Lecture #16

Proof of Theorem 2 (from Lecture 15)

▷ $\Rightarrow$: If $\mathfrak{g}$ - solvable, then by Lie thm there is a basis in which all $ad(x)$ are upper-Δ. But then, any $ad(y)$ with $y \in [\mathfrak{g}, \mathfrak{g}]$ is strictly upper-Δ. Therefore, $K(x,y) = tr(ad\,x \circ ad\,y) = 0 \quad \forall x \in \mathfrak{g},\ y \in [\mathfrak{g},\mathfrak{g}]$

$\Leftarrow$: Assume $K([\mathfrak{g},\mathfrak{g}], \mathfrak{g}) = 0$. Let $\mathfrak{a} := ad(\mathfrak{g}) \subseteq End(\mathfrak{g})$. Then, we have $tr(xy) = 0 \quad \forall x \in [\mathfrak{a},\mathfrak{a}], y \in \mathfrak{a}$. As $\mathfrak{g}$ fits into the short exact sequence

$$0 \longrightarrow \underbrace{Z(\mathfrak{g})}_{\text{clearly solvable}} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{a} \longrightarrow 0$$

it suffices to prove $\mathfrak{a}$ is solvable (thus, reducing to subalg. of $gl(?)$). This follows from the following general result:

> **Claim**: Let V be a fin.dim. $\mathbb{C}$ v.space, $\mathfrak{a} \subseteq gl(V)$ - Lie subalgebra s.t. $tr(xy) = 0 \quad \forall x \in [\mathfrak{a},\mathfrak{a}], y \in \mathfrak{a}$. Then: $\mathfrak{a}$ - solvable.

▷ Pick any $x \in [\mathfrak{a},\mathfrak{a}]$ and consider its Jordan decomposition

$$x = x_s + x_n.$$

We also consider $\overline{x_s}$ as in Thm 4c). Then:

$$tr(x \cdot \overline{x_s}) \underset{\substack{\uparrow \\ \text{follows from proof of Thm 4}}}{=} tr(x_s \cdot \overline{x_s}) = \sum |\lambda_i|^2, \text{ where } \lambda_i \text{ are eigenvalues of } x_s$$
(i.e. gen. eigenv. of x)

But as $x \in [\mathfrak{a},\mathfrak{a}]$, we can write it as $x = \sum [y_j, z_j]$, so that

$$tr(x \cdot \overline{x_s}) = tr\Big(\sum_j [y_j,z_j]\overline{x_s}\Big) = \sum_j tr(y_j[z_j,\overline{x_s}]) = -\sum_j tr(y_j[\overline{x_s}, z_j])$$

However, $[\overline{x_s}, -] = ad(\overline{x_s}) = q(ad\,x)$ by Thm 4c), with $q \in t\mathbb{C}[t]$.

Hence, $[\overline{x_s}, z_j] \in [\mathfrak{a},\mathfrak{a}]$ and so

$$tr(y_j[\overline{x_s}, z_j]) = 0 \Rightarrow tr(x \cdot \overline{x_s})$$

By above, get $\sum |\lambda_i|^2 = 0 \Rightarrow$ all $\lambda_i = 0 \Rightarrow x_s = 0 \Rightarrow x = x_n$ - nilpotent.

Then, $[\mathfrak{a},\mathfrak{a}]$ - nilpotent Lie algebra (Engel's thm) $\Rightarrow \mathfrak{a}$ - solvable. □

■

Rmk: While the above proof was over $\mathbb{C}$, the result also holds over $\mathbb{R}$, since both properties are preserved under $\otimes_{\mathbb{R}} \mathbb{C}$.

Lecture #16

Exercise: Let V be a fin. dim. complex vector space.

a) $A: V \to V$ is semisimple $\iff$ A - diagonalizable

b) If $A: V \to V$ is semisimple and $V' \subseteq V$ satisfies $A(V') \subseteq V'$, then then the corresponding operators $V' \to V'$ and $V/V' \to V/V'$ are s.s.

c) If $A, B: V \to V$ are s.s. and $AB = BA$, then $A+B$ is also s.s

d) If $A, B: V \to V$ are nilpotent and $AB = BA$, then $A+B$ is nilpotent

Let us finally prove Theorem 4 (Jordan decomposition) - see Lecture 15

- Know $V = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$, V_λ = generalized eigenspace of A with eigenvalue λ, i.e. $(A - \lambda \cdot \mathrm{Id})|_{V_\lambda}$ is nilpotent

Then, we set $A_s: V \to V$ via $A_s|_{V_\lambda} = \lambda \cdot \mathrm{Id}$, and $A_n := A - A_s$.

Clearly: A_s - s.s. (see Exercise above), A_n - nilpotent, $[A_s, A_n] = 0$.

- If $A = A_s' + A_n'$ is any other such decomposition, then A_s', A_n' commute with A and hence with A_s, A_n (which uses that $A_s = p(A)$ established below).

Then: $\underbrace{A_s - A_s'}_{\text{s.s. by Ex c)}} = \underbrace{A_n' - A_n}_{\text{nilpotent by Ex d)}} \Rightarrow A_s' = A_s,\ A_n' = A_n \Rightarrow$ uniqueness!

- By Chinese remainder thm, $\exists\, p(t) \in \mathbb{C}[t]$ s.t. $p(t) \equiv \lambda_i \mod (t - \lambda_i)^{\dim V_{\lambda_i}}$ for any $\lambda_i \in \mathbb{C}$ s.t. $V_{\lambda_i} \neq 0$. Then: $(A - \lambda_i)^{\dim V_{\lambda_i}} = 0$ on V_{λ_i}

Therefore, $\boxed{A_s = p(A)}$

- So far, we have established the unique decomposition of A into $A = A_s + A_n$ with A_s - s.s, A_n - nilp, $[A_s, A_n]$, and proved that both A_s, A_n are pol-s in A. Moreover, if $V_0 \neq 0$, then $p(t) \in t\,\mathbb{C}[t]$.

- Finally, note that $\mathrm{ad}(A) = \mathrm{ad}(A_s) + \mathrm{ad}(A_n): \mathrm{End}(V) \to \mathrm{End}(V)$.

Here, $\mathrm{ad}(A_s)$ & $\mathrm{ad}(A_n)$ - commute, $\mathrm{ad}(A_s)$ - s.s, $\mathrm{ad}(A_n)$ - nilpotent.

[**Exercise**: Check this!

By uniqueness of Jordan decomp. $\mathrm{ad}(A_s) = (\mathrm{ad}(A))_s$, $\mathrm{ad}(A_n) = (\mathrm{ad}(A))_n$.

Moreover, as $\mathrm{Ad}(A) A = 0$, can write $\mathrm{ad}(A_s) = p(\mathrm{ad} A)$ with $p \in t\,\mathbb{C}[t]$.

[**Exercise**: Prove part c) of Thm 4

Lecture #16

- Last time:
  - semisimplicity through invariant forms
  - all classical Lie algebras are reductive
  - Killing form
  - Cartan's Criteria for solvability
  - Cartan's Criteria for semisimplicity
  - Jordan decomposition

- Finish the proof of parts (a,b) of Jordan decomposition.

- Properties of semisimple Lie algebras

As an immediate corollary of Cartan's criterion for semisimplicity, we have:

Lemma 1: If $\mathrm{char}(\mathbb{k})=0$, then a f.d. dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{k}$ is semisimple iff $\mathfrak{g}\otimes_{\mathbb{k}}\bar{\mathbb{k}}$ is semisimple.

Another important result is that every ideal admits a complimentary ideal:

Lemma 2: Let $\mathfrak{g}$ be a semisimple Lie algebra, $I\subseteq\mathfrak{g}$ - ideal. Then there is an ideal $J\subseteq\mathfrak{g}$ such that $\mathfrak{g}=I\oplus J$

► Let $I^\perp$ be the orthogonal complement of I w.r.t Killing form. As discussed last time: $I^\perp$ - ideal as well. We claim that $\mathfrak{g}=I\oplus I^\perp$. To this end, it suffices to verify $I\cap I^\perp=0$ (for dimension reasons). But $I\cap I'$ is an ideal of $\mathfrak{g}$ with zero Killing form ([Hwk 6, Problem 2b]) hence it is solvable by Cartan's criterion for solvability.
But $\mathfrak{g}$ being semisimple implies then that $I\cap I^\perp=0$ ■

As an immediate corollary of the result above, we get:

Corollary 1: A Lie algebra is semisimple iff it is a direct sum of simple Lie alg.

► $\Leftarrow$: Use Cartan's criterion on the nose
$\Rightarrow$: If $\mathfrak{g}$ is not simple, find an ideal of least possible dimension, apply Lemma 2, and proceed by induction. ■

Corollary 2: For semisimple $\mathfrak{g}$, we have $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$

► Apply Corollary 1 and the equality $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ for simple $\mathfrak{g}$ ■

Lecture #16

Proposition 1: Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \dots \oplus \mathfrak{g}_k$ be a semisimple Lie algebra, with all $\mathfrak{g}_i$ being simple. Then, any ideal $I \subseteq \mathfrak{g}$ is of the form $I = \bigoplus_{i \in S} \mathfrak{g}_i$ for some subset $S \subseteq \{1, \dots, k\}$

▶ The proof is by induction on k. Base case ($k=1$) is obvious.
Let $\pi_k : \mathfrak{g} \to \mathfrak{g}_k$ be the natural projection. Consider $\pi_k(I)$ - ideal of $\mathfrak{g}_k$

Case 1: $\pi_k(I) = 0 \Rightarrow I \subseteq \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{k-1}$, and we can use the induction hypothesis

Case 2: $\pi_k(I) = \mathfrak{g}_k$. But then $[\mathfrak{g}_k, I] = [\mathfrak{g}_k, \mathfrak{g}_k] = \mathfrak{g}_k \Rightarrow \mathfrak{g}_k \subseteq I$.
In this case, $I = I' \oplus \mathfrak{g}_k$ for a subspace $I' \subseteq \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{k-1}$ which is easily seen to be an ideal of $\mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_{k-1}$. The result follows from the induction hypothesis. ∎

In particular, we note:

Corollary 3: The ideal J in Lemma 2 is unique

As another important corollary, we get:

Corollary 4: a) Any ideal in a semisimple Lie algebra is semisimple
b) Any quotient of a semisimple Lie algebra is semisimple.

Let $\mathrm{Der}(\mathfrak{g})$ be the Lie algebra of derivations of a Lie algebra $\mathfrak{g}$ (cf. [Hwk3, Prob#4f])
We have a natural map $\mathfrak{g} \xrightarrow{ad} \mathrm{Der}(\mathfrak{g})$, with $\mathrm{Ker}(ad) = \mathfrak{z}(\mathfrak{g})$ - center of $\mathfrak{g}$, and image $ad(\mathfrak{g})$ is an ideal of $\mathrm{Der}(\mathfrak{g})$, due to ([Hwk 3, Problem 4d]).

$$\boxed{[d, ad(x)] = ad(d(x)) \quad \forall d \in \mathrm{Der}(\mathfrak{g}) \ \forall x \in \mathfrak{g}}$$

Proposition 2: If $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = \mathrm{Der}(\mathfrak{g})$

▶ Consider the invariant symmetric bilinear form

$$K(a,b) := tr_{\mathfrak{g}}(ab) \quad \forall a, b \in \mathrm{Der}(\mathfrak{g}).$$

It's an extension of the Killing form on $\mathfrak{g} \overset{ad}{\hookrightarrow} \mathrm{Der}(\mathfrak{g})$. Let $I = \mathfrak{g}^{\perp}$ be the orthogonal complement of $\mathfrak{g}$ in $\mathrm{Der}(\mathfrak{g})$ under K. Then: I - ideal, $I \cap \mathfrak{g} = 0$, and $\mathrm{Der}(\mathfrak{g}) = I \oplus \mathfrak{g}$. We then have $[\mathfrak{g}, I] = 0$ (as both $ad(\mathfrak{g}) = \mathfrak{g}$ & I - ideals)
But then $\forall d \in I, \forall x \in \mathfrak{g}$: $0 = [d, ad(x)] = ad(d(x)) \Rightarrow d(x) \in \mathfrak{z}(\mathfrak{g}) \Rightarrow d(x) = 0 \ \forall x \Rightarrow d = 0$.
Therefore: $I = 0$ and so $\mathfrak{g} \simeq \mathrm{Der}(\mathfrak{g})$ ∎