Lecture #30

Goal for this week: Develop the theory of finite-dimensional representations over $\mathbb{C}$ semisimple Lie algebras $\mathfrak{g}$ (by [Lecture 9, Prop 1] they can be exponentiated to give a repr-n of the connected simply connected Lie gp G with $\mathfrak{g} = \mathrm{Lie}(G)$)

Recall that every fin. dim-l $\mathfrak{g}$-module V is completely reducible (which was a consequence of Whitehead thm, see [Lect 18, Thm 2]). Therefore, we aim at:

$\boxed{\text{classification of irreducible finite-dimensional } \mathfrak{g}\text{-modules}}$

Def 1: Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, $\lambda \in \mathfrak{h}^*$, and V be a $\mathfrak{g}$-module.
Then $v \in V$ has a weight λ if $h(v) = \lambda(h) \cdot v \ \forall h \in \mathfrak{h}$, v is called a weight vector.
Let $V[\lambda] := \{v \in V \mid h(v) = \lambda(h) v \ \forall h \in \mathfrak{h}\}$ – weight subspace of V of weight λ.
If $V[\lambda] \neq 0$, then λ is called a weight of V. Set $P(V) = \{\lambda \in \mathfrak{h}^* \mid V[\lambda] \neq 0\}$
set of weights of V

The following properties are obvious (simple exercise to check at home):
1) $\mathfrak{g}_\alpha V[\lambda] \subseteq V[\lambda + \alpha] \quad \forall \lambda \in \mathfrak{h}^*, \alpha \in R \cup \{0\}$
2) vectors of different weights are lin. independent

Given a $\mathfrak{g}$-module V, set $V' = \mathrm{span}\{\text{all weight vectors}\} = \bigoplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$.

Def 2: A $\mathfrak{g}$-module V has a weight decomposition (w.r.t. $\mathfrak{h} \subseteq \mathfrak{g}$) if $V' = V$

Exercise: Provide an example of ∞-dim $\mathfrak{g}$-module V which doesn't have weight decomp.

Lemma 1: Any fin-dim. $\mathfrak{g}$-module V has a weight decomposition.
Furthermore, all weights of V are integral i.e. $\lambda(h_i) \in \mathbb{Z} \quad \forall \lambda \in P(V), \forall i$

► Consider $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i} \subseteq \mathfrak{g}$. Viewing V as a fin. dim. module over $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$, we see that h_i acts semisimply with integer eigenvalues. But $\{h_i\}_{i=1}^{rk(\mathfrak{g})}$ is a basis of $\mathfrak{h}$, hence, $\mathfrak{h} \curvearrowright V$ semisimply $\Rightarrow V = \bigoplus_{\lambda \in \mathfrak{h}^*} V[\lambda]$ ■

So: V - fin. dim. $\mathfrak{g}$-module $\Rightarrow P(V)$ - finite subset of $P \subseteq \mathfrak{h}^*$
weight lattice, see Lecture #25

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Def 3: A vector $v \overset{\neq 0}{\in} V$ is called a highest weight vector of weight $\lambda \in \mathfrak{h}^*$ if
$$\begin{cases} e_i(v) = 0 \ \forall i, \text{ equivalently } \mathfrak{n}_+(v) = 0 \\ h(v) = \lambda(h) \cdot v \ \forall h \in \mathfrak{h}, \text{ equivalently } v \in V[\lambda]. \end{cases}$$
A $\mathfrak{g}$-module V is called a highest weight representation with highest weight λ if it is generated by such a vector.

Lemma 2: Any finite-dimensional $\mathfrak{g}$-representation $V \neq 0$ contains a nonzero highest weight vector of some weight $\lambda \in P$

▸ According to Lemma 1, we have $V = \bigoplus_{\lambda \in P(V)} V[\lambda]$ with $P(V) \subseteq P$ - finite set. As noted after Def 1, we have $e_i(V[\lambda]) \subseteq V[\lambda + \alpha_i]$. But then picking $\lambda_{max} \in P(V)$ maximal in the sense of the pairing $\lambda(\rho^\vee) \in \mathbb{Z}$, we get $e_i(V[\lambda_{max}]) = 0 \ \forall i$, hence any $v \in V[\lambda_{max}]$ is a highest weight vector. ■

Corollary 1: Any irreducible fin. dim. $\mathfrak{g}$-module is a highest weight representation

▸ Follows from Lemma 2, since V is generated by any nonzero vector ■

We shall now take some detour and consider highest weight modules which are not necessarily fin. dim. In fact, the largest of those, so-called Verma modules, are key to the theory of finite-dimensional $\mathfrak{g}$-representations.

Def 4: Let $I_\lambda \subseteq U(\mathfrak{g})$ be the left ideal generated by $\{e_i\}_{i=1}^r \cup \{h - \lambda(h) \mid h \in \mathfrak{h}\}$
The Verma module M_λ is the quotient $U(\mathfrak{g})/I_\lambda$, viewed as a $\mathfrak{g}$-module

Remark: Alternatively, evoking $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, we can define M_λ as the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda =: \mathrm{Ind}_{U(\mathfrak{b}_+)}^{U(\mathfrak{g})}(\mathbb{C}_\lambda)$, where $\mathbb{C}_\lambda$ denotes a 1-dim representation of $\underbrace{\mathfrak{h} \oplus \mathfrak{n}_+}_{=: \mathfrak{b}_+}$ (hence $U(\mathfrak{b}_+)$) with $\mathfrak{n}_+$ acting by zero, $\mathfrak{h}$ acting via λ.

! For $\mathfrak{g} = \mathfrak{sl}_2$, we already encountered Verma modules M_λ back in Lecture 11, where it was shown that M_λ is irreducible unless $\lambda \in \mathbb{Z}_{\geq 0}$, and in the latter case $M_\lambda / M_{-\lambda-2} \simeq V_\lambda$ - fin. dim. $\mathfrak{sl}_2$-module.

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In fact, as a module over $U(\mathfrak{n}_-) \subseteq U(\mathfrak{g})$ Verma modules are free rank 1 modules:

Proposition 1: The map $U(\mathfrak{n}_-) \xrightarrow{\phi} M_\lambda$ given by $\phi(x) = x(\underbrace{v_\lambda})$ is an isomorphism of left $U(\mathfrak{n}_-)$-modules
class of $1 \in U(\mathfrak{g})$

▶ Consider the vector space decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{n}_- \oplus \mathfrak{b}_+$
PBW theorem implies that the multiplication map $U(\mathfrak{n}_-) \otimes U(\mathfrak{b}_+) \xrightarrow[m]{} U(\mathfrak{g})$ is a vector space isomorphism, see [Lecture 11, Corollary 4].

Consider $\varphi_\lambda : U(\mathfrak{b}_+) \longrightarrow \mathbb{C}$, given by $\varphi_\lambda(h) = \lambda(h)$, $\varphi_\lambda(e_i) = 0$, and set $K_\lambda = \mathrm{Ker}\, \varphi_\lambda$.

Then: $m^{-1}(I_\lambda) = U(\mathfrak{n}_-) \otimes K_\lambda$

Hence: $M_\lambda = U(\mathfrak{g})/I_\lambda \xleftarrow{\sim} U(\mathfrak{n}_-) \otimes U(\mathfrak{b}^+) / U(\mathfrak{n}_-) \otimes K_\lambda \simeq U(\mathfrak{n}_-)$ ∎

As an immediate corollary of the above result, we get:

Corollary 2: M_λ has a weight decomposition with $P(M_\lambda) = \{\lambda - \sum_{i=1}^{r} n_i \alpha_i \mid n_i \in \mathbb{Z}_{\geq 0}\}$
Moreover, $\dim M_\lambda[\lambda] = 1$ and $\dim M_\lambda[\mu] < \infty$ $\forall \mu$

Notation: $Q_+ = \{\sum_{i=1}^{r} n_i \alpha_i \mid n_i \in \mathbb{Z}_{\geq 0}\}$, so that we get $P(M_\lambda) = \lambda - Q_+$

In fact, the Verma modules can be characterized by their universal property.

Proposition 2: If V is a $\mathfrak{g}$-module and $v \in V[\lambda]$ is a highest weight vector, then there is a unique $\mathfrak{g}$-module homomorphism $M_\lambda \xrightarrow{\phi_v} V$, $v_\lambda \longmapsto v$

In particular, if V is a highest weight representation with highest weight λ, then it is a quotient of M_λ.

▶ To construct ϕ_v, note that we have $U(\mathfrak{g}) \to V$ given by $x \mapsto x(v)$, which contains I_λ in its kernel, hence gives rise to the desired $\phi_v : M_\lambda \to V$.
The uniqueness is clear as M_λ is generated by v_λ.
Finally, if V is generated by such v, then we get $M_\lambda \twoheadrightarrow V$ ∎

Exercise: A quotient of any module with a highest weight decomposition must have a weight space decomposition

Corollary 3: Any highest weight $\mathfrak{g}$-module has a weight decomposition with fin.dim. weight subspaces

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In what follows, we shall use the partial order $\leq$ on $\mathfrak{h}^*$:

$$\boxed{\lambda \leq \mu \iff \mu - \lambda \in Q_+}$$

Lemma 3: Any highest weight $\mathfrak{g}$-module V has a unique highest weight and unique up to scaling highest weight vector

► Indeed, if λ is a highest weight of V, then $M_\lambda \twoheadrightarrow V$ and so $P(V) \subseteq P(M_\lambda)$ $\Rightarrow \forall \mu \in P(V)$ we have $\mu \leq \lambda$. Therefore, if μ is a highest weight of V too, then $\mu \leq \lambda$ & $\lambda \leq \mu \Rightarrow \lambda = \mu$.

Finally, $\dim M_\lambda[\lambda] = 1 \Rightarrow \dim V[\lambda] = 1 \Rightarrow$ highest weight vector is unique up to scaling ■

Proposition 3: For any $\lambda \in \mathfrak{h}^*$, the Verma module M_λ has a unique irreducible quotient L_λ. Furthermore, L_λ is a quotient of any highest weight $\mathfrak{g}$-module V with highest weight λ

► Consider any $\mathfrak{g}$-submodule $W \subsetneq M_\lambda$. By the previous exercise, W has a weight decomposition $W = \bigoplus_{\mu \in \lambda - Q_+} W[\mu]$. But $W[\lambda] = 0$ as otherwise $v_\lambda \in W \Rightarrow W = M_\lambda$.

Define $J_\lambda := \sum_{W \subsetneq M_\lambda - \text{submodule}} W$ = the (non-direct) sum of all proper $\mathfrak{g}$-submodules of M_λ.

Then: J_λ is also a proper submodule, namely the largest one.

We set $L_\lambda := M_\lambda / J_\lambda$. First, we claim it is an irreducible $\mathfrak{g}$-module. Indeed, if $0 \subsetneq N_\lambda \subsetneq J_\lambda$, then preimage of N_λ under $\pi: M_\lambda \twoheadrightarrow L_\lambda$ would be a proper submodule strictly larger than J_λ, a contradiction. Second, given any highest weight $\mathfrak{g}$-module V of highest weight λ, we have by Prop 2: $\phi_v: M_\lambda \twoheadrightarrow V$. Then $\mathrm{Ker}(\phi_v)$ is a proper $\mathfrak{g}$-submodule of $M_\lambda \Rightarrow \mathrm{Ker}(\phi_v) \subseteq J_\lambda$, hence the epimorphism $M_\lambda \twoheadrightarrow L_\lambda$ descends through $V \twoheadrightarrow L_\lambda$ ■

As an immediate corollary, we get:

Corollary 4: Irreducible highest weight $\mathfrak{g}$-modules are classified by $\lambda \in \mathfrak{h}^*$ via $\mathfrak{h}^* \ni \lambda \rightsquigarrow L_\lambda$ – from Prop 3

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As every irreducible finite-dimensional representation of $\mathfrak{g}$ is highest weight (Corollary 1), our original problem is equivalent to:

$$\boxed{\text{classification of } \lambda\in\mathfrak{h}^* \text{ such that } \dim(L_\lambda)<\infty}$$

Def 5: A weight $\lambda\in\mathfrak{h}^*$ is called dominant integral if $\underbrace{\alpha_i^\vee(\lambda)}_{=\lambda(h_i)}\in\mathbb{Z}_{\geq 0}$ $\forall i$.
$P_+ = \{\text{all dominant integral weights}\}$

Lemma 4: If L_λ is finite-dimensional, then $\lambda\in P_+$

▶ The vector $v_\lambda\in L_\lambda$ is the highest weight vector for $sl(2,\mathbb{C})_{\alpha_i}$ with the highest weight $\alpha_i^\vee(\lambda)=\lambda(h_i)$. Since L_λ - fin. dim, the $sl(2,\mathbb{C})_{\alpha_i}$-submodule generated by v_λ is also.
So: $\lambda\in P_+$ ∎

Lemma 5: If $\lambda\in P_+$, then we have $f_i^{\lambda(h_i)+1}v_\lambda=0$ in L_λ

▶ By the sl_2-theory, applied to $sl(2,\mathbb{C})_{\alpha_i}$, we have $e_i f_i^{\lambda(h_i)+1}v_\lambda=0$.
Also, we have $e_j f_i^{\lambda(h_i)+1}v_\lambda = f_i^{\lambda(h_i)+1}e_j v_\lambda=0$ for $j\neq i$, as $[e_j,f_i]=0$.
Finally, $f_i^{\lambda(h_i)+1}v_\lambda\in L_\lambda[\lambda-(\lambda(h_i)+1)\alpha_i]$.

Thus: the vector $f_i^{\lambda(h_i)+1}v_\lambda$ is a highest weight vector in L_λ. Therefore, it generates a proper submodule of L_λ (for degree reasons it cannot contain v_λ)
But: L_λ - irreducible $\Rightarrow$ this submodule is zero $\Rightarrow f_i^{\lambda(h_i)+1}v_\lambda=0$ ∎

Exercise: Verify that the corresponding $\mathfrak{g}$-module homomorphism of Verma modules $M_{\lambda-(\lambda(h_i)+1)\alpha_i}\to M_\lambda$ is injective

Lemma 6: Let V be a $\mathfrak{g}$-module with weight decomposition, with all $V[\lambda]$ being fin. dim. If V is a sum of fin. dim. $sl(2,\mathbb{C})_{\alpha_i}$-modules $\forall i$, then $\forall\lambda\in P, w\in W$, we have

$$\dim V[\lambda]=\dim V[w(\lambda)]$$

In particular, $P(V)$ is a W-invariant set.

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Proof of Lemma 6

▶ As W is generated by simple reflections, it suffices to prove $\forall \lambda\ \forall i$: $\dim V[\lambda] = \dim V[s_i(\lambda)]$. But being locally finite $\exists$ $\mathfrak{sl}(2)_{\alpha_i}$-submodule W of V s.t. $\dim W < \infty$ and W contains both $V[\lambda]$ and $V[s_i(\lambda)]$.

But by $\mathfrak{sl}_2$-theory, if $k_i := \lambda(h_i) \in \mathbb{Z}_{\geq 0}$ then $V[\lambda] \xrightarrow{f_i^{k_i}} V[s_i(\lambda)]$ and $V[s_i(\lambda)] \xrightarrow{e_i^{k}} V[\lambda]$ are both injective (actually bijective). If $k_i < 0$, then replace the roles $\lambda \leftrightarrow s_i(\lambda)$. ∎

Our final result provides a classification of all fin. dim. simple $\mathfrak{g}$-mod

Theorem 1: $\forall \lambda \in P_+$, L_λ is finite-dimensional. Moreover $\dim L_\lambda[\mu] = \dim L_\lambda[w(\mu)]\ \forall w, \mu$

Corollary/UPSHOT: We get a bijection

$$\begin{array}{ccc} P_+ & \longleftrightarrow & \{\text{simple fin. dim } \mathfrak{g}\text{-modules}\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \lambda & \longmapsto & L_\lambda \end{array}$$

Idea: We will first prove that L_λ is locally finite with respect to each $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$ so that we can use symmetry of Lemma 6 and then deduce that actually L_λ is fin. dimensional

▶ By Lemma 5, $v_\lambda \in L_\lambda$ is locally finite w.r.t. each $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$.
(by PBW thm, submodule is spanned by $f_i^a h_i^b e_i^c(v_\lambda)$ but $e_i(v_\lambda)=0$, $h_i(v_\lambda)$ is a multiple of v_λ, and $f_i^{\lambda(h_i)+1}(v_\lambda)=0$ by Lemma 5 above)

Also each of generators $\{e_j, f_j, h_j\}$ is locally finite w.r.t. adjoint action of $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$ due to Serre relations – see Lecture 29.

As L_λ is generated by v_λ over $U\mathfrak{g}$, we conclude that each vector of L_λ is locally finite w.r.t. $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$ (proof by induction: if $v \in L_\lambda$ is loc. finite and $x \in \mathfrak{g}$ is $\mathrm{ad}(\mathfrak{sl}(2,\mathbb{C})_{\alpha_i})$ loc. finite, then $x(v) \in L_\lambda$ is loc. finite w.r.t. $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$ – indeed if $v \in W$ - f.d. $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$-submodule, $x \in U$ - f.d. $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$-submodule $\Rightarrow x(v) \in \underline{U(W)}$ f.d. subm.

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(Continuation of proof)

By Lemma 6, the weights $P(L_\lambda)$ of L_λ are W-invariant and $\forall \mu\in P(L_\lambda), w\in W$:

$$\boxed{\dim L_\lambda[\mu] = \dim L_\lambda[w(\mu)].}$$

But: $\forall \mu\in P$ its W-orbit $W(\mu)$ is finite and contains (unique) element in P_+ by [Lecture 26, Proposition 1].

Thus: $\dim L_\lambda = \sum\limits_{\mu\in P(L_\lambda)} \dim L_\lambda(\mu) = \sum\limits_{\mu\in P(L_\lambda)\cap P_+} \dim L_\lambda[\mu]\cdot \underbrace{\# |W\mu|}_{\text{cardinality of the orbit } W\mu.}$

As $P(L_\lambda)\subseteq \lambda - Q_+$, it thus suffices to prove:

$\boxed{\text{Claim: For any } \lambda\in P_+ : \text{the set } P_+\cap(\lambda-Q_+) \text{ is finite}}$

Any $\nu\in P_+\cap(\lambda-Q_+)$ satisfies $\lambda-\nu\in Q_+$ but also $\lambda+\nu\in P_+$, hence, $(\lambda-\nu,\lambda+\nu)\geq 0 \Rightarrow (\nu,\nu)\leq(\lambda,\lambda)$. But the intersection of lattice P and closed ball $\{\nu\in\mathfrak{h}^*: (\nu,\nu)\leq(\lambda,\lambda)\}$ is clearly finite □

This completes the proof of $\dim(L_\lambda)<\infty$. The part about $\dim L_\lambda[\mu] = \dim L_\lambda[w(\mu)]$ is due to Lemma 6. ■