

Lecture #7-8

1

Last time:

- vector bundles (clutching f-s)
- tangent bundle & vector fields
- 1-parametric subgroups $\gamma_x: \mathbb{R} \rightarrow G \quad \forall x \in \mathfrak{g} := T_1 G$ (via sol-n of ODE)
- the exponential map $\exp: \mathfrak{g} \rightarrow G$ (noted: $\gamma_x(t) = \exp(tx)$)

$$x \mapsto \gamma_x(1)$$

Start by finishing Prop 2, Cor 3, Lemma 2, C-setup from pp. 5-6 of previous lecture notes.

So far, $\mathfrak{g} = T_1 G$ was viewed only as a vector space, but ultimately we wish to endow it with a Lie algebra structure. To this end, the Lie bracket will naturally arise from the product on G .

Consider a small nbhd $0 \in U \subseteq \mathfrak{g}$ s.t. $\log(\exp(x)\exp(y))$ is defined $\forall x, y \in U$, which gives rise to some smooth map

(expresses product on group G in local chart near 1) $\leadsto$
$$\mu: U \times U \rightarrow \mathfrak{g}$$

$$(x, y) \mapsto \log(\exp(x) \cdot \exp(y))$$
 i.e. $\exp(\mu(x, y)) = \exp(x) \cdot \exp(y)$

Lemma 1: The Taylor series for μ is given by

$$\mu(x, y) = x + y + \lambda(x, y) + \dots$$

where $\lambda: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is bilinear skew-symmetric, ... denote terms of order ≥ 3

As $\mu(0, 0) = 0$, it can be written as

$$\mu(x, y) = \underbrace{b_1(x) + b_2(y)}_{\text{linear terms}} + \underbrace{q_1(x) + q_2(y)}_{\text{quadratic terms}} + \underbrace{\lambda(x, y)}_{\text{bilinear}} + \text{higher order terms.}$$

But $\exp(0) = 1 \Rightarrow \mu(x, 0) \equiv x \quad \forall x \in U \Rightarrow b_1(x) \equiv x, q_1(x) = 0$
 $\Rightarrow \mu(0, y) \equiv y \quad \forall y \in U \Rightarrow b_2(y) \equiv y, q_2(y) = 0 \quad \left. \vphantom{\mu(x, 0)} \right\} \Rightarrow \mu(x, y) = x + y + \lambda(x, y) + \dots$

Also: $\exp(x) \cdot \exp(x) \stackrel{\text{Prop 2 \& Lecture 6}}{=} \exp(2x) \Rightarrow \mu(x, x) = 2x \Rightarrow \lambda(x, x) \equiv 0 \Rightarrow \lambda \text{ - skew-symmetric}$

Def 1: The map
$$[-, -]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$
 is called the commutator.

$$(-, -) \mapsto 2\lambda(-, -)$$

Thus,
$$\exp(x) \cdot \exp(y) = \exp\left(x + y + \frac{1}{2}[x, y] + \dots\right)$$
 $\leftarrow$ the reason for factor $\frac{1}{2}$ will be clear soon.

Example: For $G = GL_n(\mathbb{K})$, $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{K})$, we have

$$\begin{aligned} \exp(x) \cdot \exp(y) &= \left(1 + x + \frac{x^2}{2!} + \dots\right) \left(1 + y + \frac{y^2}{2!} + \dots\right) = 1 + x + y + \left(\frac{x^2}{2} + xy + \frac{y^2}{2}\right) + \dots \\ &= 1 + (x+y) + \frac{(x+y)^2}{2} + \frac{xy - yx}{2} + \dots = \exp\left(x+y + \frac{xy - yx}{2} + \dots\right) \end{aligned}$$

Hence: $[x, y] = x \cdot y - y \cdot x \quad \forall x, y \in \mathfrak{gl}_n(K)$ - the usual commutator of $n \times n$ matrices

As an immediate consequence of this example, we get:

Corollary 1: If G is a Lie subgroup of $GL_n(\mathbb{K})$, then $\mathfrak{g} = T_1 G \subseteq \mathfrak{gl}_n(\mathbb{K})$ and the commutator $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ is the restriction of that on $\mathfrak{gl}_n(\mathbb{K})$.

Let's list some basic properties of τ, β .

Proposition 1:

a) for a Lie gp homomorphism $\varphi: G_1 \rightarrow G_2$, its differential $\varphi_*: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ preserves $[\cdot, \cdot]$.

$$\varphi_*([x, y]_{\in \mathfrak{g}_1}) = [\underbrace{\varphi_*(x)}_{\in \mathfrak{g}_2}, \underbrace{\varphi_*(y)}_{\in \mathfrak{g}_2}] \quad \forall x, y \in \mathfrak{g}_1$$

b) The adjoint action $G \xrightarrow{\text{Ad}} \mathfrak{g} = T_1 G$ preserves $[\cdot, \cdot]$:

$$\text{Ad } g([x, y]) = [\text{Ad } g(x), \text{Ad } g(y)] \quad \forall x, y \in \mathfrak{g}$$

c) $\exp(x) \exp(y) \exp(-x) \exp(-y) = \exp([x, y] + \text{higher order terms})$

d) G -commutative $\Rightarrow [x, y] = 0 \quad \forall x, y \in \mathfrak{g}$.

a) Follows from the def'n of $[\cdot]$ and Prop 2c) of Lecture 6

b) special case of a) with $G_1 = G = G_2$ and $\varphi = \text{Ad}_g$

c) $\left. \begin{aligned} \exp(x) \cdot \exp(y) &= \exp\left(x+y + \frac{1}{2}[x,y] + \dots\right) \\ \exp(-x) \cdot \exp(-y) &= \exp\left(-x-y + \frac{1}{2}[x,y] + \dots\right) \end{aligned} \right\} \Rightarrow$

$\exp(x)\exp(y)\exp(-x)\exp(-y)$
" ← as $[x+y, -x-y]=0$
 $\exp([x,y] + \dots)$

d) Follows immediately from c)

Given a Lie gp G , we have a natural adjoint action $G \curvearrowright \mathfrak{g}$, that is

$$\text{Ad}: G \rightarrow \text{GL}(\mathfrak{g}) \text{ - Lie gp homomorphism}$$

Def 2: Define $\text{ad} := \text{Ad}_* = d_1 \text{Ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ - "adjoint action"

First, we note that ad preserves $[\cdot, \cdot]$ by Prop 1a), i.e.

$$\text{ad}([x, y]) = \text{ad}(x) \circ \text{ad}(y) - \text{ad}(y) \circ \text{ad}(x) \in \text{End}(\mathfrak{g}) \quad \forall x, y \in \mathfrak{g}$$

Lecture #7-8

3

The following are key properties of ad :

Lemma 2:

a) $\text{ad } x(y) = [x, y]$

b) $\text{Ad}(\exp(x)) = \exp(\text{ad } x) : \mathfrak{g} \rightarrow \mathfrak{g}$

(Remark: One can thus define $[x, y]$ as $\text{ad}(x)(y)$. However, even skew-symmetry wouldn't be obvious)

a) Let's decode the definition of Ad :

$$\text{Ad } g(y) = \left. \frac{d}{dt} \right|_{t=0} (g \exp(ty) g^{-1})$$

$\Downarrow$

$$\text{ad } x(y) = \left. \frac{d}{ds} \right|_{s=0} \left. \frac{d}{dt} \right|_{t=0} (\exp(sx) \exp(ty) \exp(-sx)) = [x, y]$$

$\exp(tsx, y) + \dots \exp(ty)$ by Prop 1c)

← in general you can take any path $\gamma(t)$ with $\gamma(0)=1$, $\gamma'(0)=y$, but $\gamma(t)=\exp(ty)$ is easy to work with

b) Follows from [Prop 2c) of Lecture 6] applied to $G \xrightarrow{\text{Ad}} \text{GL}(\mathfrak{g})$.

Alternatively, could argue that both $\exp(t \text{ad } x)$ & $\text{Ad}_{\exp(tx)}$ satisfy same ODE.

Theorem 1: For any real/complex Lie group G , $(\mathfrak{g}, [\cdot, \cdot])$ is a Lie algebra over $\mathbb{R}/\mathbb{C}$.

As $[\cdot, \cdot]$ is skew-symmetric, we just need to check Jacobi identity.

But $\text{ad}: \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$ preserves commutators and we know $[A, B] = AB - BA$
 $\forall A, B \in \mathfrak{gl}(\mathfrak{g})$

So: $\text{ad}([x, y]) = \text{ad } x \cdot \text{ad } y - \text{ad } y \cdot \text{ad } x$

$\Downarrow$

$$[x, y], z = [x, [y, z]] - [y, [x, z]] \quad \forall x, y, z \in \mathfrak{g} \quad \Rightarrow \text{Jacobi}$$

This finally establishes the long-promised goal

$$\text{Lie groups } G \rightsquigarrow \text{Lie algebras } \mathfrak{g} = \text{Lie}(G)$$

Moreover, as follows from Prop 1a), we have

$$\text{Hom}_{\text{Lie gps}}(G_1, G_2) \xrightarrow{\varphi} \text{Hom}_{\text{Lie alg}}(\mathfrak{g}_1, \mathfrak{g}_2)$$

$\varphi \mapsto \varphi^*$

a linear map of Lie algebras $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is homomorphism if it preserves Lie bracket

and this map is injective if G -connected by [Lemma 2 of Lecture 6]

UPSHOT: Have a functor $\text{Lie}: \{ \text{category of Lie gps} \} \rightarrow \{ \text{category of Lie algebras} \}$

Lecture #7-8

Def 3: For any Lie algebra $\mathfrak{g}$ over any field $\mathbb{K}$:

- a) a $\mathbb{K}$ -subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called a Lie subalgebra if $[x, y] \in \mathfrak{h} \quad \forall x, y \in \mathfrak{h}$
- b) a $\mathbb{K}$ -subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is called an ideal if $[x, y] \in \mathfrak{h} \quad \forall x \in \mathfrak{g}, y \in \mathfrak{h}$.

Proposition 2: Let G be a real/cpx Lie gp with Lie algebra $\mathfrak{g}$.

- a) If H is a Lie subgroup in G , then $\mathfrak{h} = T_1 H$ is a Lie subalgebra in $\mathfrak{g}$.
- b) If H is a normal closed Lie subgp in G , then $\mathfrak{h} = T_1 H$ is an ideal in $\mathfrak{g}$.

- a) Follows immediately from $\exp(tx) \in H \quad \forall t \in \mathbb{R}$ for $x \in T_1 H$, combined with Prop 1c), see the proof of Lemma 2a) (i.e. look at $\lim_{s, t \rightarrow 0} \frac{\log(\exp(sx) \exp(ty) \exp(-sx) \exp(-ty))}{s \cdot t}$)
- b) If $x \in \mathfrak{g}, y \in T_1 H$, then H being normal, we have $\exp(x) \cdot \exp(y) \cdot \exp(-x) \in H$, hence result follows.

Exercise: Show that if H is a closed Lie subgp of G , both H, G are connected, and $T_1 H$ is an ideal in $\mathfrak{g}$, then H is normal

Let us now state 3 fundamental theorems of Lie theory

1st Fundamental Thm: For any real/complex Lie group G , there is a bijection between connected Lie subgroups $H \subseteq G$ and Lie subalgebras $\mathfrak{h} \subseteq \mathfrak{g}$, given by $H \mapsto T_1 H$

2nd Fundamental Thm: If G_1, G_2 are real/complex Lie gps and G_1 is simply-connected (+ connected) then the map $\text{Hom}_{\text{Lie gps}}(G_1, G_2) \rightarrow \text{Hom}_{\text{alg}}(\text{Lie}(G_1), \text{Lie}(G_2)), \varphi \mapsto \varphi_*$, is bijective

3rd Fundamental Thm: Any finite-dimensional real/complex Lie algebra is the Lie algebra of a real/complex Lie group.

We shall skip their proofs for now on (and maybe will return to in the end of the course, if time remains).

Corollary 2: The categories of {finite-dimensional Lie algebras} and {connected, simply-connected Lie groups} are equivalent.