HOMEWORK 9 (DUE MARCH 27)

1. Let $\mathfrak{h}$ be a Cartan subalgebra in a complex semisimple Lie algebra $\mathfrak{g}$. This naturally gives rise to a reduced root system $R \subset \mathfrak{h}^*_{\mathbb{R}}$ (as discussed in class).
(a) For $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ one gets the so-called $\mathbf{C}_n$**-type** root system [Homework 7, Problem 6(b)]

$$R_{C_n} = \{\pm 2e_i \,|\, 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \,|\, 1 \leq i < j \leq n\} \subset \mathbb{R}^n\,.$$

Describe the simple roots $\Pi \subset R_{C_n}$ for the polarization given by $t \in (\mathbb{R}^n)^*$ such that

$$t(e_1) \gg t(e_2) \gg \cdots \gg t(e_n) > 0\,.$$

(b) For $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$ one gets the so-called $\mathbf{D}_n$**-type** root system [Homework 7, Problem 6(c)]

$$R_{D_n} = \{\pm e_i \pm e_j \,|\, 1 \leq i < j \leq n\} \subset \mathbb{R}^n\,.$$

Describe the simple roots $\Pi \subset R_{D_n}$ for the polarization given by $t \in (\mathbb{R}^n)^*$ as in (a).
(c) For $\mathfrak{g} = \mathfrak{so}_{2n+1}(\mathbb{C})$ one gets the $\mathbf{B}_n$**-type** root system [Homework 7, Problem 6(d)]

$$R_{B_n} = \{\pm e_i \,|\, 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j \,|\, 1 \leq i < j \leq n\} \subset \mathbb{R}^n\,.$$

Describe the simple roots $\Pi \subset R_{B_n}$ for the polarization given by $t \in (\mathbb{R}^n)^*$ as in (a).

2. (a) The F_4 root system $R_{F_4} \subset \mathbb{R}^4$ is defined as the union of the B_4 root system R_{B_4} and

$$\pm\tfrac{1}{2}e_1 \pm \tfrac{1}{2}e_2 \pm \tfrac{1}{2}e_3 \pm \tfrac{1}{2}e_4 \in \mathbb{R}^4$$

for all 16 choices of signs. Verify that it is a reduced root system. Describe the simple roots $\Pi \subset R_{F_4}$ for the polarization given by $t \in (\mathbb{R}^4)^*$ with $t(e_1) \gg t(e_2) \gg t(e_3) \gg t(e_4) > 0$.
(b) The E_8 root system $R_{E_8} \subset \mathbb{R}^8$ is the union of the root system R_{D_8} and

$$\tfrac{1}{2}\big(\underbrace{\pm e_1 \pm e_2 \pm \cdots \pm e_8}_{\text{even number of } - \text{ signs}}\big) \in \mathbb{R}^8$$

(there are 128 of such vectors). Verify that it is a reduced root system. Describe the simple roots $\Pi \subset R_{E_8}$ for the polarization given by $t \in (\mathbb{R}^8)^*$ with $t(e_1) \gg \cdots \gg t(e_8) > 0$.
(c) The E_7 root system R_{E_7} is defined as the intersection of R_{E_8} with the hyperplane $\{(x_1, \ldots, x_8) \in \mathbb{R}^8 \,|\, x_1 + x_2 = 0\}$ in $\mathbb{R}^8$. Verify that it is a reduced root system. Describe the simple roots $\Pi \subset R_{E_7}$ w.r.t. the polarization induced from (b).

Hint: The simple roots for R_{E_7} form a subset of simple roots for R_{E_8}.

(d) The E_6 root system R_{E_6} is defined as the intersection of R_{E_8} with the codimension two subspace $\{(x_1, \ldots, x_8) \in \mathbb{R}^8 \,|\, x_1 + x_2 = 0, x_2 - x_3 = 0\}$ in $\mathbb{R}^8$. Verify that it is a reduced root system. Describe the simple roots $\Pi \subset R_{E_6}$ w.r.t. the polarization induced from (b).

Hint: The simple roots for R_{E_6} form a subset of simple roots for R_{E_7}.

As you will learn after the break, any irreducible reduced root system is isomorphic to one of:

$$A_r\,(r \geq 1)\,,\quad B_r\,(r \geq 2)\,,\quad C_r\,(r \geq 3)\,,\quad D_r\,(r \geq 4)\,,\quad E_6\,,\quad E_7\,,\quad E_8\,,\quad F_4\,,\quad G_2\,.$$

This will also yield a complete classification of complex finite dimensional simple Lie algebras.

3. (a) Show that the assignment of a positive Weyl chamber to any polarization of a root system R provides a bijection between all polarizations $R = R_+ \cup R_-$ and all Weyl chambers.

(b) Show that if two Weyl chambers C, C' are separated by a hyperplane L_α, then $s_\alpha(C) = C'$.

4. Let $Q \subset P$ be the root and weight lattices of an abstract rank r root system $R \subset E$. Pick a polarization $R = R_+ \cup R_-$ and let $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset R_+$ denote the set of simple roots.

(a) Show that the index $|P/Q|$ equals the determinant of the Cartan matrix $A = \big(\alpha_i^\vee(\alpha_j)\big)_{i,j=1}^r$.

(b) Compute the index $|P/Q|$ for the root systems of types A_r, B_r, C_r, D_r.

(c) Describe the quotient group P/Q explicitly for the root systems of types A_r, B_r, C_r, D_r.

5. (a) Describe explicitly the Weyl groups W for the root systems of types B_r, C_r, D_r.

(b) Describe explicitly the longest element $w_0 \in W$ for the root systems from (a).

6. (a) Let $w = s_{i_1} \cdots s_{i_\ell}$ be a reduced expression of $w \in W$. Show that then

$$\{\alpha \in R_+ \,|\, w(\alpha) \in R_-\} = \{\beta_1, \ldots, \beta_\ell\} \qquad \text{with} \qquad \beta_k = s_{i_\ell} \cdots s_{i_{k+1}}(\alpha_{i_k})$$

and

$$\{\alpha \in R_+ \,|\, w^{-1}(\alpha) \in R_-\} = \{\widetilde{\beta}_1, \ldots, \widetilde{\beta}_\ell\} \qquad \text{with} \qquad \widetilde{\beta}_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})\,.$$

(b) Suppose $s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \in R_-$. Show that $s_{i_1} \cdots s_{i_{k-1}} s_{i_k}$ is not a reduced expression.

7. Given positive roots $\alpha, \beta, \alpha', \beta' \in R_+$ such that $\alpha + \beta = \alpha' + \beta'$, verify that then

$$\alpha' = \alpha + \gamma \;\text{ and }\; \beta' = \beta - \gamma \qquad \text{or} \qquad \alpha' = \beta + \gamma \;\text{ and }\; \beta' = \alpha - \gamma$$

for some $\gamma \in R \cup \{0\}$.

8. Let $w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}$ (with $N = |R_+|$) be a reduced decomposition of the longest element $w_0 \in W$. By Problem 6 above, we have $R_+ = \{\widetilde{\beta}_1, \widetilde{\beta}_2, \ldots, \widetilde{\beta}_N\}$ with $\widetilde{\beta}_k = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. This gives a natural order on R_+ via: $\widetilde{\beta}_i < \widetilde{\beta}_j$ iff $i < j$. Show that this order is **convex**:

$$\forall\, \alpha < \beta \in R_+ \quad \text{such that} \quad \alpha + \beta \in R_+ \quad \text{we have} \quad \alpha < \alpha + \beta < \beta\,.$$

In fact, any convex order on R_+ arises via a unique reduced decomposition of w_0 in this way:

$$\big\{\text{reduced decompositions of } w_0 \in W\big\} \overset{\text{bijection}}{\longleftrightarrow} \big\{\text{convex orders on } R_+\big\}\,.$$