Lectures #34-36

We start with the following simple observation:

Proposition 1: Characters $\{\chi_{L_\lambda} \mid \lambda \in P_+\}$ form a basis in the algebra $\mathbb{C}[P]^W$.

▶ For any $\lambda \in P_+$, define its average over W-orbit:

$$\boxed{m_\lambda = \sum_{\mu \in W\lambda} e^\mu = \frac{1}{|\mathrm{Stab}_W \lambda|} \sum_{w \in W} e^{w\lambda} \in \mathbb{C}[P]^W}$$

Clearly, $\{m_\lambda\}_{\lambda \in P_+}$ is a basis of $\mathbb{C}[P]^W$ by [Lecture 26, Proposition 1].

On the other hand, $\chi_{L_\lambda} \in \mathbb{C}[P]^W$ and it has the form

$$\boxed{\chi_{L_\lambda} = \sum_{\substack{\mu \le \lambda \\ \mu \in P_+}} c^\mu_\lambda \cdot m_\mu \quad \text{with } c^\lambda_\lambda = 1, \quad c^\mu_\lambda \in \mathbb{Z}}$$

Due to above property of the matrix $(c^\bullet_\bullet)$ being upper-triangular, we conclude that $\{\chi_{L_\lambda}\}_{\lambda \in P_+}$ - also a basis of $\mathbb{C}[P]^W$ ■

[Rmk: The above argument actually shows that it's also a basis of $\mathbb{Z}[P]^W$

For the rest of the present notes, we shall focus on the simplest case

$$\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) \leftarrow \text{"}A_{n-1}\text{-type"}$$

In this case, the root system is

$$\boxed{R = \{e_i - e_j \mid i \ne j\} \subseteq \mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}^n / \mathbb{R}(1, \dots, 1)}$$

For the polarization given by $(t_1, \dots, t_n) \in \mathbb{R}^n$ with $t_1 > t_2 > \dots > t_n$, we get

$$\boxed{R_+ = \{e_i - e_j \mid i < j\}, \quad \text{simple roots: } \alpha_i = e_i - e_{i+1} \ (1 \le i \le n-1)}$$

Moreover, the weight lattice P is:

$$\boxed{P = \{(\lambda_1, \dots, \lambda_n) \in \mathfrak{h}^*_{\mathbb{R}} \mid \lambda_i - \lambda_j \in \mathbb{Z}\}}$$

while dominant integral weights are:

$$\boxed{P_+ = \{(\lambda_1, \dots, \lambda_n) \in \mathfrak{h}^*_{\mathbb{R}} \mid \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\ge 0}\}}$$

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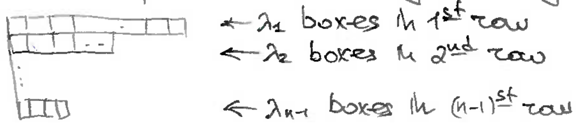
As $\mathfrak{h}^*_{\mathbb{R}} = \mathbb{R}^n/\mathbb{R}(1,\ldots,1)$, adding a multiple of $(1,\ldots,1)$ doesn't change weights.
Hence, we can identify P_+ with

$$\boxed{P_+ = \{(\lambda_1,\ldots,\lambda_{n-1},0)\in\mathbb{Z}^n_{\geq 0} \mid \lambda_1\geq\lambda_2\geq\ldots\geq\lambda_{n-1}\geq 0\}}$$

Such weights can be represented by Young diagrams:



Example: Based on [Homework 4, Problem 6] and [Homework 11, Problem 4]:

a) $S^k\mathbb{C}^n$ is an irreducible highest weight $\mathfrak{sl}_n$-module with highest weight $(k,0,\ldots,0)=k\omega_1$ for any $k\geq 1$, corresponding to Young diagram $\underbrace{\square\square\cdots\square}_{k \text{ boxes}}$.
If $e_1,e_2,\ldots,e_n$ - standard basis of $\mathbb{C}^n$, then the highest weight vector of $S^k\mathbb{C}^n$ is $e_1^{\otimes n}$.

b) $\Lambda^k\mathbb{C}^n$ is an irreducible highest weight $\mathfrak{sl}_n$-module with highest weight $\omega_k = (\underbrace{1,\ldots,1}_{k},0,\ldots,0)$ for $1\leq k\leq n-1$, corresponding to $\left.\begin{matrix}\square\\ \vdots\\ \square\end{matrix}\right\}$ k rows of 1 box each
(k^{th} fundamental weight) while $\Lambda^n\mathbb{C}^n$ is the trivial 1dim representation (h.wt. vector above: $e_1\wedge\ldots\wedge e_k$)

Let's now review the Weyl denominator formula in this context.
To this end, we identify $\mathbb{C}[P]\simeq\mathbb{C}[x_1^{\pm1},\ldots,x_n^{\pm1}]/(x_1\cdot\ldots\cdot x_n-1)$ with $x_i=e^{(0,\ldots,\underset{i^{th}\text{ spot}}{1},\ldots,0)}$

Lemma 1: The Weyl denominator formula for $\mathfrak{sl}_n(\mathbb{C})$ is just Vandermonde f-la:

$$\prod_{i<j}(x_i-x_j) = \sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)\cdot x_{\sigma(1)}^{n-1}x_{\sigma(2)}^{n-2}\ldots x_{\sigma(n)}^{0} \quad \text{with } \mathrm{sgn}(\sigma)=(-1)^{\ell(\sigma)}$$

► As $\rho=\omega_1+\ldots+\omega_{n-1}=(n-1,n-2,\ldots,1,0)$, we get

$$\text{LHS}=\prod_{i<j}(x_i-x_j) = x_1^{n-1}x_2^{n-2}\ldots x_n^0\prod_{i<j}\left(1-\frac{x_j}{x_i}\right) = e^{\rho}\prod_{\alpha\in R_+}(1-e^{-\alpha}) = \Delta \overset{\text{Weyl denom.}}{=\!=} \sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)e^{\sigma(\rho)} = \text{RHS}$$

Using the same argument, we can now see what Weyl char. f-la is:

Thm 1: For $\lambda=(\lambda_1,\ldots,\lambda_n)\in\mathbb{Z}^n_{\geq0}$ with $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_{n-1}\geq\lambda_n=0$, the χ_{L_λ} is given by

$$\boxed{\chi_{L_\lambda} = \frac{\det\left(x_i^{\lambda_j+n-j}\right)_{1\leq i,j\leq n}}{\det\left(x_i^{n-j}\right)_{1\leq i,j\leq n}}} \longleftarrow \text{the RHS is so-called Schur polynomial } S_\lambda(x_1,\ldots,x_n)$$

Remark: By Proposition 1, we see that $\{s_\lambda\}$ form a basis of symmetric polynomials in n variables (very important basis for combinatorics!)

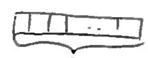
Recall: Any fin. dim. G-module is also a $\mathrm{Lie}(G)$-module, while for the inverse implication we need G to be simply connected.

Exercise: Show that $SL_n(\mathbb{C})$ are simply connected.
Hint: You may wish to relate $SL_n(\mathbb{C})$ to $SU(n)$ and use [Homework 2, Problem 7]

$$\boxed{\text{Thus, finite dimensional } SL_n(\mathbb{C})\text{-modules are completely reducible, and the irreducible ones are } \{L_\lambda \mid \lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{n-1},0)\in\mathbb{Z}^n \text{ with } \lambda_1\geq\lambda_2\geq\ldots\geq\lambda_{n-1}\geq 0\}}$$

(Proposition 2)

Recall Example from the previous page:

a) $S^k\mathbb{C}^n$ is an irreducible highest weight sl_n-module with the highest vector $e_1^{\otimes k}$, where $\{e_1,\ldots,e_n\}$ is the standard basis of $\mathbb{C}^n$, and its weight is $(k,0,\ldots,0) = k\cdot\underbrace{\omega_1}_{1^{st}\text{ fundamental weight}}$ which we depict by 

k boxes

b) $\Lambda^k\mathbb{C}^n$ $(1\leq k\leq n-1)$ is irreducible with $e_1\wedge e_2\wedge\ldots\wedge e_k$ being the highest vector, and its weight is $(\underbrace{1,\ldots,1}_{k},0,\ldots,0) = \underbrace{\omega_k}_{k^{th}\text{ fund. weight}}$, i.e. $(\omega_k,\alpha_i)=\delta_{ki}$ which we depict by 

k

Therefore: Any $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_{n-1},0)\in\mathbb{Z}^n$ with $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_{n-1}\geq 0$ can be written as

$$\lambda = (\lambda_1-\lambda_2)\cdot(1,0,0,0,\ldots) + (\lambda_2-\lambda_3)(1,1,0,0,\ldots) + (\lambda_3-\lambda_4)(1,1,1,0,\ldots)+\ldots$$

$$\Downarrow$$

$$\boxed{\lambda = (\lambda_1-\lambda_2)\omega_1 + (\lambda_2-\lambda_3)\omega_2 + \ldots + (\lambda_{n-1}-\underbrace{\lambda_n}_{0})\omega_{n-1}}$$

so λ as above correspond to $\boxed{\{m_1\omega_1+\ldots+m_{n-1}\omega_{n-1} \mid m_k\in\mathbb{Z}_{\geq 0}\} \text{ with } m_k=\lambda_k-\lambda_{k+1}}$

Moreover, as we recalled above each fundamental representation L_{ω_k} is just $\Lambda^k\mathbb{C}^n$, i.e. the exterior power of the tautological repr-n $\mathbb{C}^n$.
In particular, we note $\Lambda^n\mathbb{C}^n$ is the trivial repr-n of $sl_n(\mathbb{C})$ and $SL_n(\mathbb{C})$

(**Exercise** (easy): Show $\Lambda^k V^* \simeq \Lambda^{n-k}V$ $\forall 1\leq k\leq n$

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We have the following general result:

Proposition 3: Let $\mathfrak{g}$ be a s.s. Lie algebra, and $\lambda = \sum m_i \omega_i \in P_+$ — dominant integral weight. Then L_λ can be realized as $\mathfrak{g}$-submodule of $\bigotimes_i L_{\omega_i}^{\otimes m_i}$ generated by $v = \bigotimes_i v_{\omega_i}^{\otimes m_i}$ (here, v_{ω_i} is the highest weight vector in L_{ω_i})

► Let $V := U(\mathfrak{g}) v \subseteq \bigotimes L_{\omega_i}^{\otimes m_i}$ be the above $\mathfrak{g}$-submodule. Then it's completely reducible, and v is a highest weight vector $\Rightarrow V \simeq L_\lambda \oplus \bigoplus_k L_{\nu_k}$. But similarly to the proof of [Lecture 33, Thm 1 b)] we immediately deduce there are in fact no L_{ν_k}'s (as v generates both V and L_λ). ■

In our context from p.3, we thus obtain:

Corollary 1: For $\lambda = \sum_{k=0}^{n-1} m_k \omega_k = (m_1 + \ldots + m_{n-1}, m_2 + m_3 + \ldots + m_{n-1}, \ldots, m_{n-1}, 0)$, the irreducible $SL_n(\mathbb{C})$-module L_λ is generated inside $\bigotimes_{k=1}^{n-1} (\Lambda^k \mathbb{C}^n)^{\otimes m_k}$ by the tensor product $\bigotimes_{k=1}^{n-1} (e_1 \wedge \ldots \wedge e_k)^{\otimes m_k}$. In particular, all irreducible fin.dim representations of $SL_n(\mathbb{C})$ can be always realized inside $(\mathbb{C}^n)^{\otimes(\sum k m_k)} = (\mathbb{C}^n)^{|\lambda|}$

Let us now move to representations of $GL_n(\mathbb{C})$. Note that $\mathbb{C}^* = \{\lambda \cdot \mathrm{Id} \mid \lambda \neq 0\}$ is a subgp of $GL_n(\mathbb{C})$ and any $A \in GL_n(\mathbb{C})$ can be written as $A = \lambda \cdot B$, $B \in SL_n(\mathbb{C})$, $\lambda \in \mathbb{C}^*$.
However, this decomposition is not unique as:

$$\mathbb{C}^* \cap SL_n(\mathbb{C}) = \{\lambda \cdot \mathrm{Id} \mid \lambda^n = 1\} = \underbrace{\mu_n}_{\text{gp of roots of 1 of order } n} \simeq \mathbb{Z}/n\mathbb{Z}$$

Therefore, we have the following short exact sequence:

$$\boxed{\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_n & \longrightarrow & \mathbb{C}^* \times SL_n(\mathbb{C}) & \xrightarrow{(\lambda, B) \mapsto \lambda B} & GL_n(\mathbb{C}) & \longrightarrow & 1 \\ & & z & \longmapsto & (z^{-1}, z \cdot \mathrm{Id}) & & & & \end{array}}$$

Hence, any representation of $GL_n(\mathbb{C})$ is a repr-n of $\mathbb{C}^* \times SL_n(\mathbb{C})$, and any repr. of $\mathbb{C}^* \times SL_n(\mathbb{C})$ on which μ_n acts trivially descends to a repr. of $GL_n(\mathbb{C})$

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To this end, we shall first consider repr-s of $\mathbb{C}^*$. Note that $Lie(\mathbb{C}^*)=\mathbb{C}$ is an abelian 1dim Lie algebra, hence its representation is just a pair (V,α) where V- vect. space, $\alpha\in End(V)$ is the image of $1\in\mathbb{C}$. On the other hand, the exponential map $\mathbb{C}\longrightarrow\mathbb{C}^*$, $x\longmapsto e^{2\pi i x}$ is surjective with the kernel $\mathbb{Z}$.

Therefore, a pair (V,α) with fin.dim V exponentiates to a representation of $\mathbb{C}^*$ if and only if $\boxed{e^{2\pi i\alpha}=Id_V}$

(Exercise (easy): $e^{2\pi i\alpha}=Id \iff \alpha$ is diagonalizable with $\mathbb{Z}$ eigenvalues.

Corollary 2: Finite dimensional repr-s of $\mathbb{C}^*$ are completely reducible with irreducible $\chi_r:\mathbb{C}^*\longrightarrow End(\mathbb{C})$, $z\longmapsto z^r$ being 1-dim, labeled by $r\in\mathbb{Z}$

Combining the above discussion, we thus obtain:

Proposition 4: Irreducible finite dimensional representations of $\mathbb{C}^*\times SL_n(\mathbb{C})$ are $\chi_r\otimes L_\lambda$ with $r\in\mathbb{Z}$, λ as above.
Moreover, those that factor through $GL_n(\mathbb{C})$ satisfy $r-\sum_{i=1}^{n-1}\lambda_i\in n\mathbb{Z}$

[Rmk: To see the latter divisibility condition we note that generator $e^{\frac{2\pi i}{n}}$ of μ_n acts via $e^{-\frac{2\pi i}{n}\cdot r+\frac{2\pi i}{n}(\lambda_1+\dots+\lambda_{n-1})}=e^{\frac{2\pi i}{n}(\sum\lambda_i-r)}$ which must be 1!

If we write λ as $\lambda=(m_1+\dots+m_{n-1}, m_2+\dots+m_{n-1},\dots, m_{n-1},0)$ and define $m_n:=\frac{r-\sum\lambda_i}{n}$ (which is integer by Prop 4) we can easily see that the highest weight of above $\chi_r\otimes L_\lambda$ equals $(m_1+\dots+m_{n-1}+m_n, m_2+\dots+m_{n-1}+m_n,\dots, m_{n-1}+m_n, m_n)$.

Corollary 3: Highest weights of irred fin.dim. $GL_n(\mathbb{C})$-modules are $\{\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{Z}^n \mid \lambda_1\geq\lambda_2\geq\dots\geq\lambda_n\}$.

In other words they have the form:

$\{\lambda=m_1\omega_1+\dots+m_{n-1}\omega_{n-1}+m_n\omega_n \mid m_1,\dots,m_{n-1}\in\mathbb{Z}_{\geq 0},\ m_n\in\mathbb{Z}\}$ ← $\omega_k (k<n)$ as above and $\omega_n=(1,\dots,1)$

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Let us note that while $\Lambda^n\mathbb{C}^n$ was a trivial $SL_n(\mathbb{C})$-module, as a $GL_n(\mathbb{C})$-module it's precisely L_{ω_n} ("determinant repr-n"). Moreover, its dual $(\Lambda^n\mathbb{C}^n)^*$ is also irreducible, namely $L_{-\omega_n}$.

Thus, generalizing Corollary 1, we get:

Corollary 4: For $\lambda = m_1\omega_1 + \ldots + m_{n-1}\omega_{n-1} + m_n\omega_n$ ($m_1, \ldots, m_{n-1} \in \mathbb{Z}_{\geq 0}$, $m_n \in \mathbb{Z}$), the $GL_n(\mathbb{C})$-module L_λ can be realized inside $\bigotimes_{k=1}^{n} (\Lambda^k\mathbb{C}^n)^{\otimes m_k}$.

if $m_n < 0$, then $(\Lambda^n\mathbb{C}^n)^{\otimes m_n} = ((\Lambda^n\mathbb{C}^n)^*)^{\otimes(-m_n)}$

Def 1: Representations L_λ with $\lambda = \sum_{k=1}^{n} m_k\omega_k$ and all $m_k \in \mathbb{Z}_{\geq 0}$ are called polynomial representations of $GL_n(\mathbb{C})$.

The terminology depicts the fact that all matrix coefficients of $\rho: GL_n(\mathbb{C}) \to \mathrm{End}(L_\lambda)$ are polynomials in matrix entries of $A \in GL_n(\mathbb{C})$, and thus can be continuously extended to a representation of a semigroup $\mathrm{Mat}_{n\times n}(\mathbb{C})$.

(Exercise: $m_n \geq 0 \iff L_\lambda$ occurs inside $V^{\otimes N}$ for some N.

Thus: irreducible polynomial finite dimensional representations of $GL_n(\mathbb{C})$ are parametrized by

$$\boxed{\{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0} \mid \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}}$$

Moreover, L_λ occurs inside $(\mathbb{C}^n)^{\otimes|\lambda|}$, where $|\lambda| = \lambda_1 + \ldots + \lambda_n = \#$ boxes in corresponding Young diagram.

On the other hand, we know that any f.dim. $GL_n(\mathbb{C})$-module is completely reducible. We can thus, decompose $(\mathbb{C}^n)^{\otimes N}$ into irreducible $GL_n(\mathbb{C})$-mod:

$$\forall N \geq 1: \quad (\mathbb{C}^n)^{\otimes N} = \bigoplus_{\substack{|\lambda| = N \\ \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0} \\ \lambda_1 \geq \ldots \geq \lambda_n}} L_\lambda \otimes U_\lambda, \text{ where } U_\lambda := \mathrm{Hom}_{GL_n(\mathbb{C})}(L_\lambda, (\mathbb{C}^n)^{\otimes N})$$

are just vector spaces for now.

Equivalently, we can say that the above summation is over all size N Young diagrams of length $\leq n$ (i.e. at most n nonzero rows). We shall now equip U_λ with an additional structure of S_N-modules, where S_N denotes the symmetric group in $\{1, \ldots, N\}$.

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Consider an action $S_N \curvearrowright (\mathbb{C}^n)^{\otimes N}$ via

$$\boxed{\sigma.(v_1\otimes \dots \otimes v_N) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \dots \otimes v_{\sigma^{-1}(N)}}$$

This action obviously commutes with the diagonal action $GL_n(\mathbb{C}) \curvearrowright (\mathbb{C}^n)^{\otimes N}$

$$\boxed{g.(v_1\otimes \dots \otimes v_N) = (gv_1)\otimes (gv_2) \otimes \dots \otimes (gv_N).}$$

We shall depict this by

$$\boxed{GL_n(\mathbb{C}) \curvearrowright (\mathbb{C}^n)^{\otimes N} \circlearrowleft S_N}$$

As they commute, we obtain a natural action of S_N on each multiplicity space

$$\boxed{S_N \curvearrowright U_\lambda}$$

Def 2: Let $A := \mathrm{Im}\left(U(\mathfrak{gl}_n(\mathbb{C})) \longrightarrow \mathrm{End}((\mathbb{C}^n)^{\otimes N})\right)$ – "Schur algebra"

$B := \mathrm{Im}\left(\mathbb{C}[S_N] \longrightarrow \mathrm{End}((\mathbb{C}^n)^{\otimes N})\right)$ – "centralizer algebra"

Theorem 2 (Schur–Weyl duality):

a) The centralizer of A is B and the centralizer of B is A.

b) If a partition λ of N has at most n parts, then the representation U_λ of B is irreducible, nonzero, and pairwise non-isomorphic all irreducible representations of B.

c) If $n \geq N$, then $\{U_\lambda\}_{|\lambda|=N}$ exhaust all irreducible representations of S_N.

Note that the centralizer of B is $Z_B = S^N(\mathrm{End}(\mathbb{C}^{\otimes n}))$. We shall thus start with two general results on symmetric powers $S^N U$.

Lemma 2: For any $\mathbb{C}$-vector space U, $S^N U$ is spanned by $\{x^{\otimes N} = x\otimes \dots \otimes x \mid x \in U\}$

For example, if $N=2$ this follows from $x\otimes y + y \otimes x = \frac{1}{2}\left((x+y)^{\otimes 2} - x^{\otimes 2} - y^{\otimes 2}\right)$

By [Hwk 11, Problem 4] = [Hwk 4, Problem 6], $S^N U$ is an irreducible $GL(U)$-module. But $\mathrm{span}\{x^{\otimes N} \mid x \in U\}$ is clearly a nonzero $GL(U)$-submodule. Hence, it equals all $S^N U$.

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While the above lemma provided a set of elements that linearly span $S^N\mathcal{U}$, we wish to refine it to a set of elements that generate $S^N\mathcal{U}$ as algebra (given $\mathcal{U}$ is an algebra itself).

Lemma 3: If $\mathcal{U}$ is an associative $\mathbb{C}$-algebra, then $S^N\mathcal{U}$ is generated by
$$\Delta^{(N)}(x) := x\otimes 1\otimes \ldots \otimes 1 + 1\otimes x\otimes \ldots \otimes 1 + \ldots + 1\otimes \ldots \otimes 1\otimes x, \quad x\in\mathcal{U}$$

Let's illustrate it for $N=2$: $x\otimes x = \frac{1}{2}\left((x\otimes 1 + 1\otimes x)^2 - (x^2\otimes 1 + 1\otimes x^2)\right) = \dfrac{\Delta^{(2)}(x)^2 - \Delta^{(2)}(x^2)}{2}$

▶ Due to Lemma 2, it suffices to express $x^{\otimes N}$ via $\Delta^{(N)}(?)$. This follows from the following explicit formula:

$$x^{\otimes N} = p(\Delta^{(N)}(x), \Delta^{(N)}(x^2), \longrightarrow \Delta^{(N)}(x^N))$$
where p is a polynomial in N variables satisfying
$$y_1\ldots y_N = p\left(\textstyle\sum y_i, \sum y_i^2, \longrightarrow \sum y_i^N\right)$$

← this p is called Newton polynomial

We leave details as a simple exercise ■

Combining Lemma 3 with $Z_B = S^N(\mathrm{End}(\mathbb{C}^n))$ and the observation that $\Delta^{(N)}(x)$ for $x\in \mathrm{End}(\mathbb{C}^n)$ is precisely how x acts on $(\mathbb{C}^n)^{\otimes N}$, we get:

$$\boxed{A = \text{centralizer of } B}$$

proving one part of Thm 2a). In fact, all the rest follows from the general algebraic statement:

Proposition 5 (The Double Centralizer Lemma): Let V be a fin. dim. vector space, $A, B\subseteq \mathrm{End}(V)$ be subalgebras, s.t. B is isomorphic to a direct sum of matrix algebras and A = centralizer of B. Then: A is also isomorphic to a direct sum of matrix algebras, $V = \bigoplus_{i=1}^m W_i\otimes \mathcal{U}_i$ where W_i runs through all irred. A-modules, $\mathcal{U}_i$ - irred. B-modules, and B = centralizer of A. In particular, we get bijection btw {irred. A-mod} $\overset{1:1}{\longleftrightarrow}$ {irred. B-mod}

Remark: (general result from Algebra 1) A fin.dim. algebra A is isomorphic to a direct sum of matrix algebras iff A is semisimple (i.e. no nonzero elt of A acts trivially on all irreducible A-modules) or equivalently if any f.d. A-module is completely reducible.

The result follows immediately from decomposing V as B-module

$$V = \bigoplus_i W_i \otimes U_i \quad \text{with } W_i = \mathrm{Hom}_B(U_i, V)$$

as then $A = \oplus\,\mathrm{End}(W_i)$, $B = \oplus\,\mathrm{End}(U_i)$. ∎

In our case, B is the quotient of $\mathbb{C}[S_N]$. As every f.d. S_N-module is completely reducible (by Maschke's theorem), the same holds for B-modules, and hence by above Rmk the algebra B is isomorphic to a direct sum of matrix algebras.

__Hence:__ Prop 5 applies and we get:

1) B = centralizer of A, finishing the proof of part a) of Theorem 2

2) $\{U_\lambda\}_{\substack{\text{length}(\lambda)\le n \\ |\lambda| = N}}$ are pairwise non-isomorphic and provide all irreducible B-modules, thus establishing part b) of Theorem 2.

Finally, to prove Thm 2c) it suffices to show $B \simeq \mathbb{C}[S_N]$, i.e. the map $\mathbb{C}[S_N] \longrightarrow \mathrm{End}((\mathbb{C}^n)^{\otimes N})$ is injective for $n \geq N$. To this end, we note that $\{\sigma.(e_1 \otimes e_2 \otimes \ldots \otimes e_N) \mid \sigma \in S_N\}$ are clearly lin. indep. decomposable tensors.

This completes our proof of the <u>Schur-Weyl duality</u>! ∎

__Remark:__ We also have $A = \mathrm{span}\{g^{\otimes N} \mid g \in \mathrm{End}(\mathbb{C}^n)\}$, i.e.

$$\mathrm{Im}\left(U(\mathfrak{gl}_n(\mathbb{C})) \longrightarrow \mathrm{End}((\mathbb{C}^n)^{\otimes N})\right) = \mathrm{span}\left(\mathrm{Im}\left(GL_n(\mathbb{C}) \longrightarrow \mathrm{End}((\mathbb{C}^n)^{\otimes N})\right)\right)$$

Thus one can view Schur-Weyl duality as the simultaneous decomposition w.r.t. $(\mathfrak{gl}_n(\mathbb{C}), S_N)$ __or__ $(GL_n(\mathbb{C}), S_N)$

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· UPSHOT: Schur – Weyl duality

$$\boxed{\begin{matrix} GL(V) \curvearrowright V^{\otimes N} \curvearrowleft S_N \\ gl(V) \end{matrix}}$$

← double centralizing images
$A = \mathrm{Im}(U(gl(V)) \to \mathrm{End}(V^{\otimes N})) = $ span of $\mathrm{Im}(GL(V) \to GL(V^{\otimes N}))$
$B = \mathrm{Im}(\mathbb{C}[S_N] \to \mathrm{End}(V^{\otimes N}))$

1) $$\boxed{V^{\otimes N} \simeq \bigoplus_{|\lambda|=N} L_\lambda \otimes U_\lambda}$$
λ – Young diagram of length $\leq \dim V$
isomorphism of $A\otimes B$ - modules

and we get bijection

2) $\{$ irreducible polynomial $GL(V)$-representations $\} \xleftrightarrow{1:1} \{$ irreducible B-repr-s $\}$
$L_\lambda \longleftrightarrow U_\lambda$

3) If $\underbrace{\dim V}_{=:n} \geq N \Rightarrow B = \mathbb{C}[S_N] \Rightarrow$ $\boxed{\{U_\lambda\}_{|\lambda|=N} = \text{all irreducible } S_N\text{-modules}}$

• Let's see what the above theorem implies on the level of characters.

Recall (from Theorem 1): Character of L_λ, viewed as usual character of the group $GL(V) \simeq GL_n(\mathbb{C})$ module L_λ is given by Schur polynomial

$$\boxed{S_\lambda(\underbrace{x_1, \ldots, x_n}_{\text{eigenvalues of } g\in GL_n(\mathbb{C})}) = \frac{\det(x_i^{\lambda_j+n-j})_{1\leq i,j\leq n}}{\det(x_i^{n-j})_{1\leq i,j\leq n}}}$$

← Exercise: While Thm 1 was for $sl_n(\mathbb{C})$ modules, same formula holds in the context of $GL_n(\mathbb{C})$ – prove it.

Pick a diagonal element $x = \begin{pmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & \ddots \ x_n \end{pmatrix} \in GL_n(\mathbb{C})$. Also pick any permutation $\sigma \in S_N$, and let it have m_k cycles of length k. We shall now compute the trace of $x\otimes\sigma$-action on $V^{\otimes N}$ in two different ways:

(i) just from def-n : $\mathrm{tr}_{V^{\otimes N}}(x\otimes\sigma) = \prod_k (x_1^k + \ldots + x_n^k)^{m_k}$ ← explain in class!

(ii) from Schur-Weyl: $\mathrm{tr}_{V^{\otimes N}}(x\otimes\sigma) = \sum_\lambda S_\lambda(x_1, \ldots, x_n)\cdot \chi_{U_\lambda}(\sigma)$

$$\Rightarrow \boxed{\sum_\lambda \chi_{U_\lambda}(\sigma)\cdot \det(x_i^{\lambda_j+n-j}) = \det(x_i^{n-j})\cdot \prod_k (x_1^k + \ldots + x_n^k)^{m_k} = \prod_{i<j}(x_i - x_j)\prod_k (x_1^k + \ldots + x_n^k)^{m_k}}$$

Note that if we order monomials lexicographically by first comparing powers of x_1, then x_2, etc, then the max monomial of $\det(x_i^{\lambda_j+n-j})$ is $x_1^{\lambda_1+n-1} \cdots x_n^{\lambda_n}$.

Lectures #34-36

The above implies the well-known Frobenius character formula:

Theorem 3 (Frobenius character formula): The character value $\chi_{U_\lambda}(\sigma)$ equals the coefficient of $x_1^{\lambda_1+n-1}\cdots x_n^{\lambda_n}$ in

$$\prod_{1\le i<j\le n}(x_i-x_j)\cdot\prod_{k\ge1}(x_1^k+\ldots+x_n^k)^{m_k}$$

• Another beautiful application of the Schur-Weyl duality is the following result. Let V,W be two finite-dimensional $\mathbb{C}$ vector spaces.

Theorem 4 (Howe duality): The symmetric power $S^N(V\otimes W)$ decomposes as

$$S^N(V\otimes W)\simeq\bigoplus_{\substack{\lambda\text{-Young diagram}\\ \text{of size } |\lambda|=N}} S^\lambda V\otimes S^\lambda W$$

as $GL(V)\times GL(W)$-modules, where $S^\lambda V=\mathrm{Hom}_{\mathbb{C}[S_N]}(U_\lambda, V^{\otimes N})=\begin{cases}L_\lambda, & \text{if length}(\lambda)\le\dim V\\ 0, & \text{otherwise}\end{cases}$ and similarly for $S^\lambda W$.

► By definition: $S^N(V\otimes W)=\left[(V\otimes W)^{\otimes N}\right]^{S_N}\simeq\left[V^{\otimes N}\otimes W^{\otimes N}\right]^{S_N}$ ← here, S_N diagonally acts on $V^{\otimes N}, W^{\otimes N}$

Schur-Weyl $\Rightarrow$ $\left.\begin{aligned}V^{\otimes N}&=\bigoplus_{|\lambda|=N}S^\lambda V\otimes U_\lambda\\ W^{\otimes N}&=\bigoplus_{|\lambda|=N}S^\lambda W\otimes U_\lambda\end{aligned}\right\}\Rightarrow \boxed{S^N(V\otimes W)\simeq\bigoplus_{\substack{|\lambda|=N\\|\mu|=N}}S^\lambda V\otimes S^\mu W\otimes(U_\lambda\otimes U_\mu)^{S_N}}$

But: $U_\lambda\simeq U_\lambda^*$ $\forall\lambda$, since $\chi_{U_\lambda^*}=\overline{\chi_{U_\lambda}}=\chi_{U_\lambda}$ as χ_{U_λ} takes $\mathbb{Z}$-values by Theorem 3.

$\Rightarrow (U_\lambda\otimes U_\mu)^{S_N}\simeq(U_\lambda^*\otimes U_\mu)^{S_N}\simeq\mathrm{Hom}_{S_N}(U_\lambda,U_\mu)\simeq\begin{cases}\mathbb{C}, & \text{if } \lambda=\mu\\ 0, & \text{otherwise}\end{cases}$ by Schur lemma ∎

[Rmk: The above definition of $S^\lambda V$ is such that Schur-Weyl duality becomes: $V^{\otimes N}=\bigoplus_\lambda S^\lambda V\otimes U_\lambda$ as $\mathfrak{gl}(V)\times S_N$-module (i.e. we lifted "length$(\lambda)\le\dim V$)

As an immediate corollary of Theorem 4, we get:

Corollary 5 (Cauchy identity): Consider two tuples of variables: $(x_1,\ldots,x_k)$, $(y_1,\ldots,y_\ell)$

Then:
$$\prod_{1\le i\le k}^{1\le j\le \ell}\frac{1}{1-zx_iy_j}=\sum_\lambda S_\lambda(x_1,\ldots,x_k)S_\lambda(y_1,\ldots,y_\ell)z^{|\lambda|}.$$

► Compute trace of the action of $x\otimes y$ with $x=\begin{pmatrix}x_1&&0\\&\ddots&\\0&&x_k\end{pmatrix}\in GL(\mathbb{C}^k)$, $y=\begin{pmatrix}y_1&&0\\&\ddots&\\0&&y_\ell\end{pmatrix}\in GL(\mathbb{C}^\ell)$ and use the Molien formula $\sum_{m\ge0}\mathrm{tr}(S^m\alpha)z^m=\frac{1}{\det(1-z\cdot\alpha)}$ for any linear $\alpha: U\to U$. ∎