Lecture #15

* **Last time**

Last time we discussed line integrals of vector fields $\int_C \vec{F}\,d\vec{r}$.
and we saw that it is the same as $\int_C P\,dx + Q\,dy$ or $\int_C P\,dx + Q\,dy + R\,dz$,
where $\vec{F} = \langle P, Q\rangle$ (if we are in the plane $\mathbb{R}^2$) or $\vec{F} = \langle P, Q, R\rangle$ (if we are in the space $\mathbb{R}^3$)

And what we saw in the first two Exercises last time is that usually $\int_C \vec{F}\,d\vec{r}$ depends not only on the end & start-points of C, but also on C itself, while for $\vec{F} = \langle x, y\rangle$ we obtained the same answer for 3 different curves with the same end/start points!

Question: What is special about $\langle x, y\rangle$?

Answer: It is a gradient vector field as $\langle x,y\rangle = \nabla\left(\frac{1}{2}(x^2+y^2)\right)$

Def: A vector field $\vec{F}$ is called *conservative* if
(1) $\vec{F} = \nabla f$ for some f $\leftarrow$ such f is called a *potential function*
(2) the components of $\vec{F}$ are defined and have continuous partials everywhere

THIS MAKES OUR DEFINITION a little bit different from the one in the textbook

Ex1: Is the vector field $\vec{F} = \langle -y, x\rangle$ conservative?

Actually it is not, but let's assume at the moment we found $f(x,y)$ s.t. $\vec{F} = \nabla f \Leftrightarrow \begin{cases} f_x = -y \\ f_y = x \end{cases}$

Recall: $(f_x)_y = (f_y)_x$

$$\Rightarrow \underbrace{(-y)_y}_{-1} = \underbrace{(x)_x}_{1} \Rightarrow \text{Contradiction!}$$

So: There is no such f $\Rightarrow \vec{F}$ is not conservative

The above reasoning implies:

$$\boxed{\textbf{Thm 1: } \text{Let } \vec{F} = \langle P, Q\rangle \text{ be a vector field s.t. } P_y \neq Q_x, \text{ then } \vec{F} \text{ is } \underline{\text{not}} \text{ conservative.}}$$

Lecture #15

Ex2: Is the vector field $\vec{F} = \langle x+y, \sin(y) + 2x + 3 \rangle$ conservative?

In this case $P = x+y$, $Q = \sin(y) + 2x + 3$

$$\Downarrow \qquad\qquad \Downarrow$$
$$P_y = 1 \qquad\qquad Q_x = 2.$$

As $P_y(x,y) \neq Q_x(x,y) \Rightarrow \vec{F}$ is not conservative ∎

Actually the converse is also true, namely we have:

Thm2: If $\vec{F} = \langle P, Q \rangle$ is a vector field whose components are defined and have continuous partials everywhere, and $P_y = Q_x$, then $\vec{F}$ is conservative.

Note: This theorem doesn't tell how to find potentials!
We shall illustrate the approach by the following problem.

Ex3: (a) Is the vector field $\vec{F} = \langle x+y, x+y^2 \rangle$ conservative?
(b) If yes, find a potential of $\vec{F}$.

(a) $x+y$ & $x+y^2$ clearly have continuous partials
$(x+y)_y = 1 = (x+y^2)_x$ $\Bigg\} \Rightarrow \vec{F}$ is conservative.

(b) By part (a), we know that there exists $f(x,y)$ s.t. $\nabla f = \vec{F}$, i.e.
$f_x = x+y$, $f_y = x+y^2$.

Step 1

Integrating $x+y$ w.r.t. x, we see that $f(x,y) = \frac{1}{2}x^2 + xy + g(y)$ ← for some function of y

Step 2

$$\underbrace{f_y(x,y)}_{x+y^2} = 0 + x + g'(y) \Rightarrow g'(y) = y^2 \Rightarrow g(y) = \frac{y^3}{3} + C_0 \leftarrow \text{constant.}$$

So: For any constant C_0: $\boxed{f(x,y) = \frac{1}{2}x^2 + xy + \frac{1}{3}y^3 + C_0}$ is a potential of $\vec{F}$. ∎

! It is a good idea to check at the end if indeed $\vec{F} = \nabla f$ to make sure you didn't make mistakes

Lecture #15

Ex 4: Is the vector field $\vec{F} = \langle x+z, y-z, x+y \rangle$ conservative?

Assume yes, i.e. there is some function $f(x,y,z)$ s.t. $\nabla f = \vec{F}$, i.e.

$f_x = \underbrace{x+z}_{P}$, $f_y = \underbrace{y-z}_{Q}$, $f_z = \underbrace{x+y}_{R}$.

Similarly to our reasoning in Ex 1, recall that

$$(f_x)_y = (f_y)_x, \quad (f_x)_z = (f_z)_x, \quad (f_y)_z = (f_z)_y.$$

$$\Downarrow \qquad\qquad \Downarrow \qquad\qquad \Downarrow$$

$$P_y = Q_x \qquad P_z = R_x \qquad Q_z = R_y$$

In our case
$$(x+z)_y = 0 = (y-z)_x \quad \checkmark$$
$$(x+z)_z = 1 = (x+y)_x \quad \checkmark$$
$$(y-z)_z = -1 \neq 1 = (x+y)_y \quad -$$

So: there is no such $f(x,y,z) \Rightarrow \vec{F}$ is not conservative. ∎

Theorem 3: Let $\vec{F} = \langle P, Q, R \rangle$ be a vector field s.t.
$P_y \neq Q_x$ or $P_z \neq R_x$ or $Q_z \neq R_y$,
then $\vec{F}$ is not conservative.

The converse turns out to be also true:

Theorem 4: If $\vec{F} = \langle P, Q, R \rangle$ is a vector field whose components are defined and have continuous partials everywhere, and
$P_y = Q_x$, $P_z = R_x$, $Q_z = R_y$,
then $\vec{F}$ is conservative.

Once again, let me point out that this theorem does not specify how to find potential.

To find it we shall follow the strategy of Ex 3(b), but we will need 3 steps now.

Lecture #15

Ex5: (a) Is the vector field $\vec{F} = \langle y\cos(xy)+z,\ x\cos(xy)+2yz,\ x+y^2 \rangle$ conservative?

(b) If yes, find a potential.

(a) Just need to verify three equalities

$$(y\cos(xy)+z)_y = \cos(xy) - xy\sin(xy) = (x\cos(xy)+2yz)_x$$

$$(y\cos(xy)+z)_z = 1 = (x+y^2)_x$$

$$(x\cos(xy)+2yz)_z = 2y = (x+y^2)_y$$

As also partial derivatives of components are continuous everywhere, we see that $\vec{F}$ is conservative.

(b) Want: Find $f(x,y,z)$ s.t. $f_x = y\cos(xy)+z$, $f_y = x\cos(xy)+2yz$, $f_z = x+y^2$.

Step1: $f_x = y\cos(xy)+z$

Treating y,z as constants and integrating w.r.t. x, we get

$$\boxed{f(x,y,z) = \sin(xy) + xz + g(y,z)}$$

Step2: $\left.\begin{array}{l} f_y = x\cos(xy)+2yz \\ \quad \parallel \\ x\cos(xy)+0+g_y \end{array}\right\} \Rightarrow g_y = 2yz \Rightarrow \boxed{g(y,z) = y^2 z + h(z)}$

$$\Rightarrow \boxed{f(x,y,z) = xz + \sin(xy) + y^2 z + h(z)}$$

Step3: $\left.\begin{array}{l} f_z = x+y^2 \\ \quad \parallel \\ x+y^2+h'(z) \end{array}\right\} \Rightarrow h'(z) = 0 \Rightarrow h(z) = C_0 \leftarrow$ constant.

Hence: Potentials are: $\boxed{f(x,y,z) = xz + y^2 z + \sin(xy) + C_0}$

Ex6: Same two questions for the vector field
$$\vec{F} = \langle e^{x+y}+2x+y^2,\ e^{x+y}+2xy+y^2 \rangle.$$

Answer: $f(x,y) = e^{x+y} + x^2 + xy^2 + y^3/3 + C_0$

Lecture # 15

Recall: Fundamental theorem of Calculus from high-school says

$$\boxed{\int_a^b G'(x)\,dx = G(b) - G(a)}$$

The following generalization is known under the name "Fundamental Theorem of Line integrals" (FTLI)

Thm 5: If $\vec{F}$ is a conservative vector field with $\vec{F} = \nabla f$, and C is a curve from point A to point B, then
$$\int_C \vec{F}\,d\vec{r} = f(B) - f(A)$$

Ex 7: Let $\vec{F} = \langle x^2, y\rangle$ and C be a line segment from $(0,0)$ to $(1,1)$. Evaluate $\int_C \vec{F}\,d\vec{r}$.

▸ $\vec{F}$ is conservative and $\vec{F} = \nabla(\underbrace{\tfrac{1}{3}x^3 + \tfrac{1}{2}y^2}_{f(x,y)})$

Hence: FTLI $\Rightarrow \int_C \vec{F}\,d\vec{r} = f(1,1) - f(0,0) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ ▪

Note: In this simple case, you could also compute $\int_C \vec{F}\,d\vec{r}$ in a straightforward way.

Ex 8: If $\vec{F} = \langle x+y,\ x+y^2\rangle$ and C is a counterclockwise portion of $x^2+y^2=1$ from $(1,0)$ to $(-1,0)$ followed by the line segment from $(-1,0)$ to $(-1,1)$, evaluate $\int_C \vec{F}\,d\vec{r}$.

▸ According to Ex 3(b) $f(x,y) = \frac{1}{2}x^2 + xy + \frac{1}{3}y^3$ is a potential of $\vec{F}$.

Hence: FTLI $\Rightarrow \int_C \vec{F}\,d\vec{r} = f(-1,1) - f(1,0) = \frac{1}{2} - 1 + \frac{1}{3} - \frac{1}{2} = -\frac{2}{3}$ ▪

! When using FTLI, we pick any potential, so you may ignore the constant C_0 in the potential.

Lecture #15

Short proof of FTLI

► If C is parametrized via $\vec{r}(t)=\langle x(t), y(t), z(t)\rangle$, $a\le t\le b$ and $\vec{F}=\nabla f=\langle f_x, f_y, f_z\rangle$ } then by the very definition:

$$\int_C \vec{F}d\vec{r} = \int_a^b \nabla f(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_a^b \underbrace{\left(\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial f}{\partial z}\cdot\frac{dz}{dt}\right)}_{\overset{\text{Chain Rule}}{=\!=}\frac{d}{dt}f(\underbrace{x(t),y(t),z(t)}_{\vec{r}(t)})}dt \underset{\text{Fundamental Theorem of Calculus}}{=} f(\vec{r}(b))-f(\vec{r}(a))$$

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Summary: If we are given a line integral $\int_C \vec{F}d\vec{r}$ which is hard to compute in a straightforward way, it is a good idea to check if $\vec{F}$ is conservative (using the mixed partials test) and if it is, then FTLI applies. Hence, in that case:

- Route 1: Find explicitly the potential and evaluate it at start/end points
- Route 2: Replace the path C by a simpler path C' between the same 2 points and compute $\int_{C'}\vec{F}d\vec{r}$ in a straightforward way.

Consequences: 1) If $\vec{F}$ is conservative, then $\int_C \vec{F}d\vec{r}$ is path-independent (i.e. depends only on start/end points)

2) If $\vec{F}$ is conservative, then $\int_{\substack{\text{closed}\\ \text{path } C}} \vec{F}d\vec{r}=0$.

Ex 9: If $\vec{F}=\langle e^{x^2}, e^{y^2}\rangle$ and $C: \vec{r}(t)=\langle t\sqrt{2}\cos(\frac{7\pi}{4}t), t\sqrt{2}\sin(\frac{7\pi}{4}t)\rangle$, $0\le t\le 1$, evaluate $\int_C \vec{F}d\vec{r}$.

► As $(e^{x^2})_y=0=(e^{y^2})_x$ and components have continuous partials everywhere, we immediately see that $\vec{F}$ is conservative! However, it is impossible to find an explicit formula for a potential f since no closed f-la for $\int e^{x^2}dx$ is known.

Instead: replace C by C' — line segment b/w $\vec{r}(0)=\langle 0,0\rangle$ and $\vec{r}(1)=\langle 1,-1\rangle$.

$\vec{F}$ - conservative $\Rightarrow \int_C \vec{F}d\vec{r}=\int_{C'}\vec{F}d\vec{r}$, and we compute the latter in a straightforward way

Parametrize C': $\vec{r}(t)=\langle t,-t\rangle$, $0\le t\le 1$. Then:

$$\int_{C'}\vec{F}d\vec{r} = \int_0^1 \langle e^{t^2}, e^{t^2}\rangle\cdot\langle 1,-1\rangle dt = 0 \implies \boxed{\int_C \vec{F}d\vec{r}=0}$$

■