HOMEWORK 10 (DUE APRIL 3)

1. Given any irreducible root system $R \subset E$, verify that E is an irreducible representation of the corresponding Weyl group W.

2. (a) Verify that if two vertices of a Dynkin diagram are connected by a single edge, then the corresponding simple roots are in the same W-orbit.
(b) Show that for a reduced irreducible root system R, the Weyl group W acts transitively on the set of all roots of the same length.

3. (a) Verify that the *classical* reduced root systems of types A_n, B_n, C_n, D_n (from Problem 1 of Homework 9) indeed have the same named Dynkin diagrams.
(b) Verify that the *exceptional* reduced root systems of types E_6, E_7, E_8, F_4, G_2 (see Problem 2 of Homework 9) indeed have the same named Dynkin diagrams.

4. Complete the proof of the Main Theorem from Lecture 27 by verifying that none of the following graphs can appear as a subgraph of a Dynkin diagram (see pictures in the notes):

- analogue of $\widetilde{D}_n$ with some multiple edges
- $\widetilde{G}_2$, $D_4^{(3)}$ as well as their analogues with a multiple edge instead of the simple one
- $\widetilde{B}_n$, $A_{2n-1}^{(2)}$ as well as their analogues with one/two of their two "legs" being multiple
- $\widetilde{E}_6$, $\widetilde{E}_7$, $\widetilde{E}_8$
- $\widetilde{C}_n, D_{n+1}^{(2)}, A_{2n}^{(2)}$
- $\widetilde{F}_4, E_6^{(2)}$

Hint: Verify that the corresponding symmetrized Cartan matrices are not positive definite.

5. Let G be a connected $\mathbb{C}$ Lie group such that $\mathfrak{g} = \mathrm{Lie}(G)$ is semisimple. Fix a root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$. For any $\alpha \in R$, consider the embedding $\iota_\alpha \colon \mathfrak{sl}(2, \mathbb{C}) \hookrightarrow \mathfrak{g}$ [Lecture 20, Lemma 2], and lift it to $\iota_\alpha \colon SL(2, \mathbb{C}) \to G$. Define

$$S_\alpha := \iota_\alpha(\exp(f_\alpha)\exp(-e_\alpha)\exp(f_\alpha)) \in G.$$

Show that the dual of the adjoint action of S_α on $\mathfrak{g}^*$ preserves $\mathfrak{h}^*$, and $\mathrm{Ad}^*(S_\alpha)|_{\mathfrak{h}^*}$ coincides with the reflection $s_\alpha \colon \mathfrak{h}^* \to \mathfrak{h}^*$. Deduce that the Weyl group W acts on $\mathfrak{h}^*$ by inner automorphisms, i.e. for any $w \in W$ there is a (non-unique!) element $\tilde{w} \in G$ such that $\mathrm{Ad}^*(\tilde{w})|_{\mathfrak{h}^*} = w$.

6. Given a reduced root system $R \subset E$ with a polarization $R = R_+ \cup R_-$ and the set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset R_+$, recall the element $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha = \sum_{i=1}^r \varpi_i \in E$, and let $\rho^\vee \in E^*$ be such an element for the dual root system $R^\vee \subset E^*$.
(a) Compute $\rho, \rho^\vee$ for the *classical* root systems of types A_n, B_n, C_n, D_n.
(b) Compute $\rho, \rho^\vee$ for the *exceptional* root systems of types E_6, E_7, E_8, F_4, G_2.