lambda$).

But if not, then $\exists\gamma\in Q_+\setminus\{0\}$ s.t. $(L[\lambda-\gamma])^{\mathrm{sing}}\neq 0$, hence have $M_{\lambda-\gamma}\xrightarrow[\mathfrak{g}\text{-homom.}]{}L$.
Comparing the Casimir operator action, we get $(\lambda+\rho)(h_\gamma)=\frac{1}{2}(\gamma,\gamma)\Rightarrow\gamma=\beta$.
But by the above construction $(L(\lambda-\beta))^{\mathrm{sing}}=0\Rightarrow$ ↯ ∎

Note: In the above setup, zeroes of $\langle,\rangle^\lambda$ evaluated at λ will be arising only from the factors $h_\beta+\rho(h_\beta)-\frac{1}{2}(\beta,\beta)$; and to control the order of vanishing we shall rather look at "R-deformed" setup and track max power of t dividing $\langle\widetilde{,}\rangle^\lambda$ at the lifts
$$\boxed{\tilde\lambda=\lambda+t\nu,\quad \nu\in\mathfrak{h}^*\text{ – "generic" in the sense } \nu(h_\gamma)\neq 0\ \forall\gamma\in Q_+\setminus\{0\}}$$
so that
$$\boxed{\nu_t\left((\tilde\lambda+\rho)(h_\gamma)-\tfrac{1}{2}(\gamma,\gamma)\right)=\begin{cases}1 & \text{if }\gamma=\beta\\ 0 & \text{if }\gamma\in Q_+\setminus\{0,\beta\}.\end{cases}}$$

Upshot: For any quasiroot β, the multiplicity of $h_\beta+\rho(h_\beta)-\frac{1}{2}(\beta,\beta)$ in $\det\langle,\rangle^\lambda$ equals $\nu_t\left(\langle\widetilde{\ ,\ }\rangle^{\lambda}_{\tilde\lambda}\right)$ with $\tilde\lambda$ = generic linear deformation as above of $\lambda\in\mathfrak{h}^*$ as in Exercise on top.

Lecture #29

Evoking Lemma 1, we see that for λ as in loc.cit, we have $M^{(1)} \simeq M_{\lambda-\beta}^{\oplus N_1} \simeq L_{\lambda-\beta}^{\oplus N_1}$

Hence, any module $M^{(k)}$ in Jantzen filtration has the form $M^{(k)} \simeq M_{\lambda-\beta}^{\oplus N_k}$ for some $N_k \in \mathbb{Z}_{\geq 0}$ (and $N_k = 0$ for $k \gg 0$ just by looking at $M_\lambda[\lambda-\beta] \cap M^{(k)}$)

Set $N_\beta := \sum_{k\geq 1} N_k$. Then $\sum_{k\geq 0} \dim M^{(k)}[\lambda-\eta] = N_\beta \cdot \dim M_{\lambda-\beta}[\lambda-\eta] = N_\beta \cdot P(\eta-\beta)$

Thus: $\det(\widetilde{\langle\, ,\rangle}^{\eta}_{\tilde\lambda})$ is divisible precisely by $N_\beta \cdot P(\eta-\beta)$-th power of t, i.e.

$\nu_t(\det(\widetilde{\langle\, ,\rangle}^{\eta}_{\tilde\lambda})) = N_\beta \cdot P(\eta-\beta) \Rightarrow \det(\langle\, ,\rangle^\eta)$ is divisible precisely by $N_\beta \cdot P(\eta-\beta)$-th power of $h_\beta + \rho(h_\beta) - \frac{1}{2}(\beta,\beta)$

But then the multiplicity of h_α in the leading term of $\det\langle\, ,\rangle^\eta$ equals $\sum_{\beta \in \mathbb{Q}_{>0}\alpha \cap Q} N_\beta \cdot P(\eta-\beta)$, while by Lemma 3 of Lecture 28 it equals $\sum_{n>0} P(\eta - n\alpha)$

[Exercise: The functions $\{\phi_\beta : Q_+ \to \mathbb{Z}_{\geq 0},\ \eta \mapsto P(\eta-\beta)\}_{\beta \in \mathbb{Q}_{>0}\alpha \cap Q_+}$ are lin. indep.

Therefore, we conclude that

$$\boxed{N_\beta = \begin{cases} 0 & \text{if } \beta \in \mathbb{Q}_{>0}\alpha \setminus \mathbb{Z}_{>0}\alpha \\ \dim \mathfrak{g}_\alpha & \text{if } \beta \in \mathbb{Z}_{>0}\cdot\alpha \end{cases}}$$

As noted last time, this determinant formula immediately yields:

$$\boxed{M_\lambda \text{- irreducible} \iff (\lambda+\rho)(h_\alpha) \neq \tfrac{n}{2}(\alpha,\alpha) \quad \forall \alpha \in \Delta^+, n \in \mathbb{Z}_{>0}}$$

However, the same technique of Jantzen's filtrations also allows to classify all irreducible subquotients of M_λ when M_λ is reducible.

[Exercise: Show that each irreducible subquotient of M_λ is one of L_μ, and that all M_λ admit a (unique) finite Jördan-Holder series.

Definition 3: For any λ, define $\Delta_+(\lambda) := \{\alpha \in \Delta^+ \mid (\lambda+\rho)(h_\alpha) = \frac{n}{2}(\alpha,\alpha) \text{ for some } n \in \mathbb{Z}_{>0}\}$

For any $\alpha \in \Delta_+(\lambda)$, set $Z_\alpha := \{n \in \mathbb{Z}_{>0} \mid (\lambda+\rho)(h_\alpha) = \frac{n}{2}(\alpha,\alpha)\}$ ($|Z_\alpha| = 1$ if $(\alpha,\alpha) \neq 0$)

Lecture #29

Theorem (BGG–Jantzen–Kac–Kazhdan): L_μ is an irreducible subquotient of M_λ iff $\exists\, \beta_1,\dots,\beta_k \in \Delta^+$, $n_1,\dots,n_k \in \mathbb{Z}_{>0}$ $(k\geq 0)$ s.t.

$$\mu = \lambda - n_1\beta_1 - n_2\beta_2 - \dots - n_k\beta_k \qquad (*)$$

$$(\lambda+\rho-n_1\beta_1-\dots-n_{i-1}\beta_{i-1})(h_{\beta_i}) = \frac{n_i}{2}(\beta_i,\beta_i) \quad \forall\, 1\leq i\leq k.$$

The proof of this result again utilizes $\nu_t(\det \widetilde{\langle,\rangle}^{\eta}_{\tilde\lambda}) = \sum_{k\geq 1} M^{(k)}(\lambda-\eta)$

► Consider the Jantzen filtration $M_\lambda = M^{(0)} \supsetneq M^{(1)} \supseteq M^{(2)} \supseteq \dots$ By above:

$$\sum_{k\geq 1} ch\, M^{(k)} = \sum_{\eta\in Q_+}\sum_{k\geq 1} \dim M^{(k)}[\lambda-\eta]\cdot e^{\lambda-\eta} = \sum_{\eta\in Q_+} \nu_t(\det \widetilde{\langle,\rangle}^{\eta}_{\tilde\lambda})\cdot e^{\lambda-\eta}$$

But: applying the determinant formula of the previous theorem to $\widetilde{\langle,\rangle}^{\eta}$ we get precisely: $\boxed{\nu_t(\det(\widetilde{\langle,\rangle}^{\eta}_{\tilde\lambda}) = \sum_{\substack{\alpha\in\Delta_+(\lambda)\\ n\in\mathbb{Z}_\alpha}} P(\eta-n\alpha)}$

So: $$\sum_{k\geq 1} ch\, M^{(k)} = \sum_{\alpha\in\Delta_+(\lambda)}^{n\in\mathbb{Z}_\alpha}\sum_{\eta\in Q_+} P(\eta-n\alpha)e^{\lambda-\eta} = \sum_{\alpha,n}\sum_{\eta\in Q_+} \dim M_{\lambda-n\alpha}[\lambda-\eta]\, e^{\lambda-\eta} =$$
$$= \sum_{\alpha\in\Delta_+(\lambda)}^{n\in\mathbb{Z}_\alpha} ch\, M_{\lambda-n\alpha}$$

Corollary 1: $$\sum_{k\geq 1} ch\, M^{(k)} = \sum_{\substack{\alpha\in\Delta_+(\lambda)\\ n\in\mathbb{Z}_\alpha}} ch\, M_{\lambda-n\alpha}$$

In particular, any irreducible L_μ appears as a subquotient of some $M^{(k)}$ (and hence of $M^{(1)}$) iff it appears as a subquotient of some $\{M_{\lambda-n\alpha}\}_{n\in\mathbb{Z}_\alpha}^{\alpha\in\Delta_+(\lambda)}$. With this observation at hand, we are ready to prove theorem:

$\boxed{\Rightarrow}$ If L_μ is an irreducible subquotient of M_λ, then either it coincides with $L_\lambda = M_\lambda/M^{(1)}$, which corresponds to $k=0$ above, or is a subquotient of $M^{(1)}$. But then, by above, it's also a subquotient of some $M_{\lambda-n\alpha}$ $(\alpha\in\Delta_+(\lambda),\ n\in\mathbb{Z}_\alpha)$. As $|\lambda-n\alpha-\mu| < |\lambda-\mu|$ ($|\dots|$ denotes height) we know by induction the result $(*)$ for the pair $(\lambda-n\alpha,\mu)$. It remains to note that $\alpha\in\Delta_+(\lambda)$, $n\in\mathbb{Z}_\alpha$ exactly imply we take $\beta_1=\alpha$, $n_1=n$.

Lecture #29

(Continuation)

$\boxed{\Leftarrow}$ If $k=0 \Rightarrow \mu=\lambda \Rightarrow$ done

If $k\geq 1$, fix $\alpha:=\beta_1$, $n:=n_1$, so that the pair $(\lambda-n\alpha, \mu)$ also satisfies $(*)$

The latter, by induction, implies L_μ is a subquotient of $M_{\lambda-n\alpha}$.

But then, evoking Corollary 1, we get that L_μ is a subquotient of some $M^{(k)}$, hence of M_λ ∎

Exercise: a) Show that if $\mathfrak{g}=\mathfrak{g}(A)$ - simple f.dim., then

$$\dim \mathrm{Hom}_{\mathfrak{g}}(M_\mu, M_\lambda)\leq 1 \quad \forall \lambda, \mu.$$

b) Provide a counterexample to above inequality when $\mathfrak{g}$ is Kac-Moody

* * *

Let's provide some more details on the argument of replacing $\mathbb{C} \rightsquigarrow \mathbb{C}[[t]]=R$ that was crucially used in the above definition of Jantzen filtration.

For a fixed $\tilde{\lambda}$ and any $\eta\in Q_+$, the weight component $\tilde{M}_{\tilde{\lambda}}[\tilde{\lambda}-\eta]$ is a free R-module of finite rank, equipped with nondegenerate form $\langle , \rangle^{\eta}_{\tilde{\lambda}}$.

Lemma 2: a) There exist two R-bases $\{f_1, \ldots, f_r\}$, $\{e_1, \ldots, e_r\}$ of $\tilde{M}:=\tilde{M}_{\tilde{\lambda}}[\tilde{\lambda}-\eta]$ and $\{a_1, \ldots, a_r\}\subseteq R\setminus\{0\}$ s.t. $\langle f_i, -\rangle_{\tilde{\lambda}} = a_i\cdot e_i^*(-)$ $\forall i$.

b) In terms of notations in part a), we have:

$$\boxed{\tilde{M}^{(k)}=\bigoplus_{i:\nu_t(a_i)<k} R\cdot t^{k-\nu_t(a_i)}f_i \oplus \bigoplus_{i:\nu_t(a_i)\geq k} Rf_i}$$

c) Applying $\phi: R\xrightarrow{t\mapsto 0}\mathbb{C}$, we get

$$\boxed{M^{(k)}=\bigoplus_{i:\nu_t(a_i)\geq k}\mathbb{C}\phi(f_i)}$$

▸ Part a) follows from a general result of commutative algebra on the structure of submodules of free modules over P.I.D. To this end, take $\tilde{N}:=\{\langle v, -\rangle_{\tilde{\lambda}}\}_{v\in\tilde{M}}$ which is a free R submodule of $\tilde{M}^*$ of rank $r=\mathrm{rank}_R \tilde{M}$ as $\langle , \rangle_{\tilde{\lambda}}$ is nondeg. Then $\tilde{N}=\bigoplus_{i=1}^{r} Ra_ie_i^*$ for some $a_i\in R$. Then $a_ie_i^*=\langle f_i, -\rangle_{\tilde{\lambda}}$

Lecture #23

(Continuation)

Parts b), c) follow immediately now ∎

> Corollary 2: For any basis $\{e_1, \ldots, e_r\}$ of $\widetilde{M}$, set $\widetilde{D} := \det(\widetilde{\langle e_i, e_j\rangle}_{\tilde\lambda})$. Then:
> $$\nu_t(\widetilde{D}) = \sum_{k\geq 1} \dim_{\mathbb{C}} M^{(k)}$$

▷ Pick basis $\{e_1, \ldots, e_r\}$ as in Lemma above, and $\{f_1, \ldots, f_r\}$ as well.

Then $\exists\, X = (x_{ij}) \in GL_r(R)$ s.t. $e_a = \sum_b x_{ab} f_b$. Then:

as in Lemma 2

$$\widetilde{\langle e_i, e_j\rangle}_{\tilde\lambda} = \sum_p x_{ip} \cdot \widetilde{\langle f_p, e_j\rangle}_{\tilde\lambda} = x_{ij} \cdot a_j \Rightarrow \widetilde{D} = \det(X) \cdot \prod_{j=1}^{n} a_j \Rightarrow \boxed{\nu_t(\widetilde{D}) = \sum_j \nu_t(a_j)}$$

But $M^{(k)} \underset{\text{Lemma 2(c)}}{=} \bigoplus_{i : \nu_t(a_i)\geq k} \mathbb{C}\,\phi(f_i) \Rightarrow \sum_{k\geq 1} \dim_{\mathbb{C}} M^{(k)} = \sum_{k\geq 1} \#\{i \mid \nu_t(a_i) \geq k\} = \sum_j \nu_t(a_j)$ ∎

Finally, we note that similarly to how $M^{(1)}$ is the kernel of some form on $M^{(0)} = M_\lambda$, one can think of each $M^{(k+1)}$ as the kernel of some form on $M^{(k)}$

> Proposition 1: There is a symmetric bilinear form $\langle\, ,\rangle^{(m)}_\lambda$ on $M^{(k)}$ s.t.
> $$M^{(k+1)} = \text{radical } \langle\, ,\rangle^{(m)}_\lambda = \{v \in M^{(k)} \mid \langle v, w\rangle^{(m)}_\lambda = 0 \ \forall w \in M^{(k)}\}$$

The construction of such form is quite natural given the above R-deformation

▶ For $v_1, v_2 \in \widetilde{M}^{(k)}$, we know that $\widetilde{\langle v_1, w\rangle}_{\tilde\lambda}$, $\widetilde{\langle w, v_2\rangle}_{\tilde\lambda}$ are divisible by t^k.

We thus define:

$$\boxed{\langle \phi(v_1), \phi(v_2)\rangle^{(k)}_\lambda := \phi\big(\underbrace{t^{-k} \cdot \widetilde{\langle v_1, v_2\rangle}_{\tilde\lambda}}_{\in R}\big) \in \mathbb{C}}$$

- First, we need to show this is well-defined, i.e. if $\phi(v_1') = \phi(v_1)$, $\phi(v_2') = \phi(v_2)$ for $v_1', v_2', v_1, v_2 \in \widetilde{M}^{(k)}$, then the above right-side is the same for v_1', v_2'. Equivalently, we need to show that if $v_1 \in \widetilde{M}^{(k)} \cap t\widetilde{M}$ or $v_2 \in \widetilde{M}^{(k)} \cap t\widetilde{M}$, then the result is zero. But if $v_1 = tu_1$, then $\widetilde{\langle v_1, v_2\rangle}_{\tilde\lambda} = t \cdot \widetilde{\langle u_1, v_2\rangle}_{\tilde\lambda} \in t \cdot t^k \cdot R$ $\Rightarrow \phi(t^{-k} \cdot \widetilde{\langle v_1, v_2\rangle}_{\tilde\lambda}) = 0 \in \mathbb{C}$; and similarly for $v_2 = tu_2$.
- The pairing $\langle\, ,\rangle^{(k)}_\lambda$ is clearly symmetric.

Lecture #29
(Continuation)

- The inclusion $M^{(k+1)} \subseteq \mathrm{rad}\langle\, ,\, \rangle^{(k)}_{\tilde{\lambda}}$ is clear, since if $v_1 \in \widetilde{M}^{(k+1)}$, $v_2 \in \widetilde{M}^{(k)}$, then $\langle \widetilde{v_1, v_2} \rangle_{\tilde{\lambda}} \vdots t^{k+1} \Rightarrow \phi(t^{-k} \cdot \langle \widetilde{v_1, v_2} \rangle_{\tilde{\lambda}}) = 0 \in \mathbb{C}$.
- Finally, the opposite inclusion $\mathrm{rad}\langle\, ,\, \rangle^{(k)}_{\tilde{\lambda}} \subseteq M^{(k+1)}$ follows from Lemma 2(b) ∎