

# Lecture #4

To prove Prop 1 below, we start with the following definition:

Def 1: A closed Lie subgroup of a (real/complex) Lie group  $G$  is a subgroup  $H$  which is also an embedded submanifold.

recall: that means we have an immersion  $X \xrightarrow{f} G$  with  $\text{image}(f) = \text{our subgroup } H$  such that  $f: X \rightarrow H$  is a homeomorphism.

Note: In the above definition, we do not require  $H$  to be closed. However, the above terminology is justified by the following result:

Lemma 1: A closed Lie subgroup  $H$  of  $G$  is closed in  $G$ .

Exercise: a) Show that the closure  $\bar{H}$  of  $H$  in  $G$  is a subgroup of  $G$ .  
 b) Show that each coset  $Hx$  for  $x \in \bar{H}$  is open and dense in  $\bar{H}$ .  
 c) Deduce that  $\bar{H} = H$ , i.e.  $H$  is closed in  $G$ .

The reverse is also true, but is much harder (so we'll just state it):

Theorem: Any closed subgroup of a real Lie group  $G$  is a closed Lie subgroup.

As an important application of Lemma 1, we have ("local-to-global"):

Proposition 1: a) If  $G$  is a connected Lie group and  $U$  is a nbhd of  $1 \in G$ , then  $U$  generates the entire  $G$ .  
 b) If  $f: G \rightarrow H$  is a homomorphism of Lie groups,  $H$  is connected, and  $df: T_1 G \rightarrow T_1 H$  is surjective, then  $f$  is surjective.

a) Let  $K = \langle U \rangle$  denote the subgp of  $G$  generated by  $U$ . Then  $\forall k \in K, k \cdot U \subseteq K$ , hence  $K$  is open in  $G$ . Therefore,  $K$  is an embedded submanifold of  $G$ , hence  $K$  is a closed Lie subgp of  $G$  (see Def 1). But then  $K$  is also closed by Lemma. As  $G$  is connected and  $\emptyset \neq K$  - open & closed  $\Rightarrow K = G \Rightarrow U$  generates  $G$ .  
 b) Surjectivity of  $df: T_1 G \rightarrow T_1 H$  implies that  $f(G)$  contains a nbhd of  $1 \in H$  (here, we actually use the implicit function theorem). Thus, by part a), and the fact that  $f(G)$  - subgroup, we get  $f$  - surjective.

One may ask how to obtain surjections of Lie gps from Prop 1(b). To this end, recall that for discrete subgps  $H \subseteq G$  one has the notion of coset space  $G/H$ , which is equipped with a gp structure if  $H$  is normal. Let's now see how this can be adapted to the realm of Lie groups.

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Def 2: A regular map of manifolds  $\pi: E \rightarrow B$  is called a fiber bundle with base  $B$ , total space  $E$ , and fiber being a manifold  $F$  if  $\forall x \in B$  has a neighborhood  $U$  s.t.  $\pi^{-1}(U) \simeq U \times F$  with  $\pi$  being projection  $U \times F \rightarrow U$ .

Note that for  $\dim(F)=0$ , this recovers the notion of covering (Lecture 2).

Here is the key result we care about in the context of Lie groups:

Proposition 2: a) Let  $G$  be a (real/complex) Lie group of dimension  $n$  and  $H \subseteq G$  a closed Lie subgroup of dimension  $k$ . Then the coset space  $G/H$  (called a homogeneous space of  $G$ ) has a natural structure of a manifold of dimension  $n-k$ , and the map  $\pi: G \rightarrow G/H$  is a fiber bundle with fiber  $H$ .

b) The tangent space at  $\bar{x} = \pi(1)$  is given by  $T_{\bar{x}}(G/H) = T_1 G / T_1 H$ .

c) If moreover  $H$  is normal in  $G$ , then  $G/H$  is a (real/complex) Lie group.

► Pick  $\bar{g} \in G/H$  and choose any  $g \in \pi^{-1}(\bar{g})$ . Then  $g \cdot H = L_g(H)$  is an embedded submanifold of  $G$  (i.e. image of  $H$  under  $L_g$ ). Choose a small (so that we can work in local chart) transversal submanifold  $M \subseteq G$  passing through  $g$ , i.e.  $T_g G = T_g(gH) \oplus T_g M$ .

In particular,  $\dim M = n-k$ . Choosing  $M$  small enough ensures that

$$M \cdot H = \{m \cdot h \mid m \in M, h \in H\} \text{ - open in } G$$

← To this end, we again apply the inverse function theorem to  $m: M \times H \rightarrow G$

Set  $\bar{M} := \pi(M)$ . Then  $\bar{M}$  is an open nbhd of  $\bar{g} \in G/H$  (as  $\pi^{-1}(\bar{M}) = M \cdot H$  is open). It's also clear that  $\pi: M \rightarrow \bar{M}$  is a homeomorphism. This gives a local chart near  $\bar{g} \in G/H$ .

The transition maps are regular and so we get a structure of a manifold on  $G/H$ . (independent of the choices made).

Furthermore, since  $M \times H \rightarrow M \cdot H$  is a diffeomorphism,  $\pi$  is a fiber bundle with the fiber  $H$ . This proves part a).

Evoking  $T_g G = T_g(gH) \oplus T_g M$ , we also get  $T_{\bar{x}}(G/H) \simeq T_1 G / T_1 H$ . This proves part b).

Finally, we leave part c) as a simple exercise (it suffices to check regularity of multiplication at 1).

Exercise (\*): a) Prove part c) of Prop 2.

b) Verify that manifold structure constructed above is independent of both  $g \in \pi^{-1}(\bar{g})$  and the small transversal submanifold  $M$ .

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As an immediate corollary of Prop 2, we get:

Corollary 1: Let  $H$  be a closed Lie subgroup of a Lie group  $G$ .

- a) If  $H$  is connected, then the map  $\pi_0(G) \rightarrow \pi_0(G/H)$  between the sets of connected components is a bijection. In particular, if  $H, G/H$  are connected then so is  $G$ .
- b) If both  $H, G$  are connected, then there is an exact sequence of fundam. gps:
- $$\pi_1(H) \rightarrow \pi_1(G) \rightarrow \pi_1(G/H) \rightarrow \{1\}$$

Recall: exact here means that  $\text{Ker}(\pi_1(G) \rightarrow \pi_1(G/H)) = \text{Im}(\pi_1(H) \rightarrow \pi_1(G))$   
and  $\pi_1(G) \rightarrow \pi_1(G/H)$  is surjective

The above is just the very end of the famous long exact sequence of (higher) homotopy groups of a fibration. It's a good exercise to prove the above:

Exercise<sup>(\*)</sup>: Prove Corollary 1.

[A nice application of this result is the computation of  $\pi_1$  for classical gps.]  
- to be discussed later on

- For many purposes, the notion of a closed Lie subgroup (Def 1) is quite restrictive.

Example ("Irrational winding on the torus"): Consider  $\mathbb{R} \xrightarrow[\text{torus}]{T^2 = \mathbb{R}^2 / \mathbb{Z}^2}$   
Then the image is dense in  $T^2$ .  $t \mapsto (t \bmod \mathbb{Z}, \pi t \bmod \mathbb{Z})$

For this reason, it is useful to consider the following notion:

Def 3: A minimal Lie subgroup of a Lie group  $G$  is a subgroup  $H$  which is also an immersed submanifold (i.e. there is a mfd structure on  $H$  and immersion  $H \hookrightarrow G$ )  
 $\uparrow$  does not need to be an embedded submanifold or a closed subset

In particular, the above "irrational winding of the torus" or " $(\mathbb{Q}, +) \subseteq (\mathbb{R}, +)$ " are Lie subgps

The relation to the above discussion is provided by the following result:

Proposition 3: Let  $f: G_1 \rightarrow G_2$  be a homomorphism of Lie groups. Then,  $H := \text{Ker}(f)$  is a normal closed Lie subgroup in  $G_1$ , while  $\text{Im}(f)$  is a Lie subgroup in  $G_2$  (Def 3). If furthermore  $\text{Im}(f)$  is an embedded submanifold of  $G_2$ , then it is a closed Lie subgroup in  $G_2$  and  $f$  gives rise to an isomorphism of Lie groups  $G_1/H \cong \text{Im}(f)$ .

We shall prove this result after some basics on Lie algebras.

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• We conclude today's lecture by discussing the primary reason why Lie groups are so important (arising as symmetries of various geometric objects).

Def 4: Let  $X$  be a manifold,  $G$  be a Lie group, and  $a: G \times X \rightarrow X$  be a regu map.

We call this a regular (left) action of  $G$  on  $X$  if  $(g_1 \cdot g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$ , i.e.

$$a(m(g_1, g_2), x) = a(g_1, a(g_2, x)): G \times G \times X \rightarrow X$$

In other words  $\forall g \in G$  we have a diffeomorphism  $p(g) = a(g, \cdot) \in \text{Diff}(X)$  with

$$p(1) = \text{Id} \text{ and } p(g_1 g_2) = p(g_1) \circ p(g_2)$$

Examples: a)  $GL_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$

b)  $O_n(\mathbb{R}) \curvearrowright S^{n-1} = \text{unit sphere in } \mathbb{R}^n \leftarrow O_n(\mathbb{R}) = \text{orthogonal } n \times n \text{ matrices} = \{A: AA^T = \text{Id}\}$

Closely related with the above notion is the notion of a representation:

Def 5: A representation of a (real or complex) Lie group  $G$  is a vector space  $V$  (complex if  $G$  is complex; real or complex if  $G$  is real) together with a group morphism  $\rho: G \rightarrow \text{End}(V)$ . When  $V$  is finite-dimensional, we require  $\rho$  to be analytic, so that it is a morphism of Lie groups.

A homomorphism of  $G$ -representations  $V$  and  $W$  is a linear map  $f: V \rightarrow W$  that commutes with  $G$ -action, i.e.  $f(\rho_V(g)v) = \rho_W(g)(f(v)) \quad \forall g \in G, v \in V$ .

• Given  $G \curvearrowright V$  one has a natural repr-n  $G \curvearrowright V^*$ , called the dual repr-n:

$$\rho_{V^*}(g) = \rho_V(g^{-1})^*: V^* \rightarrow V^*$$

• Given  $G \curvearrowright^{\rho_V} V$  and  $G \curvearrowright^{\rho_W} W$ , one has the tensor product repr-n  $G \curvearrowright V \otimes W$ :

$$\rho_{V \otimes W}(g) = \rho_V(g) \otimes \rho_W(g): V \otimes W \rightarrow V \otimes W.$$

• Given  $G \curvearrowright^{\rho_V} V$  and subspace  $W \subseteq V$  s.t.  $\rho_V(g)W \subseteq W \quad \forall g$ , one has a natural quotient representation  $G \curvearrowright V/W$ .

Finally, the most relevant to us is the adjoint action of any Lie group  $G$  on  $\mathfrak{g} = T_1 G$ . To this end, consider the composition of the left & right translations

$$\begin{aligned} G \times G &\rightarrow G & \text{and} & & G \times G &\rightarrow G \\ (g, h) &\mapsto gh & & & (g, h) &\mapsto hg^{-1} \end{aligned}$$

to get the adjoint action  $G \times G \xrightarrow{\text{Ad}} G$  given by  $\text{Ad}_g(x) = g x g^{-1}$ . Since

$\text{Ad}_g(1) = 1 \quad \forall g$ , we get  $d_1(\text{Ad}_g): \mathfrak{g} \rightarrow \mathfrak{g}$  that is commonly denoted just by  $\text{Ad}_g$ .