Lecture 26

- Last time:
  - root/coroot and weight/coweight lattices
  - Weyl chambers, their walls
  - positive Weyl chamber C_+ for any polarization $R = R_+ \cup R_-$

- We shall now also provide an affirmative answer to Q2 from [Lect25, p.3] which asked if the entire root system R can be recovered from Π, the set of simple roots. $\boxed{\text{We set } s_i := s_{\alpha_i} \ \forall \alpha_i \in \Pi \text{, called simple reflections.}}$

  To this end, we start from the following simple result:

Lemma 1: Let R be a root system with a fixed polarization $R = R_+ \cup R_-$, and let $\Pi = \{\alpha_1, \ldots, \alpha_r\} \subset R_+$ be the set of simple roots. Then for every Weyl chamber C, there exist $i_1, \ldots, i_\ell \in \{1, \ldots, r\}$ s.t. $C = s_{i_1} s_{i_2} \ldots s_{i_\ell}(C_+)$, where C_+ = positive Weyl chamber

Pick two points $t \in C$ and $t_+ \in C_+$ generically enough, connect them by a line segment, and consider its intersections with root hyperplanes (the "generic" condition is to guarantee that all these intersections are "simple") Let N be the number of such intersections, and we will argue by induction on N

Base ($N=1$): In this case, C is adjacent to C_+, but all walls of C_+ are $L_{\alpha_1}, \ldots, L_{\alpha_r}$, hence $\exists i_1: C = s_{i_1}(C_+)$ by [Lect 25, Lemma 3a)]

Induction Step ($N-1 \rightsquigarrow N$): Let C' be the chamber first entered as we move from t along the above segment, so that $C = s_\alpha(C')$ where L_α is a wall of C'. By induction hypothesis, C' can be written as $\boxed{C' = u(C_+) \text{ for some } u = s_{i_1} s_{i_2} \ldots s_{i_k}(C_+)}$ with $i_1, \ldots, i_k \in \{1, \ldots, r\}$.

In particular, the wall $L_\alpha = u(L_{\alpha_j})$ for some $1 \le j \le r \Rightarrow s_\alpha = u s_j u^{-1}$

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(continuation)

Thus: $C = s_\alpha(C') = s_\alpha(u(C_+)) = us_j \cancel{u^{-1}}(C_+) = us_j(C_+)$

This implies the result, with $l = k+1$ and $s_l = j$ ∎

With this result, we can immediately prove:

Theorem 1: In the notations of Lemma 1:
a) The simple reflections $\{s_i\}_{i=1}^r$ generate W = Weyl gp
b) $W(\Pi) = R$

▶ a) $\forall \alpha \in R$, the hyperplane L_α is a wall of some Weyl chamber C.
By Lemma 1, $C = s_{i_1} \dots s_{i_l}(C_+)$ for some $i_1, \dots, i_l$ $\Rightarrow L_\alpha = s_{i_1} \dots s_{i_l}(L_{\alpha_j})$
for some j. Hence: $\boxed{\alpha = \pm w(\alpha_j)}$ with $w = s_{i_1} \dots s_{i_l}$
AND $\boxed{s_\alpha = w s_j w^{-1}}$ $\Rightarrow$ a)

note: $s_\beta(\beta) = -\beta$ $\Rightarrow$ b) ∎

Upshot: The whole root system R can be reconstructed from Π as $W(\Pi)$, where $W = \langle s_1, \dots, s_r \rangle$ — subgp of $O(E)$ generated by simple reflections.

Example (A_{n-1}-type root system): $W = S_n$, $s_i = (i, i+1)$. Thus part a) of Thm 1 says that any permutation can be written as a product of transpositions.

• **Length function**

We shall say that a root hyperplane L_α separates two Weyl chambers C, C' if they lie on different sides of L_α, i.e. $\alpha(C)$ & $\alpha(C')$ have different signs.
[Warning: We don't assume L_α to be a wall of either C or C']

Def 1: The length $l(w) \in \mathbb{Z}_{\geq 0}$ of $w \in W$ is the number of root hyperplanes separating C_+ and $w(C_+)$

As $\alpha(C_+) > 0$ $\forall \alpha \in R_+$ and $\alpha(w(C_+)) = (w^{-1}\alpha)(C_+)$, we can equivalently write

$$\boxed{l(w) = \#\{\alpha \in R_+ \mid w^{-1}(\alpha) \in R_-\}}$$

Furthermore, if α is as above, then $\beta = -w^{-1}(\alpha)$ satisfies $\beta(C_+) > 0$ & $w(\beta)(C_+) < 0$ and vice versa, so that $l(w) = l(w^{-1}) \Rightarrow \boxed{l(w) = \#\{\alpha \in R_+ \mid w(\alpha) \in R_-\}}$

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<u>Example</u>: $w = s_i$ – simple reflection. In this case, C_+ and $s_i(C_+)$ are clearly separated only by L_{α_i} $\Rightarrow$ $\ell(s_i) = 1$. Therefore:

$$\{\alpha \in R_+ \mid s_i(\alpha) \in R_-\} = \{\alpha_i\} \Rightarrow \boxed{s_i \text{ permutes the set } R_+ \setminus \{\alpha_i\}}$$

Lemma 2: Let $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$. Then $\alpha_i^\vee(\rho) = 1$ $\forall i$. Hence, $\rho = \sum_{i=1}^{r} \omega_i$

This weight will be important in the repr. theory (as we will soon see)

▶ $\rho = \frac{1}{2}\alpha_i + \frac{1}{2}\sum_{\alpha \in R_+ \setminus \{\alpha_i\}} \alpha \overset{\text{Example}}{\Longrightarrow} s_i(\rho) = -\frac{1}{2}\alpha_i + \frac{1}{2}\sum_{\alpha \in R_+ \setminus \{\alpha_i\}} \alpha = \rho - \alpha_i \Rightarrow \alpha_i^\vee(\rho) = 1 \Rightarrow \rho = \sum_{i=1}^{r} \omega_i$ ■

We shall now derive an equivalent definition of the length $\ell(w)$:

Theorem 2: Let $w = s_{i_1} s_{i_2} \dots s_{i_\ell}$ be a representation of $w \in W$ as a product of simple reflections that has minimal possible length. Then $\ell = \ell(w)$.

▶ As in the proof of Thm 1, consider a chain of Weyl chambers $C_k = s_{i_1} \dots s_{i_k}(C_+)$ so that $C_0 = C_+$, $C_\ell = w(C_+)$, and $0 \le k \le \ell$. As <u>we already saw</u>, C_k & C_{k-1} are <u>adjacent</u> Weyl chambers (separated by L_{β_k} with $\beta_k = s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k})$). This implies the inequality $\underline{\ell(w) \le \ell}$, as every separating hyperplane should be precisely one of L_{β_k} above. ↑CHECK

On the other hand, picking generic points $t_+ \in C_+$ and $t_+^w \in w(C_+)$, the line segment connecting them will intersect every separating root hyperplane and no other root hyperplanes. Then, the proof of Thm 1 shows that w can be written as a product of $\ell(w)$ simple reflections $\Rightarrow \ell \le \ell(w) \Rightarrow \ell = \ell(w)$ ■

Def 2: An expression $w = s_{i_1} s_{i_2} \dots s_{i_\ell}$ is called <u>reduced</u> if $\ell = \ell(w)$

Corollary 1: The action $W \curvearrowright \{\text{Weyl chambers}\}$ is simply transitive.

▶ We know this action is transitive (see Thm 1 of Lect 25). Hence, it suffices to show $w(C_+) = C_+ \Rightarrow w = 1$. But by definition, $\ell(w) = 0 \overset{\text{Thm 2}}{\Longrightarrow} w = 1$ ■

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Let $\overline{C}_+$ be the closure of the positive Weyl chamber C_+. Then, by above every W-orbit of $W \curvearrowright E$ has an element from $\overline{C}_+$. In fact a stronger result holds:

Proposition 1: $E/W \simeq \overline{C}_+$, i.e. every W-orbit of $W \curvearrowright E$ contains exactly 1 elt in $\overline{C}_+$

▶ Assume the contrary, i.e. $\exists\, \lambda, \mu \in \overline{C}_+$ $\exists w \in W$ s.t. $\mu = w(\lambda)$. We assume that w is the shortest possible, and $w \neq 1$. Consider a reduced decomposition $w = s_{i_1} \dots s_{i_l}$. Note that $s_{i_1} w = s_{i_2} \dots s_{i_l} \Rightarrow l(s_{i_1} w) \leq l-1 < l(w)$.

Thus, $\exists \gamma \in R_+$ s.t. $w(\gamma) \in R_-$ but $s_{i_1} w(\gamma) \in R_+ \Rightarrow w(\gamma) = -\alpha_{i_1} \Rightarrow$

$$\Rightarrow \gamma = -s_{i_l} s_{i_{l-1}} \dots s_{i_2} \underbrace{s_{i_1}(\alpha_{i_1})}_{-\alpha_{i_1}} = s_{i_l} \dots s_{i_2}(\alpha_{i_1}) \Rightarrow s_\gamma = s_{i_l} \dots s_{i_2} s_{i_1} s_{i_2} \dots s_{i_l}$$

Then: $\overset{\lambda \in \overline{C}_+}{0 \leq} (\lambda, \gamma) = (\underbrace{w(\lambda)}_{\mu}, w(\gamma)) = (\mu, \underbrace{w(\gamma)}_{\in R_-}) \leq 0 \Rightarrow (\lambda, \gamma) = 0 \Rightarrow s_\gamma(\lambda) = \lambda$.

But then, we get:

$$\mu = w(\lambda) = s_{i_1} \dots s_{i_l}(\lambda) = \underbrace{s_{i_2} \dots s_{i_l} s_{i_l} \dots s_{i_2}}_{=id} s_{i_1} s_{i_2} \dots s_{i_l}(\lambda) = s_{i_2} \dots s_{i_l} s_\gamma(\lambda) = s_{i_2} \dots s_{i_l}(\lambda)$$

Hence, contradiction with w being of minimal length! ∎

Given a polarization, $C_- := -C_+$ is called the <u>negative Weyl chamber</u>.

Corollary 2: $\exists!\, w_0 \in W$ s.t. $C_- = w_0(C_+)$.
Moreover, $l(w_0) = \# R_+$ and $l(w) < l(w_0)$ $\forall w \in W \setminus w_0$.
Finally, $w_0^2 = 1$.
hence the name w_0 - <u>longest element of the Weyl gp</u>

▶
- uniqueness & existence of w_0 follow from Cor 1.
- $l(w_0) = \# \{\alpha \in R_+ \mid w_0(\alpha) \in R_-\} = \# R_+$
- $l(w) \leq l(w_0)$ and equality means $w(R_+) = R_- \Rightarrow w(C_+) = C_- \Rightarrow w = w_0$
- $w_0^2(C_+) = C_+ \Rightarrow w_0^2 = 1$ by Cor 1 ∎

Example (A_{n-1}-root system): w_0 is the permutation $\begin{pmatrix} 1 & 2 & 3 & \dots & n-1 & n \\ n & n-1 & n-2 & & 2 & 1 \end{pmatrix}$.

Lemma 3: $\forall w \in W$: $l(w) + l(w_0 w) = l(w_0) = l(w) + l(w w_0)$

▶ $|R_+| = |\{\alpha \in R_+ : w(\alpha) \in R_-\}| + |\{\alpha \in R_+ : w(\alpha) \in R_+\}| = l(w) + |\{\alpha \in R_+ : w_0 w(\alpha) \in R_-\}| = l(w) + l(w_0 w)$

Also: $|\{\alpha \in R_+ : w(\alpha) \in R_+\}| = |\{\alpha \in R_- : w w_0(\alpha) \in R_+\}| = |\{\alpha \in R_+ : w w_0(\alpha) \in R_-\}| = l(w w_0)$ ∎