Lecture #12

* Free Lie Algebras

Def.: Let V be a vector space over $\mathbb{k}$. The *free Lie algebra* $L(V)$ generated by V is the Lie subalgebra of $T(V)$ generated by V, under the usual commutator $[a,b]=a\cdot b-b\cdot a$ $\forall a,b\in T(V)$.

Evoking $\mathbb{Z}_{\geq 0}$-grading on $T(V)$ from last time with $\deg(V)=1$, we see that $L(V)$ is a $\mathbb{Z}_{>0}$-graded vector space (in fact, $\mathbb{Z}_{>0}$-graded Lie algebra):

$$\boxed{L(V)=\bigoplus_{n>0} L_n(V) \text{ with } L_n(V)=\text{span of commutators of } n \text{ elements of } V \text{ inside } T(V)}$$

Rmk: One can also speak of free Lie algebras generated by elements $\{x_i\}$, which is just the above $L(V)$ with V having basis $\{x_i\}$. Basis of these algebras is parametrized by so-called *Lyndon words* in $\{x_i\}$

Proposition 1: a) Embedding $L(V)\hookrightarrow T(V)$ gives rise to isom. $\varphi: \mathcal{U}(L(V)) \xrightarrow{\sim} T(V)$

b) $L(V)$ has the universal property that $\forall$ Lie alg $\mathfrak{g}$:

$$\mathrm{Hom}_{Lie}(L(V),\mathfrak{g}) \underset{\text{restrict to } V}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathbb{k}}(V,\mathfrak{g}) \text{ is isom.}$$

universal property characterizes $L(V)$ uniquely, but existence requires above explicit construction.

▶ a) Linear map $V\hookrightarrow TV$ gives rise to $L(V)\xrightarrow{\text{Lie hom.}}T(V)$, hence, we get $\varphi: \mathcal{U}(L(V))\xrightarrow{\text{alg. hom.}}T(V)$. On the other hand, $V\xrightarrow{\text{linear}}L(V)\hookrightarrow\mathcal{U}(L(V))$ gives rise to $\psi: T(V)\xrightarrow{\text{alg. hom.}}\mathcal{U}(L(V))$. We claim that φ,ψ are inverse to each other, hence isom, which follows from checking both $\varphi\circ\psi$ and $\psi\circ\varphi$ are identities on V, as V generates both $\mathcal{U}(L(V)), T(V)$

b) Any $\alpha: V\xrightarrow{\text{linear}}\mathfrak{g}$ gives rise to $V\xrightarrow{\alpha}\mathfrak{g}\hookrightarrow\mathcal{U}(\mathfrak{g})$, hence, alg. homom $T(V)\longrightarrow\mathcal{U}(\mathfrak{g})$. By a), this gives $\mathcal{U}(L(V))\xrightarrow{\text{alg. hom}}\mathcal{U}(\mathfrak{g}) \Rightarrow \alpha: L(V)\xrightarrow{\text{Lie alg. hom.}}\mathcal{U}(\mathfrak{g})$

But $L(V)$ is generated by V, $\alpha(V)\subseteq\mathfrak{g} \Rightarrow \alpha(L(V))\subseteq\mathfrak{g} \Rightarrow \alpha: L(V)\to\mathfrak{g}$

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A simple corollary of Prop 1a) and PBW theorem is:

Exercise: If $n = \dim_{\Bbbk}(V)$ and $d_m(n) = \dim_{\Bbbk}(L_m(V))$, then
$$\prod_{m\geq 1}(1-q^m)^{d_m(n)} = 1 - nq \quad , \quad q\text{- formal variable}$$

* Baker-Campbell-Hausdorff theorem & formula

Recall from Lectures 6-8 how $[\,,\,]$ was introduced on $\mathfrak{g} = T_1 G$:

$$\boxed{\mu: \mathcal{U}\times\mathcal{U} \longrightarrow \mathfrak{g} \qquad \mu(x,y) = \log(\exp(x)\cdot\exp(y)) = x+y+\tfrac{1}{2}[x,y] + \sum_{n\geq 3}\mu_n(x,y)}$$

$\mathcal{U}$ – small nbhd of $0\in\mathfrak{g}$; $\mu_n(x,y)$ – degree n terms

Question: Did we loose any information on $\mathfrak{g}$ arising from $\{\mu_n \mid n\geq 3\}$?

Surprisingly not, due to the following:

Theorem 1 (BCH theorem): For any $n\geq 3$, $\mu_n(x,y)$ is a Lie polynomial in x,y of degree n with $\mathbb{Q}$-coefficients (i.e. $\mathbb{Q}$-linear combination of iterated commutators in x,y) which is universal (i.e. G-independent)

To prove this result (without finding explicit formulas for μ_n), we shall first take a quick detour to coproducts. To this end, we start with algebra homomorphism

"coproduct" $\Delta: T(\mathfrak{g}) \longrightarrow T(\mathfrak{g})\otimes T(\mathfrak{g})$ uniquely defined by $\Delta(x) = x\otimes 1 + 1\otimes x \quad \forall x\in\mathfrak{g}$

Lemma 1: Δ descents to $\Delta: \mathcal{U}(\mathfrak{g}) \xrightarrow[\text{alg. hom}]{} \mathcal{U}(\mathfrak{g})\otimes\mathcal{U}(\mathfrak{g})$

► $\forall x,y\in\mathfrak{g}$: $\Delta(x\otimes y - y\otimes x - [x,y]) = (x\otimes 1 + 1\otimes x)(y\otimes 1 + 1\otimes y) - (y\otimes 1 + 1\otimes y)(x\otimes 1 + 1\otimes x) - ([x,y]\otimes 1 + 1\otimes [x,y]) = (xy - yx - [x,y])\otimes 1 + 1\otimes(xy - yx - [x,y])$

Thus $\Delta(I) \subseteq I\otimes T(\mathfrak{g}) + T(\mathfrak{g})\otimes I$ with $I = (xy - yx - [x,y]) \Rightarrow \Delta$ descends. ▪

$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/I$

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Remark: In general, note that coproduct on associative algebra A allows to take $\otimes$-product of A-modules (V_1, ρ_1), (V_2, ρ_2) via $a.(v_1 \otimes v_2) = \sum \rho_1(a_{(1)})v_1 \otimes \rho_2(a_{(2)})v_2$ with $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.
Evoking our identification $\{\mathfrak{g}\text{-modules}\} = \{A = U(\mathfrak{g})\text{-modules}\}$, we see that this $\otimes$-product matches $\otimes$-product of $\mathfrak{g}$-mod.

Def: An element x of $T(\mathfrak{g})$ or $U(\mathfrak{g})$ (or some of their completions w.r.t. filtration) is

a) primitive if $\Delta(x) = x \otimes 1 + 1 \otimes x$

b) group-like if $\Delta(x) = x \otimes x$

Exercise: a) Show that x-primitive $\Leftrightarrow$ $\underbrace{\exp(x)}_{\sum_{n\geq 0} \frac{x^n}{n!}}$-group-like

b) Prove that if char $\mathbb{k} = 0$, then $\underbrace{\mathrm{Prim}(U\mathfrak{g})}_{\text{set of all primitive elts of } U\mathfrak{g}} = \mathfrak{g}$.

c) Is part b) true for char$(\mathbb{k}) = p > 0$?

Proof of Theorem 1 ("Baker–Campbell–Hausdorff Theorem")

The proof is completely formal and doesn't need to appeal to G!
Consider the largest Lie algebra generated by two elts x, y, i.e. $L(V)$ with $V = \mathbb{k}x \oplus \mathbb{k}y$ – 2dim v.space. Then, all we want to show is that all terms μ_n below are Lie polynomials, i.e. in $L_n(V)$:

$$\log(\exp(x) \cdot \exp(y)) = \sum_{n\geq 1} \mu_n(x,y)$$

formal power series; equality in $T(V) = \mathbb{k}\langle x,y\rangle$ – free assoc algebra

But x, y – primitive $\overset{\text{Ex a)}}{\Longrightarrow}$ $\exp(x), \exp(y)$ – group-like $\Rightarrow$ $\exp(x)\cdot\exp(y)$ – group like
$\overset{\text{Ex a)}}{\Longrightarrow}$ $\log(\exp(x)\cdot\exp(y))$ – primitive $\xrightarrow{\text{Ex b)}}$ $\mu_n(x,y) \in L_n(V)$ ∎

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Remark: In fact, there is an explicit formula for all μ_n above, known as the "Baker-Campbell-Hausdorff formula":

$$\boxed{\log(e^x \cdot e^y) = \sum_{k\geq 1} \sum_{\substack{m_1+n_1>0 \\ \dots \\ m_k+n_k>0}} \frac{(-1)^{k-1}}{k\sum_{i=1}^{k}(m_i+n_i)\prod_{i=1}^{k} m_i!\,n_i!} [\underbrace{X,\dots,X}_{m_1},[\underbrace{y,\dots,y}_{n_1},\dots,[\underbrace{y,\dots,y}_{n_k}]]]}$$

* Nilpotent & Solvable Lie Algebras

Recall that a subspace I of a Lie algebra $\mathfrak{g}$ is called an ideal if $[\mathfrak{g}, I] \subseteq I$ i.e. $[x,y] \in I \ \forall x \in \mathfrak{g}, y \in I$.

Exercise: a) Given two ideals I_1, I_2 of $\mathfrak{g}$ verify that I_1+I_2, $I_1 \cap I_2$, and $[I_1, I_2] = \mathrm{span}\{[x,y] \mid x \in I_1, y \in I_2\}$ - ideals of $\mathfrak{g}$.

b) For any Lie homomorphism $\varphi: \mathfrak{g}_1 \to \mathfrak{g}_2$ verify that $\mathrm{Ker}\,\varphi$ - ideal of $\mathfrak{g}_1$, $\mathrm{Im}\,\varphi$ - subalgebra of $\mathfrak{g}_2$, and $\mathfrak{g}_1/_{\mathrm{Ker}\,\varphi} \xrightarrow[\sim]{\text{Lie alg}} \mathrm{Im}\,\varphi$.

Def: A commutant of a Lie algebra $\mathfrak{g}$ is the ideal $[\mathfrak{g}, \mathfrak{g}]$.

Note that $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ - abelian Lie algebra (i.e. $[\,,\,] \equiv 0$ on it) and moreover $[\mathfrak{g},\mathfrak{g}]$ is the minimal of all ideals I s.t. $\mathfrak{g}/I$ - abelian

Exercise: Compute commutant of $\mathfrak{gl}_n(\mathbb{k})$

Def: a) For any Lie algebra $\mathfrak{g}$, the derived sequence of ideals is:

$$\mathfrak{g} = D^0\mathfrak{g} \supseteq D^1\mathfrak{g} \supseteq D^2\mathfrak{g} \supseteq \dots \quad \text{with } D^{k+1}\mathfrak{g} = [D^k\mathfrak{g}, D^k\mathfrak{g}] \ \forall k.$$

b) $\mathfrak{g}$ is called solvable if $D^n\mathfrak{g} = 0$ for some $n \geq 0$.

c) For any Lie algebra $\mathfrak{g}$, the lower central series of ideals is:

$$\mathfrak{g} = D_0\mathfrak{g} \supseteq D_1\mathfrak{g} \supseteq D_2\mathfrak{g} \supseteq \dots \quad \text{with } D_{k+1}\mathfrak{g} = [\mathfrak{g}, D_k\mathfrak{g}]$$

d) $\mathfrak{g}$ is called nilpotent if $D_n\mathfrak{g} = 0$ for some $n \geq 0$.

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<u>Exercise</u>: a) Verify that $\mathfrak{b} = \left\{ \begin{matrix} \text{upper-triangular} \\ n \times n \text{ matrices} \end{matrix} \right\}$ - solvable <u>BUT</u> not nilpotent.

b) Verify that $\mathfrak{n} = \left\{ \begin{matrix} \text{strictly upper-triangular} \\ n \times n \text{ matrices} \end{matrix} \right\}$ is nilpotent.

c) Is $\mathfrak{sl}_2$ either solvable or nilpotent?

<u>Exercise</u>: a) Show that any subalgebra and quotient of solvable Lie algebra are solvable

b) If ideal I of $\mathfrak{g}$ and quotient $\mathfrak{g}/I$ are solvable Lie algebras, does $\mathfrak{g}$ have to be solvable?

c) Show that any subalgebra and quotient of nilpotent Lie algebra are nilpotent.

d) If ideal I & quotient $\mathfrak{g}/I$ are nilpotent, does $\mathfrak{g}$ have to be nilpotent too?