HOMEWORK 6 (DUE FEBRUARY 27)

1. (a) Given a finite-dimensional $\mathfrak{g}$-module (V, ρ) over k, verify that the following defines an action of $\mathfrak{g}$ on the space Bil_V of all k-bilinear maps $V \times V \to \mathsf{k}$:

$$(xB)(v,w) = -B(\rho(x)v, w) - B(v, \rho(x)w) \qquad \forall\, x \in \mathfrak{g}\,,\ v, w \in V$$

(b) Identifying Bil_V with $\mathrm{Hom}_{\mathsf{k}}(V, V^*)$ via $B \mapsto \varphi_B$ where

$$\varphi_B \colon V \to V^* \quad \text{is given by} \quad v \mapsto B(v, -)$$

Verify that $B \in \mathrm{Bil}_V^{\mathfrak{g}}$ iff the corresponding linear map φ_B is a $\mathfrak{g}$-module homomorphism.

(c) If V is an irreducible $\mathfrak{g}$-module, show that $\dim(\mathrm{Bil}_V^{\mathfrak{g}}) \leq 1$.

(d) Let $(V, \rho) = (\mathfrak{g}, \mathrm{ad})$ and B be a $\mathfrak{g}$-invariant form on $\mathfrak{g}$. Show that for any ideal $\mathfrak{a} \subset \mathfrak{g}$ its orthogonal complement $\mathfrak{a}^{\perp} = \{x \in \mathfrak{g} \,|\, B(x,y) = 0\ \forall\, y \in \mathfrak{a}\}$ is also an ideal.

(e) For $\mathfrak{g} = \mathfrak{sl}_2$ and any $n \geq 0$, show that the irreducible $\mathfrak{g}$-module V_n has an invariant non-degenerate bilinear form which is symmetric for even n and skew-symmetric for odd n.

2. (a) For any filtration $0 = F_0V \subset F_1V \subset \cdots \subset F_NV = V$ by $\mathfrak{g}$-submodules verify that

$$B_V = \sum_{1 \leq k \leq N} B_{F_kV/F_{k-1}V} \colon \mathfrak{g} \times \mathfrak{g} \to \mathsf{k}$$

(b) For any ideal $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$, verify the equality $K^{\mathfrak{a}}(x,y) = K^{\mathfrak{g}}(x,y)$ for any $x, y \in \mathfrak{a}$, where $K^{\mathfrak{a}}$ and $K^{\mathfrak{g}}$ denote the Killing forms of $\mathfrak{a}$ and $\mathfrak{g}$, respectively.

3. (a) Show that $\mathfrak{sp}_{2n}(\mathsf{k}), \mathfrak{so}(p,q), \mathfrak{u}_n, \mathfrak{su}_n$ are reductive ($\mathsf{k} = \mathbb{R}$ or $\mathbb{C}$).

(b) Verify that $\mathfrak{sl}_n(\mathsf{k}), \mathfrak{so}_n(\mathsf{k}), \mathfrak{sp}_{2n}(\mathsf{k}), \mathfrak{su}_n$ are semisimple ($\mathsf{k} = \mathbb{R}$ or $\mathbb{C}$).

(c) Verify that $\mathfrak{gl}_n(\mathsf{k}) = \mathsf{k} \cdot \mathrm{Id} \oplus \mathfrak{sl}_n(\mathsf{k})$ and $\mathfrak{u}_n = i\mathbb{R} \cdot \mathrm{Id} \oplus \mathfrak{su}_n$ ($\mathsf{k} = \mathbb{R}$ or $\mathbb{C}$).

4. Let V be a finite-dimensional complex vector space.

(a) Verify that a linear operator $A \colon V \to V$ is semisimple iff A is diagonalizable.

(b) Show that if $A \colon V \to V$ is semisimple and a subspace $V' \subset V$ is A-stable, i.e. $A(V') \subset V'$, then the corresponding linear operators $V' \to V'$ and $V/V' \to V/V'$ are also semisimple.

(c) Show that the sum of two commuting semisimple operators $V \to V$ is also semisimple.

(d) Show that the sum of two commuting nilpotent operators $V \to V$ is also nilpotent.

(e) Prove part (c) of the Jordan decomposition.

5. Let $\mathfrak{g}$ be a real Lie algebra with a positive definite Killing form. Show that then $\mathfrak{g} = 0$.

6. Verify that a short exact sequence of $\mathfrak{g}$-modules

$$0 \longrightarrow V \longrightarrow U \longrightarrow W \longrightarrow 0$$

gives rise to an exact sequence $H^1(\mathfrak{g}, V) \to H^1(\mathfrak{g}, U) \to H^1(\mathfrak{g}, W)$ of vector spaces.