HOMEWORK 11 (DUE APRIL 10)

1. This problem is aimed at completing the details in the proof of [Lecture 29, Theorem 2]. Let R be a reduced root system with a Cartan matrix $A = (a_{ij})_{i,j=1}^r$.

(a) Let $\widetilde{\mathfrak{g}}(R)$ be a Lie algebra generated by $\{e_i, h_i, f_i\}_{i=1}^r$ subject to Chevalley relations:

$$[h_i, h_j] = 0\,, \quad [h_i, e_j] = a_{ij} e_j\,, \quad [h_i, f_j] = -a_{ij} f_j\,, \quad [e_i, f_j] = \delta_{ij} h_i \qquad \forall\, i, j\,.$$

Let $\widetilde{\mathfrak{n}}_+(R)$ be the subalgebra generated by $\{e_i\}_{i=1}^r$, $\widetilde{\mathfrak{n}}_-(R)$ be the subalgebra generated by $\{f_i\}_{i=1}^r$, $\widetilde{\mathfrak{h}}_+(R)$ be the subalgebra generated by $\{h_i\}_{i=1}^r$. Verify that as vector spaces:

$$\widetilde{\mathfrak{g}}(R) = \widetilde{\mathfrak{n}}_-(R) \oplus \widetilde{\mathfrak{h}}(R) \oplus \widetilde{\mathfrak{n}}_+(R)\,.$$

(b) Prove that $\widetilde{\mathfrak{n}}_+(R)$ is a free Lie algebra in $\{e_i\}_{i=1}^r$, $\widetilde{\mathfrak{n}}_-(R)$ is a free Lie algebra in $\{f_i\}_{i=1}^r$, and $\widetilde{\mathfrak{h}}_+(R)$ has a basis $\{h_i\}_{i=1}^r$.

Hint: Consider a Lie algebra $\mathfrak{a} = \bar{\mathfrak{h}} \ltimes L_r$, *where* $\bar{\mathfrak{h}}$ *is an abelian Lie algebra with a basis* $\{\bar{h}_i\}_{i=1}^r$, L_r *is a free Lie algebra generated by* $\{\bar{f}_i\}_{i=1}^r$, *and the semidirect product is with respect to* $[\bar{h}_i, \bar{f}_j] = -a_{ij}\bar{f}_j$ *([Homework 5, Problem 5]). Consider the universal enveloping algebra*

$$U(\mathfrak{a}) = \mathbb{C}[\bar{h}_1, \ldots, \bar{h}_r] \ltimes \mathbb{C}\langle \bar{f}_1, \ldots, \bar{f}_r\rangle\,.$$

Construct the action of $\widetilde{\mathfrak{g}}(R)$ *on* $U(\mathfrak{a})$ *such that* $h_i(H \otimes 1) = (\bar{h}_i H) \otimes 1$, $f_i(H \otimes F) = H \otimes (\bar{f}_i F)$ *for any* $H \in \mathbb{C}[\bar{h}_1, \ldots, \bar{h}_r]$ *and* $F \in \mathbb{C}\langle \bar{f}_1, \ldots, \bar{f}_r\rangle$. *Consider the linear map*

$$\varphi\colon \widetilde{\mathfrak{g}}(R) \to U \quad \text{given by} \quad x \mapsto x(1)\,.$$

Deduce that $\{h_i\}_{i=1}^r \subset \widetilde{\mathfrak{g}}(R)$ *are linearly independent (since so are* $\varphi(h_i) \in U(\mathfrak{a})$*), hence form a basis of* $\widetilde{\mathfrak{h}}(R)$. *As the restriction of* φ *to* $\widetilde{\mathfrak{n}}_-(R)$ *sends any commutator of* f_i*'s to the corresponding commutator of* $\bar{f}_i$*'s, deduce that* $\widetilde{\mathfrak{n}}_-(R)$ *is indeed a free Lie algebra in* $\{f_i\}_{i=1}^r$.

2. Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Show that any submodule and quotient of a $\mathfrak{g}$-module that admits a weight decomposition also admit such decompositions.

3. Let $\mathfrak{g}$ be a semisimple Lie algebra with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Prove that any nonzero $\mathfrak{g}$-module homomorphism $\phi\colon M_\lambda \to M_\mu$ of Verma modules is injective.

Hint: Use the fact that $U(\mathfrak{n}_-)$ *is a domain.*

4. This problem revises [Homework 4, Problem 6] in the context of Lectures 30–31.

(a) For any $k \geq 0$, consider the representation $V = S^k(\mathbb{C}^n)$ of $\mathfrak{sl}_n(\mathbb{C})$. Compute all weights of V and describe the corresponding weight subspaces. Prove that V is irreducible by showing that the space of highest weight vectors is one-dimensional (describe it explicitly).

(b) For any $1 \leq k \leq n$, consider the representation $V = \Lambda^k(\mathbb{C}^n)$ of $\mathfrak{sl}_n(\mathbb{C})$. Compute all weights of V and describe the corresponding weight subspaces. Prove that V is irreducible by showing that the space of highest weight vectors is one-dimensional (describe it explicitly).

5. Let $R \subset E$ be a reduced irreducible simply laced root system. Up to rescaling R, we shall assume that $(\alpha, \alpha) = 2$ for any root $\alpha \in R$. Let $Q \subset E$ be the root lattice of R.

(a) Verify that $R = \{\alpha \in Q \,|\, (\alpha, \alpha) = 2\}$.

(b) Consider a vector space V with a basis $\{u_i\}_{i \in I} \cup \{v_\alpha\}_{\alpha \in R}$, where $\{\alpha_i\}_{i \in I}$ are simple roots of R with respect to some polarization. Verify that linear operators $e_i, f_i, h_i \colon V \to V$ given by

$$h_i(u_j) = 0\,, \qquad h_i(v_\alpha) = (\alpha, \alpha_i) v_\alpha\,,$$

$$e_i(u_j) = |(\alpha_i, \alpha_j)| v_{\alpha_i}\,, \qquad e_i(v_\alpha) = \begin{cases} v_{\alpha+\alpha_i} & \text{if } \alpha + \alpha_i \in R \\ u_i & \text{if } \alpha = -\alpha_i \\ 0 & \text{if } \alpha + \alpha_i \notin R \cup 0 \end{cases}\,,$$

$$f_i(u_j) = |(\alpha_i, \alpha_j)| v_{-\alpha_i}\,, \qquad f_i(v_\alpha) = \begin{cases} v_{\alpha-\alpha_i} & \text{if } \alpha - \alpha_i \in R \\ u_i & \text{if } \alpha = \alpha_i \\ 0 & \text{if } \alpha - \alpha_i \notin R \cup 0 \end{cases}$$

give rise to an action $\mathfrak{g} \curvearrowright V$. Identify this $\mathfrak{g}$-module with the adjoint representation of $\mathfrak{g}$.

This reconstructs the simply laced simple Lie algebras from their Cartan matrices.