Lecture # 11

Last time: - proof of classification of fin. dim. $\mathfrak{sl}_2$-modules
- notion of the universal enveloping algebra $U\mathfrak{g}$

Remark 1: Recall that on each irreducible fin. dim. $\mathfrak{sl}_2$-module V_n, the operator $\rho(h)$ acts with eigenvalues $n, n-2, \ldots, -n+2, -n$, all eigenspaces being 1-dimensional. Moreover, f^k maps eigenspace corresponding to eigenvalue k isomorphically onto eigenspace corresponding to $-k$, while e^k maps opposite way. As any fin. dim. $\mathfrak{sl}_2$-module is $\bigoplus V_{n_i}$, we get:

> Corollary 1 ($\mathbb{Z}_2$-symmetry): Any fin. dim. $\mathfrak{sl}_2$-module (V,ρ) admits a "weight decomposition" $V = \bigoplus_{n\in\mathbb{Z}} V(n)$ with $V(n) = \{v\in V \mid \rho(h)v = n\cdot v\}$ and $\rho(e)^n : V(-n) \xrightarrow{\sim} V(n)$, $\rho(f)^n : V(n) \xrightarrow{\sim} V(-n)$ $\forall n\geq 1$. In particular, $\dim V(n) = \dim V(-n)$ for all $n\in\mathbb{Z}$

later in the course we'll derive a similar symmetry for all simple Lie alg.

Remark 2: As noticed last time, each V_n had a so-called highest weight vector v_0 ($= x^n$ in our realization V_n = deg n pol's in x,y) s.t. $\rho(e)v_0 = 0$, $\rho(h)v_0 = n\cdot v_0$, and V_n was spanned by $\{\rho(f)^k v_0\}_{k=0}^{n}$

In fact, there are similar ∞-dim modules M_λ $\forall \lambda\in\mathbb{C}$, called Verma modules. Explicitly, consider a vector space M_λ with basis $\{v_k \mid k\geq 0\}$ and operators

$$H(v_k) = (\lambda-2k)v_k,\quad F(v_k) = (k+1)v_{k+1},\quad E(v_k) = (\lambda-k+1)v_{k-1}$$

assume $v_{-1}=0$.

Then $[H,E]=2E$, $[H,F]=-2F$, $[E,F]=H$, so that they define $\mathfrak{sl}_2 \curvearrowright M_\lambda$ with $\rho(e)=E$, $\rho(f)=F$, $\rho(h)=H$

Note: $\rho(h)$ still acts diagonally with eigenvalues $\lambda, \lambda-2, \lambda-4, \ldots$ and all these eigenspaces being 1-dim, BUT we no longer have $\mathbb{Z}_2$-symmetry as in Corollary 1.

Lecture #11

Exercise: a) M_λ - irreducible if $\lambda \notin \mathbb{Z}_{\geq 0}$

b) If $\lambda = n \in \mathbb{Z}_{\geq 0}$, then span$\{v_{n+1}, v_{n+2}, \ldots\}$ is an sl_2-submodule $\simeq M_{-n-2}$

c) If $\lambda = n \in \mathbb{Z}_{\geq 0}$, we get short exact sequence of sl_2-modules

$$\boxed{0 \to \underbrace{M_{-n-2} \to M_n}_{\text{Verma } \infty\text{-dim}} \to \underbrace{V_n}_{\text{irr. fin. dim}} \to 0}$$

this is the simplest example of the BGG (Bernstein-Gelfand-Gelfand) resolution

In particular, computing $\underbrace{\mathrm{tr}_\cdot(z^h)}_{= \text{character } \chi_\cdot(z)}$ which is additive on short exact seq-s:

$$\boxed{\chi_{V_n}(z) = \chi_{M_n}(z) - \chi_{M_{-n-2}}(z) = \frac{z^n}{1-z^{-2}} - \frac{z^{-n-2}}{1-z^{-2}} = \frac{z^{n+1} - z^{-n-1}}{z - z^{-1}}}$$

this is the simplest example of the Weyl character formula that we'll discuss later

* Universal enveloping and its properties

$\mathfrak{g}$ - Lie algebra $\rightsquigarrow$ $U\mathfrak{g} = T\mathfrak{g} / (x\otimes y - y \otimes x - [x,y])_{x,y\in\mathfrak{g}}$

$\mathfrak{g}$-modules = $U\mathfrak{g}$-modules

In particular, the key actor last time in the proof of Theorem 1(d) was the Casimir operator C_V which, under the above identification, is the action of the Casimir element $C = e\cdot f + f \cdot e + \frac{1}{2}h^2 \in U(sl_2)$

Exercise: Prove that the center of $U(sl_2)$ is a polynomial algebra in C. (over $\mathbb{C}$!)

Recall: Jacobi identity $\iff \mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g} \Rightarrow \mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g}^{\otimes k}$ $\forall k \geq 1 \Rightarrow \mathfrak{g} \overset{ad}{\curvearrowright} T\mathfrak{g}$

Explicitly $ad(x)(y_1 \otimes \ldots \otimes y_k) = [x,y_1]\otimes y_2 \otimes \ldots \otimes y_k + y_1 \otimes [x, y_2] \otimes \ldots \otimes y_k + \ldots$

Lemma 1: $\mathfrak{g} \overset{ad}{\curvearrowright} T\mathfrak{g}$ descends to $\mathfrak{g} \overset{ad}{\curvearrowright} U\mathfrak{g}$.

► $ad(x)(y\otimes z - z \otimes y - [y,z]) = [x,y]\otimes z + y \otimes [x,z] - [x,z]\otimes y - z \otimes [x,y] - [x,[y,z]]$

$= \underbrace{([x,y]\otimes z - z\otimes [x,y] - [[x,y],z])}_{-//-} + \underbrace{(y \otimes [x,z] - [x,z]\otimes y - [y,[x,z]])}_{\text{of the form } a\otimes b - b\otimes a - [a,b]}$

$+ \underbrace{([[x,y],z] + [y,[x,z]] - [x,[y,z]])}_{=0 \text{ by Jacobi} + [\,,\,] \text{ skew symmetry}}$

Lecture # 11

Moreover, as $\mathrm{ad}(x)(y)=[x,y]=x\cdot y-y\cdot x$ in $U\mathfrak{g}$ $\forall x,y\in\mathfrak{g}$, we note that (in contrast to $\mathfrak{g}\curvearrowright T\mathfrak{g}$), the action $\mathrm{ad}:\mathfrak{g}\curvearrowright U\mathfrak{g}$ is actually by inner derivations, i.e. $\boxed{\mathrm{ad}(x)(y)=x\cdot y-y\cdot x \;\; \forall x\in\mathfrak{g},\, y\in U\mathfrak{g}}$. Hence:

Corollary 2: $\underbrace{Z(U\mathfrak{g})}_{\text{center of } U\mathfrak{g}} \underset{\text{v.space}}{\simeq} \underbrace{(U\mathfrak{g})^{\mathfrak{g}}}_{\mathfrak{g}\text{-invariants of the } \mathfrak{g}\text{-module } U\mathfrak{g}}$

To state next the PBW theorem and its corollaries we need to recall a general notion of filtered algebras and their associated graded:

Def 1: a) A $\mathbb{Z}_{\geq 0}$-graded algebra A is an algebra A with vector space decomposition $A=\bigoplus_{k\geq 0} A_k$ s.t. $A_k\cdot A_\ell\subseteq A_{k+\ell}$ $\forall k,\ell\geq 0$, $1\in A_0$.

b) A $\mathbb{Z}_{\geq 0}$-filtered algebra A is an algebra A with a filtration of A as vector space: $0\subseteq F_0A\subseteq F_1A\subseteq F_2A\subseteq\ldots$ such that:
$F_kA\cdot F_\ell A\subseteq F_{k+\ell}A$ $\forall k,\ell\geq 0$, $1\in F_0A$, $A=\bigcup_{k\geq 0}F_kA$

c) Given a $\mathbb{Z}_{\geq 0}$-filtered algebra A as in b), the associated graded algebra, denoted $\mathrm{gr}_{F_\bullet}A$ or simply $\mathrm{gr}\,A$, is a vector space
$$\boxed{\mathrm{gr}\,A=\bigoplus_{k\geq 0}\mathrm{gr}_k A \text{ with } \mathrm{gr}_k A = F_kA/F_{k-1}A} \quad (\text{assume } F_{-1}A=0)$$
with the product defined via
$$\bar{x}\cdot\bar{y}:=\overline{x\cdot y} \qquad \forall\, \bar{x}\in\mathrm{gr}_kA,\ \bar{y}\in\mathrm{gr}_\ell A$$
where $x\in F_kA$ is a non-unique lift of $\bar{x}$, $y\in F_\ell A$ – a lift of $\bar{y}$

Exercise: a) Verify that the product on $\mathrm{gr}\,A$ is well-defined, and makes it into a $\mathbb{Z}_{\geq 0}$-graded algebra

b) Verify that if $\mathrm{gr}_{F_\bullet}A$ is a domain, then so is A.

c) For a linear map $\varphi:A\to B$ of $\mathbb{Z}_{\geq 0}$-filtered algebras s.t. $\varphi(F_kA)\subseteq F_kB$, construct $\mathrm{gr}_\bullet\varphi:\mathrm{gr}_\bullet A\to\mathrm{gr}_\bullet B$. Show that if $\mathrm{gr}_\bullet\varphi$ – isomorphism, then so is φ.

Lecture # 11

- Note that any $\mathbb{Z}_{\geq 0}$-graded algebra $A = \bigoplus_{k\geq 0} A_k$ is naturally $\mathbb{Z}_{\geq 0}$-filtered with $F_k A = \bigoplus_{s=0}^{k} A_s$, and moreover $gr_{F_\cdot} A \simeq A$ as graded algebras.

  For example, for any vector space V, the tensor algebra $T(V)$ is naturally $\mathbb{Z}_{\geq 0}$-graded with $A_k = V^{\otimes k}$, and hence $\mathbb{Z}_{\geq 0}$-filtered with $F_k T(V) = \bigoplus_{s=0}^{k} V^{\otimes s}$.

- If A is a $\mathbb{Z}_{\geq 0}$-graded algebra and $I \subseteq A$ is a graded 2-sided ideal, i.e. $I = \bigoplus_{k\geq 0} (I \cap A_k)$, then the algebra A/I is also graded with $(A/I)_k = A_k / I \cap A_k$.

  However, if I is not graded, then A/I is only a $\mathbb{Z}_{\geq 0}$-filtered algebra with $F_k(A/I)$ being the image of $F_k A = \bigoplus_{s=0}^{k} A_s$ under $A \twoheadrightarrow A/I$.

Upshot: Combining the above two bullets with $V = \mathfrak{g}$ - Lie algebra, and I being the 2-sided ideal generated by $\{x\otimes y - y\otimes x - [x,y]\}_{x,y\in\mathfrak{g}}$ we get:

$\boxed{U\mathfrak{g} \text{ is } \mathbb{Z}_{\geq 0}\text{-filtered algebra with } F_k(U\mathfrak{g}) \text{ being the image of } \bigoplus_{s=0}^{k} \mathfrak{g}^{\otimes s} \subseteq T\mathfrak{g}}$

As $x\cdot y - y\cdot x = [x,y] \in F_1 U\mathfrak{g}$ $\forall x,y\in\mathfrak{g}$, it's easy to see $[x,y] \in F_{k+l-1}(U\mathfrak{g})$ $\forall x\in F_k(U\mathfrak{g}), y\in F_l(U\mathfrak{g})$

$\Rightarrow gr_{F_\cdot} U\mathfrak{g}$ is a commutative algebra

Moreover, $\mathfrak{g}$ generates $T\mathfrak{g} \Rightarrow \mathfrak{g}$ generates $U\mathfrak{g} \Rightarrow \mathfrak{g}$ generates $gr_{F_\cdot} U\mathfrak{g}$

$\boxed{\text{Corollary 3: } \exists\ \mathbb{Z}_{\geq 0}\text{-graded algebra epimorphism } \phi : S^{\cdot}\mathfrak{g} \to gr\, U\mathfrak{g}}$

The following is the key result (we shall skip the proof for now, and leave it to undergraduate's presentation at the end of term):

$\boxed{\text{Theorem 1 (PBW theorem = Poincare-Birkhoff-Witt): } \phi \text{ is an algebra isomorphism}}$

$\Updownarrow$ One can get a more familiar reformulation, which is basis-dependent:

$\boxed{\text{Theorem 1': For any ordered basis } \{x_i\} \text{ of } \mathfrak{g}, \text{ the ordered monomials } \overrightarrow{\prod} x_i^{n_i} \text{ form a basis of } U\mathfrak{g}}$

Lecture #11

Let's now discuss some corollaries of PBW theorem.

Corollary 3: The linear map $\mathfrak{g} \to U\mathfrak{g}$ is injective

Corollary 4: If $\mathfrak{g}_1, \ldots, \mathfrak{g}_s$ are Lie subalgebras of $\mathfrak{g}$ s.t. $\mathfrak{g} = \bigoplus_{j=1}^{s} \mathfrak{g}_j$ as v.space then mult: $U(\mathfrak{g}_1) \otimes \ldots \otimes U(\mathfrak{g}_s) \to U(\mathfrak{g})$ is a v.space isomorphism.

Corollary 5: $U\mathfrak{g}$ is a domain ← see Exercise on p.3

Natural Question: Can ϕ be upgraded to v. space isom. $S\mathfrak{g} \xrightarrow{\sim} U\mathfrak{g}$ with some properties?

Recall. Both $S\mathfrak{g}$ and $U\mathfrak{g}$ are equipped with natural $\mathfrak{g}$-action.

Def 2: Assuming $\mathrm{char}(\mathbb{k})=0$, define the symmetrization map
$$\sigma: S\mathfrak{g} \longrightarrow U\mathfrak{g} \quad \text{via} \quad x_1 \cdot x_2 \cdot \ldots \cdot x_n \longmapsto \frac{1}{n!} \sum_{\tau \in S(n)} x_{\tau(1)} \cdot x_{\tau(2)} \cdot \ldots \cdot x_{\tau(n)} \quad \forall x_i \in \mathfrak{g}$$

[Exercise: Verify that σ is a $\mathfrak{g}$-module morphism

Lemma 2: $\sigma: S\mathfrak{g} \to U\mathfrak{g}$ is a $\mathfrak{g}$-module isomorphism

► σ is clearly compatible with filtrations and $\mathrm{gr}.\sigma = \phi$. As ϕ - isom, we see that σ - v. space isom (by Exercise on p.3). It remains to use the above Exercise to conclude σ - $\mathfrak{g}$-module isomorphism ∎

Combining Lemma 2 & Corollary 2, we get:

Corollary 6: $Z(U\mathfrak{g}) \underset{\text{v. space}}{\simeq} (S\mathfrak{g})^{\mathfrak{g}}$