

HOMEWORK 2 (DUE JANUARY 30)

Disclaimer: Problems with * refer to such that require some knowledge of manifold theory or topology. If you have not taken these courses, view this as an opportunity to learn a bit of each (please come to office hours, or you can find most of these results online or in books). Failure to do these problems will not affect your course grade.

1. Let H be a closed Lie subgroup of a Lie group G .
 - (a) Show that the closure \overline{H} of H in G is a subgroup of G .
 - (b) Show that each coset Hx (for $x \in \overline{H}$) is open and dense in \overline{H} .
 - (c) Deduce that $\overline{H} = H$, thus proving that H is closed in G .
- 2*. Show that if H is a normal closed Lie subgroup of a Lie group G , then G/H is a Lie group (this is the last part of Proposition 2 from Lecture 4).
- 3*. (a) If H is a connected closed Lie subgroup of a Lie group G , then $\pi_0(G) = \pi_0(G/H)$.
 - (b) If both H and G are connected, construct an exact sequence of fundamental groups

$$\pi_1(H) \rightarrow \pi_1(G) \rightarrow \pi_1(G/H) \rightarrow \{1\}.$$
4. (a) For any sequence $\underline{d} = (d_1, \dots, d_k) \in \mathbb{Z}^k$ with $0 < d_1 < d_2 < \dots < d_k = n$, consider the set $\mathcal{F}_{\underline{d}}(\mathbb{R})$ of partial flags, i.e. sequences of subspaces $\{0\} \subset V_1 \subset V_2 \subset \dots \subset V_{k-1} \subset V_k = \mathbb{R}^n$ with $\dim_{\mathbb{R}}(V_i) = d_i$ for all i . Show that the natural action of $GL_n(\mathbb{R})$ on $\mathcal{F}_{\underline{d}}(\mathbb{R})$ is transitive, thus equipping $\mathcal{F}_{\underline{d}}(\mathbb{R})$ with a manifold structure. Identify the fibers of this fiber bundle.
 - (b) Construct a transitive action of $SU(2)$ on $\mathbb{CP}^1 \simeq S^2$ with stabilizers $U(1) \simeq S^1$, thus recovering the **Hopf fibration** $S^1 \hookrightarrow S^3 \rightarrow S^2$.
5. Verify all the properties of \exp and \log maps stated in Lecture 5.
6. (a) Describe explicitly the Lie algebras of $Sp(2n, \mathbb{K})$, $O(p, q)$, $SO(p, q)$, $U(p, q)$, $SU(p, q)$.
 - (b) Compute dimensions of all classical groups.
7. Compute π_0 and π_1 (the fundamental group) of the following classical Lie groups:

$$SU(n), \quad U(n), \quad SO(m, \mathbb{R}), \quad O(m, \mathbb{R}) \quad \text{for all } n \geq 1, m \geq 2$$