

• Last time

- complete reducibility of fin. dim \mathfrak{g} -modules for semisimple \mathfrak{g} (and Cartan's element)

• For the rest of this week we shall develop an analogue of the

• Toy Example: $\text{ad}(h): \mathfrak{sl}_2 \rightarrow \mathfrak{sl}_2$ acting with 1dim eigenspaces $\mathfrak{C}_e, \mathfrak{C}_f, \mathfrak{C}_h$.

• Semisimple elements and toral algebras

Def 1: a) An element $x \in \mathfrak{g}$ is called semisimple if $\text{ad } x: \mathfrak{g} \rightarrow \mathfrak{g}$ is a semisimple operator

b) An element $x \in \mathfrak{g}$ is called nilpotent if $\text{ad } x$ is a nilpotent operator.

(Exercise: Show that for $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ these definitions agree with the usual ones.

The following result generalizes Jordan decomposition to semisimple Lie algebras:

Proposition 1: Let \mathfrak{g} be a semisimple complex Lie algebra. Then every element $x \in \mathfrak{g}$ admits a unique decomposition

$$x = x_s + x_n$$

with x_s -semisimple, x_n -nilpotent, and $[x_s, x_n] = 0$

Moreover, if $[x, y] = 0$ for some $y \in \mathfrak{g}$, then $[x_s, y] = [x_n, y] = 0$.

▶ Let us write \mathfrak{g} as a direct sum of generalized eigenspaces for $\text{ad } x$

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}_\lambda, \quad \mathfrak{g}_\lambda = \{y \in \mathfrak{g} \mid (\text{ad } x - \lambda)^n y = 0 \text{ for } n \gg 0\}$$

Lemma 1: $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$

▶ By Jacobi identity $(\text{ad } x)[y, z] = [(\text{ad } x)y, z] + [y, (\text{ad } x)z]$

$$\Rightarrow (\text{ad } x - \lambda - \mu)[y, z] = [(\text{ad } x - \lambda)y, z] + [y, (\text{ad } x - \mu)z]$$

$$\text{Thus: } (\text{ad } x - \lambda - \mu)^N [y, z] = \sum_{k=0}^N \binom{N}{k} [(\text{ad } x - \lambda)^k y, (\text{ad } x - \mu)^{N-k} z]$$

Hence, if $y \in \mathfrak{g}_\lambda, z \in \mathfrak{g}_\mu, N > \dim \mathfrak{g}_\lambda + \dim \mathfrak{g}_\mu$, each term in the right-hand side

vanishes and so $[y, z] \in \mathfrak{g}_{\lambda+\mu}$ ◻

Back to the proof of Prop 1, consider the Jordan decomposition in $\mathfrak{gl}(V)$: $\text{ad}(x) = \text{ad}(x)_s + \text{ad}(x)_n$. By explicit construction: $\text{ad}(x)_s|_{\mathfrak{g}_\lambda} = \lambda \cdot \text{Id}$. But the above Lemma implies then that $\text{ad}(x)_s \in \text{Der}(\mathfrak{g})$. However, $\text{Der}(\mathfrak{g}) = \mathfrak{g}$ by [Lecture 16, Prop 2] as \mathfrak{g} -semisimple, hence $\exists x_s \in \mathfrak{g}$ s.t. $\text{ad}(x)_s = \text{ad}(x_s)$. Take $x_n = x - x_s$. This proves existence. The uniqueness follows from $\mathfrak{g} \hookrightarrow \mathfrak{gl}(\mathfrak{g})$ and uniqueness in $\mathfrak{gl}(V)$.

• Finally, if $[x, y] = 0$, then $\text{ad}(y)$ preserves each $\mathfrak{g}_\lambda \Rightarrow \text{ad}([x_s, y]) = 0 \stackrel{\text{b/c } \mathfrak{g}(\mathfrak{g})=0}{\Rightarrow} [x_s, y] = 0$
 $\Rightarrow [x_n, y] = 0$

Lecture #19

As an immediate corollary, we get:

Corollary 1: In any semisimple complex Lie algebra \mathfrak{g} , there exist non-zero semisimple elts

► If not, then by Proposition 1, every $x \in \mathfrak{g}$ is nilpotent

But then \mathfrak{g} -nilpotent by Engel's theorem, hence solvable, contradiction. ■

But in what follows, we shall need a whole family of commuting semisimple elts.

Def 2: A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is called toral if it's commutative and consists of semisimple elements

Example: For $\mathfrak{g} = \mathfrak{sl}_n$, any subspace of diagonal traceless matrices forms a toral subalg.

Proposition 2: Let \mathfrak{g} be a complex semisimple Lie algebra, $\mathfrak{h} \subseteq \mathfrak{g}$ -toral subalg, and (\cdot, \cdot) be a non-degenerate symmetric bilinear form on \mathfrak{g} (e.g. Killing)

Then:

a) $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$, \mathfrak{g}_α -common eigenspace for all $\text{ad}(h) |_{\mathfrak{h} \subseteq \mathfrak{h}}$, i.e.

$$\mathfrak{g}_\alpha = \{y \in \mathfrak{g} \mid [h, y] = \alpha(h) \cdot y \quad \forall h \in \mathfrak{h}\}$$

and $\mathfrak{h} \subseteq \mathfrak{g}_0$.

b) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$

c) if $\alpha + \beta \neq 0$, then $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$

d) $\forall \alpha$, the form (\cdot, \cdot) gives rise to a non-degenerate pairing $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \rightarrow \mathbb{C}$

► a) Is just the statement that a commuting family of diagonalizable operators can be simultaneously diagonalized (basic linear algebra).

b) Follows from Lemma 1 (actually, it's a very special case of it).

c) Pick any α, β and $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$. Then for any $h \in \mathfrak{h}$:

$$0 = ([h, x], y) + (x, [h, y]) = (\alpha(h) + \beta(h)) \cdot (x, y) \quad \left. \vphantom{0} \right\} \Rightarrow (x, y) = 0.$$

If $\alpha + \beta \neq 0 \Rightarrow \exists h \in \mathfrak{h}$ s.t. $\alpha(h) + \beta(h) \neq 0$

d) Follows from c) and nondegeneracy of (\cdot, \cdot) : if for some α

there is $x \in \mathfrak{g}_\alpha \setminus \{0\}$ s.t. $(x, \mathfrak{g}_{-\alpha}) = 0 \stackrel{c)}{\Rightarrow} (x, \mathfrak{g}) = 0 \Rightarrow (\cdot, \cdot)$ is degenerate $\Rightarrow \nabla$ ■

As an immediate corollary, we get:

Corollary 2: a) \mathfrak{g}_0 is a reductive subalgebra
 b) if $x \in \mathfrak{g}_0$, then $x_s, x_n \in \mathfrak{g}_0$

► a) Consider $\text{ad}: \mathfrak{g}_0 \rightarrow \mathfrak{g}_0$. The corresponding trace form $(x_1, x_2) = \text{tr}_{\mathfrak{g}_0}(\text{ad } x_1 \circ \text{ad } x_2)$ on \mathfrak{g}_0 is precisely $K_{\mathfrak{g}_0 \times \mathfrak{g}_0}^{\mathfrak{g}}$, which is non-degenerate by Prop 2d). Hence, \mathfrak{g}_0 is reductive by [Lecture 15, Lemma 2].

b) if $x \in \mathfrak{g}_0 \Rightarrow [h, x] = 0 \quad \forall h \in \mathfrak{h} \xrightarrow{\text{Prop 1}} [x_s, h] = 0 = [x_n, h] \quad \forall h \in \mathfrak{h} \Rightarrow x_s, x_n \in \mathfrak{g}_0$

• Cartan subalgebras

One would like to work with maximal toral algebras (for the reason that will become clear next time).

Def 3: Let \mathfrak{g} be a complex semisimple Lie algebra. A Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a toral subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ s.t. $\mathfrak{g}_0 = \mathfrak{h}$

Example: For $\mathfrak{g}_0 = \mathfrak{sl}_n$, take $\mathfrak{h} = \{ \text{all traceless diagonal matrices} \}$. Then: \mathfrak{h} -Cartan subalg. of \mathfrak{g}

Existence of such subalgebras follows from the following simple result:

Proposition 3: Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a maximal toral subalgebra (i.e. not properly contained in another toral subalgebra). Then \mathfrak{h} is a Cartan subalgebra.

► Consider the decomposition $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_n$ from Prop 1. As $\mathfrak{h} \subseteq \mathfrak{g}_0$, it suffices to show that \mathfrak{g}_0 is toral, hence $\mathfrak{h} = \mathfrak{g}_0$ by maximality.

• Let $x \in \mathfrak{g}_0$, then $x_s, x_n \in \mathfrak{g}_0$ by Corollary 2b). By maximality of \mathfrak{h} , we must have $x_s \in \mathfrak{h}$. But then $\text{ad}(x_s)|_{\mathfrak{g}_0} = 0$ and so $\text{ad}(x)|_{\mathfrak{g}_0} = \text{ad}(x_n)|_{\mathfrak{g}_0}$ - nilpotent $\forall x \in \mathfrak{g}_0 \xrightarrow{\text{Engel's thm}} \mathfrak{g}_0$ is nilpotent

However, by Corollary 2a), \mathfrak{g}_0 -reductive. Hence: \mathfrak{g}_0 -abelian.

• To prove that any $x \in \mathfrak{g}_0$ is semisimple, need to show $x_n = 0$. Then as \mathfrak{g}_0 is abelian:

$\forall y \in \mathfrak{g}_0 \quad [x_n, y] = 0 \Rightarrow \underbrace{\text{ad } x_n}_{\text{nilpotent}} \circ \underbrace{\text{ad } y}_{\text{commutes with ad}(x_n)}: \mathfrak{g} \rightarrow \mathfrak{g}$ is nilpotent $\Rightarrow \text{tr}_{\mathfrak{g}_0}(\text{ad } x_n \circ \text{ad } y) = 0 \quad \forall y \in \mathfrak{g}_0$

But this form is nondegenerate on \mathfrak{g}_0 , hence, $x_n = 0$.

We will see later that all Cartan subalgebras of \mathfrak{g} are conjugate under $\text{Aut}(\mathfrak{g})$. Thus, they all have the same dimension, called the rank of \mathfrak{g} : $\dim(\mathfrak{h}) = \text{rank}(\mathfrak{g})$.