

HOMEWORK 7 (DUE MARCH 6)

1. (a) Classify all 2-dimensional Lie algebras over any field \mathbf{k} .
- (b) Classify all finite-dimensional Lie algebras \mathfrak{g} for which $\dim Z(\mathfrak{g}) = \dim \mathfrak{g} - 2$.
Hint: If $\dim \mathfrak{g} = n$ show that either $\mathfrak{g} \simeq \text{Ab}_{n-3} \oplus \text{Heis}_3$ or $\mathfrak{g} \simeq \text{Ab}_{n-2} \oplus \mathfrak{a}$, where Ab_m denotes the abelian Lie algebra of dimension m , \mathfrak{a} is the non-abelian 2-dimensional Lie algebra from part (a), and Heis_3 is the Heisenberg algebra from [Homework 5, Problem 4(a)].
- (c) Show that any non-abelian 3-dimensional nilpotent Lie algebra is isomorphic to the Heisenberg algebra Heis_3 .
2. (a) Show that all derivations of the 2-dimensional non-abelian Lie algebra are inner.
- (b) Find all derivations of the Heisenberg algebra Heis_3 . Show that not all of them are inner.
3. Let \mathfrak{g} be a semisimple Lie algebra with a non-degenerate invariant symmetric bilinear form (\cdot, \cdot) , and let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . For any root α of \mathfrak{g} , verify that the element

$$h_\alpha = \frac{2}{(\alpha, \alpha)} H_\alpha \in \mathfrak{h}$$

is independent of the pairing.

4. Let \mathfrak{g} be a semisimple Lie algebra with a non-degenerate invariant symmetric bilinear form (\cdot, \cdot) , and \mathfrak{h} be a Cartan subalgebra of \mathfrak{g} . Let $R \subset \mathfrak{h}^*$ be the root system of \mathfrak{g} . Prove the following **string property** for any $\alpha, \beta \in R$:

$$\{\beta + n\alpha \mid n \in \mathbb{Z}\} \cap (R \cup \{0\}) = \{\beta - p\alpha, \beta - (p-1)\alpha, \dots, \beta + (q-1)\alpha, \beta + q\alpha\}$$

for some $p, q \in \mathbb{Z}_{\geq 0}$ satisfying $p - q = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$.

5. Let \mathfrak{g} be a complex finite-dimensional Lie algebra which has a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

where R is a finite subset of $\mathfrak{h}^* \setminus \{0\}$, \mathfrak{h} is an abelian Lie subalgebra, and we have

$$[h, x] = \alpha(h)x \quad \text{for any } h \in \mathfrak{h}, x \in \mathfrak{g}_\alpha, \alpha \in R.$$

Assume further that for any $\alpha \in R \cup \{0\}$, the Killing form of \mathfrak{g} gives rise to a

$$\text{non-degenerate pairing } \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \longrightarrow \mathbb{C}.$$

Show that \mathfrak{g} is semisimple and \mathfrak{h} is its Cartan subalgebra.

6. *Explicit description of root systems and root subspaces in classical types.*

(a) Let $\mathfrak{h} \subset \mathfrak{g}$ be a subspace of diagonal matrices in $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Verify that \mathfrak{h} is a Cartan subalgebra. Describe explicitly the root system $R \subset \mathfrak{h}^*$ and the corresponding root subspaces.

(b) Let $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ be the symplectic Lie algebra realized as

$$\mathfrak{g} = \left\{ A \in \text{Mat}_{2n \times 2n}(\mathbb{C}) \mid AJ + JA^t = 0 \right\} \quad \text{with} \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where $I_n \in \text{Mat}_{n \times n}(\mathbb{C})$ is the identity matrix. Let $\mathfrak{h} \subset \mathfrak{g}$ be a subspace of diagonal matrices. Verify that \mathfrak{h} is a Cartan subalgebra. Describe explicitly the root system $R \subset \mathfrak{h}^*$ and the corresponding root subspaces of \mathfrak{g} .

(c) Let $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$ be the orthogonal Lie algebra realized as

$$\mathfrak{g} = \left\{ A \in \text{Mat}_{2n \times 2n}(\mathbb{C}) \mid AJ + JA^t = 0 \right\} \quad \text{with} \quad J = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subspace of diagonal matrices. Verify that \mathfrak{h} is a Cartan subalgebra. Describe explicitly the root system $R \subset \mathfrak{h}^*$ and the corresponding root subspaces of \mathfrak{g} .

(d) Let $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$ be the orthogonal Lie algebra realized as

$$\mathfrak{g} = \left\{ A \in \text{Mat}_{(2n+1) \times (2n+1)}(\mathbb{C}) \mid AJ + JA^t = 0 \right\} \quad \text{with} \quad J = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}.$$

Let $\mathfrak{h} \subset \mathfrak{g}$ be a subspace of diagonal matrices. Verify that \mathfrak{h} is a Cartan subalgebra. Describe explicitly the root system $R \subset \mathfrak{h}^*$ and the corresponding root subspaces of \mathfrak{g} .