Lecture #3

- Discuss the notions of:
  - universal coverings (+ there classical construction)
  - fundamental group $\pi_1(X,x)$

from the end of page 5 of Lecture #2.

- After this general detour to the theory of coverings, let us return to realm of Lie groups. Let G be a connected (real/complex) Lie group and $\tilde{G}$ be the universal covering of G, so that $\tilde{G} = \{\text{homotopy classes of paths } \gamma:[0,1]\to G \mid \gamma(0)=1\}$ (unit of G)
By above: $\tilde{G}$ is a manifold. But it is also a group w.r.t. the following product:

$$\boxed{(\gamma_1\cdot\gamma_2)(t) := \underbrace{\gamma_1(t)\cdot\gamma_2(t)}_{\text{product in } G} \quad \text{for } t\in[0,1]}$$

Lemma 1: $\tilde{G}$ is a simply-connected Lie group, and the covering $\pi:\tilde{G}\to G$ is a homomorphism of Lie groups.

► The only non-trivial part of this result is that $\tilde{G}$ – Lie gp, i.e. $\tilde{m}:\tilde{G}\times\tilde{G}\to\tilde{G}$ – regular. But to this end, we note the commutative diagram

$$\begin{array}{ccc} \tilde{G}\times\tilde{G} & \xrightarrow{\tilde{m}} & \tilde{G} \\ \downarrow{\scriptstyle \pi\times\pi} & & \downarrow{\scriptstyle \pi} \\ G\times G & \xrightarrow{m} & G \end{array}$$

Thus, $\tilde{m}$ is a lifting of the map $m\circ(\pi\times\pi): \underbrace{\tilde{G}\times\tilde{G}}_{\text{simply-connected}} \to G$. Hence, by part (3) of the Homotopy Lifting Property (from the end of Lecture 2), $\tilde{m}$ is regular! ■

Basic Example: The universal cover of the Lie group $(S^1,\cdot)$ is $(\mathbb{R},+)$.

Moreover, the universal covering $\tilde{G}$ of G is just a central extension of G, due to

Lemma 2: $\underbrace{\mathrm{Ker}(\pi:\tilde{G}\to G)}_{\simeq \pi_1(G,1) \text{ by general property of universal coverings.}}$ is a central subgroup of $\tilde{G}$

Exercise: a) Show that every discrete normal subgroup of a connected Lie gp is central.
b) Deduce the proof of Lemma 2 from part a).

Upshot: Last time we explained how the case of any Lie group can be reduced to connected ones, while by above we can further reduce to simply-connected ones.