Lecture #10

Last time: - category of fin. dim. $\mathfrak{g}$-modules
- quotients, $\oplus$, $\otimes$, dual of $\mathfrak{g}$-modules
- Schur lemma
- Main Theorem on fin. dim sl_2-modules

Let's recall this theorem:

Theorem 1: a) V_n constructed last time are irreducible sl_2-modules

b) If $V\neq 0$ is a fin. dim. sl_2-module, then e and f act nilpotently on V. Moreover $U := \mathrm{Ker}(e)\neq 0$ is stable under h-action, which acts diagonally with $\mathbb{Z}_{\geq 0}$ eigenvalues

c) Any irreducible fin. dim. sl_2-module is $\simeq V_n$ for some n

d) Any fin. dim. sl_2-module is completely reducible

Before giving a proof, let's discuss two important corollaries.

Corollary 1 (Jacobson-Morozov theorem): For any fin. dim. vector space W over $\mathbb{C}$ and a nilpotent operator $E: W\to W$ $\exists\, sl_2 \overset{\rho}{\curvearrowright} W$ with $\rho(e)=E$.

▶ As E-nilpotent on W, one can choose a basis of W in which E has Jordan form with blocks $\begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ 0 & & & 0\end{pmatrix}_{n_i\times n_i}$. But evoking irreducible sl_2-module V_{n_i-1} we note that $\rho_{V_{n_i-1}}(e)$ acts also by the same single Jordan block. Thus, $W\simeq \bigoplus_i V_{n_i-1}$ as sl_2-modules, with $\rho(e)=E$ ■

Corollary 2 (Clebsch-Gordon formula): $V_n\otimes V_m \simeq \bigoplus_{i=0}^{\min(n,m)} V_{2i+|n-m|}$ as sl_2-modules (for any $n,m\geq 0$)

The simplest (non-constructive) proof uses notion of characters.

Def: For any fin. dim. sl_2-module V, its character is $\boxed{\chi_V(z)=\mathrm{tr}_V(z^h)}$ Laurent pol-l by Thm 1

Exercise: Prove above Corollary!

[Hint for above exercise: verify that $\chi_{V_n}(z) = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}$ are linearly independent, use $\chi_{V\oplus W}(z) = \chi_V(z) + \chi_W(z)$, $\chi_{V\otimes W}(z) = \chi_V(z)\chi_W(z)$ and Theorem

Proof of Theorem 1

a) Recall that $\rho_{V_n}(h)$ acts diagonally in the basis $\{x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\}$ of V_n with pairwise distinct eigenvalues. Thus (by Vandermonde f-la) any $\rho_{V_n}(h)$-invariant subspace $V_n' \neq 0$ must be spanned by a subset of above basis. However, applying $\rho_{V_n}(e)$, $\rho_{V_n}(f)$ to any single $x^k y^{n-k}$ clearly generates all other basis elements $\Rightarrow V_n' = V_n$. So V_n - irreducible.

b) Given any fin. dim. sl_2-module V, consider the respective three operators $H = \rho_V(h)$, $E = \rho_V(e)$, $F = \rho_V(f)$ which satisfy sl_2-relations:

$$[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F.$$

In particular, $HE = E(H+2) \Rightarrow H^N E = E(H+2)^N$, $HF = F(H-2) \Rightarrow H^N F = F(H-2)^N$ $\forall N$.

Therefore, if $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$ with $V_\lambda = \{v \in V : (H-\lambda)^N v = 0 \text{ for some } N \geq 1\}$ ← generalized eigenspace for H-action

then $E(V_\lambda) \subseteq V_{\lambda+2}$, $F(V_\lambda) \subseteq V_{\lambda-2}$ $\forall \lambda \in \mathbb{C}$. Since $\dim V < \infty$, we conclude that E, F are nilpotent. In particular, $U := \mathrm{Ker}(E)$ is nonzero, and it's H-invariant as $\forall v \in U$: $E(Hv) = (H-2)Ev = 0$. Finally, the fact that H acts diagonally on U with $\mathbb{Z}_{\geq 0}$-eigenvalues follows from:

[*Exercise*: a) Verify $EF^N v = F^{N-1} \cdot N \cdot (H - N + 1)v$ $\forall N \geq 1$, $v \in U$

b) Deduce $E^N F^N v = N!\, H(H-1) \ldots (H-N+1)v$ $\forall N \geq 1$, $v \in U$

c) As $F^N \equiv 0$ for $N \geq \dim V$, above implies $H(H-1)\ldots(H-N+1) \underset{\text{on } U}{\equiv} 0$

Deduce H acts diagonalizably with eigenvalues $\in \mathbb{Z}_{\geq 0}$

c) Assume now that sl_2-module V is irreducible, $\dim V < \infty$. Following notations of b), we then have $0 \neq \underset{\mathrm{Ker}(E)}{U} \subseteq V$ which is H-invariant.

Pick any H-eigenvector in U: $v \in U \setminus \{0\}$ s.t. $E(v) = 0$, $H(v) = \lambda v$ $(\lambda \in \mathbb{Z}_{\geq 0})$.

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Proof - Continuation

Consider a sequence of vectors in V: $v_0=v$, $v_1=F(v)$, $v_2=F^2(v)$, ...
Using commutation relations among E, F, H, we see that:

$$F: v_k \mapsto v_{k+1}, \quad H: v_k \mapsto (\lambda-2k)v_k, \quad E: v_k \mapsto k(\lambda-k+1)v_{k-1}$$

see Exercise a) above.

We also know that F is nilpotent on V by b) $\Rightarrow \exists N: v_N=0$ but $v_{N-1}\neq 0$.
Then $0=E(v_N)=N(\lambda-N+1)v_{N-1} \Rightarrow N=\lambda+1\in\mathbb{Z}_{>0}$. Therefore,
$\underbrace{V':=\text{span}\{v_0, v_1, \dots, v_{N-1=\lambda}\}}$ is an sl_2-submodule of V, hence equal to V as V is irreducible.

It is now easy to see that it is isomorphic to V_λ (here $\lambda\in\mathbb{Z}_{\geq 0}$!).
Indeed, consider linear isomorphism $\varphi: V' \xrightarrow{\sim} V_\lambda$ such that

$$\boxed{v_0 \overset{\varphi}{\mapsto} x^\lambda, \quad v_1 \overset{\varphi}{\mapsto} y\partial_x(x^\lambda)=\lambda\cdot x^{\lambda-1}y, \quad v_2 \overset{\varphi}{\mapsto} y\partial_x(\lambda x^{\lambda-1}y)=\lambda(\lambda-1)x^{\lambda-2}y^2, \dots}$$

As $x\partial_y(\lambda(\lambda-1)\dots(\lambda-k+1)x^{\lambda-k}y^k)=k(\lambda-k+1)\cdot\lambda(\lambda-1)\dots(\lambda-k+2)x^{\lambda-k+1}y^{k-1}$ we see that φ is indeed an sl_2-module isomorphism. This completes c).

d) The last part of the theorem requires another key idea

Def: Given an sl_2-module $sl_2 \overset{\rho}{\curvearrowright} V$, the Casimir operator C_V is:

$$\boxed{C_V=\rho_V(f)\rho_V(e)+\rho_V(e)\rho_V(f)+\frac{1}{2}\rho_V(h)^2}$$

Exercise: a) Verify that $C_V: V\to V$ is an sl_2-module morphism, i.e. $[C_V, \rho_V(x)]=0 \quad \forall x\in sl_2$

b) Verify that $C_{V_n}=Id_{V_n}\cdot\left(\frac{n^2}{2}+n\right)$ (the fact that it's a scalar operator follows from Schur l.)

Now given any fin. dim. sl_2-module V, we can first split it into $\oplus$ indecomposables. Thus, we may assume from beginning V-indecomposable. Then, by Lemma 1 of previous Lecture, $(C_V-\lambda)^N\equiv 0$ on V, i.e. there is only one eigenvalue of $C_V \curvearrowright V$ (as otherwise $V=\oplus$ generalized C_V-eigenspaces)

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Proof - Continuation

▶ Pick any Jordan-Hölder filtration of V, which is by part c) a flag of $\mathfrak{sl}_2$-submodules $0 = V_0' \subsetneq V_1' \subsetneq V_2' \subsetneq \ldots \subsetneq V_s' = V$ such that $V_j'/V_{j-1}' \simeq V_{n_j}$ as $\mathfrak{sl}_2$-modules. But comparing C_V-action to $C_{V_j'/V_{j-1}'}$, we see that $\lambda = \frac{1}{2}n_j^2 + n_j$ for all j (by Exercise above) $\Rightarrow n_1 = n_2 = \ldots = n_s =: n$.

[Exercise: Prove that $V \simeq sV_n = V_n^{\oplus s}$

Hint: Pick a basis $v^{(1)}, \ldots, v^{(s)}$ of $V(n) = \rho_V(h)$-eigenspace w/ eigenvalue n and check that $\forall k \in \{1, \ldots, s\}$ $U_k := \operatorname{span}\{v^{(k)}, \rho_V(f)v^{(k)}, \ldots, \rho_V(f)^n v^{(k)}\}$ is $\simeq V_n$ as $\mathfrak{sl}_2$-modules, and $\bigoplus_{k=1}^{s} U_k \hookrightarrow V$, which must be isomorphism by comparison of dimensions. ∎

* Universal enveloping algebra

Recall the usual construction of the tensor algebra of a $\mathbb{k}$-vector space V

$$\boxed{\begin{gathered} T(V) = \bigoplus_{n\geq 0} V^{\otimes n} = \mathbb{k} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \ldots \\ \text{with multiplication } V^{\otimes n} \times V^{\otimes m} \longrightarrow V^{\otimes(n+m)} \end{gathered}}$$

In particular, a choice of a basis $\{v_i\}$ of V identifies $T(V)$ with a free algebra $\mathbb{k}\langle v_i \rangle$.

Def: For a Lie algebra $\mathfrak{g}$ over a field $\mathbb{k}$, the universal enveloping algebra of $\mathfrak{g}$, denoted $U(\mathfrak{g})$, is the quotient of $T(\mathfrak{g})$ by 2-sided ideal generated by $\{x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{g}\}$.

$$\boxed{U\mathfrak{g} = T\mathfrak{g} / (x\otimes y - y \otimes x - [x,y])_{x,y\in\mathfrak{g}}}$$

The terminology is explained by the following universal property of the algebra $U\mathfrak{g}$: ↷

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Lemma 1: For any algebra A over $\Bbbk$, the map

$$\mathrm{Hom}_{\text{Ass.alg.}}(U(\mathfrak{g}), A) \longrightarrow \mathrm{Hom}_{\text{Lie Alg}}(\mathfrak{g}, A)$$

viewed as Lie algebra via $[a,b]=a\cdot b-b\cdot a$

(obtained by a composition with $\mathfrak{g} \hookrightarrow T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$) is bijective

▷ Indeed, any algebra homomorphism $U\mathfrak{g} \to A$ is determined by a linear map $\varphi: \mathfrak{g} \to A$ satisfying $\varphi(x)\varphi(y)-\varphi(y)\varphi(x)-\varphi([x,y])=0$ which precisely means that φ is a Lie algebra homomorphism ◁

Taking $A=\mathrm{End}(V)$ with V - vector space over $\Bbbk$, we thus get:

Corollary 3: Any $\mathfrak{g}$-module has a natural structure of $U(\mathfrak{g})$-module and vice versa.