

Lecture #6

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- Last time, we discussed
 - stabilizer-orbit thm for Lie gps
 - homogeneous spaces and their mfd structure
 - classical gps and Lie gp structure on them via exp-map
 - notion of Lie algebras in general

Let me start by making a few remarks

1) The Lie alg-s of corresponding to $O(n, \mathbb{K})$ and $SO(n, \mathbb{K})$ coincide:

$$\mathfrak{o}_n(\mathbb{K}) = \{x \in \mathfrak{gl}_n(\mathbb{K}) \mid x + x^T = 0\}$$

$$\mathfrak{so}_n(\mathbb{K}) = \{x \in \mathfrak{gl}_n(\mathbb{K}) \mid x + x^T = 0, \text{tr}(x) = 0\}$$

This is not surprising as $SO(n, \mathbb{K})$ is a connected component of $O(n, \mathbb{K})$

2) In contrast, $\mathfrak{su}_n \neq \mathfrak{u}_n$.

3) To describe Lie alg-s of for classical gps corresponding to bilinear form $B(-, -)$, you may wish to take $\frac{d}{dt} \Big|_{t=0} B(e^{tx} u, e^{tx} v)$ (this relevant to Hwk 2!)

4) The fiber bundles $H \rightarrow G$ may be used to compute $\pi_1(X)$ via $X = G/H$ Hwk 2, Problem #3.

• Goal for today: Analogue of exp-map for general Lie groups.

To do so, we shall first recall the notion of vector fields, which are just sections of tangent bundles.

Def 1: Let X be a real mfd, and let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A $\mathbb{K}$ -vector bundle of rank n on X is a mfd E with $\pi: E \rightarrow X$ and $\mathbb{K}$ -vector space structure on each $\pi^{-1}(x)$ so that $\forall x \in X \exists$ open nbhd $x \in U \subseteq X$ with $\boxed{\pi^{-1}(U) \xrightarrow[\cong]{\pi^{-1}} U \times \mathbb{K}^n}$ where projection is on 1st component, fiber's vector space structure is on the 2nd cpt (compare to fiber bundles in general), i.e. g_u - linear _{n fibers}.

Covering X by such open charts $\{U_\alpha\}$, we get transition maps which give rise to:

$$\boxed{\text{clutching functions } U_\alpha \cap U_\beta \xrightarrow{h_{\alpha\beta}} GL_n(\mathbb{K}) \text{ s.t. } g_{U_\alpha} \circ g_{U_\beta}^{-1}: (x, v) \mapsto (x, h_{\alpha\beta}(v))}$$

for any pair $U_\alpha \cap U_\beta \neq \emptyset$. Here, $g_{U_\alpha}: \pi^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times \mathbb{K}^n$ as in Def 1. Moreover:

$$\boxed{h_{\alpha\beta} = h_{\beta\alpha}^{-1} \text{ and } h_{\alpha\beta} \circ h_{\beta\gamma} = h_{\alpha\gamma}}$$

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Exercise: Revert the above construction, i.e. to any open cover $\{U_\alpha\}_{\alpha \in I}$ of X with a collection $\{h_{\alpha\beta}\}_{\alpha, \beta \in I}$ satisfying above properties construct the corresponding vector bundle.

Example (important for what follows): Take an atlas $\{(U_\alpha, \phi_\alpha)\}$ defining mfd structure on X , and define $h_{\alpha\beta}(x)$ as the differential of $\tau_{\alpha\beta}$ at x , i.e.

$$h_{\alpha\beta}(x) = d_{\phi_\beta(x)}(\tau_{\alpha\beta}) \quad \text{with} \quad \tau_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}: \phi_\beta(U_\alpha \cap U_\beta) \rightarrow \phi_\alpha(U_\alpha \cap U_\beta)$$

Exercise: Check the above compatibility conditions for these $h_{\alpha\beta}$.

This defines a vector bundle, called the tangent bundle $TX \xrightarrow{\pi} X$, so that $\pi^{-1}(x) = T_x X$, the tangent space of X at point x .

Def 2: A section of a vector bundle $\pi: E \rightarrow X$ is a map $s: X \rightarrow E$ s.t. $\pi \circ s = \text{Id}_X$

Notation: $\Gamma(X, E) = \{\text{vector space of all sections of } E \rightarrow X\}$

More generally, $\Gamma(U, E) = \{\text{sections of } E \rightarrow X \text{ over } U\}$.

Exercise: Show that a rank n $\mathbb{K}$ -vector bundle $E \xrightarrow{\pi} X$ is trivial (i.e. $E \cong X \times \mathbb{K}^n$) iff $\exists s_1, \dots, s_n \in \Gamma(X, E)$ s.t. $\{s_1(x), \dots, s_n(x)\}$ - basis of $\pi^{-1}(x) \forall x$.

Finally, we can now define the vector fields:

Def 3: A vector field on X is a section of the tangent bundle $TX \rightarrow X$

(Notation: $\text{Vect}(X)$)

Down-to-earth, we can think of a vector field V locally as expression $V = \sum_{i=1}^n v_i(x) \frac{\partial}{\partial x_i}$ in local coordinates $\{x_i\}$, where v_i is a regular f-n, and switching to a different chart with local coordinates $\{x'_i\}$, we get

$$v'_i = \sum_{j=1}^n \frac{\partial x_j}{\partial x'_i} v_j$$

Note: Every such V gives rise to a derivation of $\mathcal{O}(U)$ and germ $\mathcal{O}_x$.

Vice versa, derivations of $\mathcal{O}(U)$ compatible with $U' \subseteq U$ give rise to v.field.

Finally, if a Lie group G acts on mfd X , then it also acts on TX

$$G \curvearrowright X \rightsquigarrow G \curvearrowright TX \rightsquigarrow G \curvearrowright \text{Vect}(X)$$

In particular, $L_g, R_g: G \curvearrowright G$ give rise to $G \curvearrowright \text{Vect}(G)$, also denoted L_g or R_g .

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- Def 4: a) A vector field $V \in \text{Vect}(G)$ is left-invariant if $L_g V = V \forall g \in G$.
 b) — // — is right-invariant if $R_g V = V \forall g \in G$.
 c) — // — is bi-invariant if $L_g V = V = R_g V \forall g \in G$.

Lemma 1: a) The map $V \mapsto V(\overset{\text{unit of } G}{1})$ gives rise to a vector space isomorphism
 $\{\text{left-invariant vector fields on } G\} \cong T_1 G$.
 b) Similarly for right-inv. vector fields

- a) $\forall x \in g = T_1 G$, define $V \in \text{Vect}(G)$ via $V(g) = L_g(x)$, thus spreading x all over G . It's clear that this V is left-invariant. Moreover, it's unique such with $V(1) = x$.
 b) Analogous, but use R_g instead of L_g

Exercise: Show that $V \mapsto V(1)$ gives rise to a vector space isom.
 $\{\text{bi-invariant vector fields on } G\} \cong (T_1 G)^{\text{Ad } G} = \{ \overset{T_1 G}{x} \mid \text{Ad } g(x) = x \}$

Exercise: Verify that if $G = GL_n(\mathbb{R}) \subseteq \text{Mat}_{n \times n}(\mathbb{R})$ so that each $T_g G$ is canonically identified with $\text{Mat}_{n \times n}(\mathbb{R})$, then the left-invariant vector field on G is given by $g \mapsto g \cdot g^{-1}$ (usual product of matrices); state analogous f.l.a for right-invariant.

Corollary 1: For any Lie group G , the tangent bundle $TG \rightarrow G$ is trivial.

Remark: The above can be generalized from vector fields to tensor fields. To this end, let T^*X be the dual of the TX , called cotangent bundle. A tensor field of rank (n, m) on mnpd X is a section of $(TX)^{\otimes n} \otimes (T^*X)^{\otimes m}$. In particular, tensor fields of rank $(0, 1)$ are called differential 1-forms. (Locally they can be written as $\Omega = \sum_{i=1}^n w_i dx_i$, but changing coordinates we have $w_i' = \sum_j \frac{\partial x_j}{\partial x_i'} w_j$, cf. v_i' vs v_j)
 (In particular, $\forall f \in \mathcal{O}(X)$ get 1-form $df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$)
 Then, similarly to Lemma 1, we have
 $\{\text{left (or right) invariant tensor fields of rank } (n, m) \text{ on } G\} \cong (T_1 G)^{\otimes n} \otimes (T_1 G)^{* \otimes m}$

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The key in the general construction of the exponential map $T_1 G =: \mathfrak{g} \rightarrow G$ is the construction of one-parameter subgroups:

Proposition 1: Let G be a real Lie group and set $\mathfrak{g} := T_1 G$.
 Then $\forall x \in \mathfrak{g} \exists!$ morphism of Lie groups $\gamma = \gamma_x: \mathbb{R} \rightarrow G$ with $\gamma'(0) = x$
 $= d_x \gamma(1)$

• For classical groups, such γ was constructed last time via $\gamma(t) = \exp(tx)$.
 In general, this doesn't make sense as there is no product on $T_1 G$. However, similarly to calculus we note that such γ should satisfy

$$\boxed{\gamma'(t) = \gamma(t) \cdot \gamma'(0) = (L_{\gamma(t)})_* \gamma'(0)} \quad (*)$$

(to see this, write $\gamma(t+s) = \gamma(t) \cdot \gamma(s)$ and take $\frac{d}{ds}|_{s=0}$ of both sides)

By existence & uniqueness theorem for ODE with initial condition, the eq-n (*) has a unique solution with $\gamma'(0) = x$ in the nbhd of 0, i.e. for $-\varepsilon < t < \varepsilon$.

• First, we claim that this γ satisfies group morphism on this nbhd, i.e.

$$\boxed{\gamma(t+s) = \gamma(t) \cdot \gamma(s) \text{ when } s, t, s+t \in (-\varepsilon, \varepsilon)} \quad (**)$$

Indeed, fix t and consider $\begin{cases} \gamma_1(s) := \gamma(t+s) \\ \gamma_2(s) := \gamma(t) \cdot \gamma(s) \end{cases}$. Then $\begin{cases} \gamma_1(0) = \gamma_2(0) = \gamma(t) \\ \gamma_1'(0) = \gamma'(t) \stackrel{(*)}{=} \gamma(t) \cdot \gamma'(0) \end{cases} \Rightarrow$

$\Rightarrow$ by uniqueness $\gamma_1 \equiv \gamma_2 \Rightarrow \gamma(t+s) = \gamma(t) \gamma(s)$ as above in (**)

• Also note that $\boxed{\gamma(t) \cdot x = x \cdot \gamma(t) \text{ for } t \in (-\varepsilon, \varepsilon)} \quad (***)$

To this end, compute $\frac{d}{ds}|_{s=0}$ in $\gamma(s) \gamma(t) = \gamma(t+s) = \gamma(t) \gamma(s)$

• It remains to extend γ from $(-\varepsilon, \varepsilon)$ to $\mathbb{R}$, i.e. $\gamma: \mathbb{R} \rightarrow G$.

(There is at most one way to do so to preserve gp homomorphism, and we need to check consistency $\forall t, s$.)

We do so by extending iteratively from $(-2^k \varepsilon, 2^k \varepsilon)$ to $(-2^{k+1} \varepsilon, 2^{k+1} \varepsilon)$ via:

$$\boxed{\gamma(t) = \gamma\left(\frac{t}{2}\right) \cdot \gamma\left(\frac{t}{2}\right) \quad \forall t \in (-2^{k+1} \varepsilon, 2^{k+1} \varepsilon)} \quad \leftarrow \begin{matrix} \gamma\left(\frac{t}{2}\right) \text{ is defined} \\ \text{as } \frac{t}{2} \in (-2^k \varepsilon, 2^k \varepsilon). \end{matrix}$$

For $|t| < 2^k \varepsilon$ this agrees with previous step, so that we indeed get extension at each step.

To see that $\gamma(t+s) = \gamma(t) \gamma(s)$ as long as $t, s, t+s \in (-2^{k+1} \varepsilon, 2^{k+1} \varepsilon)$, it suffices to check $\gamma'(t) = \gamma(t) \cdot x$ for $|t| < 2^{k+1} \varepsilon$, which is straightforward:
 $\gamma'(t) = \frac{1}{2} \gamma'\left(\frac{t}{2}\right) \gamma\left(\frac{t}{2}\right) + \frac{1}{2} \gamma\left(\frac{t}{2}\right) \gamma'\left(\frac{t}{2}\right) = \frac{1}{2} \gamma\left(\frac{t}{2}\right) x \gamma\left(\frac{t}{2}\right) + \frac{1}{2} \gamma\left(\frac{t}{2}\right) \gamma\left(\frac{t}{2}\right) x \stackrel{(***)}{=} \gamma\left(\frac{t}{2}\right)^2 x = \gamma(t) x$

$\Rightarrow$ This provides claimed $\gamma_x: \mathbb{R} \rightarrow G$. Uniqueness is clear from above

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Def 5: Let G be a real Lie group and $\mathfrak{g} = T_1 G$. Then, the exponential map
 $\exp: \mathfrak{g} \rightarrow G$ is given by $\exp(x) = \gamma_x(1)$ with γ_x from Prop 1

Note: If $\gamma(t)$ satisfies $\gamma'(t) = \gamma(t) \cdot \gamma'(0)$, then for any $c \in \mathbb{R}$ the function $\bar{\gamma}(t) := \gamma(ct)$ satisfies $\bar{\gamma}'(t) = \gamma(ct) \cdot c\gamma'(0) = \bar{\gamma}(t) \cdot c\gamma'(0)$, i.e. by uniqueness:

$$\gamma_{cx}(t) = \gamma_x(ct) \quad \forall x \in \mathfrak{g}, c \in \mathbb{R}, t \in \mathbb{R}$$

In particular, we get:

Corollary 2: $\gamma_x(t) = \exp(tx)$

Examples: 1) For $G = GL_n(\mathbb{R})$ or classical subgroup of $GL_n(\mathbb{R})$, we have
 $\gamma_x(t) = \exp(tx)$ in the sense of usual exp from Lecture 5

Hence, for classical gps get the same map as in Lecture 5.

2) For $G = (\mathbb{R}^n, +)$, we have $\gamma_{\vec{x}}(t) = t \cdot \vec{x} \quad \forall \vec{x} \in \mathbb{R}^n$, so that $\exp(\vec{x}) = \vec{x}$.

The key properties of exp are summarized in the following result:

Proposition 2: Let G be a real Lie gp and $\mathfrak{g} = T_1 G$.

a) $\exp: \mathfrak{g} \rightarrow G$ is a regular map with $\exp(0) = 1$, $d_0 \exp = \text{Id}_{\mathfrak{g}}$ and it's a diffeomorphism of nbhd of $0 \in \mathfrak{g}$ onto a nbhd of $1 \in G$.

b) $\exp((t+s)x) = \exp(tx) \cdot \exp(sx) \quad \forall x \in \mathfrak{g}, \forall s, t \in \mathbb{R}$

c) $\forall$ morphism of Lie groups $\varphi: G_1 \rightarrow G_2$ and $\forall x \in T_1(G_1) = \mathfrak{g}_1$, we have

$$\exp(\underbrace{\varphi_*(x)}_{\in \mathfrak{g}_2 = T_1(G_2)}) = \varphi(\exp(x))$$

d) For any $g \in G, x \in \mathfrak{g}$, we have $g \cdot \exp(x) \cdot g^{-1} = \exp(\text{Ad}_g(x))$

Corollary 3: As an immediate corollary of part a), we see that exp has an inverse map $\log: \underbrace{U}_{\text{some nbhd of } 1 \in G} \rightarrow \mathfrak{g}$, called logarithm map

(For classical gps, this coincides with matrix logarithm (Lect 5))
 Thus, any basis of $\mathfrak{g}$ gives a local coordinate system near $1 \in G$

Proof of Proposition 2

a) Regularity follows from general fact as our ODE depends regularly on x .

$$\exp(0) = \gamma_0(1) = 1 \quad \text{as } \gamma_0(t) = 1 \quad \forall t$$

$$\exp'(0)x = \left. \frac{d}{dt} \right|_{t=0} \exp(tx) \stackrel{\text{def}}{=} x \Rightarrow \exp'(0) = d_0 \exp = \text{Id}_{\mathfrak{g}}$$

Thus, by the inverse function theorem, $\exp$ is a diffeomorphism near 0.

b) Follows from

$$\exp((t+s)x) = \gamma_x(t+s) = \gamma_x(t) \gamma_x(s) = \exp(tx) \exp(sx)$$

c) Both $\varphi(\exp(tx))$ and $\exp(\varphi_*(tx))$ define 1-parametric subgps of G_2 with the same tangent vector at $1 \in G_2$, equal to $\varphi_*(x)$. By uniqueness, they coincide.

d) Follows from c) with $G_1 = G = G_2$ and $\varphi = \text{Ad}_g$.

Lemma 2: Let G_1, G_2 be real Lie gps and G_1 - connected.

Then any Lie gp homomorphism $\varphi: G_1 \rightarrow G_2$ is uniquely determined by $\varphi_*: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$

By Prop 2(c), φ_* determines φ on a nbhd of $1 \in G_1$.

But by [Lect 4, Prop 1], any open nbhd of connected G_1 generates G_1 .

$\Rightarrow \varphi$ is uniquely determined

! In fact, all the above results also hold for $\mathbb{C}$ Lie gps (just need correspondingly version of ODE), so we have $\exp: \mathfrak{g} \rightarrow G$ with all properties above.