Lecture #33

- Last 3 lectures
  - classified all irreducible (hence all) finite-dimensional modules over semisimple $\mathfrak{g}$ (Lect 30)
  - established character formula for χ_{L_λ} (Lect 31-32)

- Today we are going to further comment on these both results.

Lemma 1: Let $\lambda \in P_+$ and M_λ be the Verma module of highest weight λ with $v_\lambda \overset{\neq 0}{\in} M_\lambda[\lambda]$. Then $\forall i$, the vector $f_i^{\lambda(h_i)+1}(v_\lambda)$ is a highest weight vector of M_λ. Moreover, the submodule M_i of M_λ generated by $f_i^{\lambda(h_i)+1} v_\lambda$ is isomorphic to Verma module $M_{\lambda-(\lambda(h_i)+1)\alpha_i}$.

▶ The equality $e_j f_i^{\lambda(h_i)+1}(v_\lambda)=0 \ \forall j$ was already checked in [Lecture 30, Lemma 5] (even though loc.cit. was about L_λ not M_λ, the argument is the same). Also it's clear that $f_i^{\lambda(h_i)+1}(v_\lambda) \in M_\lambda[\lambda-(\lambda(h_i)+1)\alpha_i]$. Finally, $f_i^{\lambda(h_i)+1}(v_\lambda) \neq 0$ as $U(\mathfrak{n}_-) \xrightarrow{\sim} M_\lambda$ [Lecture 30 Prop 1].

Hence: there is a nonzero $\mathfrak{g}$-module homomorphism [Lecture 30 Proposition 2]

$$\boxed{M_{\lambda-(\lambda(h_i)+1)\alpha_i} \longrightarrow M_\lambda}$$

$$\text{hf wt. vector } v_{\lambda-(\lambda(h_i)+1)\alpha_i} \longmapsto f_i^{\lambda(h_i)+1}(v_\lambda)$$

But by [Hwk 11, Problem 3], any nonzero $\mathfrak{g}$-module homomorphism between Verma modules is injective. Hence, $M_{\lambda-(\lambda(h_i)+1)\alpha_i} \xrightarrow{\sim} M_i \subsetneq M_\lambda$

generated by $f_i^{\lambda(h_i)+1}(v_\lambda)$ ■

- For $\lambda \in P_+$ the result above provides a family of submodules $\{M_i\}_{i=1}^{r=\mathrm{rk}(\mathfrak{g})}$ of M_λ. Since each of them is proper, so is their (not direct!) sum $\sum M_i$. Define

$$\boxed{\tilde{L}_\lambda := M_\lambda \Big/ \sum_{i=1}^{r} M_i}$$

Theorem 1: a) $\tilde{L}_\lambda$ is finite-dimensional

b) $\tilde{L}_\lambda$ is irreducible $\mathfrak{g}$-module, hence $\tilde{L}_\lambda \simeq L_\lambda$

[**Example 1:** For $\mathfrak{g}=\mathfrak{sl}_2$, this reduces to $V_n \simeq M_n / M_{-n-2}$, established in Lecture 11.

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Proof of Theorem 1

b) If $\tilde{L}_\lambda$ is already shown to be fin. dim, with clearly $P(\tilde{L}_\lambda) \subseteq \lambda - Q_+$, we can decompose $\tilde{L}_\lambda$ into the direct sum of irreducibles:

$$\boxed{\tilde{L}_\lambda = \bigoplus_{\substack{\mu \leq \lambda \\ \mu \in P_+}} L_\mu^{\oplus m_\mu}}$$

As $\dim \tilde{L}_\lambda[\lambda] = 1$, we see that $m_\lambda = 1 \Rightarrow \tilde{L}_\lambda = L_\lambda \oplus \bigoplus_{\substack{\mu < \lambda \\ \mu \in P_+}} L_\mu^{\oplus m_\mu}$ with L_λ gen-d by $0 \neq v_\lambda \in \tilde{L}_\lambda[\lambda]$

But then on one hand v_λ generates all $\tilde{L}_\lambda$, while on the other hand - only L_λ.

Thus: all $m_\mu = 0$ for $\mu < \lambda$ AND $\tilde{L}_\lambda = L_\lambda$.

a) The proof of part a) saying that $\dim(\tilde{L}_\lambda) < \infty$ is actually analogous to that of [Lecture 30, Theorem 1].

- First, we note that $\forall v \in \tilde{L}_\lambda$ $\forall 1 \leq i \leq r$, the $\mathfrak{sl}(2,\mathbb{C})_{\alpha_i}$-submodule generated by v (given explicitly by $U(\mathfrak{sl}(2,\mathbb{C})_{\alpha_i})v$) is finite-dimensional. The argument is the same as it was for L_λ. Namely, the result is true for $v_\lambda \in \tilde{L}_\lambda$ due to Lemma 1. Hence, it also holds for any $v \in \tilde{L}_\lambda$.

- B/c of the above, [Lecture 30, Lemma 6] applies, and we get:

$$\boxed{\tilde{L}_\lambda = \bigoplus_{\substack{\mu \leq \lambda \\ \mu \in P}} \tilde{L}_\lambda[\mu] \quad \text{with } \dim \tilde{L}_\lambda[\mu] = \dim \tilde{L}_\lambda[w\mu] \quad \forall \mu\ \forall w \in W}$$

Now using the same argument as in [Lecture 30, Theorem 1 & Claim] we get $P(\tilde{L}_\lambda)$ is a finite set. But weight subspaces $\tilde{L}_\lambda[\mu]$ are all fin. dimensional (as so they are in M_λ). This implies $\dim(\tilde{L}_\lambda) < \infty$ ∎

In fact, the condition from [Lecture 30 Lemma 6] is very important and deserves a special treatment as a definition.

Def 1: A $\mathfrak{g}$-module V is called *integrable* if $\forall v \in V$ $\forall 1 \leq i \leq r$, we have

$$\boxed{\dim(U(\mathfrak{sl}(2,\mathbb{C}))_{\alpha_i} v) < \infty}$$

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Define the shifted (a.k.a. dot) action of the Weyl group W on $\mathfrak{h}^*$:

Def 2: $\boxed{w.\lambda := w(\lambda+\rho)-\rho}$ with $\rho=\frac{1}{2}\sum_{\alpha\in R_+}\alpha = \sum_{i=1}^{r}\omega_i$ ← check it's indeed an action!

Note: $s_i(\rho)=\rho-\alpha_i$ by [Lecture 26, Lemma 2)
$\Downarrow$
$s_i.\lambda = s_i(\lambda+\rho)-\rho = s_i(\lambda)-\alpha_i = \lambda-(\lambda(h_i)+1)\alpha_i$

In particular, $\boxed{f_i^{\lambda(h_i)+1}v_\lambda \in M_\lambda[s_i.\lambda]}$ in Lemma 1.

Thus, by Theorem 1, we have a short exact sequence $\bigoplus_i M_{s_i.\lambda}\to M_\lambda\to L_\lambda\to 0$.

The following beautiful result was established by Bernstein-Gelfand-Gelfand:

Theorem 2 (BGG resolution): For $\lambda\in P_+$, there exists a long exact sequence

$$\boxed{0\to M_{w_0.\lambda}\to\dots\to\bigoplus_{\substack{w\in W\\ \ell(w)=k}} M_{w.\lambda}\to\dots\to\bigoplus_{i=1}^{r} M_{s_i.\lambda}\to M_\lambda\to L_\lambda\to 0}$$

called the BGG resolution of L_λ

where $\ell: W\to\mathbb{Z}_{\geq 0}$ is the length, $w_0\in W$ – the longest element from Lect 26.

The proof is out of scope of the present course

Example 1: For $\mathfrak{g}=\mathfrak{sl}_3(\mathbb{C})$, $W=S_3=\{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1=s_2s_1s_2\}$, we get

$$0\to M_{s_1s_2s_1.\lambda}\to (M_{s_1s_2.\lambda}\oplus M_{s_2s_1.\lambda})\to (M_{s_1.\lambda}\oplus M_{s_2.\lambda})\to M_\lambda\to L_\lambda\to 0$$

Remark: Evaluating characters in the BGG resolution, we get

$$\boxed{\chi_{L_\lambda}=\sum_{w\in W}(-1)^{\ell(w)}\chi_{M_{w.\lambda}}=\sum_{w\in W}(-1)^{\ell(w)}\cdot\frac{e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in R_+}(1-e^{-\alpha})}=\frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\prod_{\alpha\in R_+}(e^{\alpha/2}-e^{-\alpha/2})}}$$

Therefore, BGG resolution can be thought of as a "categorification" of the Weyl character f-la.

We note that Thm 2 greatly refines Claim 2 from our proof of Theorem 1 in Lecture 31-32 (which only used a family of iterative short exact sequences)