Lecture #15

- Last time:
  - finished proof of Lie theorem
  - proved Engel's theorem
  - introduced notions of semisimple, simple, reductive Lie alg-s
  - radical $\mathrm{rad}(\mathfrak{g})$ and Levi theorem
  - finished with decomposition of reductive Lie alg-s (as direct sum!)

$$\mathfrak{g} \simeq \underbrace{\mathfrak{z}(\mathfrak{g})}_{\text{abelian}} \oplus \underbrace{\mathfrak{g}_{s.s}}_{\text{semisimple}}$$

(we shall get a different proof of this later)

Example: $\mathfrak{g} = \mathfrak{gl}_n \rightsquigarrow \mathfrak{g} = \mathbb{C}\cdot \mathrm{Id} \oplus \mathfrak{sl}_n$

- Semisimplicity through invariant bilinear forms

Last time we established simplicity of $\mathfrak{sl}_2$ by brute force (using $\mathrm{ad}(h) \curvearrowright \mathfrak{sl}_2$)
We shall now see how one could check semisimplicity in general.

Exercise: a) Verify that given any $\mathfrak{g}$-module V, the following defines a $\mathfrak{g}$-action on the space of all bilinear forms $V\times V \to \Bbbk$:

$$\boxed{xB(v,w) = -B(x.v, w) - B(v, x.w) \quad \forall x\in\mathfrak{g}}$$

We call B to be $\mathfrak{g}$-invariant if $B(x.v,w) + B(v,x.w) = 0 \ \forall x\in\mathfrak{g}\ \forall v,w$

b) Verify that B is $\mathfrak{g}$-invariant if the corresponding linear map

$$V \longrightarrow V^*, \quad v \longmapsto B(v,-)$$

is a $\mathfrak{g}$-module homom. (Check that identifying bil. maps $V\times V \to \Bbbk$ with linear maps $V\to V^*$, action in a) agrees with that on $\mathrm{Hom}(V,V^*)$)

c) If V is an irreducible $\mathfrak{g}$-module$/_{\mathbb{C}}$, show that the space of $\mathfrak{g}$-inv. bilinear forms $V\times V\to\mathbb{C}$ is zero or 1-dimensional

d) If $V=\mathfrak{g}$ w.r.t. adjoint action of $\mathfrak{g}$, and $I\subseteq \mathfrak{g}$ - ideal, then its orthogonal complement $I^\perp = \{x\in\mathfrak{g} \mid B(x,y)=0\ \forall y\in I\}$ is also ideal.

Example: Let $\mathfrak{g} = \mathfrak{gl}_n$ and $B(x,y) := \mathrm{tr}(xy)$. It is clearly symmetric and invariant:

$$B([x,y],z) + B(y,[x,z]) = \mathrm{tr}(xyz - yxz + yxz - yzx) = \mathrm{tr}([x,yz]) = 0$$

Generalizing this example with the same proof, we have:

Lemma 1. For any $\mathfrak{g}$-module V, the following is a symmetric inv. bilinear form on $\mathfrak{g}$:

$$B_V(x,y) = \mathrm{tr}_V\,(\rho_V(x)\rho_V(y))$$

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The reason why the forms from Lemma 1 are important is the following:

Lemma 2: Let $\mathfrak{g}$ be a Lie alg, V – f. dim. $\mathfrak{g}$-module, s.t. $B_V : \mathfrak{g}\times\mathfrak{g} \to \mathbb{K}$ is nondegenerate.
Then: $\mathfrak{g}$ – reductive

Exercise: Given any filtration $0 = F_0V \subseteq F_1V \subseteq \ldots \subseteq F_NV = V$ of a $\mathfrak{g}$-module V by submodules, we have $B_V(x,y) = \sum_{k=1}^{N} B_{F_kV/F_{k-1}V}(x,y)$ $\forall x,y \in \mathfrak{g}$

Proof of Lemma 2

▶ To prove $\mathfrak{g}$ – reductive, it suffices to show that $[\mathfrak{g}, \mathrm{rad}(\mathfrak{g})] = 0$.
Pick any $x \in [\mathfrak{g}, \mathrm{rad}(\mathfrak{g})]$. Then by [Lecture 14, Prop 2], $\rho_W(x) = 0$ for any irreducible $\mathfrak{g}$-module (W, ρ_W). Thus, $B_W(x,y) = 0$ $\forall y \in \mathfrak{g}$ $\forall$ irred. W.
But any fin. dim $\mathfrak{g}$-module V admits a (Jordan–Holder) filtration $0 = F_0V \subseteq F_1V \subseteq \ldots \subseteq F_NV = V$ s.t. each $F_kV/F_{k-1}V$ is an irreducible $\mathfrak{g}$-mod.
Then $B_V(x, -) \equiv 0$ by the above Exercise $\Rightarrow$ contradiction! ■

Now we are ready to state our first key result for today:

Theorem 1: All classical Lie algebras are reductive.

▶ a) For $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{K})$, consider the natural $\mathfrak{g}$-action $\mathfrak{gl}_n(\mathbb{K}) \curvearrowright \mathbb{K}^n =: V$
Then $B_V(x,y) = \sum_{i,j=1}^{n} x_{ij} y_{ji}$ is clearly non-degenerate (with $\{E_{ji}\}$ being dual to $\{E_{ij}\}$)

b) For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{K})$, we take the same $V = \mathbb{K}^n$. Then
$B_V(x,y) = 0$ for $x \in \mathfrak{sl}_n$, $y \in \mathbb{C}\cdot\mathrm{Id}$, hence, non-degeneracy follows from a)

c) For $\mathfrak{g} = \mathfrak{so}_n(\mathbb{K})$, take $V = \mathbb{K}^n$. Then:
$B_V(x,y) = \sum_{i,j} x_{ij} y_{ji} \underset{y_{ij} = -y_{ji}}{\overset{x_{ij} = -x_{ji}}{=}} -2 \sum_{i>j} x_{ij} y_{ij}$, which is clearly nondegenerate ■

Exercise: a) Finish above proof by treating $\mathfrak{g} = \mathfrak{sp}_n(\mathbb{K})$, $\mathfrak{g} = \mathfrak{u}_n$, $\mathfrak{g} = \mathfrak{su}_n$ (explain why other cases follow as well)
b) Verify that $\mathfrak{sl}_n(\mathbb{K})$, $\mathfrak{so}_n(\mathbb{K})$ with $n>2$, $\mathfrak{su}_n$, $\mathfrak{sp}_{2n}$ are semisimple, while $\mathfrak{gl}_n(\mathbb{K}) = \mathbb{K}\cdot\mathrm{Id} \oplus \mathfrak{sl}_n(\mathbb{K})$, $\mathfrak{u}_n = i\mathbb{R}\cdot\mathrm{Id} \oplus \mathfrak{su}_n$

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As an important example of the above setup, we can consider $\mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g} =: V$.

Def 1: The Killing form on $\mathfrak{g}$ is the bilinear form $B_\mathfrak{g}$ from above, i.e

$$\boxed{K(x,y) = \mathrm{tr}_\mathfrak{g}(\mathrm{ad}\, x \circ \mathrm{ad}\, y)}$$ ← it's clearly symmetric

! **Warning**: If $\mathfrak{a} \leq \mathfrak{g}$ is a Lie subalgebra, then the Killing form on $\mathfrak{a}$ is NOT the restriction of the on $\mathfrak{g}$ to $\mathfrak{a} \times \mathfrak{a}$. Hence, if needed, we shall use $K^\mathfrak{a}$ for the former and $K^\mathfrak{g}$ to the latter.

(**Exercise**: If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, then $K^\mathfrak{a} = K^\mathfrak{g}|_{\mathfrak{a}\times\mathfrak{a}}$.

Example: Let's work out example of $\mathfrak{sl}_2$ first of all. Pick a basis e, h, f:

$$\mathrm{ad}(e) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}(f) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}, \quad \mathrm{ad}(h) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

Thus: $K(e,f) = K(f,e) = 4$, $K(h,e) = K(e,h) = 0$, $K(h,f) = K(f,h) = 0$

$K(h,h) = 8$, $K(e,e) = 0$, $K(f,f) = 0$.

So: $\boxed{K(x,y) = 4\,\mathrm{tr}(xy)}$

↑ by Exercise 1(c) on p.1, this is not surprising as $\mathfrak{sl}_2$ – simple.

Our other two Key Results for today are Cartan's Criteria for solvability and semisimplicity of Lie algebras

Theorem 2 (Cartan's Criteria for solvability): $\mathfrak{g}$ – solvable iff $K([\mathfrak{g},\mathfrak{g}], \mathfrak{g}) = 0$

Theorem 3 (Cartan's Criteria for semisimplicity):

$\mathfrak{g}$ – semisimple $\Leftrightarrow$ Killing form K is non-degenerate

It is instructive to compare Thm 3 to Lemma 2.

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Let us first deduce Theorem 3 from Theorem 2.

Proof of Theorem 3

$\Leftarrow$: K-nondeg, hence, $\mathfrak{g}$-reductive by Lemma 2. Hence, remains to prove $\mathfrak{z}(\mathfrak{g})=0$.
Pick $x\in\mathfrak{z}(\mathfrak{g})$, then $\mathrm{ad}(x)\equiv 0 \Rightarrow K(x,-)\equiv 0 \Rightarrow x\in \mathrm{Ker}\, K \Rightarrow$ ↯

$\Rightarrow$: Let $\mathfrak{g}$-semisimple, and set $I:=\mathrm{Ker}\, K$. By Exercise 1d), with $I=\mathfrak{g}$,
get $I\subseteq\mathfrak{g}$-ideal $\Rightarrow$ Killing form of I is the restriction of the one on $\mathfrak{g}$.
Thus: $K^I\equiv 0 \overset{\text{Thm 2}}{\Longrightarrow} I$-solvable, hence, $I=0$ as $\mathfrak{g}$-semisimple ∎

The proof of Theorem 2 is based on the following general result in linear alg:

Theorem 4 (Jordan decomposition): Let V be a fin. dim. complex vector space

a) Any linear operator A can be uniquely written as

$$A = A_s + A_n, \quad \text{with} \quad \begin{array}{l} A_n\text{-nilpotent} \\ A_s\text{-semisimple (a.k.a. diagonalizable)} \\ A_s\cdot A_n = A_n\cdot A_s \end{array}$$

b) Define $\mathrm{ad}(A): \mathrm{End}(V)\circlearrowleft$ via $\mathrm{ad}(A)B := AB-BA=[A,B]$. Then:

$$\mathrm{ad}(A)_s=\mathrm{ad}(A_s),\quad \mathrm{ad}(A)_n=\mathrm{ad}(A_n)$$

and $\mathrm{ad}(A_s)$ can be written as

$$\mathrm{ad}(A_s)=p(\mathrm{ad}(A)) \quad \text{for some } p\in t\,\mathbb{C}[t]$$

c) Define $\bar{A}_s$ that has the same eigenspaces as A_s, but conjugate eigenvalues

$$A_s v=\lambda v \implies \bar{A}_s v:=\bar{\lambda} v.$$

Then $\mathrm{ad}(\bar{A}_s)$ can also be written as

$$\mathrm{ad}(\bar{A}_s)=q(\mathrm{ad}(A)) \quad \text{for some } q\in t\,\mathbb{C}[t]$$

We shall now first deduce Thm 2 from Thm 4, and then prove Thm 4

Terminology: The operator $A: V\to V$ is called ~~semisimple~~ if $\forall$ subspace $V'\subseteq V$
s.t. $A(V')\subseteq V'$ $\exists\, V''$ s.t. $A(V'')\subseteq V''$ and $V=V'\oplus V''$.

Remark: As we shall see next time, the result also holds over $\mathbb{R}$, where
semisimple $\neq$ diagonalizable, but is deduced from version over $\mathbb{C}$