Lecture #25

- Last time
  - root systems $R \subseteq \mathfrak{h}^*_{\mathbb{R}}$ arising from semisimple Lie algebras
  - Weyl group $W \subset O(E)$ of any root system $R \subset E$
  - complete classification of rank 2 root systems
    ↳ Corollary: $\alpha, \beta \in R$ - linearly indep. roots with $(\alpha,\beta)<0 \Rightarrow \alpha+\beta \in R$.
  - polarization of $R = R_+ \cup R_-$ AND simple roots $\Pi \subseteq R_+$
    ↳ Work out example of A_n-type root system
  - we ended last time with the following key result left as a Hwk Exercise:

$$\boxed{\text{Thm: } \Pi \subset R_+ \text{ is a basis of } E}$$

- Discuss Root systems of types B_n, C_n, D_n, their relation to orthogonal/symplectic Lie algebras in the context of [Hwk 7, Problem 6] (may postpone till Dynkin diagrams's discussion) (see also [Hwk 9, Problem 1])

- Root and Weight lattices

A lattice in a real vector space E is a subgroup $L \subset E$ generated by a basis of E (hence, by a suitable change of basis, any lattice can be identified with $\mathbb{Z}^n \subseteq \mathbb{R}^n$). Moreover, given a lattice $L \subset E$ one has a natural dual lattice

$$\boxed{E^* \supset L^* := \{f \in E^* \mid f(x) \in \mathbb{Z} \ \forall x \in L\}}$$

Clear: If L is generated (over $\mathbb{Z}$) by a basis $\{e_i\}_{i=1}^n$, then L^* is generated by the dual basis $\{e_i^*\}_{i=1}^n$ defined via $e_i^*(e_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \neq j \end{cases}$

In the context of abstract root systems $R \subset E$, there are several lattices that are very important:

1) root lattice $Q \subset E$ is the lattice generated (over $\mathbb{Z}$) by R

If we pick a polarization $R = R_+ \cup R_-$ and $\Pi \subset R_+$ is the set of simple roots, then $\boxed{Q = \bigoplus_{\alpha_i \in \Pi} \mathbb{Z}\alpha_i.}$

Lecture #25
(continuation)

2) coroot lattice $Q^\vee \subset E^*$ generated by $\{\alpha^\vee \mid \alpha \in R\}$, which is just the root lattice of the dual root system $R^\vee \subset E^*$, see [Hwk 8, Prob 4]

$$\boxed{Q^\vee = \bigoplus_{\alpha_i \in \Pi} \mathbb{Z}\alpha_i^\vee}$$

3) weight lattice $P \subset E$ is defined as the dual of $Q^\vee$:

$$\boxed{P = (Q^\vee)^* = \{\lambda \in E \mid \alpha^\vee(\lambda) \in \mathbb{Z} \quad \forall \alpha \in R\}}$$

4) coweight lattice $P^\vee \subset E^*$ is defined as the dual of Q or equivalently as a weight lattice of the dual root system

$$\boxed{P^\vee = Q^* = \{\lambda \in E^* \mid \lambda(\alpha) \in \mathbb{Z} \quad \forall \alpha \in R\}}$$

Note: $\forall \alpha, \beta \in R$: $\alpha^\vee(\beta) = n_{\alpha\beta} \in \mathbb{Z} \Rightarrow \boxed{Q \subseteq P \text{ and } Q^\vee \subseteq P^\vee}$

If we fix a polarization $R = R_+ \cup R_-$ and denote simple roots by $\Pi = \{\alpha_i\}_{i=1}^{r}$, then $\boxed{P = \bigoplus_{i=1}^{r} \mathbb{Z}\omega_i}$, where the fundamental weights ω_i are s.t. $\omega_i = (\alpha_i^\vee)^*$

$\boxed{P^\vee = \bigoplus_{i=1}^{r} \mathbb{Z}\omega_i^\vee}$, where the fundamental coweights $\omega_i^\vee$ are s.t. $\omega_i^\vee = (\alpha_i)^*$,

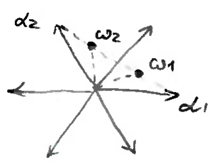
i.e. $\boxed{\alpha_j^\vee(\omega_i) = \omega_i^\vee(\alpha_j) = \delta_{ij}.}$

Example: a) A_1-type root system (α - simple root)

$R = \{\pm e_1 = \pm\alpha\} \subset E \leftarrow$ 1dim $\mathbb{R}$-v.space with basis $\{\alpha\}$. $\Rightarrow \begin{cases} Q = \mathbb{Z}\cdot\alpha \\ P = \mathbb{Z}\cdot\omega = \frac{1}{2}\mathbb{Z}\cdot\alpha \end{cases}$

Recall that $\alpha^\vee(\alpha) = 2 \Rightarrow \omega = \frac{1}{2}\alpha$

b) A_2-type root system



$Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$

$P = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$

! Discuss the precise location of ω_1, ω_2 in class

Exercise: a) Show that $|P/Q| = \det(A)$, $A = (\alpha_i^\vee(\alpha_j))$ — the Cartan matrix

b) Compute $|P/Q|$ for root systems of types A_n, B_n, C_n, D_n

c)* Identify explicitly the finite group P/Q for types A_n, B_n, C_n, D_n.

Lecture #25

• Weyl chambers

Last time we saw that each $t \in E^*$ s.t. $t(\alpha) \neq 0$ $\forall \alpha \in R$ gives rise to a polarization $R = R_+ \cup R_-$ and consequently to the set of simple roots $\Pi \subset R_+$.

There are two natural questions:

Q1: Do different polarizations give rise to equivalent sets of simple roots?

Q2: Can we recover all root system R just from the set Π of simple roots?

Note that our definition of a polarization depends on a choice of

$$\boxed{t \in E \setminus \bigcup_{\alpha \in R} L_\alpha, \quad \text{where } L_\alpha = \{ \lambda \in E \mid (\lambda, \alpha) = 0 \}}$$

[Here, we already identified $E \simeq E^*$ through the inner product on E]

Clearly, the polarization of R does not change when t continuously deforms without crossing any of the hyperplanes L_α. With this in mind, we introduce:

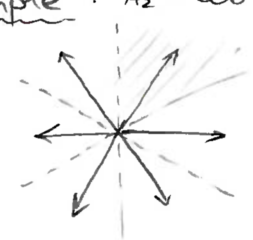
Def 1: A Weyl chamber is a connected component of $E \setminus \bigcup_{\alpha \in R} L_\alpha$

Therefore, a Weyl chamber is defined by a system of strict inequalities

$$\boxed{\pm(\alpha, \lambda) > 0 \quad \text{with } \alpha \in R.}$$

Any such system of inequalities determines either an empty set or a Weyl chamber.

Example: A_2-root system



dotted lines are our L_α's

describe chambers in class!

The following result is geometrically obvious:

Lemma 1: a) The closure $\overline{C}$ of a Weyl chamber C is a convex cone

b) The boundary of $\overline{C}$ is a union of codimension 1 faces F_i, which are convex cones inside one of the root hyperplanes L_α

The root hyperplanes L_α containing the faces F_i are called the walls of C.

Lecture #25

Having fixed a polarization $R = R_+ \cup R_-$ of the root system R determines the corresponding positive Weyl chamber C_+

$$\boxed{\begin{aligned} C_+ &= \{\lambda \in E \mid (\lambda,\alpha) > 0 \ \forall \alpha \in R_+\} = \{\lambda \in E \mid (\lambda,\alpha_i) > 0 \ \forall \alpha_i \in \Pi\} \\ &= \{\sum_{\alpha_i \in \Pi} c_i \omega_i \mid c_i \in \mathbb{R}_{>0}\} \text{ where } \omega_i \text{ are the fundamental weights} \end{aligned}}$$

Clearly, C_+ has $r = rk(R)$ faces $L_{\alpha_1} \cap \bar{C}_+, \ldots, L_{\alpha_r} \cap \bar{C}_+ \Rightarrow$ walls are $\{L_{\alpha_i}\}_{i=1}^{r}$.

Lemma 2: This defines a bijection between all polarizations of R and the set of Weyl chambers

(Exercise: Prove Lemma 2

Let's now recall the Weyl group W. We know $w(\alpha) \in R \ \forall \alpha \in R, w \in W$
Hence, W-action on E maps root hyperplanes to root hyperplanes, giving rise an action of W on the set of Weyl chambers.

Theorem 1: The Weyl group W acts transitively on the set of Weyl chambers

The proof is based on the notion of adjacent chambers:

Def 2: Two Weyl chambers C, C' are *adjacent* if they have a common codimension 1 face F.

The proof of theorem is based on the following result (geometrically clear):

Lemma 3: a) If C, C' are adjacent Weyl chambers separated by a hyperplane L_α, then $s_\alpha(C) = C'$

b) Any two chambers C, C' can be connected through a sequence of adjacent chambers, i.e. $C_0 = C, C_1, \ldots, C_k = C'$ s.t. C_i is adjacent to C_{i+1}

Lemma 3 $\Rightarrow$ Theorem 1 is obvious now.

Lecture #25

Corollary 1: Every Weyl chamber has $r = rk(R)$ walls

► For the positive Weyl chamber C_+, we saw that walls are $\{L_{\alpha_i} \mid \alpha_i \in \Pi\}$ and there are exactly r of those. As every Weyl chamber C can be written as $C = w(C_+)$ for some $w \in W$ by Thm 1, all of them have the same number of walls ∎

Corollary 2: Any two polarizations of R are related by the action of an element $w \in W$. Thus, if Π and Π' are systems of simple roots corresponding to two polarizations $R = R_+ \cup R_-$ and $R = R'_+ \cup R'_-$, then $\exists w \in W$ s.t. $w(\Pi) = \Pi'$

► Due to the bijection

$$\{\text{polarizations of } R\} \xleftrightarrow{1:1} \{\text{Weyl chambers}\}$$

from Lemma 2, and the fact (Theorem 1) that W acts transitively on the set of Weyl chambers, we get $W \curvearrowright \{\text{polarizations of } R\}$ transitively. It is clear that if $w(R_+) = R'_+$, then $w(\Pi) = \Pi'$ ∎

Remark 1: Next time we shall see that actually the action of W on Weyl chambers is simply transitive! (illustrated by A_2-example)

Remark 2: The answer to Q2 on p.3 is also positive, as we shall see next time, which is primarily based on the following Lemma:

Lemma 4: Let R be a reduced root system with fixed polarization $R = R_+ \sqcup R_-$, and $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset R_+$ - simple roots. Then for any Weyl chamber C there is a (non-unique) sequence $i_1, \ldots, i_\ell \in \{1, \ldots, r\}$ s.t.

$$C = s_{i_1} \ldots s_{i_\ell}(C_+), \quad C_+ = \text{positive Weyl chamber}$$