HOMEWORK 5 (DUE FEBRUARY 20)

1. Let V be an n-dimensional vector space over a field $\mathbf{k}$, and $L(V) = \bigoplus_{m\geq 1} L_m(V)$ be the free Lie algebra generated by V with the natural $\mathbb{Z}_{>0}$-grading. Prove that the dimensions of its graded components $d_m(n) := \dim_{\mathbf{k}} L_m(V)$ are uniquely determined by (q–formal variable)

$$\prod_{m\geq 1} (1-q^m)^{d_m(n)} = 1 - nq\,.$$

2. Recall the coproduct map Δ on (appropriate completions of) $T(\mathfrak{g})$ or $U(\mathfrak{g})$.

(a) Show that an element x is primitive if and only if $\exp(x)$ is group-like.

(b) Show that if $\mathrm{char}(\mathbf{k}) = 0$, then any primitive element of $U(\mathfrak{g})$ is an element of $\mathfrak{g} \subset U(\mathfrak{g})$.

Hint: Reduce to $\mathrm{gr}\, U(\mathfrak{g}) \simeq S(\mathfrak{g}) = \oplus_{n\geq 0} S^n(\mathfrak{g})$ *and use* $\mathrm{mult} \circ \Delta = 2^n \mathrm{Id}$ *on each* $S^n(\mathfrak{g})$*.*

(c) Provide a counterexample to (b) when $\mathrm{char}(\mathbf{k}) > 0$.

3. For a $\mathfrak{g}$-module (V, ρ), the space of **coinvaraints** is defined as

$$V_{\mathfrak{g}} = V/\mathfrak{g}V \qquad \text{where} \quad \mathfrak{g}V = \mathrm{span}\big\{\rho(x)v \,|\, x \in \mathfrak{g}, v \in V\big\}\,.$$

(a) Prove that for completely reducible V, the composition $V^{\mathfrak{g}} \to V \to V_{\mathfrak{g}}$ is an isomorphism.

(b) Show that in general, it is not so.

4. (a) Let $\mathfrak{g}$ be a 3-dimensional real Lie algebra with basis x, y, z and commutation relations

$$[x,y] = z\,, \qquad [x,z] = 0\,, \qquad [y,z] = 0\,,$$

called the **Heisenberg algebra**. Construct explicitly the connected, simply-connected Lie group corresponding to $\mathfrak{g}$, and verify without using Baker-Campbell–Hausdorff formula that $\exp(tx)\exp(sy) = \exp(tsz)\exp(sy)exp(tx)$.

(b) Generalize the previous part to the Lie algebra $\mathfrak{g} = V \oplus \mathbb{R}z$, where V is a real vector space with a non-degenerate skew-symmetric form $\omega\colon V \otimes V \to \mathbb{R}$ and the commutation relations

$$[u,v] = \omega(u,v)z\,, \qquad [z,v] = 0 \qquad \forall\; u,v \in V\,.$$

5. Given a Lie algebra $\mathfrak{h}$ and another Lie algebra $\mathfrak{g}$ acting on $\mathfrak{h}$ by derivations, one defines the **semidirect product** Lie algebra $\mathfrak{g} \ltimes \mathfrak{h}$ which is $\mathfrak{g} \oplus \mathfrak{h}$ as a vector space with the commutator

$$[(x_1,h_1),(x_2,h_2)] = \big([x_1,x_2], [h_1,h_2] + x_1(h_2) - x_2(h_1)\big)\,.$$

Verify this indeed produces a Lie algebra.

6. For $n_1, \ldots, n_k \in \mathbb{Z}_{>0}$, set $n := n_1 + \ldots + n_k$, and consider the **parabolic subalgebra**

$$\mathfrak{g} = \big\{A \in \mathfrak{gl}(n) \,|\, A_{ij} = 0 \text{ for } j \leq n_1 + \ldots + n_s < i \text{ and any } 1 \leq s < k\big\}$$

consisting of block upper triangular matrices with diagonal blocks of size $n_1 \times n_1, \ldots, n_k \times n_k$.

(a) Find the radical $\mathrm{rad}(\mathfrak{g})$.

(b) Provide an example of the Levi decomposition for this $\mathfrak{g}$.