Lecture #14

Theorem 1 (Engel's theorem): Lie algebra $\mathfrak{g}$ - nilpotent $\iff$ $\mathrm{ad}\,x : \mathfrak{g} \circlearrowleft$ - nilpotent operator $\forall x \in \mathfrak{g}$

Rmk: We clearly cannot hope for anything like Lie's theorem, i.e. it's not true that any fin. dim. module of a nilpotent Lie alg. $\mathfrak{g}$ has a basis in which $\rho(\mathfrak{g})$ - strictly upper-triangular (e.g. take diagonal action of $\mathbb{k}$)

Last time we noted that it suffices to prove the following result:

Proposition 1: Let V - fin. dim. v. space over any field $\mathbb{k}$, and $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ - a Lie subalgebra (i.e. closed under $[\,,\,]$) consisting of nilpotent operators. Then $\exists\, v \in V \setminus \{0\}$ s.t. $x(v) = 0 \ \forall x \in \mathfrak{g}$

▸ Proof is by induction on $\dim \mathfrak{g}$: base case $\dim \mathfrak{g} = 1$ is clear.
The induction step is crucially based on the following:

Claim 1: $\exists$ ideal $\mathfrak{h} \subseteq \mathfrak{g}$ of codimension 1.

Let's first finish the proof of Proposition using this Claim
By induction hypothesis, applied to $\mathfrak{h} \subseteq \mathrm{End}(V)$, we have $V^{\mathfrak{h}} \neq 0$ where $V^{\mathfrak{h}} = \{v \in V \mid h(v) = 0 \ \forall h \in \mathfrak{h}\}$. Pick any $x \in \mathfrak{g} \setminus \mathfrak{h}$ s.t. $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{k}x$ as v. spaces.
Note that $[x, \mathfrak{h}] \subseteq \mathfrak{h}$ as $\mathfrak{h}$-ideal $\Rightarrow$ $V^{\mathfrak{h}}$ is x-invariant $\left(hxv = x\underbrace{hv}_{=0} + \underbrace{[h,x]v}_{=0} \ \forall v \in V^{\mathfrak{h}} \right)$
Thus, $V^{\mathfrak{h}} \subseteq V$ is a $\mathfrak{g}$-submodule. Hence, it remains to find $v \in V^{\mathfrak{h}}$ s.t. $x(v) = 0$. But x - nilpotent operator $V^{\mathfrak{h}} \to V^{\mathfrak{h}}$ $\Rightarrow$ $\exists n$: $x^n|_{V^{\mathfrak{h}}} \neq 0$, $x^{n+1}|_{V^{\mathfrak{h}}} = 0$
Then any nonzero $v \in x^n(V^{\mathfrak{h}})$ satisfies: $x(v) = 0$, $\mathfrak{h}(v) = 0 \Rightarrow \mathfrak{g}(v) = 0$

Proof of Claim 1

▸ Pick max proper Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ (non unique choice!) We claim that then $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ with $\dim \mathfrak{h} = \dim \mathfrak{g} - 1$. Indeed, $\forall y \in \mathfrak{h}$ a nilpotent operator $\mathrm{ad}(y) : \mathfrak{g} \circlearrowleft$ descends to nilpotent $\mathrm{ad}(y) : \mathfrak{g}/\mathfrak{h} \circlearrowleft$
By induction hypothesis for $\mathfrak{h}$, or rather its image in $\mathrm{End}(\mathfrak{g}/\mathfrak{h})$, $\exists\, \bar{x} \in \mathfrak{g}/\mathfrak{h}$ s.t. $[\mathfrak{h}, \bar{x}] = 0 \in \mathfrak{g}/\mathfrak{h}$ $\Rightarrow$ any lift x of $\bar{x}$ satisfies $[x, \mathfrak{h}] \subseteq \mathfrak{h}$.

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(Continuation)

But then $\mathrm{span}\{\mathfrak{h}, x\} \subseteq \mathfrak{g}$ is a larger Lie subalgebra than $\mathfrak{h}$ $\Rightarrow$ equals $\mathfrak{g}$.

So: $\dim \mathfrak{h} = \dim \mathfrak{g} - 1$ and $\mathfrak{h}$ - ideal of $\mathfrak{g}$ $\square$

* For the rest of today, we shall discuss semisimple and reductive Lie algebras. To this end, we start with 2 definitions:

|| Def 1: Lie algebra $\mathfrak{g}$ - semisimple if it has no nonzero solvable ideals

|| Def 2: Lie algebra $\mathfrak{g}$ - simple if it has no proper ideals and is not abelian

Remark: a) If $\mathfrak{g}$ - semisimple, then its center $\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x,y] = 0 \ \forall y\}$ is 0.

b) The condition "not abelian" in Def 2 is made to exclude 1-dimensional Lie algebras

c) Note that notion of Def 2 is stronger than Def 1, i.e $\mathfrak{g}$ - simple $\Rightarrow$ $\mathfrak{g}$ - semisimple (the only possible nonzero ideal is $\mathfrak{g}$ but if it was solvable $\Rightarrow$ $\mathfrak{g}$ - abelian $\lightning$)

Example: $\mathfrak{sl}_2$ - simple.

Let I be a proper ideal of $\mathfrak{sl}_2$. In particular, $[h, I] \subseteq I$. But $\mathrm{ad}(h) = [h, -]$ acts on $\mathfrak{sl}_2$ with pairwise distinct eigenvalues: $[h,e] = 2e$, $[h,h] = 0$, $[h,f] = -2f$. Thus, any $\mathrm{ad}(h)$-invariant subspace is spanned by some of $\{h, e, f\}$. But using $[e, I] \subseteq I$, $[f, I] \subseteq I$ we can now easily see that $I = \mathfrak{sl}_2$ $\Rightarrow$ $\lightning$. $\blacksquare$

Remark: In fact, we shall soon see that all classical Lie algebras $\mathfrak{sl}_n$, $\underset{(n>4)}{\mathfrak{so}_n}$, $\mathfrak{sp}_{2n}$ are simple

For the rest of today, we shall discuss how any fin. dim. Lie algebra can be "split" into solvable and semisimple parts.

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Lemma 1: In any fin. dim. Lie algebra $\exists!$ max solvable ideal.

▶ Uniqueness is clear, while existence is due to the following result:

(*) $\boxed{\text{if } I_1, I_2 \text{ - solvable ideals of } \mathfrak{g} \Rightarrow I_1+I_2 \text{ - also solvable}}$

which implies that the sum of all solvable ideals will be max solv. ideal.

To prove (*), we note $0 \to I_1 \to I_1+I_2 \to (I_1+I_2)/I_1 \simeq I_2/I_1\cap I_2 \to 0$

But $\underset{\text{solvable}}{I_2} \twoheadrightarrow I_2/I_1\cap I_2 \Rightarrow I_1, (I_1+I_2)/I_1$ - solvable $\Rightarrow I_1+I_2$ - solvable
Lemma 3 of Lecture 13 ∎

Def 3: The max solvable ideal of $\mathfrak{g}$ (see Lemma 1) is called radical of $\mathfrak{g}$ and is denoted $\mathrm{rad}(\mathfrak{g})$.

[Rmk: $\mathrm{rad}(\mathfrak{g})=0 \Leftrightarrow \mathfrak{g}$ - semisimple.

Lemma 2: a) For any $\mathfrak{g}$, the quotient $\mathfrak{g}_{ss} := \mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$ is semisimple

b) Vice versa, if $I \subseteq \mathfrak{g}$ - solvable ideal s.t. $\mathfrak{g}/I$ - semisimple $\Rightarrow \underset{\mathrm{rad}\,\mathfrak{g}}{\overset{I}{\shortparallel}}$

▶ a) For any nonzero solvable ideal $\bar{I} \subseteq \mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$, consider its preimage $I=\pi^{-1}(\bar{I})$ for $\pi: \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathrm{rad}\,\mathfrak{g}$. Then, $0 \to \mathrm{rad}\,\mathfrak{g} \to I \to \bar{I} \to 0 \Rightarrow I$ - solvable, a contradiction with $\mathrm{rad}\,\mathfrak{g}$ - max solvable.
solvable

b) On one hand, by definition of radical: $I \subseteq \mathrm{rad}\,\mathfrak{g}$, but on the other hand $\mathrm{rad}\,\mathfrak{g}/I \subseteq \mathfrak{g}/I$ - solvable $\Rightarrow \mathrm{rad}\,\mathfrak{g}/I = 0 \Rightarrow I = \mathrm{rad}\,\mathfrak{g}$ ∎

The part a) of above Lemma implies that any f.d. Lie algebra $\mathfrak{g}$ has:

$$\boxed{\text{short exact sequence} \quad 0 \to \underset{\text{solvable}}{\mathrm{rad}\,\mathfrak{g}} \hookrightarrow \mathfrak{g} \overset{\pi}{\twoheadrightarrow} \underset{\text{semisimple}}{\mathfrak{g}_{ss}} \to 0} \quad (\dagger)$$

In fact, the above can be upgraded to (proof is omitted for right now):

Theorem 2 (Levi theorem): Any fin. dim. Lie algebra $\mathfrak{g}$ over $\mathbb{k}=\bar{\mathbb{k}}$, $\mathrm{char}\,\mathbb{k}=0$ can be decomposed into $\mathfrak{g} = \underset{\text{ideal}}{\mathrm{rad}(\mathfrak{g})} \oplus \underset{\text{subalgebra NOT ideal}}{\mathfrak{g}_{ss}}$ as v. spaces, $\mathfrak{g}_{ss}$ - semisimple

In other words, projection $\pi: \mathfrak{g} \twoheadrightarrow \mathfrak{g}_{ss}$ admits a $\overset{\text{(non-unique)}}{\text{section}}$ $s: \mathfrak{g}_{ss} \hookrightarrow \mathfrak{g}$.

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Def 4: Decomposition $\mathfrak{g} \overset{\text{v.sp.}}{=} \mathrm{rad}\,\mathfrak{g} \oplus \mathfrak{g}_{s.s}$ of Theorem 2 is called a *Levi decomposition*.

[Rmk: After solving Problem 5 of Homework 5, recast above as $\mathfrak{g} \simeq \mathfrak{g}_{ss} \ltimes \mathrm{rad}\,\mathfrak{g}$.

Exercise: Find a Levi decomposition for $\mathfrak{g} = \begin{pmatrix} *_{n_1 \times n_1} & & * \\ & *_{n_2 \times n_2} & \\ & & \ddots \;\; * \end{pmatrix}$ – block upper triangular (with blocks $n_1 \times n_1$, $n_2 \times n_2$, ...)

Proposition 2: Let V be an irreducible $\mathfrak{g}$-module (over $k = \bar{k}$, char $k = 0$).

Then: a) $\forall h \in \mathrm{rad}(\mathfrak{g})$, $\rho(h)$ is a scalar operator on V.

b) $\forall h \in [\mathfrak{g}, \mathrm{rad}\,\mathfrak{g}]$ acts by zero on V.

▷ a) Consider $\mathrm{rad}\,\mathfrak{g} \curvearrowright V \overset{\text{Lie's Thm}}{\Longrightarrow} \exists\, v \in V \setminus \{0\}$ s.t. $\rho(h)v = \lambda(h)v \;\; \forall h \in \mathrm{rad}\,\mathfrak{g}$, with $\lambda \in (\mathrm{rad}\,\mathfrak{g})^*$.

Pick any $x \in \mathfrak{g} \setminus \mathrm{rad}\,\mathfrak{g}$ and consider $\tilde{\mathfrak{g}} = \mathrm{rad}\,\mathfrak{g} \oplus kx$ – Lie subalgebra of $\mathfrak{g}$.

Following proof of Prop 1 from last time, consider the same subspace

$$W = \mathrm{span}\{V_0 = v,\ V_1 = x(V_0),\ V_2 = x(V_1),\ \ldots\}$$

We proved that $\forall h \in \mathrm{rad}(\mathfrak{g})$ acts by $\lambda(h)\cdot \mathrm{Id}_W$ on W, i.e. $W \subseteq V_\lambda = \{\lambda\text{-eigensp. of } \mathrm{rad}(\mathfrak{g})\}$.

But as the choice of $v \in V_\lambda$ and $x \in \mathfrak{g} \setminus \mathrm{rad}\,\mathfrak{g}$ was arbitrary, this implies that $V_\lambda \subseteq V$ is a nonzero $\mathfrak{g}$-submodule $\overset{V\text{-simple}}{\Longrightarrow} V_\lambda = V$, which proves a).

b) Obvious from a). ∎

As $[\mathfrak{g}, \mathrm{rad}\,\mathfrak{g}]$ acts trivially on all f.d. simple $\mathfrak{g}$-modules, it's natural to consider the class of those $\mathfrak{g}$ s.t. $[\mathfrak{g}, \mathrm{rad}\,\mathfrak{g}] = 0 \iff \mathrm{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$.

Def 5: A Lie algebra $\mathfrak{g}$ is *reductive* if $\mathrm{rad}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g})$ (= center of $\mathfrak{g}$)

In the case of reductive Lie algebras, Theorem 2 can be recast as:

Proposition 3: $\mathfrak{g}$ - reductive $\iff \mathfrak{g} \cong \underbrace{\mathfrak{z}(\mathfrak{g})}_{\text{abelian}} \oplus \underbrace{\mathfrak{g}_{s.s.}}_{\text{semisimple}}$ – direct sum of Lie alg-s

We shall provide a direct proof of this result next week.