# Lecture #37

We have seen that Casimir element $C$ played the key role both in the proof of Whitehead's theorem (Lectures 17–18) as well as Weyl character formula (Lect 31-32). It is thus natural to ask the following question:

$\boxed{\underline{Q}: \text{Describe } Z\mathfrak{g} = Z(U\mathfrak{g}), \text{ the center of universal enveloping for semisimple } \mathfrak{g}}$

(As a closely related question, one can also ask if different simple finite dimen. $\mathfrak{g}$-modules have different central characters?)

The answer to above $\underline{Q}$ is provided by the Harish-Chandra theorem. To state the result, consider the triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ which implies by PBW theorem (see [Lecture 11, Corollary 4]) that multiplication map

$$\boxed{m: U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \xrightarrow{\sim} U\mathfrak{g} \quad \text{is a v. space isom.}}$$

Moreover, we have natural "counit" algebra homomorphisms

$$\boxed{\begin{array}{ll} \varepsilon_+ : U(\mathfrak{n}_+) \longrightarrow \mathbb{C} & \text{with } 1 \mapsto 1,\ e_i \mapsto 0 \\ \varepsilon_- : U(\mathfrak{n}_-) \longrightarrow \mathbb{C} & \text{with } 1 \mapsto 1,\ f_i \mapsto 0 \end{array}}$$

This gives us a natural linear map ("HC" stays for Harish-Chandra):

$$\boxed{HC: U\mathfrak{g} \xrightarrow{m^{-1}} U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+) \xrightarrow{\varepsilon_- \otimes id \otimes \varepsilon_+} U(\mathfrak{h}) = S(\mathfrak{h}) = \mathbb{C}[\mathfrak{h}^*]}$$

While $HC$ is not an algebra homomorphism, its restriction to $Z\mathfrak{g} \subseteq U\mathfrak{g}$ gives rise to algebra homom. $HC: Z\mathfrak{g} \longrightarrow \mathbb{C}[\mathfrak{h}^*]$

$\boxed{\underline{\text{Lemma 1}}: HC: Z\mathfrak{g} \longrightarrow \mathbb{C}[\mathfrak{h}^*] \text{ is algebra homom}}$

► This follows immediately from the fact that any central element of $U\mathfrak{g}$ has $Q$-degree 0, so that is a sum of element in $U(\mathfrak{h})$ and elements in $Ker(\varepsilon_-) \cdot U(\mathfrak{h}) \cdot Ker(\varepsilon_+)$ ■

Recall the shifted/dot action $W_\circ \curvearrowright \mathfrak{h}^*$ from [Lecture 33, Definition 2]:

$$\boxed{w \circ \lambda = w(\lambda+\rho) - \rho \quad \forall w \in W, \lambda \in \mathfrak{h}^*}$$

This gives rise to $W_\circ \curvearrowright \mathbb{C}[\mathfrak{h}^*]$.

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Fix any $\lambda\in P$ and $V$ – highest weight $\mathfrak{g}$-module of highest weight $\lambda$.

$\boxed{\text{Lemma 2: } \forall z\in Z\mathfrak{g} \text{ acts on } V \text{ as a multiplication by } HC(z)(\lambda).}$

▸ Let $v_\lambda\in V$ be a highest weight vector, so that $e_i(v_\lambda)=0$, $h(v_\lambda)=\lambda(h)v_\lambda$.

Then $z(v_\lambda)=HC(z)(v_\lambda)=\underbrace{HC(z)}_{\in\mathbb{C}[\mathfrak{h}^*]}(\lambda)\cdot v_\lambda$

On the other hand, $z$ – central and $V$ is generated by $v_\lambda$

$\Rightarrow z(v)=HC(z)(\lambda)\cdot v \quad \forall v\in V$ ∎

Our next result establishes certain $W_\bullet$ – symmetry

$\boxed{\text{Proposition 1: } HC(Z\mathfrak{g})\subseteq \mathbb{C}[\mathfrak{h}^*]^{W_\bullet}}$

While above we use $W_\bullet$ – action, one can also use the usual $W$ – action. To do so, define $\chi_z\in\mathbb{C}[\mathfrak{h}^*]$ for $z\in Z\mathfrak{g}$ via $\chi_z(\lambda+\rho)=HC(z)(\lambda)$, i.e. we "shift" by $\rho$. Then above result is equivalent to

$$\boxed{\chi_z\in\mathbb{C}[\mathfrak{h}^*]^W \quad \forall z\in Z\mathfrak{g}}$$

▸ As $W$ is generated by simple reflections, it suffices to show that

$$\boxed{HC(z)(\lambda)=HC(z)(s_i\circ\lambda)} \qquad \forall\lambda\in\mathfrak{h}^*,\ z\text{ – central},\ s_i\text{ – simple reflection}$$

Moreover, the above suffices to check for $\lambda\in P_+=\{\text{integral dominant weights}\}$ as the latter is Zariski dense set in $\mathfrak{h}^*$ (essentially, we check equality of polyn-s)

However, by Lecture 33, we know that $\forall\lambda\in P_+$ there is embedding of Verma modules

$$\boxed{M_{s_i\circ\lambda}\hookrightarrow M_\lambda}$$

and thus $HC(z)(\lambda)=HC(z)(s_i\circ\lambda)$ by Lemma 2. This completes the proof ∎

Thus, we obtain the Harish-Chandra algebra homomorphism

$$\boxed{HC: Z\mathfrak{g}\longrightarrow\mathbb{C}[\mathfrak{h}^*]^{W_\bullet}} \quad (\text{or after } \rho\text{-shift } \chi: Z\mathfrak{g}\to\mathbb{C}[\mathfrak{h}^*]^W)$$

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Recall that $U\mathfrak{g}$ is $\mathbb{Z}$-filtered, hence, so is $Z\mathfrak{g} \subseteq U\mathfrak{g}$. Likewise, $\mathbb{C}[\mathfrak{h}^*]$ is $\mathbb{Z}$-graded, hence, so is $\mathbb{C}[\mathfrak{h}^*]^W$, and thus $\mathbb{C}[\mathfrak{h}^*]^{W_\bullet}$ is $\mathbb{Z}$-filtered.

Now we are ready to state the key result (answer to $\underline{Q}$ on p. 1):

> <u>Theorem 1</u> (Harish-Chandra theorem)
>
> The map $HC: Z\mathfrak{g} \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^{W_\bullet}$ is an isomorphism of filtered algebras

We already know that HC - algebra homomorphism. It's also easy to check that it's compatible with filtrations. Thus, by [Homework 4, Problem 3b], it suffices to verify that $gr_\bullet(HC)$ is an isomorphism, where $gr_\bullet$ denotes the associated graded. Here, we clearly have $gr_\bullet(\mathbb{C}[\mathfrak{h}^*]^{W_\bullet}) \simeq \mathbb{C}[\mathfrak{h}^*]^{\underset{\uparrow}{W}}$ usual $W$-action.

On the other hand, by [Lecture 11, Lemma 2, Corollary 2, Corollary 6]: $gr_\bullet(Z\mathfrak{g}) \simeq (S\mathfrak{g})^{\mathfrak{g}}$. Thus, Theorem 1 follows from another classical result:

> <u>Theorem 2</u> (Chevalley theorem): For any $f \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ its restriction to $\mathfrak{h}$ is $W$-inv., and furthermore this gives rise to <u>isomorphism</u> of graded algebras
>
> $$res: \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W$$

Indeed, $gr_\bullet(HC) = res$ and so $res$ being isomorphism $\Rightarrow$ HC - isomorphism.

Furthermore, another classical result of Chevalley is:

> <u>Theorem 3</u>: $(S\mathfrak{h})^W = \mathbb{C}[\mathfrak{h}]^W \simeq \mathbb{C}[\mathfrak{h}^*]^{W_\bullet}$ are polynomial algebras in $r = rk(\mathfrak{g})$ generators

As an immediate corollary of Thm 1 & 3, we get:

> <u>Corollary 1</u>: The center $Z\mathfrak{g}$ of $U\mathfrak{g}$ for semisimple $\mathfrak{g}$ is a polynomial algebra in $r = rk(\mathfrak{g})$ generators (called sometimes "higher order Casimirs")

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We shall conclude with a short proof of Theorem 2 as it uses the ideas that we already encountered over the last week.

### Proof of Theorem 2

- Let $G$ be the connected simply-connected Lie gp with Lie algebra $\mathfrak{g}$. Then $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$. Evoking [Homework 10, Problem 5], we then see that $\mathrm{res}(f) \in \mathbb{C}[\mathfrak{h}]$ is $W$-invariant $\forall f \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$. This shows that the natural restriction map $\mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$ gives rise to $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W$. It clearly preserves $\mathbb{Z}_{\geq 0}$-gradings and is compatible with products.
- We also claim that this map is injective, i.e. if $f \in \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \simeq \mathbb{C}[\mathfrak{g}]^G$ and $f|_{\mathfrak{h}} = 0$ then $f \equiv 0$. To this end, we note that by Lectures 21-22, the set $\mathfrak{g}^{sr}$ of strongly regular elements is dense in $\mathfrak{g}$ and any $x \in \mathfrak{g}^{sr}$ is $G$-conjugate to some $h \in \mathfrak{h}$ $\Rightarrow f|_{\mathfrak{g}^{sr}} \equiv 0 \Rightarrow f|_{\mathfrak{g}} \equiv 0$
- It thus remains to prove that $\mathrm{res}: \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W$ is surjective.

For any $\lambda \in P_+$, $n \in \mathbb{Z}_{\geq 0}$, consider

$$\boxed{f_{\lambda,n}(x) := \mathrm{tr}_{L_\lambda}(x^n) \quad \forall x \in \mathfrak{g}}$$

which are clearly $G$-invariant. It remains to prove:

$$\boxed{\underline{\text{Claim}}: \mathrm{res}(f_{\lambda,n}) \text{ span } (S^n\mathfrak{h}^*)^W = \text{degree } n \text{ component of } \mathbb{C}[\mathfrak{h}]}$$

▷ Evoking [Lectures 34-36, Proposition 1], we know $\chi_\lambda = \sum_{\substack{\mu \leq \lambda \\ \mu \in P_+}} c_\lambda^\mu \cdot m_\mu$ with $c_\lambda^\lambda = 1$, $c_\lambda^\mu \in \mathbb{Z}$, $m_\mu = W\text{-orbit average} = \sum_{\nu \in W\mu} e^\nu$.

As the matrix $(c_\cdot^\cdot)$ is upper-triangular, we can revert above by expressing $m_\lambda = \sum_{\substack{\mu \leq \lambda \\ \mu \in P_+}} \tilde{c}_\lambda^\mu \cdot \chi_\mu$, with $\tilde{c}_\lambda^\lambda = 1$, $\tilde{c}_\lambda^\mu \in \mathbb{Z}$. Hence, it suffices to show $m_{\lambda,n}$ span $(S^n\mathfrak{h}^*)^W$, where $m_{\lambda,n}(h) = \sum_{\nu \in W\lambda} \nu(h)^n$. However, as $W$-averaging gives a surjective map $S^n\mathfrak{h}^* \to (S^n\mathfrak{h}^*)^W$ it suffices to show $\{\lambda^n \mid \lambda \in P_+\}$ span $S^n\mathfrak{h}^*$. Again, as $P_+$ is Zariski dense in $\mathfrak{h}^*$, it suffices to prove $\{\lambda^n \mid \lambda \in \mathfrak{h}^*\}$ span $S^n\mathfrak{h}^*$ but this was already established in [Lect 34-36, Lemma 2] ▣