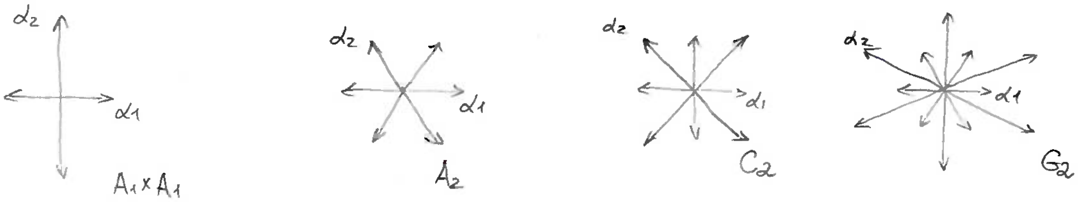
Student presentation #2 : Weyl groups are Coxeter groups

Recall that Weyl group of any reduced root system is generated by simple reflections $\{s_i\}$. Clearly $s_i^2 = Id$. Moreover, for any $i \neq j$, the root subsystem generated by (plane spanned by α_i, α_j) $\cap$ (R= root system) is a rank 2 root system. The latter ones were classified in Lecture 23-24:



$s_2 s_1 = \curvearrowright$ by π $\Downarrow$ $(s_2 s_1)^2 = Id$

$s_2 s_1 = \curvearrowright$ by $\frac{2\pi}{3}$ $\Downarrow$ $(s_2 s_1)^3 = Id$

$s_2 s_1 = \curvearrowright$ by $\frac{\pi}{2}$ $\Downarrow$ $(s_2 s_1)^4 = Id$

$s_2 s_1 = \curvearrowright$ by $\frac{\pi}{3}$ $\Downarrow$ $(s_2 s_1)^6 = Id$

These rank 2 computations show that $\forall i \neq j$ the order of $s_i s_j \in W$ is $m_{ij} \in \{2, 3, 4, 6\}$. The main result of today's talk is:

Main Theorem: W is isomorphic to abstract group with generators $\{s_i\}$ and defining relations $s_i^2 = 1$ $(\forall i)$, $(s_i s_j)^{m_{ij}} = 1$ $(\forall i \neq j)$

In other words, every Weyl group is a Coxeter group (of finite type)

The proof of this result is based on the so-called "Deletion Condition" which is vastly used in various problems on Weyl groups:

Proposition (Deletion Condition):
Given a decomposition $w = s_{i_1} \dots s_{i_r}$ of $w \in W$ as a product of simple reflections with $l(w) < r$, there exist $1 \leq j < k \leq r$ such that

$$W = s_{i_1} \dots s_{i_{j-1}} s_{i_{j+1}} \dots s_{i_{k-1}} s_{i_{k+1}} \dots s_{i_r}$$

Rmk: As the name suggests, this result implies that one can get a reduced decomposition for w from any decomposition by crossing out two factors at a time.

Proof of Deletion Condition

▶ In the proof we shall interpret the length $l(w)$ as $\#\{\alpha \in R_+ \mid w(\alpha) \in R_-\}$, established in Lecture 26. Invoking that each simple reflection s_i permutes $R_+ \setminus \{\alpha_i\}$ and maps $\alpha_i \mapsto -\alpha_i$, we conclude that

$$\boxed{l(v s_i) = \begin{cases} l(v)+1 & \text{if } v(\alpha_i) \in R_+ \\ l(v)-1 & \text{if } v(\alpha_i) \in R_- \end{cases} \qquad \forall v \in W}$$

- As $l(s_{i_1} \dots s_{i_r}) = l(w) < r \Rightarrow \exists k$ such that $l(s_{i_1} \dots s_{i_k}) = l(s_{i_1} \dots s_{i_{k-1}}) - 1$

  By above, this implies that

$$\boxed{s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k}) \in R_-}$$

- As $\alpha_{i_k} \in R_+$ and $s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k}) \in R_- \Rightarrow \exists\, 1 \le j < k$ such that $\begin{cases} s_{i_{j+1}} \dots s_{i_{k-1}}(\alpha_{i_k}) \in R_+ \\ s_{i_j} s_{i_{j+1}} \dots s_{i_{k-1}}(\alpha_{i_k}) \in R_- \end{cases}$

  Since $\alpha_{i_j} \overset{s_{i_j}}{\longmapsto} -\alpha_{i_j}$ and s_{i_j} permutes $R_+ \setminus \{\alpha_{i_j}\}$, we conclude that

$$\boxed{s_{i_{j+1}} \dots s_{i_{k-1}}(\alpha_{i_k}) = \alpha_{i_j}}$$

- The above implies that $s_{i_j} = s_{i_{j+1}} \dots s_{i_{k-1}} s_{i_k} \underbrace{s_{i_{k-1}} \dots s_{i_{j+1}}}_{= (s_{i_{j+1}} \dots s_{i_{k-1}})^{-1}}$

  Multiplying on the right with $s_{i_{j+1}}$, then $s_{i_{j+2}}, \dots, s_{i_{k-1}}$, we get:

$$\boxed{s_{i_j} s_{i_{j+1}} \dots s_{i_{k-1}} = s_{i_{j+1}} \dots s_{i_{k-1}} s_{i_k}}$$

- Finally, we obtain

$$\begin{aligned} w &= s_{i_1} \dots s_{i_{j-1}} (s_{i_j} s_{i_{j+1}} \dots s_{i_{k-1}}) s_{i_k} s_{i_{k+1}} \dots s_{i_r} \\ &= s_{i_1} \dots s_{i_{j-1}} (s_{i_{j+1}} \dots s_{i_{k-1}}) \underbrace{s_{i_k} s_{i_k}}_{=1} s_{i_{k+1}} \dots s_{i_r} \\ &= s_{i_1} \dots \widehat{s_{i_j}} \dots \widehat{s_{i_k}} \dots s_{i_r}. \end{aligned}$$

Proof of Main Theorem

Let $G = \langle s_i \rangle / \begin{smallmatrix} s_i^2 = 1 \\ (s_i s_j)^{m_{ij}} = 1 \end{smallmatrix}$ be the abstract Coxeter gp with m_{ij} specified above.

As verified in the beginning, we have $G \twoheadrightarrow W$, and we need to show it is an isomorphism! In other words, our goal is to prove that:

$$\boxed{s_{i_1} s_{i_2} \dots s_{i_r} = 1 \text{ in } W \implies s_{i_1} \dots s_{i_r} = 1 \text{ in } G \text{, i.e. is consequence of defining relations}}$$

- First, as $\det(s_k) = -1$ $\forall k$, we conclude that r must be even. Let $r = 2p$, $p \in \mathbb{Z}_{\geq 1}$. The proof proceeds by induction on p.
- Base case ($p=1$) if $s_i s_j = 1 \overset{s_i^2=1}{\Longrightarrow} s_i = s_j \Rightarrow i = j$. Thus, it's one of the defining relations
- Step of induction

First, we rewrite the given equality $s_{i_1} s_{i_2} \dots s_{i_{2p}} = 1$ in a slightly different way (so that we can apply Deletion Condition): $s_{i_1} s_{i_2} \dots s_{i_p} s_{i_{p+1}} = s_{i_{2p}} s_{i_{2p-1}} \dots s_{i_{p+2}}$ by using $s_\bullet^2 = 1$ both in W and G. As LHS has more simple reflections than RHS, we see from Proposition that $\exists$ $1 \leq j < k \leq p+1$ such that

$$\boxed{s_{i_1} s_{i_2} \dots s_{i_p} s_{i_{p+1}} = s_{i_1} \dots \widehat{s_{i_j}} \dots \widehat{s_{i_k}} \dots s_{i_{p+1}}}$$

If $j > 1$ or $k < p+1$, then this is equivalent to $s_{i_j} \dots s_{i_k} = s_{i_{j+1}} \dots s_{i_{k-1}}$ which is equivalent to $s_{i_j} \dots s_{i_k} s_{i_{k-1}} \dots s_{i_{j+1}} = 1$ containing less then $r = 2p$ terms, hence, induction assumption applies to it. Likewise, the original relation is also equivalent to $s_{i_1} \dots \widehat{s_{i_j}} \dots \widehat{s_{i_k}} \dots s_{i_{2p}} = 1$, to which induction assumption applies too!

So: we shall consider only when $j = 1$ & $k = p+1$, so that

$$s_{i_1} s_{i_2} \dots s_{i_p} s_{i_{p+1}} = s_{i_2} \dots s_{i_p}$$

or equivalently (as $s_{i_{p+1}}^2 = 1$ both in W and G):

$$\boxed{s_{i_1} s_{i_2} \dots s_{i_p} = s_{i_2} s_{i_3} \dots s_{i_{p+1}}} \quad (1)$$

the product of the first p reflections equals the product of p reflections starting from s_{i_2}.

(Continuation of the proof)

- Note that evoking $s_i^2=1$ both in W and G, the original relation is equivalent to

$$\boxed{s_{i_{a+1}} s_{i_{a+2}} \dots s_{i_{2p}} s_{i_1} s_{i_2} \dots s_{i_a} = 1 \quad \forall\ 1 \le a \le 2p-1} \qquad (2)$$

In particular, for $a=1$, we get a "length $2p$" relation

$$s_{i_2} s_{i_3} \dots s_{i_{2p}} s_{i_1} = 1$$

Applying the same analysis as on the previous page, we see that it is a consequence of defining relations in G unless

$$\boxed{s_{i_2} \dots s_{i_{p+1}} = s_{i_3} \dots s_{i_{p+1}} s_{i_{p+2}}} \qquad (3)$$

The above can be equivalently written as

$$\boxed{s_{i_3} s_{i_2} s_{i_3} \dots s_{i_{p+1}} s_{i_{p+2}} s_{i_{p+1}} \dots s_{i_4} = 1}$$

As this relation is of "length $2p$" again, it either follows from defining rel-s in G, or must satisfy:

$$\boxed{s_{i_3} s_{i_2} s_{i_3} \dots s_{i_p} = s_{i_2} s_{i_3} \dots s_{i_{p+1}}} \qquad (4)$$

- Comparing (1) & (4), we conclude

$$s_{i_1} s_{i_2} \dots s_{i_p} = s_{i_3} s_{i_2} \dots s_{i_p} \overset{s_\cdot^2=1}{\Longrightarrow} s_{i_1} = s_{i_3} \Rightarrow \boxed{i_1 = i_3} \qquad (*)$$

- Applying the same analysis to (2) above, we get that this relation holds in G by inductive assumption or $\boxed{i_{a+1} = i_{a+3} \quad \forall\ 1 \le a \le 2p-1}$

Thus: $\left.\begin{matrix} i_1 = i_3 = \dots = i_{2p-1} \\ i_2 = i_4 = \dots = i_{2p} \end{matrix}\right\} \Rightarrow$ original relation is of the form $(s_{i_1} s_{i_2})^p = 1$.

$\left.\begin{matrix}\text{As } s_{i_1} s_{i_2} \in W \text{ has order } m_{i_1 i_2} \Rightarrow p \text{ is divisible by } m_{i_1 i_2}. \\ \text{But } (s_{i_1} s_{i_2})^{m_{i_1 i_2}} = 1 \text{ is also one of defining rel-s of } G\end{matrix}\right\} \Rightarrow (s_{i_1} s_{i_2})^p = 1 \text{ in } G$

$\blacksquare$

Rmk: This elegant proof of the Main Theorem is due to R. Steinberg.