## HOMEWORK 8 (DUE MARCH 13)

1. (a) Describe all strongly regular elements $x \in \mathfrak{gl}_n(\mathbb{C})$.

(b) Describe all regular elements $x \in \mathfrak{gl}_n(\mathbb{C})$.

2. Let $V$ be a 4-dimensional complex vector space with a basis $e_1, e_2, e_3, e_4$.

(a) Define a bilinear form $B$ on $W = \Lambda^2 V$ via $w_1 \wedge w_2 = B(w_1, w_2) e_1 \wedge e_2 \wedge e_3 \wedge e_4$. Verify that $B$ is a symmetric non-degenerate form on $W$ and construct an orthonormal basis for $B$.

(b) Let $\mathfrak{g} = \{x \in \mathfrak{gl}(W) \,|\, B(xw_1, w_2) + B(w_1, xw_2) = 0 \ \forall\, w_1, w_2 \in W\}$. Show that $\mathfrak{g} \simeq \mathfrak{so}_6(\mathbb{C})$.

(c) Show that the form $B$ is invariant under the natural action of $\mathfrak{sl}_4(\mathbb{C})$ on $W$.

(d) Using the above results, construct a Lie algebra isomorphism $\mathfrak{sl}_4(\mathbb{C}) \xrightarrow{\sim} \mathfrak{so}_6(\mathbb{C})$.

3. Let $\{e_i\}_{i=1}^n$ be the standard orthonormal basis of $\mathbb{R}^n$. Verify that

$$R := \{\pm e_i, \pm 2e_i \,|\, 1 \le i \le n\} \cup \{\pm e_i \pm e_j \,|\, 1 \le i < j \le n\}$$

is a non-reduced root system in $\mathbb{R}^n$ (called **$\mathbf{BC}_n$-type** root system).

4. (a) Let $R \subset E$ be a root system. Verify that the set

$$R^\vee := \big\{\alpha^\vee \,|\, \alpha \in R\big\} \subset E^*$$

is a root system (the **dual root system** of $R \subset E$) with the **coroot** $\alpha^\vee \in E^*$ defined via

$$\alpha^\vee(x) = \frac{2(\alpha, x)}{(\alpha, \alpha)} \qquad \forall\, x \in E\,.$$

(b) Verify that the root systems $(R^\vee)^\vee \subset (E^*)^*$ and $R \subset E$ are isomorphic.

(c) Let $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset R$ be the set of simple roots of a reduced root system $R$ with respect to a polarization given by $t \in E^*$. Verify that $\Pi^\vee := \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ is the set of simple roots of $R^\vee$ with respect to the polarization given by $t^\vee = \pi^{-1}(t)$, where $\pi\colon E \to E^*$ is the isomorphism induced by the inner product.

5. (a) Given a collection of vectors $\{v_i\}_{i=1}^n$ in a Euclidean space $E$ which satisfy:

- $(v_i, v_j) \le 0$ for any $i \ne j$
- $t(v_i) > 0$ for some $t \in E^*$ and all $1 \le i \le n$

prove that the vectors $\{v_i\}_{i=1}^n$ are linearly independent.

*Hint: assuming we have a nontrivial linear combination of $v_i$'s equal to 0, rearrange the terms so that $\sum_{i \in A} c_i v_i = \sum_{i \in B} c_i v_i$ for some disjoint subsets $A, B \subset \{1, \dots, n\}$ with $c_i > 0$. Show that $A, B \ne \emptyset$. Then compute the inner product of the left-hand and right-hand sides.*

(b) Use (a) to show that the set of simple roots $\Pi$ in a root system $R \subset E$ is a basis of $E$.