Lecture #6 Today: "Banzhaf Power" 09/12/2016

As we saw, the power is not proportional to the weights. So we still have a question

Q: How to measure the power of players in the weighted voting?

- Banzhaf Power [Section 2.2 of your Textbook]

This method is named after a law professor John Banzhaf (he invented this at 25 years!)

To motivate the following approach, let us consider a simple example.

Example: Consider the weighted voting system [101: 100, 99, 2]
(i.e. there are 3 players with weights 100, 99, 2, while the quota equals 101)
Then it is easy to notice that there are 4 ways for a motion to pass

(1) P_1, P_2, P_3 vote for it
(2) P_1, P_2 vote for it
(3) P_1, P_3 vote for it
(4) P_2, P_3 vote for it.

As a result, we see that all 3 players have the same influence on the outcome (while the weights of P_1 and P_3 are very different)

To formulate the method, let us introduce some new definitions:

- Coalition: any set of players who might join forces and vote the same way. Coalition consisting of all players is called the grand coalition, while the "smallest" coalitions are those consisting of 1 player.

Notation: We will denote the coalition consisting of players $P_{i_1}, \dots, P_{i_k}$ as $\{P_{i_1}, \dots, P_{i_k}\}$, i.e. we list their names in $\{\dots\}$ (the order doesn't matter, e.g. $\{P_1, P_2\} = \{P_2, P_1\}$)

- Winning Coalition: A coalition is called winning if it has enough votes to win
(—"— is called losing if it does not have enough votes to win)

Note that the grand coalition (consisting of all players) is always winning, while a coalition consisting of 1 player is winning iff this player is a dictator.

- Critical Player: In a winning coalition, a player is said to be critical if eliminating this player from the coalition, we get a losing coalition.

- Critical Count: The critical count of a player is the number of times this player is a critical player (over all winning coalitions)

In the above example of the weighted voting system $[101; 100, 99, 3]$, we get

- The list of all coalitions : $\{P_1\}$, $\{P_2\}$, $\{P_3\}$, $\{P_1, P_2\}$, $\{P_1, P_3\}$, $\{P_2, P_3\}$, $\{P_1, P_2, P_3\}$
- The list of winning coalitions : $\{\underline{P_1}, \underline{P_2}\}$, $\{\underline{P_1}, \underline{P_3}\}$, $\{\underline{P_2}, \underline{P_3}\}$, $\{P_1, P_2, P_3\}$

  we underline all critical players in each winning coalition
- The critical count of each player equals 2 (You just count how many times each player was underlined in total).

- Now we are ready to introduce the key concept "Banzhaf power index".

Let $P_1, \ldots, P_N$ be the players in a weighted voting system, while $B_1, \ldots, B_N$ will denote their respective critical count. Their sum $T = B_1 + \ldots + B_N$ is called the "total critical count".

Def: The Banzhaf power index (BPI) of a player is a ratio of the player's critical count over the total critical count T.

In other words: BPI of P_1 equals $\beta_1 = \frac{B_1}{T}$

BPI of P_2 equals $\beta_2 = \frac{B_2}{T}$

$\vdots$

BPI of P_N equals $\beta_N = \frac{B_N}{T}$

Note that for each $1 \le i \le N$, we have $0 \le \beta_i \le 1$, while $\boxed{\beta_1 + \ldots + \beta_N = 1}$

Thus, we get a Banzhaf power distribution of the weighted voting system. This is a reasonable count to measure the players' power, though this is not a unique way (in Section 2.3 you may read about the Shapley-Shubik Power).

Ex: In the above example, we get $B_1 = B_2 = B_3 = 2 \Rightarrow T = 6 \Rightarrow \beta_1 = \beta_2 = \beta_3 = \frac{2}{6} = \frac{1}{3}$

Hence, this method prescribes the same power to each player (as it was expected).

Rmk: This example illustrates that while $w_3 = 3 \ll 100 = w_1$, the BPI of P_1 and P_3 are the same!

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- To compute the Banzhaf power distribution, you need to:
  1. List all the winning coalitions
  2. Determine all the critical players within each winning coalition (convenient notation: underline all critical players)
  3. Find the critical counts $B_1, \_\_, B_N$ $\rightsquigarrow$ Compute $T := B_1 + \_\_ + B_N$
  4. You are ready to compute the BPI : $\beta_1 = \frac{B_1}{T}$, $\beta_2 = \frac{B_2}{T}, \dots, \beta_N = \frac{B_N}{T}$.

- Let us now perform this algorithm in different settings.

Problem 1: Compute the Banzhaf power distribution of the weighted voting system $[6: 5, 3, 2]$.

- Winning coalitions : $\{\underline{P_1}, \underline{P_2}\}$, $\{\underline{P_1}, \underline{P_3}\}$, $\{\underline{P_1}, P_2, P_3\}$ (Discuss this in class)
- Critical count: $B_1 = 3$, $B_2 = 1$, $B_3 = 1$ $\Rightarrow$ $T = 3+1+1 = 5$.
- The BPI are: $\boxed{\beta_1 = \frac{3}{5}, \quad \beta_2 = \frac{1}{5}, \quad \beta_3 = \frac{1}{5}}$

(! This is an instructive, but quite time-consuming problem, so we will not cover it in class, but I expect you to work it out at home.)

Problem 2*: Compute the Banzhaf power distribution of the weighted voting system $[6: 4, 2, 1, 1, 1, 1]$.

► The most nontrivial step in this situation is to provide a complete list of all winning coalitions
! If you forget one, you will get a completely wrong answer.

- There are 6 players, hence, the total number of coalitions equals $2^6 - 1 = 63$ (we will go over that later on). Since there are so many of coalitions, we need to come up with a clever way of listing all the winning coalitions. Let us do this step-by-step considering coalitions consisting of 1, 2, 3, 4, 5, 6 players.
  - Size 1: None of the size 1 coallitions (i.e. consisting of 1 player) is winning
  - Size 2: The only winning coalition of 2 players is $\{\underline{P_1}, \underline{P_2}\}$

Size 3: Any winning coalition consisting of 3 players must have P_1 in itself. Moreover, any size 3 coalition including P_1 is winning. There are two types of such coalitions: (i) those including P_2
(ii) not including P_2.

(i) There are 4 coalitions: $\{P_1, P_2, P_*\}$, where $*$ is 3, 4, 5, or 6
In all those coalitions the critical players are P_1 and P_2

(ii) There are 6 coalitions of size 3, which include P_1, but not P_2:
$\{P_1, P_3, P_4\}$, $\{P_1, P_3, P_5\}$, $\{P_1, P_3, P_6\}$, $\{P_1, P_4, P_5\}$, $\{P_1, P_4, P_6\}$, $\{P_1, P_5, P_6$
In all these coalitions each player is critical.

Size 4: A coalition of 4 players is winning if and only if it includes P_1. Moreover, there are 2 types of those: (i) including P_2, and (ii) not including P_2.

(i) In the first case of (i), there are 6 such coalitions obtained from the above six by adding P_2 everywhere. Note that the only critical player in those coalitions is P_1.

(ii) In the second case of (ii), there are 4 such coalitions:
$\{P_1, P_3, P_4, P_5\}$, $\{P_1, P_3, P_4, P_6\}$, $\{P_1, P_3, P_5, P_6\}$, $\{P_1, P_4, P_5, P_6\}$.
Again, the only critical player in all these coalitions is P_1.

Size 5: There are six size 5 coallitions and all of them are winning:
$\{\underline{P_1}, P_2, P_3, P_4, P_5\}$, $\{\underline{P_1}, P_2, P_3, P_4, P_6\}$, $\{\underline{P_1}, P_2, P_3, P_5, P_6\}$
$\{\underline{P_1}, P_2, P_4, P_5, P_6\}$, $\{\underline{P_1}, P_3, P_4, P_5, P_6\}$, $\{\underline{P_2}, \underline{P_3}, \underline{P_4}, \underline{P_5}, \underline{P_6}\}$
where we underlined all critical players.

Size 6: The grand coalition $\{P_1, P_2, P_3, P_4, P_5, P_6\}$ is winning, but none of the players is critical

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Now we are ready to complete the computations in Problem 2.

Size 1 $\rightsquigarrow$ no critical players

Size 2 $\rightsquigarrow$ P_1, P_2 - critical players which were counted 1 time each.

Size 3(i) $\rightsquigarrow$ P_1 - critical count 4, P_2 - critical count 4, all other players - 0.

Size 3(ii) $\rightsquigarrow$ P_1 - critical count 6, P_2 - 0, P_3, P_4, P_5, P_6 - each counted 3 times.

Size 4(ii) $\rightsquigarrow$ P_1 - critical count 6, all other players - not counted at all

Size 4(iii) $\rightsquigarrow$ P_1 - critical count 4, none of the other players was critical

Size 5 $\rightsquigarrow$ P_1 - critical count 5, P_2 - 1 time, P_3, P_4, P_5, P_6 - 1 time each

Size 6 $\rightsquigarrow$ no critical players.

Upshot: P_1 has critical count $B_1 = 26$

P_2 — " — $B_2 = 6$

P_3, P_4, P_5, P_6 — " — $B_{3,4,5,6} = 4$

$\Rightarrow T = 26 + 6 + 4 + 4 + 4 + 4 = 48.$

Hence, $\boxed{\beta_1 = \frac{26}{48}, \quad \beta_2 = \frac{6}{48}, \quad \beta_3 = \beta_4 = \beta_5 = \beta_6 = \frac{4}{48}}$ ∎

Problem 3: A weighted voting system with 6 players has the following winning coalitions (with critical players underlined):

$\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_4, P_5, P_6\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_4, P_5\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, \underline{P_4}, P_6\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, \underline{P_5}, P_6\}$

$\{\underline{P_1}, \underline{P_2}, \underline{P_3}, \underline{P_4}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, \underline{P_5}\}$

Find the Banzhaf power distribution of this weighted voting system.

► The hardest part of the process is not needed as we are already given the complete list of winning coalitions and critical players. From above we immediately find the critical count:

$B_1 = 6, \; B_2 = 6, \; B_3 = 6, \; B_4 = 2, \; B_5 = 2, \; B_6 = 0 \Rightarrow T = 22$

Hence: $\boxed{\beta_1 = \beta_2 = \beta_3 = \frac{6}{22}, \quad \beta_4 = \beta_5 = \frac{2}{22}, \quad \beta_6 = 0}$ ∎

Problem 4: In a weighted voting system with 3 players the winning coalitions are $\{P_1, P_2, P_3\}$, $\{P_1, P_2\}$, $\{P_1, P_3\}$

(a) Find the critical players in each winning coalition.

(b) Find the Banzhaf power distribution of this weighted voting syst

► This is slightly more difficult than the previous problem as we have to determine all critical players. However, this is easy to do: a player is critical if removing him from a winning coalition, we get a losing coalition.

(a) Hence, the critical players are as follows:

$$\{\underline{P_1}, P_2, P_3\},\ \{\underline{P_1}, \underline{P_2}\},\ \{\underline{P_1}, \underline{P_3}\}$$

(b) $B_1 = 3$, $B_2 = B_3 = 1$ (follows immediately from a)) $\Rightarrow T = 3+1+1 = 5$.

Thus: $\boxed{\beta_1 = \frac{3}{5},\ \beta_2 = \beta_3 = \frac{1}{5}}$ (Rmk: Compare to Problem 1 from p.3)

Problem 5: A small company consists of a director (D) and four equal regular workers (P_1, P_2, P_3, P_4). They decided to have a party over weekends, but have to choose b/w Saturday/Sunday. For this, regular workers vote first and choice is made by majority. The director votes only to break a tie 2-2.

Find the Banzhaf power distribution. ("Yes" stays for Sunday, "No" – for Saturday)

► There are two kinds of winning coalitions: (i) A majority of regular workers vote "Yes", in which case the director does not participate, (ii) Only two regular workers vote "Yes" and then the director breaks a tie by voting "Yes".

Let us now write down all such coalitions and underline critical players (see next page)

(i) We have 5 such coalitions:

$\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$, $\{\underline{P_1}, \underline{P_2}, \underline{P_4}\}$, $\{\underline{P_1}, \underline{P_3}, \underline{P_4}\}$, $\{\underline{P_2}, \underline{P_3}, \underline{P_4}\}$, $\{P_1, P_2, P_3, P_4\}$

(ii) There are 6 such coalitions

$\{\underline{P_1}, \underline{P_2}, \underline{D}\}$, $\{\underline{P_1}, \underline{P_3}, \underline{D}\}$, $\{\underline{P_1}, \underline{P_4}, \underline{D}\}$, $\{\underline{P_2}, \underline{P_3}, \underline{D}\}$, $\{\underline{P_2}, \underline{P_4}, \underline{D}\}$, $\{\underline{P_3}, \underline{P_4}, \underline{D}\}$.

__So__: $B_1 = B_2 = B_3 = B_4 = 6$, while the BPI of the director is also $P_5 = 6$.

Therefore, $T = 6+6+6+6+6 = 30$ and

$$\boxed{\beta_1 = \beta_2 = \beta_3 = \beta_4 = \beta_5 = \frac{6}{30} = \frac{1}{5}}$$

In other words, all the regular workers have the same Banzhaf power index as the director ∎

- Let us complete this lecture by deducing a closed formula for the number of coalitions out of total number of N players.

  __N=1__ ⇒ there is only one coalition $\{P_1\}$

  __N=2__ ⇒ there are 3 coalitions $\{P_1\}, \{P_2\}, \{P_1, P_2\}$.

  __N=3__ ⇒ there are 7 coalitions $\{P_1\}, \{P_2\}, \{P_3\}, \{P_1, P_2\}, \{P_1, P_3\}, \{P_2, P_3\}, \{P_1, P_2, P_3\}$

  Note that $7 = 2^3 - 1$, $3 = 2^2 - 1$, $1 = 2^1 - 1$.

__Claim__: For any $N \geq 1$, there are exactly $2^N - 1$ coalitions out of $P_1, \rightarrow P_N$.

- If a coalition does not include P_N, then it is the same as a coalition out of $P_1, \rightarrow P_{N-1}$ and we assume that we already know there are $2^{N-1} - 1$ of those.

If a coalition includes P_N, then either it is just $\{P_N\}$ or crossing out the player P_N, we get a coalition out of $P_1, \rightarrow P_{N-1}$, and there are $2^{N-1} - 1$ of such by our assumption.

__Hence__, in total we get $(2^{N-1} - 1) + 1 + (2^{N-1} - 1) = 2^N - 1$ ✓ ∎

This completes our discussion of "Mathematics of Power".