Lecture #31-32

Let V be a finite-dimensional representation of a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. If G is the corresponding connected simply connected Lie group with $\mathrm{Lie}(G)=\mathfrak{g}$, then

$$\mathfrak{g} \curvearrowright V \underset{\text{exponentiated}}{\rightsquigarrow} G \curvearrowright V, \text{ by [Lecture 9, Prop 1]}$$

As we know from the case of representations of finite gps, an important quantity of a book is:

$$\boxed{\text{character } \chi_V : G \to \mathbb{C} \quad \text{given by } \chi_V(g) = tr_V(g)}$$

As χ_V is conjugacy-class invariant, the values of $\{\chi_V(e^h) \mid h \in \mathfrak{h}\}$ determine $\{\chi_V(e^x) \mid x\text{-s.s}\}$. On the other hand, the set of s.s (semisimple) elements is dense in $\mathfrak{g}$.

[Rem: One can actually appeal to "strongly regular elements" $\subsetneq$ semisimple elements, Lect 21. Since χ_V is an analytic function on G, it is determined by its values on nonempty open set $\{e^x \mid x\text{-s.s.}, \|x\| \ll 1\} \subseteq$ nbhd of $1 \in G$ which is $\simeq$ nbhd of $0 \in \mathfrak{g}$ (see Lecture 6).

$$\boxed{\underline{\underline{\text{SO}}}: \chi_V : G \to \mathbb{C} \text{ is determined by the values } \{\chi_V(e^h) \mid h \in \mathfrak{h}\}}$$

However, as noted in Lecture 30, V has a weight decomposition $V = \bigoplus_{\lambda \in P} V[\lambda]$ and by definition, we have

$$\boxed{\chi_V(e^h) = \sum_{\lambda \in P} \dim V[\lambda] \cdot e^{\lambda(h)} = \sum_{\lambda \in \underbrace{P(V)}_{\text{finite set}}} \dim V[\lambda] \cdot e^{\lambda(h)}}$$

Thus all the information is readily captured by the following invariant that "counts" dimensions of weight components

Def 1: Character of V is: $\boxed{\chi_V := \sum_{\mu \in P} \dim V[\mu] \cdot e^{\mu} \in \mathbb{Z}[P]}$

Here, $\mathbb{Z}[P]$ is the group algebra of the abelian gp P, i.e. it's a vector space with basis $\{e^{\lambda} \mid \lambda \in P\}$ and product $e^{\lambda} \cdot e^{\mu} = e^{\lambda+\mu}$.

[Rem: e^{λ} can be viewed as an analytic function on $\mathfrak{h}$ via $e^{\lambda}(h) := e^{\lambda(h)}$. Thus, we may denote e^0 simply by 1.

[Rem: As $P \simeq \bigoplus_{i=1}^{r} \mathbb{Z}\omega_i$, ω_i - fundamental weights, we have $\mathbb{Z}[P] \simeq \mathbb{Z}[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$, $x_i := e^{\omega_i}$

Note that for $\mathfrak{g} = \mathfrak{sl}_2$, we have already encountered characters in Lecture 10, where they were defined via $\chi_V(z) = \sum_{m \in \mathbb{Z}} \dim V(m) \cdot z^m$. As $P = \mathbb{Z}\omega_1$ for $\mathfrak{sl}_2$, if we match $z \leftrightarrow e^{\omega_1}$, then it precisely recovers the above notion.

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Since we noted in Lecture 30, that one really needs more general repr-s, not just finite-dimensional ones, to develop the theory of the latter, we shall now extend the characters to a bigger category of $\mathfrak{g}$-modules:

Def 2: The category $\mathcal{O}$ is the category of representation V of $\mathfrak{g}$ which admit weight decomposition into fin.dim'l weight spaces

$$\boxed{V = \bigoplus_{\mu\in P} V[\mu], \quad \dim V[\mu] < \infty}$$

s.t. $\exists$ finite set $\lambda_1, \ldots, \lambda_m \in P$ satisfying

$$\boxed{P(V) \subseteq \bigcup_{1\le i\le m} (\lambda_i - Q_+)}$$

[Rem: Usually, $\lambda_i \in \mathfrak{h}^*$, but for our purpose it suffices to consider $\lambda_i \in P$.

[Example: Any highest weight module is in $\mathcal{O}$ (recall $P(M_\lambda) = \lambda - Q_+$).

We would like now to generalize character χ_V to $V \in \mathcal{O}$. As $P(V)$ may be infinite, we will get a certain completion of $\mathbb{Z}[P]$. Explicitly, let

$$\boxed{R \text{ be the ring of } \left\{ \sum_{\mu\in P} c_\mu \cdot e^\mu \,\middle|\, c_\mu \in \mathbb{Z} \ \& \ \{\mu \mid c_\mu \neq 0\} \subseteq \bigcup (\lambda_i - Q_+) \text{ for some } \lambda_1, \ldots, \lambda_m \in P \right\}}$$

[Exercise (easy): Verify that this is indeed a ring.

Then for any $V \in \mathcal{O}$, we define its character χ_V as before:

$$\boxed{\chi_V := \sum_{\mu \in P} \dim V[\mu]\, e^\mu \in R}$$

Lemma 1: The following properties hold:

a) If $U \simeq V \oplus W$ in $\mathcal{O}$, then $\chi_U = \chi_V + \chi_W$.

b) More generally, if $0 \to V \to U \to W$ is a short exact sequence in $\mathcal{O}$, then $\chi_U = \chi_V + \chi_W$.

c) For any $V, W \in \mathcal{O}$, we have $V \otimes W \in \mathcal{O}$ and $\chi_{V\otimes W} = \chi_V \cdot \chi_W$

(Exercise (easy): prove this lemma

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Lemma 2: For any $\lambda \in P$, the character of the Verma module M_λ is given by:

$$\chi_{M_\lambda} = \frac{e^\lambda}{\prod_{\alpha\in R_+}(1-e^{-\alpha})} \quad \text{where} \quad \frac{1}{1-e^{-\alpha}} = 1+e^{-\alpha}+e^{-2\alpha}+\dots$$

▸ According to [Lect 30, Prop 1], we have $U(n_-) \xrightarrow{\sim} M_\lambda$ given by $x \mapsto x(v_\lambda)$

But according to PBW thm, monomials $\{\prod_{\alpha\in R_+} f_\alpha^{n_\alpha} \mid n_\alpha \in \mathbb{Z}_{\geq 0}\}$ form a basis of $U(n)$

Thus: $\boxed{\chi_{M_\lambda} = e^\lambda \sum_{\mu\in Q_+} e^{-\mu}\cdot P(\mu),\ P(\mu) = \#\{\text{decompositions } \mu = \sum_{\alpha\in R_+} n_\alpha\cdot\alpha \text{ with } n_\alpha\in\mathbb{Z}_{\geq 0}\}}$

$P(\mu)$ — Kostant partition function

But the latter is easily seen to be precisely $e^\lambda\cdot\prod_{\alpha\in R_+}\frac{1}{1-e^{-\alpha}}$ ■

For the latter purpose, let's rewrite the above f-la. Recall $\rho = \frac{1}{2}\sum_{\alpha\in R_+}\alpha$ [Lecture 26, Lemma 2]. Then:

$$\boxed{\chi_{M_\lambda} = \frac{e^{\lambda+\rho}}{\prod_{\alpha\in R_+}(e^{\alpha/2}-e^{-\alpha/2})} =: \frac{e^{\lambda+\rho}}{\Delta}}$$

Δ — Weyl denominator

Def 3: The sign character is a group homomorphism $\varepsilon: W \to \mathbb{Z}/2 \simeq \{\pm 1\}$ given by $\varepsilon(w) = \det(w|_{\mathfrak{h}^*}) = (-1)^{\ell(w)}$. Also, $f\in\mathbb{Z}[P]$ is anti-invariant if $w(f) = \varepsilon(w)\cdot f\ \forall w\in W$

The following simple property of Δ is key to the rest:

Lemma 3: The Weyl denominator Δ is anti-invariant, i.e. $w(\Delta) = \varepsilon(w)\cdot\Delta\ \forall w\in W$

▸ As W is generated by simple reflections s_i, it suffices to check for those.

But: $s_i(\alpha_i) = -\alpha_i$ and s_i permutes $R_+\setminus\{\alpha_i\}$, hence

$$s_i(\Delta) = (e^{-\alpha_i/2} - e^{\alpha_i/2})\cdot\prod_{\alpha\in R_+\setminus\{\alpha_i\}}(e^{\alpha/2}-e^{-\alpha/2}) = -\Delta = \varepsilon(s_i)\cdot\Delta$$ ■

Now we are ready to state the key result of today's class:

Theorem 1 (Weyl character formula): For any $\lambda\in P_+$, the character χ_{L_λ} of the irreducible finite-dimensional $\mathfrak{g}$-module L_λ is given by

$$\boxed{\chi_{L_\lambda} = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\Delta}}$$

Before presenting the proof, we shall first discuss some corollaries.

Corollary 1 (Weyl denominator formula): $\Delta = \sum_{w\in W} (-1)^{\ell(w)} e^{w\rho}$

▶ Apply Thm 1 to $\lambda=0$ and note that $\chi_{L_0}=1$ (because L_0 = trivial rep'n) ∎

Corollary 2: For any $\lambda\in P_+$, we have
$$\chi_{L_\lambda} = \frac{\sum_{w\in W}(-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\sum_{w\in W}(-1)^{\ell(w)} e^{w\rho}}$$

▶ Combine Thm 1 + Cor 1 ∎

The Weyl character formula also allows to compute $\dim(L_\lambda)=\chi_{L_\lambda}(e^0)$. However, the substitution into the Weyl char. f-la doesn't immediately provide the answer, as both numerator & denominator vanish at e^0. The standard way to resolve this indeterminancy is to first compute the q-character:

Def 4: For a finite-dimensional $\mathfrak{g}$-module V, define $\dim_q V\in \mathbb{Z}[q,q^{-1}]$ via
$$\boxed{\dim_q V = \mathrm{tr}_V(q^{2\rho}) = \sum_{\lambda\in P}{}' \dim V[\lambda]\cdot q^{(2\rho,\lambda)}}$$
where $(\cdot,\cdot)$ is rescaled to satisfy $(\lambda,\mu)\in\mathbb{Z}$ $\forall \lambda,\mu\in P$.

Thus, $\dim_q V = \pi_\rho(\chi_V)$, where $\pi_\rho: \mathbb{Z}[P]\to\mathbb{Z}[q,q^{-1}]$, $e^\mu \longmapsto q^{(2\rho,\mu)}$

Proposition 1: For $\lambda\in P_+$, we have
$$\dim_q L_\lambda = \prod_{\alpha\in R_+} \frac{q^{(\lambda+\rho,\alpha)}-q^{-(\lambda+\rho,\alpha)}}{q^{(\rho,\alpha)}-q^{-(\rho,\alpha)}}$$

▶ $$\dim_q L_\lambda = \pi_\rho(\chi_{L_\lambda}) \overset{\text{Thm 1}}{=} \frac{\sum_{w\in W}(-1)^{\ell(w)} q^{2(w(\lambda+\rho),\rho)}}{\prod_{\alpha\in R_+}(q^{(\alpha,\rho)}-q^{-(\alpha,\rho)})} \underset{\ell(w)=\ell(w^{-1})}{\overset{w\leftrightarrow w^{-1}}{=}} \frac{\sum_{w\in W}(-1)^{\ell(w)} q^{2(\lambda+\rho,w(\rho))}}{\prod_{\alpha\in R_+}(q^{(\rho,\alpha)}-q^{-(\rho,\alpha)})}$$

Here, the numerator can be rewritten using $\boxed{\pi_{\lambda+\rho}: \mathbb{Z}[P]\to\mathbb{Z}[q,q^{-1}],\ e^\mu \longmapsto e^{(2(\lambda+\rho),\mu)}}$ as:
$$\sum_{w\in W}(-1)^{\ell(w)} q^{(2(\lambda+\rho),w(\rho))} = \pi_{\lambda+\rho}\Big(\sum_{w\in W}(-1)^{\ell(w)}e^{w\rho}\Big) \underset{\text{Cor 1}}{=} \pi_{\lambda+\rho}(\Delta) = \prod_{\alpha\in R_+}\left(q^{(\lambda+\rho,\alpha)}-q^{-(\lambda+\rho,\alpha)}\right)$$
This completes the proof ∎

As $\dim V = (\dim_q V)|_{q=1}$, we obtain by L'Hôpital rule the dimension formula:

Corollary 3: For $\lambda\in P_+$, we have $\dim L_\lambda = \dfrac{\prod_{\alpha\in R_+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in R_+}(\rho,\alpha)}$

[Rmk: It's a-priori nonobvious that the right-hand side is an integer!

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Proof of Theorem 1

Let us finally prove the Weyl character formula.

- An important ingredient is again the Casimir element C, see [Lecture 18, Def 4]. Recall that $C = \sum_i a_i a^i \in U(\mathfrak{g})$, where $\{a_i\}$, $\{a^i\}$ – dual bases of $\mathfrak{g}$ w.r.t. Killing form $(\ ,\ )$.

Let $\{x_j\}_{j=1}^r$ be an orthonormal basis of $\mathfrak{h}$ w.r.t. $(\ ,\ )|_{\mathfrak{h}\times\mathfrak{h}}$. We also pick $\{e_\alpha, f_\alpha\}_{\alpha\in R_+}$ so that $(e_\alpha, f_\alpha) = \frac{2}{(\alpha,\alpha)}$, so that $\{e_\alpha, f_\alpha, h_\alpha = \frac{2}{(\alpha,\alpha)} H_\alpha\}$ satisfy sl_2-relations, see [Lecture 20, Lemma 2]. Then:

$$\boxed{C = \sum_{j=1}^{r} x_j^2 + \sum_{\alpha\in R_+} \frac{(\alpha,\alpha)}{2}(e_\alpha f_\alpha + f_\alpha e_\alpha) = \sum_{j=1}^{r} x_j^2 + \sum_{\alpha\in R_+} H_\alpha + \sum_{\alpha\in R_+} (\alpha,\alpha)\cdot f_\alpha e_\alpha}$$

Lemma 4: If V is a highest weight $\mathfrak{g}$-module with highest weight λ, then
$$C_{|V} = (\lambda, \lambda+2\rho)\cdot \mathrm{Id}_V = (|\lambda+\rho|^2 - |\rho|^2)\cdot \mathrm{Id}_V$$

Since Casimir elt is central (Lect 18, Lemma 2), and V is generated by its highest wt vector v_λ, it suffices to prove $C(v_\lambda) = (\lambda, \lambda+2\rho)\cdot v_\lambda$.

But: 1) $e_\alpha(v_\lambda) = 0 \ \forall \alpha\in R_+ \Rightarrow$ all $f_\alpha e_\alpha$ annihilate v_λ.

2) $\sum_{j=1}^{r} x_j^2(v_\lambda) = \sum_{j=1}^{r} \lambda(x_j)\cdot\lambda(x_j)\cdot v_\lambda = (\lambda,\lambda)\cdot v_\lambda$

3) $H_\alpha(v_\lambda) = \lambda(H_\alpha)\cdot v_\lambda = (\lambda,\alpha)\cdot v_\lambda \Rightarrow \sum_{\alpha\in R_+} H_\alpha(v_\lambda) = (\lambda, 2\rho)\cdot v_\lambda$

- Consider the product $\chi_{L_\lambda}\cdot\Delta \in \mathbb{Z}[P]$. Know that Δ is W-anti-invariant, while χ_{L_λ} is W-invariant ([Lect 30, Thm 1]), hence, $\chi_{L_\lambda}\cdot\Delta$ is W-anti-invariant.

<u>So</u>: $\boxed{\chi_{L_\lambda}\cdot\Delta = \sum_{\mu\in P} c_\mu\cdot e^\mu \quad \text{with } c_{w\mu} = (-1)^{l(w)} c_\mu}$ Also: $\boxed{c_\mu = 0 \text{ if } \mu\notin \lambda+\rho-Q_+}$

On the other hand, we know that $\forall \mu\in P$ its W-orbit contains a (unique!) element in P_+ (see [Lecture 26, Proposition 1]). Thus it remains to prove:

Claim 1: $\forall \mu\in P_+\cap(\lambda+\rho-Q_+)$, we have $c_\mu = 0$ unless $\mu = \lambda+\rho$

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(Continuation of the proof)

- $\boxed{\text{Claim 2: For any } V\in\mathcal{O}\text{, we have } \chi_V=\sum_{\mu\in P} b_\mu\cdot\chi_{M_\mu} \text{ with } b_\mu\in\mathbb{Z} \text{ and } \{\mu \mid b_\mu\neq 0\}\subset\bigcup_{\lambda\in P(V)}(\lambda-Q_+)}$

▷ This is proved by constructing on each step a homomorphism from $\underbrace{\oplus M_?}_{\text{direct sum of Verma modules}}$ to one of the modules constructed in the previous step, and taking Ker, Coker. The result follows from Lemma 1b) □

In our case, we get $\chi_{L_\lambda}=\sum_\mu b_\mu\cdot\chi_{M_\mu}$. But it follows from the above proof of Claim 2 that $C_{M_\mu}=C_{L_\lambda}$. In particular, $\left.\begin{array}{l} y_{L_\lambda}\cdot\Delta=\sum_\mu c_\mu e^\mu \\ \quad\| \ \sum_\mu b_\mu\cdot e^{\mu+\rho}\end{array}\right\} \Rightarrow c_\mu=b_{\mu-\rho}.$

- To establish Claim 1, it remains to show:

$\boxed{\forall\, \underset{\neq\lambda}{\mu}\in P_+\cap(\lambda-Q_+) \text{ we have } \underbrace{C_{M_\mu}}_{|\mu+\rho|^2-|\rho|^2}\neq\underbrace{C_{L_\lambda}}_{|\lambda+\rho|^2-|\rho^2|}}$ ← Indeed, it's clear that $c_\lambda=b_{\lambda-\rho}=1$ from the proof of Claim 2

If $\mu=\lambda-\beta\in P_+$, $\beta\in Q_+\setminus\{0\}$, then:

$$|\lambda+\rho|^2-|\mu+\rho|^2=2(\lambda+\rho,\beta)-|\beta|^2=(\lambda+\rho,\beta)+(\underbrace{\lambda+\rho-\beta}_{=\mu+\rho},\beta)=\underbrace{(\lambda+\mu,\beta)}_{\geq 0}+\underbrace{2(\rho,\beta)}_{>0}>0$$

This establishes Claim 1, hence, also Theorem 1 ∎