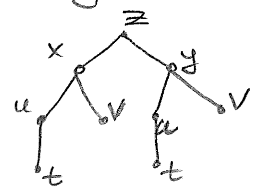
* Last time
  - Partial derivatives
  - Clairaut's Theorem: $f_{xy} = f_{yx}$ (given both are continuous)
  - Chain Rule: always start by drawing the corresponding dependence,

e.g. if



$$\Rightarrow \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial t}$$

  - Implicit differentiation: given $F(x,y)=0$, find $\frac{\partial y}{\partial x}$ or $\frac{\partial x}{\partial y}$

here x – independent variable
$y = y(x)$ – function in x

here y – independ.
x – function in y

$$0 = \frac{\partial}{\partial x} F(x, y=y(x)) = F_x \cdot \underbrace{\frac{\partial x}{\partial x}}_{1} + F_y\cdot\frac{\partial y}{\partial x} \Rightarrow \boxed{\frac{\partial y}{\partial x} = -\frac{F_x}{F_y}}$$

$$0 = \frac{\partial}{\partial y} F(x = x(y), y) = F_x\cdot\frac{\partial x}{\partial y} + F_y\cdot\underbrace{\frac{\partial y}{\partial y}}_{1} \Rightarrow \boxed{\frac{\partial x}{\partial y} = -\frac{F_y}{F_x}}$$

* Let us now see how implicit differentiation works for $F(x,y,z)=0$.

<u>Note</u>: We think of two variables as independent, while the third depends on those two

E.g. to find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ we think of x,y – independent and $z = z(x,y)$

$$0 = \frac{\partial}{\partial x} F(x,y,z=z(x,y)) = F_x\cdot\underbrace{\frac{\partial x}{\partial x}}_{1} + F_y\cdot\underbrace{\frac{\partial y}{\partial x}}_{0} + F_z\cdot\frac{\partial z}{\partial x} \Rightarrow \boxed{\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}}$$

$$0 = \frac{\partial}{\partial y} F(x,y,z=z(x,y)) = F_x\cdot\underbrace{\frac{\partial x}{\partial y}}_{0} + F_y\cdot\underbrace{\frac{\partial y}{\partial y}}_{1} + F_z\cdot\frac{\partial z}{\partial y} \Rightarrow \boxed{\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}}$$

<u>Ex 1</u>: Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ given $x^3 + y + z^4 = 2 - 3\cos(2x)\, e^{3y}\cdot z$

Rewrite this equality as $F(x,y,z)=0$ with $F(x,y,z) = x^3 + y + z^4 - 2 + 3\cos(2x) e^{3y} z$

Applying the above formulas, get:

$$\frac{\partial z}{\partial x} = -\frac{3x^2 - 6\sin(2x) e^{3y} z}{4z^3 + 3\cos(2x) e^{3y}}$$

$$\frac{\partial z}{\partial y} = -\frac{1 + 9\cos(2x) e^{3y} z}{4z^3 + 3\cos(2x) e^{3y}}$$

LECTURE #7

* Directional derivatives

Recall: $f_x(x_0, y_0)$ – rate of change of f as we move in x-direction
$f_y(x_0, y_0)$ – –//– in y-direction

Question: Given a direction vector $\vec{v}$ compute rate of change of f along $\vec{v}$.

! First: The directional derivative depends only on the direction of $\vec{v}$ and not its magnitude. From now on, we will treat the case of unit vectors $\vec{u} = \hat{v} = \frac{\vec{v}}{\|\vec{v}\|}$.

Def: The directional derivative of $f(x,y)$ in the direction of a unit vector $\vec{u} = \langle a, b\rangle$ at the point $P(x_0, y_0)$ is

$$\boxed{D_{\vec{u}} f(x_0, y_0) = \lim_{t\to 0} \frac{f(x_0+ta, y_0+tb) - f(x_0, y_0)}{t}}$$

Applying chain rule to $z = f(x,y)$, $x = x(t) = x_0 + at$, $y = y(t) = y_0 + bt$, get:

$$\boxed{D_{\vec{u}} f(x_0, y_0) = f_x(x_0, y_0)\cdot a + f_y(x_0, y_0)\cdot b}$$ ← that is how we will compute it in practice

Note: $D_{\hat{i}} f = f_x$, $D_{\hat{j}} f = f_y$, thus directional derivative generalizes 1st partials.

Rmk: Intuitively, we can think as follows. Given a unit vector $\vec{u} = \langle a, b\rangle$, moving by $\langle at, bt\rangle$ from the point (x_0, y_0) is the same as moving by at along x-axis and then by $b\cdot t$ along y-axis, hence, the rate of change is exactly $a\cdot f_x + b\cdot f_y$.

Ex2: Find the directional derivative of $f(x,y) = e^x \sin y$ at the point $P(0, \pi/2)$ in the direction indicated by the angle $\theta = \pi/4$.

▸ To apply our formula, first find unit vector $\vec{u}$ specified by the given angle:

$$\vec{u} = \langle \cos\theta, \sin\theta\rangle = \langle 1/\sqrt{2}, 1/\sqrt{2}\rangle$$

$$\partial_x f(x,y) = e^x \sin y \Rightarrow \partial_x f(0, \pi/2) = 1$$

$$\partial_y f(x,y) = e^x \cos y \Rightarrow \partial_y f(0, \pi/2) = 0$$

$$\left.\right\} \Rightarrow D_{\vec{u}} f(0, \tfrac{\pi}{2}) = 1\cdot\frac{1}{\sqrt{2}} + 0\cdot\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

LECTURE #7

* Gradient Vector

Def: If $f = f(x,y)$, then the gradient of f is the vector function

$$\boxed{\nabla f(x_0, y_0) = \langle f_x(x_0, y_0),\ f_y(x_0, y_0) \rangle}$$

If $f = f(x,y,z)$, then the gradient of f is the vector function

$$\boxed{\nabla f(x_0, y_0, z_0) = \langle f_x(x_0, y_0, z_0),\ f_y(x_0, y_0, z_0),\ f_z(x_0, y_0, z_0) \rangle}$$

Key Fact: If $\vec{u}$ is a unit vector then

$$\boxed{D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \overset{\text{dot-product}}{\cdot} \vec{u}}$$

Ex 3: (a) Find the gradient of $f(x,y) = x^2/y^3$

(b) Find the rate of change of f at $P(3,1)$ in the direction $\vec{v} = \langle 3, -4 \rangle$

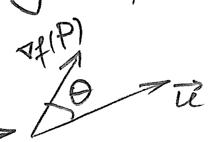
▷ (a) $\nabla f = \langle \frac{2x}{y^3},\ -\frac{3x^2}{y^4} \rangle$

(b) $$\left.\begin{array}{l} \nabla f(3,1) = \langle 6, -27 \rangle \\ \vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \langle \frac{3}{5}, -\frac{4}{5} \rangle \end{array}\right\} \Rightarrow D_{\vec{u}} f(3,1) = 6 \cdot \frac{3}{5} + (-27) \cdot \frac{-4}{5} = \frac{126}{5}$$

∎

* Max/Min of $D_{\vec{u}} f$

Given a function f (of 2 or 3 variables) and a point P, one may ask in what direction does f increases/decreases most rapidly? Equivalently, we need to maximize or minimize $D_{\vec{u}} f(P)$.



But: $D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u} = \|\nabla f(P)\| \cdot \underbrace{\|\vec{u}\|}_{1} \cdot \cos\theta$, $\quad 0 \le \theta \le \pi$

As $-1 \le \cos\theta \le 1$ with $\cos\theta = 1 \Leftrightarrow \theta = 0$
$\cos\theta = -1 \Leftrightarrow \theta = \pi$

Upshot: * The maximal value of $D_{\vec{u}} f(P)$ equals $\boxed{\|\nabla f(P)\|}$ and is achieved when $\theta = 0$, i.e. $\vec{u}$ is in the direction of $\nabla f(P)$: $\boxed{\vec{u} = \frac{\nabla f(P)}{\|\nabla f(P)\|}}$

* The minimal value of $D_{\vec{u}} f(P)$ equals $\boxed{-\|\nabla f(P)\|}$ and is achieved when $\theta = \pi$, i.e. $\vec{u}$ is in the opposite direction of $\nabla f(P)$:

$$\boxed{\vec{u} = -\frac{\nabla f(P)}{\|\nabla f(P)\|}}$$

LECTURE #7

Ex 4: Let $f(x,y,z) = \frac{x+y}{z}$ and $P(1,1,1)$. In what direction does f have the max/min directional derivative? what are the corresponding values?

$\nabla f = \langle \frac{1}{z}, \frac{1}{z}, -\frac{x+y}{z^2} \rangle \Rightarrow \nabla f(P) = \langle 1, 1, -2 \rangle$.

So: Maximal value of $D_{\vec u} f(P)$ is $\sqrt{1^2+1^2+(-2)^2} = \boxed{\sqrt{6}}$ and is achieved for $\boxed{\vec u = \langle \frac{1}{\sqrt6}, \frac{1}{\sqrt6}, \frac{-2}{\sqrt6} \rangle}$

Minimal value of $D_{\vec u} f(P)$ is $\boxed{-\sqrt{6}}$ and is achieved for $\boxed{\vec u = \langle \frac{-1}{\sqrt6}, \frac{-1}{\sqrt6}, \frac{2}{\sqrt6} \rangle}$

* Tangent Planes to level surfaces & Tangent lines to level curves

Def: Let S be a *level surface* of $F(x,y,z)$, i.e. a set of all points (x,y,z) such that $F(x,y,z) = k \leftarrow$ fixed constant, and let $P(x_0,y_0,z_0)$ be on S.

The *tangent plane* to S at point P is the plane passing through P and perpendicular to $\nabla F(P)$. Thus, it is explicitly given by

$$\boxed{F_x(x_0,y_0,z_0)\cdot(x-x_0) + F_y(x_0,y_0,z_0)\cdot(y-y_0) + F_z(x_0,y_0,z_0)\cdot(z-z_0) = 0}$$

The *normal line* to S at P is the line passing through P in the direction of $\nabla F(P)$. Thus, it is explicitly given by

$$\boxed{x = x_0 + F_x(x_0,y_0,z_0)\cdot t,\ \ y = y_0 + F_y(x_0,y_0,z_0)\cdot t,\ \ z = z_0 + F_z(x_0,y_0,z_0)\, t}$$

! If F is a function of 2 variables $F(x,y)$, we get the equation of the tangent line to the level curve S at the given point P on S:

$$\boxed{F_x(x_0,y_0)\cdot(x-x_0) + F_y(x_0,y_0)(y-y_0) = 0}$$

! **Key Property**: If we consider any curve C on S passing through P, then the tangent vector $\vec u$ to this curve at P lies in the tangent plane (i.e. is orthogonal to $\nabla F(P)$).

Indeed, as S is a level surface, the function F takes the constant value when restricted to C, which immediately implies

$$\nabla F(P) \cdot \vec u = 0 \Rightarrow \vec u \text{ is orthogonal to } \nabla F(P)$$

LECTURE #7

Ex 5: (a) Find equation of the tangent plane and the normal line to the surface given by

$$z^2 - 2xy + e^x = 5 \quad \text{at point } P(0,1,2).$$

(b) For the surface S from (a), find all the points on S at which the tangent plane is parallel to xy-plane.

(a) This surface is a level surface of $F(x,y,z) = z^2 - 2xy + e^x$.

$$\nabla F(x_0, y_0, z_0) = \langle -2y_0 + e^{x_0}, -2x_0, 2z_0 \rangle$$

$$\Downarrow$$

$$\nabla F(0,1,2) = \langle -1, 0, 4 \rangle$$

So: tangent plane is given by $\boxed{-x + 4(z-2) = 0}$

normal line is given by $\boxed{\begin{cases} x = -t \\ y = 1 \\ z = 2 + 4t \end{cases}}$

(b) Tangent plane is parallel to xy-plane iff the normal vector has form $(0, 0, *)$. But normal vector is given by ∇F.

Thus: we need $\underbrace{-2y + e^x = 0 \text{ and } \underbrace{-2x = 0}_{\Downarrow\, x=0}}_{\Downarrow\, x=0,\ y=\frac{1}{2}}$

What about z? So far it didn't really matter. BUT don't forget that we are looking for points on the surface, i.e.

$$z^2 - 2xy + e^x = 5 \xrightarrow{x=0,\, y=\frac{1}{2}} z^2 = 4 \Leftrightarrow z = \pm 2.$$

So: At points $\boxed{(0, \frac{1}{2}, \pm 2)}$