HOMEWORK 4 (DUE FEBRUARY 13)

1. (a) Verify that the formulas for the dual, the direct sum, and the tensor product of representations of a Lie algebra $\mathfrak{g}$, as defined in class, indeed produce $\mathfrak{g}$-representations.

(b) For finite-dimensional $\mathfrak{g}$-modules V, W, identifying $V^* \otimes W \simeq \mathrm{Hom}(V, W)$ as vector spaces, endow $\mathrm{Hom}(V, W)$ with a $\mathfrak{g}$-module structure.

(c) For a $\mathfrak{g}$-module (U, ρ_U), the **space of $\mathfrak{g}$-invariants** is defined as

$$U^{\mathfrak{g}} := \{u \in U \mid \rho_U(x)(u) = 0 \ \forall x \in \mathfrak{g}\}.$$

Verify that $\mathrm{Hom}(V, W)^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(V, W)$, where the right-hand side denotes $\mathfrak{g}$-module homomorphisms, while the left-hand side denotes the space of $\mathfrak{g}$-invariants of the module from (b).

(d) For $\mathfrak{g} = Lie(G)$, derive the formulas for the $\mathfrak{g}$-action on $V \oplus W$, V^*, and $V \otimes W$ from the corresponding constructions for G-modules (spell out the latter ones).

2. Recall the definition of the **character** of any finite-dimensional $\mathfrak{sl}_2$-module (V, ρ_V):

$$\chi_V(z) := tr_V(z^h) = \sum_m \dim V(m)\, z^m$$

where $V(m)$ denotes a generalized eigenspace of $\rho_V(h)$ with eigenvalue $m \in \mathbb{C}$.

(a) Show that $V(m)$ coincides with the m-eigenspace of $\rho_V(h)$ and $V(m) = 0$ unless $m \in \mathbb{Z}$.

(b) Prove the character formulas for the dual, the direct sum, and the tensor product:

$$\chi_{V^*}(z) = \chi_V(z^{-1}), \qquad \chi_{V \oplus W}(z) = \chi_V(z) + \chi_W(z), \qquad \chi_{V \otimes W}(z) = \chi_V(z)\chi_W(z).$$

(c) Deduce the Clebsch-Gordan formula by matching the characters of both sides.

3. (a) For a $\mathbb{Z}_{\geq 0}$-filtered associative algebra $\mathcal{A}$ show that if $\mathrm{gr}\,\mathcal{A}$ is a domain, then so is $\mathcal{A}$.

(b) For a linear map $\varphi\colon \mathcal{A} \to \mathcal{B}$ of $\mathbb{Z}_{\geq 0}$-filtered associative algebras satisfying $\varphi(F_k\mathcal{A}) \subset F_k\mathcal{B}$, construct $\mathrm{gr}\,\varphi\colon \mathrm{gr}\,\mathcal{A} \to \mathrm{gr}\,\mathcal{B}$. Show that if $\mathrm{gr}\,\varphi$ is a vector space isomorphism, then so is φ.

(c) Verify that the symmetrization map $\sigma\colon S(\mathfrak{g}) \to U(\mathfrak{g})$ is a $\mathfrak{g}$-module homomorphism. Deduce that it is actually a $\mathfrak{g}$-module isomorphism.

4.(a) Verify that the Casimir element $C := fe + ef + h^2/2 \in U(\mathfrak{sl}_2)$ is central.

(b) Finish the argument in the proof of complete reducibility of finite dimensional $\mathfrak{sl}_2$-modules by showing that if V has a Jordan-Hölder flag $0 = V_0' \subset V_1' \subset \cdots \subset V_s' = V$ of $\mathfrak{sl}_2$-submodules with $V_j'/V_{j-1}' \simeq V_n$ for all $1 \leq j \leq s$ and some $n \geq 0$, then in fact $V \simeq V_n^{\oplus s}$.

5. Working over $\mathbb{C}$, prove that the center $ZU(\mathfrak{sl}_2)$ of $U(\mathfrak{sl}_2)$ is a polynomial algebra in C.

6. For any finite-dimensional complex vector space V, verify that V, all its symmetric powers S^nV, and all exterior powers $\Lambda^m V$ $(m \leq \dim V)$ are irreducible representations of $GL(V)$.