Lecture #13

Last time: Solvable & Nilpotent Lie Algebras

Lemma 1: $\mathfrak{g}$-solvable $\iff$ $\exists$ chain of Lie subalgebras $\mathfrak{g}=I^0 \supseteq I^1 \supseteq I^2 \supseteq \ldots \supseteq I^n = 0$ s.t. I^{k+1} - ideal of I^k and I^k/I^{k+1} - abelian $\forall k$.

► $\Rightarrow$: take $I^k = D^k\mathfrak{g}$ $\forall k$

$\Leftarrow$: argue by induction that $D^k\mathfrak{g} \subseteq I^k$ $\forall k$: base $k=0$ is clear, while step of induction is $D^{k+1}\mathfrak{g} = [D^k\mathfrak{g}, D^k\mathfrak{g}] \subseteq [I^k, I^k] \subseteq I^{k+1}$ b/c I^k/I^{k+1} - abelian

So: $D^n\mathfrak{g} \subseteq I^n = 0 \Rightarrow D^n\mathfrak{g} = 0 \Rightarrow \mathfrak{g}$ - solvable ∎

Lemma 2: $\mathfrak{g}$ - nilpotent $\iff$ $\exists$ chain of ideals $\mathfrak{g} = I_0 \supset I_1 \supset I_2 \supset \ldots \supset I_n = 0$ s.t. $[\mathfrak{g}, I_k] \subseteq I_{k+1}$ $\forall k$

► $\Rightarrow$: take $I_k = D_k\mathfrak{g}$

$\Leftarrow$: argue by induction on k that $D_k\mathfrak{g} \subseteq I_k$ $\forall k$ $\Rightarrow D_n\mathfrak{g} = 0 \Rightarrow \mathfrak{g}$ - nilpotent ∎

As we shall also often use the last Exercise from previous lecture, namely its parts a) & b), let's prove it:

Lemma 3: a) If $\mathfrak{g}$ - solvable, then any subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ - solvable, any quotient $\mathfrak{g}/I$ - solvable too

b) If ideal I of $\mathfrak{g}$ and $\mathfrak{g}/I$ are both solvable $\Rightarrow \mathfrak{g}$ - solvable

► a) $D^k\mathfrak{h} \subseteq D^k\mathfrak{g}$, so if $D^n\mathfrak{g} = 0 \Rightarrow D^n\mathfrak{h} = 0$

$D^k\mathfrak{g} \twoheadrightarrow D^k(\mathfrak{g}/I)$, so if $D^n\mathfrak{g} = 0 \Rightarrow D^n(\mathfrak{g}/I) = 0$

b) I - solvable, $\mathfrak{g}/I$ - solvable $\Rightarrow D^n(I) = 0$, $D^m(\mathfrak{g}/I) = 0$ for some n, m.

As $D^k(\mathfrak{g}/I) = (D^k(\mathfrak{g}) + I)/I$ $\forall k$ (check this!) $\Rightarrow D^m(\mathfrak{g}) \subseteq I$

Then $\underbrace{D^k(D^m(\mathfrak{g}))}_{D^{m+k}(\mathfrak{g})} \subseteq D^k(I)$ $\forall k \Rightarrow D^{n+m}\mathfrak{g} = 0 \Rightarrow \mathfrak{g}$ - solvable ∎

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Last time we saw two prototypical examples:

1) $\mathfrak{b} = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}_{n\times n} \right\}$ (= {upper-Δ $n\times n$ matrices}) - solvable Lie alg.

2) $\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}_{n\times n} \right\}$ = {strictly upper-Δ $n\times n$ matrices} - nilpotent Lie alg.

Now we are ready to state the key results about solvable & nilpotent Lie algebras

Theorem 1 (Lie's Theorem): Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a fin. dim. representation of a solvable Lie algebra $\mathfrak{g}$, over an algebraically closed field $\mathbb{k}$ of char $=0$. Then there is a basis of V, with respect to which all $\{\rho(x) \mid x \in \mathfrak{g}\}$ are upper-triangular matrices, i.e. $\rho(\mathfrak{g}) \subseteq \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$

Exercise: Find a counterexample for char $\mathbb{k} > 0$.

Theorem 2 (Engel's Theorem): A Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\forall x \in \mathfrak{g}$ operator $ad(x): \mathfrak{g} \to \mathfrak{g}$ - nilpotent

Remarks: 1) Theorem 2 is valid over all fields, in contrast to Theorem 1.

2) While Theorem 1 is about any fin. dim. $\mathfrak{g}$-module, Theorem 2 is only about the adjoint module $\mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g}$

Corollary 1: (assume $\mathbb{k} = \bar{\mathbb{k}}$, char $\mathbb{k} = 0$)

a) Any irreducible f.d. module of solvable $\mathfrak{g}$ is 1-dimensional

b) If $\mathfrak{g}$-solvable $\Rightarrow \exists$ ideals $0 \subset I_1 \subset I_2 \subset \ldots \subset I_N = \mathfrak{g}$ s.t. I_{k+1}/I_k - 1-dimensional $\forall k$.

c) $\mathfrak{g}$ - solvable $\Leftrightarrow [\mathfrak{g}, \mathfrak{g}]$ - nilpotent

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Proof of Corollary 1

a) Apply Lie's Theorem to $\mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g}$. Then there is a basis $v_1, \ldots, v_N$ of $\mathfrak{g}$ s.t. $ad(x)(v_k) \in \mathrm{span}\{v_1, \ldots, v_k\}$ $\forall x \in \mathfrak{g}$ $\forall 1 \leq k \leq N$. In particular, the 1-dim subspace $\Bbbk \cdot v_1$ is $ad(\mathfrak{g})$-invariant $\Rightarrow$ submodule $\Rightarrow$ $V = \Bbbk v_1$ as V-simple

b) As $I \subseteq \mathfrak{g}$-ideal iff I is a submodule of the adjoint module, we get:

- $\mathfrak{g}$-solvable $\Rightarrow$ take $I_k = \mathrm{span}\{v_1, \ldots, v_k\}$ $\forall k$ in the notations of a)
- given ideals $I_1 \subset I_2 \subset \ldots \subset I_N = \mathfrak{g}$ with $\dim(I_{k+1}) = \dim(I_k) + 1$, pick a basis of $\mathfrak{g}$ so that $I_k = \mathrm{span}\{v_1, \ldots, v_k\}$ $\forall k$. Then $ad(\mathfrak{g}) \subseteq$ upper-Δ matrices
  $\Downarrow$
  $\mathfrak{g}$-solvable

c) $\Leftarrow$: $\left.\begin{array}{l} \text{If } [\mathfrak{g},\mathfrak{g}]\text{-nilpotent} \Rightarrow [\mathfrak{g},\mathfrak{g}]\text{-solvable} \\ \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\text{-abelian} \Rightarrow \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\text{-solvable} \end{array}\right\} \Rightarrow \mathfrak{g}$-solvable by Lemma 3.

$\Rightarrow$: Apply Lie's Theorem to $ad: \mathfrak{g} \curvearrowright \mathfrak{g}$ to get $ad(\mathfrak{g}) \subseteq \mathfrak{b} = \left\{\begin{array}{c}\text{upper-}\Delta \\ \text{matrices}\end{array}\right\}$, hence, $ad([\mathfrak{g},\mathfrak{g}]) \subseteq [\mathfrak{b},\mathfrak{b}] = \mathfrak{n} = \left\{\begin{array}{c}\text{strictly upper-}\Delta \\ \text{matrices}\end{array}\right\}$. This proves that $ad([\mathfrak{g},\mathfrak{g}]) = [ad\,\mathfrak{g}, ad\,\mathfrak{g}]$ is nilpotent, hence $\exists n > 0$ s.t.

$ad[x_1, [x_2, \ldots, [x_{n-1}, x_n]\ldots]] = 0$ $\forall x_i \in [\mathfrak{g},\mathfrak{g}]$ $\Rightarrow [x_0, [x_1, \ldots [x_{n-1}, x_n]\ldots]] = 0$ $\forall x_0, x_1, \ldots x_n \in [\mathfrak{g},\mathfrak{g}]$ ∎

The proof of both theorems relies on the following two results:

Proposition 1: Let $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ be a fin. dim. representation of a solvable Lie algebra $\mathfrak{g}$, over an alg. closed field $\Bbbk$ of char $\Bbbk = 0$. Then $\exists v \in V \setminus \{0\}$ - common eigenvector of all $\{\rho(x)\}_{x \in \mathfrak{g}}$.

Proposition 2: Let V be a fin. dim. vector space over any field $\Bbbk$, and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be a Lie subalgebra consisting of nilpotent operators. Then $\exists v \in V \setminus \{0\}$ s.t. $\mathfrak{g}(v) = 0$, i.e. v - common eigenvector w/ eigenvalue 0

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- Note that Prop 1 $\Rightarrow$ Thm 1 by induction on $\dim V$. Indeed, pick $v_1 = v$ as in Prop 1 and consider $\mathfrak{g} \curvearrowright \bar{V} := V/\Bbbk v_1$. By induction hypothesis, there is a basis $\bar{v}_2, \ldots, \bar{v}_N$ of $\bar{V}$ in which $\mathfrak{g}$ acts by upper-Δ matrices. Pick any lifts $v_2, \ldots, v_N$ (of $\bar{v}_2, \ldots, \bar{v}_N$) in V: then in basis $\{v_1, v_2, \ldots, v_N\}$ $\mathfrak{g}$ acts by upper-Δ matrices $\Rightarrow$ Theorem 1.
- Completely similar argument shows that Prop 2 $\Rightarrow$ Thm 2.

We shall now prove Proposition 1 (postponing Prop 2 to the next class)

Proof of Proposition 1

▷ Proceed by induction on $\dim \mathfrak{g}$. Base case $\dim \mathfrak{g} = 1$ is clear!

Step 1: $\mathfrak{g}$-solvable $\Rightarrow [\mathfrak{g}, \mathfrak{g}] \subsetneq \mathfrak{g}$. Pick any vector subspace $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}' \subsetneq \mathfrak{g}$ so that $\dim(\mathfrak{g}') = \dim(\mathfrak{g}) - 1$. Then $[\mathfrak{g}, \mathfrak{g}'] \subseteq [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}' \Rightarrow \mathfrak{g}'$-ideal of $\mathfrak{g}$. Moreover $\mathfrak{g}$-solvable $\Rightarrow \mathfrak{g}'$-solvable Lie algebra (of smaller dimension). Hence by induction hypothesis applied to $\mathfrak{g}' \curvearrowright V$:

$$\boxed{\exists v \in V \setminus \{0\} \text{ s.t. } \rho(h')v = \lambda(h') \cdot v \quad \forall h' \in \mathfrak{g}', \ \lambda \in \mathfrak{g}'^*} \qquad (*)$$

Step 2: Pick any $x \in \mathfrak{g} \setminus \mathfrak{g}'$ so that $\mathfrak{g} = \mathfrak{g}' \oplus \Bbbk x$ as vector spaces. Consider

$$\boxed{W := \mathrm{span}\{v_0 = v, \ v_1 = \rho(x)v, \ v_2 = \rho(x)v_1 = \rho(x)^2 v, \ \ldots\}}$$

$$\boxed{\text{Claim 1: } \forall h' \in \mathfrak{g}', \ \forall k \geq 0: \ \rho(h')v_k = \lambda(h')v_k + \sum_{\ell < k} \# \cdot v_\ell, \quad \#\text{- some coeff-s}}$$

▷ Induction on k. Base $k=0$ is clear by $(*)$. For induct. step:

$$\rho(h')v_{k+1} = \rho(h')\rho(x)v_k = \rho(x)(\rho(h')v_k) + \rho([h', x])v_k = \lambda(h')v_k + \sum_{\ell<k} \# \cdot v_\ell$$

By inductive hypothesis: $\rho(h')v_k = \lambda(h)v_k + \sum_{\ell<k} \# \cdot v_\ell$

$$\rho([h', x])v_k = \lambda([h', x]) \cdot v_k + \sum_{\ell<k} \tilde{\#} \cdot v_\ell$$

□

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(Continuation of Proof)

Step 3: As $\dim W \leq \dim V < \infty$ $\exists n$ s.t. $v_0, v_1, \ldots, v_n$ - lin. indep, but v_{n+1} is in span $\{v_0, v_1, \ldots, v_n\}$. As $v_{k+1} = \rho(x) v_k$, we then see that actually $\{v_0, v_1, \ldots, v_n\}$ - basis of W.

By Claim 1 above: $tr_W \, \rho(h') = (n+1) \cdot \lambda(h')$

On the other hand if $\mathfrak{g}' \ni h' = [h, x]$ with $h \in \mathfrak{g}'$, then $tr_W \, \rho(h') = tr_W([\rho(h), \rho(x)]) = 0$

$\Rightarrow \lambda([\mathfrak{g}', x]) \equiv 0$.

But then: evoking the proof of Claim 1, we actually get:

$$\boxed{\rho(h') v_k = \lambda(h') v_k \quad \forall h' \in \mathfrak{g}' \; \forall k}$$

Step 4: Pick any eigenvector $w \in W \setminus \{0\}$ of the operator $\rho(x) : W \to W$. By above w is also an eigenvector of $\rho(\mathfrak{g}')$, hence, of all $\rho(\mathfrak{g})$ as $\mathfrak{g} = \mathfrak{g}' \oplus kx$ ■

Remark: We note that while in the end of Step 3 we concluded that $\rho(\mathfrak{g}')$ acts by scalar operators on W, this wasn't immediately clear: we first proved it acts by upper-Δ matrices (Claim 1).