Lectures #17-18

- Finish the proof of $\mathrm{Der}(\mathfrak{g}) = \mathfrak{g}$ for semisimple $\mathfrak{g}$ from last time

* Extensions of $\mathfrak{g}$-modules and Whitehead's theorem

Given any Lie algebra $\mathfrak{g}$ and its modules V, W a natural question is to classify all extensions of W by V, i.e. $\mathfrak{g}$-modules U fitting into

$$\boxed{0 \to V \to U \xrightarrow{p} W \to 0}$$ – short exact sequence of $\mathfrak{g}$-mod

To do this, pick any splitting of this sequence as vector spaces, i.e. pick an injective linear map $\imath : W \to U$ such that $p \circ \imath = \mathrm{id}_W$. This gives rise to

$$\boxed{\text{v.space isom. } V \oplus W \xrightarrow{\sim} U, \quad (v, w) \mapsto v + \imath(w)}$$

Thus, action of $\mathfrak{g} \overset{\rho}{\curvearrowright} U$ can be viewed as $\mathfrak{g} \overset{\rho}{\curvearrowright} V \oplus W$. Thus:

$$\boxed{\rho(x)(v, w) = (\rho_V(x)v + a(x)(w), \rho_W(x)w) \quad \text{with } a(x) \in \mathrm{Hom}_{\Bbbk}(W, V)}$$

As ρ is linear, we see that $a : \mathfrak{g} \to E = \mathrm{Hom}_{\Bbbk}(W, V)$ is a linear map.
In particular, we note that if $a \equiv 0$, then resulting $\mathfrak{g}$-module is just $V \oplus W$.
Let's now see which conditions on a are equivalent to ρ being a module:

- $\rho([x,y])(v,w) = (\rho_V([x,y])v + a([x,y])(w), \rho_W([x,y])w) \quad \forall x, y \in \mathfrak{g} \ \forall v \in V, w \in W$
  which can be written as $\boxed{([x,y].v + a([x,y])w, [x,y].w)}$ for brevity
- $\rho(x)\rho(y)(v,w) = \rho(x)(\rho_V(y)v + a(y)(w), \rho_W(y)w) = \begin{pmatrix} \rho_V(x)\rho_V(y)v + \rho_V(x)(a(y)w) \\ + a(x)(\rho_W(y)w), \ \rho_W(x)\rho_W(y)w \end{pmatrix}$

$\Downarrow$

$$[\rho(x), \rho(y)](v,w) = (\rho(x)\rho(y) - \rho(y)\rho(x))(v,w) =$$

$$= \Big((\rho_V(x)\rho_V(y) - \rho_V(y)\rho_V(x))(v) + \rho_V(x)a(y)w - a(y)\rho_W(x)w + a(x)\rho_W(y)w - \rho_V(y)a(x)w,$$
$$(\rho_W(x)\rho_W(y) - \rho_W(y)\rho_W(x))w\Big)$$

$$= \Big([\rho_V(x), \rho_V(y)]v + (\rho_V(x)a(y) - a(y)\rho_W(x))w + (a(x)\rho_W(y) - \rho_V(y)a(x))w,$$
$$[\rho_W(x), \rho_W(y)]w\Big)$$

which can be written for brevity as

$$\boxed{([x,y].v + [x, a(y)]w + [a(x), y]w, [x,y].w)}$$

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Comparing the two boxed formulas from p.1, we see that ρ defines an action of $\mathfrak{g}$ on $V\oplus W$ iff $\forall x,y\in\mathfrak{g}$:

$$\boxed{a([x,y])=(\rho_V(x)a(y)-a(y)\rho_W(x))-(\rho_V(y)a(x)-a(x)\rho_W(y))}$$

Evoking $\mathfrak{g}$-action on $E=\mathrm{Hom}_{\mathbb{k}}(W,V)$, we see that

$$\rho_V(x)a(y)-a(y)\rho_W(x)=x.\,a(y),\quad \rho_V(y)a(x)-a(x)\rho_W(y)=x.\,\rho_W(y)$$

and so above equality reads:

$$\boxed{a([x,y])=x.\,a(y)-y.\,a(x)}$$

This brings us to the following general definition:

Def 1: For any Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-module E, all linear maps $a:\mathfrak{g}\to E$ s.t. $a([x,y])=x.a(y)-y.a(x)$ are called <u>1-cocycles of $\mathfrak{g}$ with values in E</u>. Let $Z^1(\mathfrak{g},E)=\left\{\begin{matrix}\text{all}\\ \text{such } a\end{matrix}\right\}$

<u>Examples</u>: a) $E=$ 1dim trivial module $\mathbb{k}$ $\Rightarrow Z^1(\mathfrak{g},\mathbb{k})=(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$

b) $E=\mathfrak{g}$ w.r.t. $\mathrm{ad}:\mathfrak{g}\curvearrowright\mathfrak{g}$ $\Rightarrow Z^1(\mathfrak{g},\mathfrak{g})=\mathrm{Der}(\mathfrak{g})$

<u>Upshot of above</u>: Every $a\in Z^1(\mathfrak{g},\mathrm{Hom}_{\mathbb{k}}(W,V))$ defines an extension U_a of W by V. But we note that above construction depended on a choice of $\imath: W\hookrightarrow U$. Thus, we actually want to classify our extensions up to isom. of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V & \longrightarrow & U_a & \longrightarrow & W & \longrightarrow & 0\\ & & \downarrow{\scriptstyle id_V} & & \downarrow{\scriptstyle \wr f} & & \downarrow{\scriptstyle id_W} & & \\ 0 & \longrightarrow & V & \longrightarrow & U_b & \longrightarrow & W & \longrightarrow & 0 \end{array}$$

i.e. f is an isomorphism s.t. with respect to 2-step filtration $\mathrm{gr}.f=id_{V\oplus W}$.

This forces f to be of the form

$$\boxed{f(v,w)=(v+Aw,\ w)\ \text{ with } A: W\to V \text{ - linear map}}$$

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We shall now see which conditions on A guarantee that f intertwines $\mathfrak{g}$-actions

$$\rho_{u_b}(x)\, f(v,w) = x.f(v,w) = x.(v+Aw,\, w) = (\rho_V(x)v + \rho_V(x)Aw + b(x)w,\ \rho_W(x)w)$$

$$f\,\rho_{u_a}(x)(v,w) = f(x.(v,w)) = f(\rho_V(x)v + a(x)w,\ \rho_W(x)w)$$
$$= (\rho_V(x)v + a(x)w + A\rho_W(x)w,\ \rho_W(x)w)$$

and so above expressions coincide iff

$$\boxed{\rho_V(x)A + b(x) = A\rho_W(x) + a(x)}$$

Thinking of $A \in \mathrm{Hom}_{\mathbb{K}}(W,V)$ and noting $\rho_V(x)A - A\rho_W(x) = x.A$, the above is

$$\boxed{a(x) - b(x) = x.A}$$

This brings us to the following general definition:

Def 2: For any Lie algebra $\mathfrak{g}$ and $\mathfrak{g}$-module E, a linear map $a: \mathfrak{g} \to E$ given by $x \mapsto x.v$ for some fixed $v \in E$ is called <u>1-coboundary of v</u>, denoted $a = dv$. Let $B^1(\mathfrak{g}, E) = \{\text{all such } a\}$

Note that each 1-coboundary is a 1-cocycle because E is $\mathfrak{g}$-module.
Thus $B^1(\mathfrak{g},E) \underset{\text{subspace}}{\leq} Z^1(\mathfrak{g},E)$.

Def 3: The quotient $H^1(\mathfrak{g},E) = Z^1(\mathfrak{g},E)/B^1(\mathfrak{g},E)$ is called the <u>1^{st} cohomology of $\mathfrak{g}$ with coefficients in E</u>

<u>Upshot</u>: Two 1-cocycles $a, b: \mathfrak{g} \to E = \mathrm{Hom}_{\mathbb{K}}(W,V)$ determine isomorphic $\mathfrak{g}$-module extensions (in above sense) iff $a - b \in B^1(\mathfrak{g},E)$. Thus, we obtain a bijection between all extensions of W by V and $H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{K}}(W,V))$, i.e. extensions are parametrized by

$$\boxed{\mathrm{Ext}^1(W,V) = H^1(\mathfrak{g}, \mathrm{Hom}_{\mathbb{K}}(W,V))}$$

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Key result for today is:

> **Theorem 1** (Whitehead Theorem): If $\mathrm{char}(\Bbbk)=0$, $\mathfrak{g}$-semisimple, then $H^1(\mathfrak{g},V)=0$ for any finite dimensional $\mathfrak{g}$-module V

Examples: a) For $V=\Bbbk$-trivial, this recovers $[\mathfrak{g},\mathfrak{g}]=\mathfrak{g}$ for semisimple $\mathfrak{g}$

b) For $V=\mathfrak{g}$ - adjoint repr, this recovers $\mathrm{Der}(\mathfrak{g})=\mathfrak{g}$.

We start with the following simple observation:

> **Lemma 1**: Let $\mathfrak{g}$ - Lie algebra, E - fin. dim. $\mathfrak{g}$-module, and $C\in U\mathfrak{g}$ - a central element s.t. $C|_{\Bbbk}=0$, $C|_E=\lambda\cdot Id_E$. Then $H^1(\mathfrak{g},E)=0$.

▸ Following discussion of pp. 1-3, this is equivalent to splitting extension

$$0\longrightarrow E\longrightarrow U\xrightarrow{p}\Bbbk\longrightarrow 0.$$

Pick any $\tilde{v}\in U$ s.t. $p(\tilde{v})=1$. As C-central, get $p(C\tilde{v})=C(p(\tilde{v}))=0 \overset{\text{exactness}}{\Rightarrow}$ $C\tilde{v}\in E \Rightarrow C^2\tilde{v}=\lambda\cdot C\tilde{v}$. Hence, $v:=\tilde{v}-\frac{1}{\lambda}C\tilde{v}$ satisfies $p(v)=1$, $C(v)=0$.

This v defines a splitting $\Bbbk\hookrightarrow U$, $1\longmapsto v$. Indeed, since C acts on U with two eigenvalues $\lambda, 0$ and generalized eigenspace $U_{[0]}=\{w: C^{\gg 1}(w)=0\}$ is 1dim, we see that $x(v)\in\Bbbk v$ as $Cxv\overset{C\text{-central}}{=}xCv=0$. Hence $\Bbbk v$ is a $\mathfrak{g}$-submod and $v\notin E$ as $Cv=0$, $C|_E=\lambda Id_E$, so that $\Bbbk v\underset{\mathfrak{g}\text{-mod}}{\simeq}\Bbbk$-trivial. ∎

To apply this lemma, we shall need a way to produce central elements in $U\mathfrak{g}$. Assume $(\cdot,\cdot):\mathfrak{g}\times\mathfrak{g}\longrightarrow\Bbbk$ is a nondegenerate invariant bilinear form. Pick any basis $\{x_i\}_{i=1}^{N=\dim\mathfrak{g}}$ of $\mathfrak{g}$ and let $\{x^i\}_{i=1}^N$ be the dual basis w.r.t. $(\cdot,\cdot)$, i.e. $(x_i,x^j)=\delta_{ij}$.

> **Def 4**: The Casimir element determined by $(\cdot,\cdot)$ is $C=\sum_{i=1}^{N=\dim\mathfrak{g}}x_ix^i\in U\mathfrak{g}$

Example: For $\mathfrak{g}=\mathfrak{sl}_2$ and $(\,,\,)=$ trace form get old Casimir $=ef+fe+h^2/2$.

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Lemma 2: C does not depend on the choice of $\{x_i\}$, and C is central

▸ Using $(\cdot,\cdot)$ we can identify $\mathfrak{g}\otimes\mathfrak{g} \simeq \mathfrak{g}\otimes\mathfrak{g}^* \simeq \mathrm{End}(\mathfrak{g})$. Then, $\mathrm{Id}_{\mathfrak{g}} \in \mathrm{End}(\mathfrak{g})$ exactly corresponds to the Casimir tensor $\Omega = \sum_{i=1}^{N} x_i \otimes x^i$, hence basis-independ.

Moreover, since $\mathrm{Id}_{\mathfrak{g}}$ is an intertwiner of the adjoint module, and the above isom. are $\mathfrak{g}$-module isomorphisms $\Rightarrow$ Ω is adjoint-invariant.

Finally, as $\mathfrak{g}\otimes\mathfrak{g} \xrightarrow[m]{\text{product}} U\mathfrak{g}$ is a $\mathfrak{g}$-module morphism, we get $C = m(\Omega)$ is central in $U\mathfrak{g}$ ■

Lemma 3: Let $\mathfrak{g}$-semisimple Lie algebra $(\mathrm{char}(\mathbb{k})=0)$ and V-nontrivial simple fin. dim. $\mathfrak{g}$-module. Then there is a central element $C \in U\mathfrak{g}$ s.t. $C|_{\mathbb{k}} = 0$, $C|_V = \lambda \mathrm{Id}_V$ with $\lambda \neq 0$

▸ Consider the invariant form $B_V : \mathfrak{g}\times\mathfrak{g} \to \mathbb{k}$ from last lectures:

$$B_V(x,y) = \mathrm{tr}_V(\rho(x)\rho(y))$$

Case 1: B_V-nondegenerate

Let C be corresponding Casimir. By Lemma 2 it's central, hence by Schur's lemma it acts on V by $\cdot\lambda$. But one gets explicit value of λ by evaluating $\mathrm{tr}_V(C)$ in two ways:

$$\lambda \dim V = \mathrm{tr}_V(C) = \sum_i \mathrm{tr}_V(\rho(x_i)\rho(x^i)) = \sum_i B_V(x_i, x^i) = \dim\mathfrak{g}$$

$\Rightarrow \lambda = \dfrac{\dim\mathfrak{g}}{\dim V} \neq 0$

Case 2: B_V is degenerate, and let $I = \mathrm{Ker}(B_V) \underset{\text{ideal}}{\subseteq} \mathfrak{g}$

First, we note that $I \neq \mathfrak{g}$ as otherwise $\rho(\mathfrak{g}) \subseteq \mathfrak{gl}(V)$ would be solvable by Cartan's criteria $\overset{\mathfrak{g}\text{-s.s}}{\Longrightarrow} \rho(\mathfrak{g}) \equiv 0 \Rightarrow V$-trivial $\mathfrak{g}$-module $\Rightarrow$ ↯

As $\mathfrak{g}$-semisimple, we can decompose it into $\mathfrak{g} = I \oplus \tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{g}}$ is also semisimple. By construction $B_V|_{\tilde{\mathfrak{g}}\times\tilde{\mathfrak{g}}}$ is nondegenerate

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▸ (Continuation)

Let $\tilde{C} \in U\mathfrak{g}$ be the Casimir elt w.r.t. $Br|_{\bar{\mathfrak{g}}\times\bar{\mathfrak{g}}}$. Because $[\bar{\mathfrak{g}}, I]$, we also get $[\tilde{C}, I]=0 \Rightarrow \tilde{C}$ - central element of $U\mathfrak{g}$. $\overset{\text{Schur}}{\Longrightarrow}$ acts on V by $\cdot\lambda$.

Evaluating $tr_V(\tilde{C})$ as in Case 1, we find $\lambda = \frac{\dim \bar{\mathfrak{g}}}{\dim V} \neq 0$.

■

Corollary 1: For any simple f.d. module V of semisimple $\mathfrak{g}$: $H^1(\mathfrak{g}, V)=0$

▸ If $V \not\simeq \Bbbk$, then the result follows from Lemmas 1 & 3.

If $V \simeq \Bbbk$, then our extension looks as $0 \to \Bbbk \to U \to \Bbbk \to 0$, so that $\rho_U(x) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$ - nilpotent $\Rightarrow \rho(\mathfrak{g})$-solvable & semisimple $\Rightarrow \rho(\mathfrak{g}) \equiv 0$.

(alternatively, could argue $H^1(\mathfrak{g}, \Bbbk) = (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^* = 0$)

■

Exercise: For any Lie algebra $\mathfrak{g}$ and a short exact sequence of $\mathfrak{g}$-modules $0 \to V \to U \to W$, show that $H^1(\mathfrak{g}, V) \to H^1(\mathfrak{g}, U) \to H^1(\mathfrak{g}, W)$ is exact in the middle term.

Proof of Whitehead Theorem

▸ By Corollary 1: $H^1(\mathfrak{g}, V)=0$ if V - simple

For general f.dim $\mathfrak{g}$-module V, pick a Jordan-Hölder filtration of V by $\mathfrak{g}$-submodules $0 = F_0V \subset F_1V \subset F_2V \subset \ldots \subset F_NV$ so that $F_{k+1}V/F_kV$ - simple.

Arguing by induction and using Exercise above, get $H^1(\mathfrak{g}, F_kV)=0 \ \forall k$, in particular, $H^1(\mathfrak{g}, V=F_NV)=0$

■

Theorem 2: Any f.dim. module U over semisimple $\mathfrak{g}$ is completely reducible

▸ If U-simple, the claim is tautological. Otherwise it has a submodule V. Then $0 \to V \to U \to U/V \to 0 \Rightarrow U \simeq V \oplus U/V$ by Theorem 1. Continue process for V and U/V

■

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We also obtain a proof of result stated in the end of Lecture 14:

Corollary 2: In $\mathrm{char}(k)=0$, a reductive Lie algebra $\mathfrak{g}$ admits a unique splitting as direct sum of abelian and semisimple Lie alg.

▷ Consider $\mathrm{ad}: \mathfrak{g} \curvearrowright \mathfrak{g}$. Note that $\mathfrak{z}(\mathfrak{g})$ is in its kernel and so we get: $\mathrm{ad}: \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \curvearrowright \mathfrak{g}$. But $\mathfrak{g}_{ss} = \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ is semisimple. Hence, the short exact sequence of $\mathfrak{g}_{ss}$-modules splits:

$$\begin{tikzcd} 0 \arrow[r] & \mathfrak{z}(\mathfrak{g}) \arrow[r] & \mathfrak{g} \arrow[r] & \mathfrak{g}_{ss} \arrow[r] \arrow[l, dashed, bend left, "i"] & 0 \end{tikzcd}$$

Thus, $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}$.

Uniqueness follows from observation that if $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \underset{\text{semisimple}}{\mathfrak{a}}$, then $[\mathfrak{g},\mathfrak{g}] = [\mathfrak{a},\mathfrak{a}] = \mathfrak{a}$. This shows that $\mathrm{Im}(i) =$ commutant $[\mathfrak{g},\mathfrak{g}]$. ■

Example: $\mathfrak{gl}_n(k) = k\cdot I \oplus \mathfrak{sl}_n(k)$.