Student presentation #1 : Proof of Levi theorem

• There are two standard proofs in the literature. The one presented by Ryan in class was cohomological and while it looks simpler it relies on vanishing of $H^2(\mathfrak{g}, V)=0$ when $\mathfrak{g}$-semisimple, V-fin. dim (cf. Lectures 17–18 for $H^1(\mathfrak{g}, V)=0$). Below is an alternative shorter proof.
Let's recall the formulation – see [Lecture #14, Theorem 2]

Thm (Levi thm): Any finite-dimensional Lie algebra $\mathfrak{g}$ (over a field $\mathbb{k}$ of $\mathrm{char}(\mathbb{k})=0$) can be written as a direct sum $\mathfrak{g} \overset{\text{v. space}}{=\!=} \underbrace{\mathrm{rad}(\mathfrak{g})}_{\text{ideal}} \oplus \underbrace{\mathfrak{g}_{ss}}_{\text{Lie subalgebra}}$ with $\mathfrak{g}_{ss}$ - semisimple

Rmk: a) Recall that by [Lecture 14, Lemma 2], the quotient $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semisimple and by above is isomorphic to $\mathfrak{g}_{ss}$, so that as Lie algebras $\mathfrak{g}_{ss}$ are choice independent. However, there may be different embeddings $\mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \hookrightarrow \mathfrak{g}$ as above.

b) As noted in the Remark after Levi thm in Lecture 14, the above implies
$$\boxed{\mathfrak{g} \simeq \mathfrak{g}_{ss} \ltimes \mathrm{rad}(\mathfrak{g})}$$

Proof of thm

➤ To shorten our notations, we shall use $\mathfrak{r}$ for $\mathrm{rad}(\mathfrak{g})$, while $\mathfrak{a}$ for $\mathfrak{g}_{s.s.}$

Case 1: $\mathfrak{r} = Z(\mathfrak{g})$, i.e. $\mathfrak{g}$ - reductive

This case was already treated in [Lectures 17–18, Corollary 2]

Namely the adjoint action $\mathfrak{g} \overset{ad}{\curvearrowright} \mathfrak{g}$ factorizes through $\mathfrak{g}/\mathfrak{r} \overset{ad}{\curvearrowright} \mathfrak{g}$, and $\mathfrak{r} \subseteq \mathfrak{g}$ is a subrepresentation. However, as $\mathfrak{g}/\mathfrak{r}$ is semisimple, we know that any fin. dim. module is completely reducible [Lectures 17–18, Thm 2], hence: $\mathfrak{g} \simeq \mathfrak{r} \oplus \mathfrak{a}$ as $\mathfrak{g}/\mathfrak{r}$-modules. In particular, $\mathfrak{a} \simeq \mathfrak{g}/\mathfrak{r}$ - semisimple and actually $\mathfrak{a} \subseteq \mathfrak{g}$ is not just a subalg. but an ideal. ✓

Case 2: $\begin{cases} \mathfrak{r} \neq Z(\mathfrak{g}) \Longleftrightarrow [\mathfrak{g}, \mathfrak{r}] \neq 0 \\ \mathfrak{r} \text{ contains no nonzero proper ideals} \end{cases} \Rightarrow \begin{cases} [\mathfrak{g}, \mathfrak{r}] = \mathfrak{r} \text{ as otherwise } [\mathfrak{g}, \mathfrak{r}] \text{ would be an ideal in } \mathfrak{r} \\ [\mathfrak{r}, \mathfrak{r}] = 0 \text{ as } [\mathfrak{r}, \mathfrak{r}] \subsetneq \mathfrak{r} \text{ and is an ideal in } \mathfrak{r} \end{cases}$

In this case, instead of considering adjoint action, we consider the following:

$$\boxed{\mathfrak{g} \overset{\varkappa}{\curvearrowright} \mathfrak{gl}(\mathfrak{g}), \text{ i.e. } \varkappa: \mathfrak{g} \to \mathrm{End}(\mathfrak{gl}(\mathfrak{g})), \text{ given by } \varkappa(x)\varphi = [\mathrm{ad}(x), \varphi] \quad \forall x \in \mathfrak{g},\ \varphi \in \mathrm{End}(V)}$$

Proof of Levi theorem

We shall now define the following 3 $\mathfrak{g}$-invariant subspaces of $\mathfrak{gl}(\mathfrak{g})$ under $\mathfrak{X}$:

$$\boxed{\begin{aligned} \mathcal{A} &:= \{\varphi \in \mathfrak{gl}(\mathfrak{g}) \mid \varphi(\mathfrak{g}) \subseteq \mathfrak{r},\ \varphi_{|\mathfrak{r}} = c \cdot \mathrm{Id}_{\mathfrak{r}} \text{ for some } c \in \mathbb{k}\} \\ \mathcal{B} &:= \{\varphi \in \mathcal{A} \mid \varphi_{|\mathfrak{r}} = 0\} \\ \mathcal{C} &:= \{\mathrm{ad}(x) \mid x \in \mathfrak{r}\} \end{aligned}}$$

$\mathcal{A} \supseteq \mathcal{B} \supseteq \mathcal{C}$

Properties:

1) $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are indeed $\mathfrak{X}(\mathfrak{g})$-invariant

- For $\varphi \in \mathcal{A}$, $x \in \mathfrak{g}$:

$$\left.\begin{array}{l} (\mathrm{ad}(x)\varphi - \varphi\,\mathrm{ad}(x))(y) = [x, \varphi(y)] - \varphi([x,y]) \subseteq \mathfrak{r} \quad \forall y \in \mathfrak{g} \\ \text{if } y \in \mathfrak{r} \Rightarrow \varphi(y) = cy \Rightarrow (\mathrm{ad}(x)\varphi - \varphi\,\mathrm{ad}(x))(y) = 0 \end{array}\right\} \Rightarrow \mathfrak{X}(x)\varphi \in \mathcal{B} \subseteq \mathcal{A} \quad \forall x \in \mathfrak{g},\ \varphi \in \mathcal{A}$$

- For $\varphi \in \mathcal{B}$ – same check
- For $\mathcal{C}$: $[\mathrm{ad}(x), \mathrm{ad}(y)] = \mathrm{ad}[x,y]$ ∎

2) $\dim(\mathcal{A}/\mathcal{B}) = 1$

3) $\mathfrak{X}(\mathfrak{g})\mathcal{A} \subseteq \mathcal{B}$ ← checked above

4) $\mathfrak{X}(\mathfrak{r})\mathcal{A} \subseteq \mathcal{C}$

$$\forall x \in \mathfrak{r}, \varphi \in \mathcal{A}, y \in \mathfrak{g}: (\mathfrak{X}(x)\varphi)(y) = [\underset{\in\mathfrak{r}}{x}, \underset{\in\mathfrak{r}}{\varphi(y)}] - \varphi([\underset{\in\mathfrak{r}}{x}, \underset{\in\mathfrak{g}}{y}]) = -\varphi([x,y]) = -c[x,y] = \mathrm{ad}(-cx)y$$ ∎

Let's now consider the corresponding quotient representations $\mathcal{A}/\mathcal{B}$, $\mathcal{A}/\mathcal{C}$ of $\mathfrak{g}/\mathfrak{r}$.
We have a natural projection

$$\boxed{\pi: \mathcal{A}/\mathcal{C} \twoheadrightarrow \underbrace{\mathcal{A}/\mathcal{B}}_{\text{1dim } \mathfrak{g}/\mathfrak{r}\text{-repr.}}}$$

⇝ $\mathrm{Ker}(\pi) \subsetneq \mathcal{A}/\mathcal{C}$ is a codim 1 $\mathfrak{g}/\mathfrak{r}$-submodule

But $\mathfrak{g}/\mathfrak{r}$ -semisimple ⇒ any f.dim. module is completely reducible, hence,

$$\boxed{\mathcal{A}/\mathcal{C} \simeq \mathrm{Ker}(\pi) \oplus \underbrace{\mathbb{k}(\varphi_0 + \mathcal{C})}_{\text{1dim complement to } \mathrm{Ker}(\pi)} \text{ as } \mathfrak{g}/\mathfrak{r}\text{-modules}}.$$

Rescaling φ_0 we can assume $\varphi_0 \in \mathcal{A}$ is s.t. $\varphi_{0|\mathfrak{r}} = \mathrm{Id}_{\mathfrak{r}}$.

Exercise (easy): Any 1-dim repr. of a semisimple Lie algebra is trivial!

Hence, $\mathfrak{X}(\mathfrak{g})\varphi_0 \subseteq \mathcal{C}$. Here is our choice of $\mathfrak{a}$ (clearly a subalgebra of $\mathfrak{g}$):

$$\boxed{\mathfrak{a} := \{x \in \mathfrak{g} \mid \mathfrak{X}(x)\varphi_0 = 0\}}$$

We shall now check $\mathfrak{a}$ is as needed.

Proof of Levi theorem

(Continuation of Step 2)

Properties of $\mathfrak{a}$:

1) $\mathfrak{a} \cap \mathfrak{r} = 0$

▶ If $x \in \mathfrak{r} \Rightarrow \mathfrak{X}(x)\varphi_0 = \mathrm{ad}(-x)$
If $x \in \mathfrak{a} \Rightarrow \mathfrak{X}(x)\varphi_0 = 0$
$\Big\} \Rightarrow \mathrm{ad}(x) = 0$ for $x \in \mathfrak{a} \cap \mathfrak{r}$.

Thus, if $x \neq 0$, then $\mathbb{k}x \subsetneq \mathfrak{r}$ is a nonzero proper ideal, contradicting our assumptions ∎

2) $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{r}$

▶ Pick any $x \in \mathfrak{g}$. By above: $\mathfrak{X}(x)\varphi_0 \in \mathcal{C} \Rightarrow \exists y \in \mathfrak{r}$ s.t. $\mathfrak{X}(x)\varphi_0 = \mathrm{ad}(y)$
But $\mathfrak{X}(y)\varphi_0 = \mathrm{ad}(-y)$
$\Big\} \Rightarrow \mathfrak{X}(x+y)\varphi_0 = 0$.

Thus: $x = (x+y) + (-y)$ with $x+y \in \mathfrak{a}$, $-y \in \mathfrak{r}$

The uniqueness follows from 1) above ∎

This establishes the theorem in Case 2. ✓

Case 3: general case

(not to get into Cases 1-2, can assume $\dim(\mathfrak{r}) > 1$ and there exists a proper nonzero ideal $0 \subsetneq \mathfrak{r}' \subsetneq \mathfrak{r} \subseteq \mathfrak{g}$)

We shall argue by induction on $\dim(\mathfrak{g})$.

Exercise (easy): $\mathrm{rad}(\mathfrak{g}/\mathfrak{r}') = \mathfrak{r}/\mathfrak{r}'$.

As $\dim(\mathfrak{g}/\mathfrak{r}') < \dim(\mathfrak{g})$, the induction hypothesis implies $\boxed{\mathfrak{g}/\mathfrak{r}' = \mathfrak{r}/\mathfrak{r}' \oplus \underbrace{\mathfrak{a}'}_{\text{subalg. in } \mathfrak{g}/\mathfrak{r}'}}$

To get back to $\mathfrak{g}$, consider the natural quotient map $\varpi : \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\mathfrak{r}'$ and let's look at $\varpi^{-1}(\mathfrak{a}')$. As $\mathfrak{r}/\mathfrak{r}' \cap \mathfrak{a}' = 0$, we see $\boxed{\varpi^{-1}(\mathfrak{a}') \cap \mathfrak{r} = \mathfrak{r}'}$. Also:

$\boxed{\mathrm{rad}(\varpi^{-1}(\mathfrak{a}')) = \mathfrak{r}'}$. Indeed, $\mathfrak{r}' \subseteq \varpi^{-1}(\mathfrak{a}')$ is solvable but $\varpi^{-1}(\mathfrak{a}')/\mathfrak{r}' \simeq \mathfrak{a}'$ - s.s.

Applying the induction hypothesis to $\varpi^{-1}(\mathfrak{a}')$, we have $\boxed{\varpi^{-1}(\mathfrak{a}') = \mathfrak{r}' \oplus \underbrace{\mathfrak{a}}_{\text{subalgebra}}}$
(note $\mathfrak{a} \simeq \mathfrak{a}'$ as Lie algebras)

Hence: $\boxed{\mathfrak{g} \simeq \mathfrak{r} \oplus \underbrace{\mathfrak{a}}_{\text{subalgebra}}}$

This completes the induction step, hence also our proof of the theorem ∎