FUSION AND SPECIALIZATION FOR TYPE ADE SHUFFLE ALGEBRAS

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Abstract. Root vectors in quantum groups (of finite type) generalize to fused currents in quantum loop groups ([5]). In the present paper, we construct fused currents as duals to specialization maps of the corresponding shuffle algebras ([7, 8, 9]) in types ADE, an approach which has potential for generalization to arbitrary Kac-Moody types. Both root vectors and fused currents depend on a convex order of the positive roots, and the choice we make in the present paper is that of the Auslander-Reiten order ([25]) corresponding to an orientation of the type ADE Dynkin diagram.

1. Introduction

1.1. **Lie algebras of finite type.** Consider a finite-dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, which admits a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$$

into nilpotent parts $\mathfrak{n}^\pm$ and a Cartan part $\mathfrak{h}$. We have canonical decompositions

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^+} \mathbb{C} \cdot X_{\pm\alpha}$$

(Δ^+ denotes a henceforth fixed set of positive roots of the Lie algebra $\mathfrak{g}$). The reason the one-dimensional direct summands are canonical is that they are determined by their commutation relations with the Cartan subalgebra $\mathfrak{h}$, in the sense that

$$[h, X_{\pm\alpha}] = \pm\alpha(h) \cdot X_{\pm\alpha}, \qquad \forall\, h \in \mathfrak{h}$$

However, picking a basis of the one-dimensional summands is non-canonical, because the $X_{\pm\alpha}$'s are only determined up to constant multiples. Numerous interesting choices exist, notably that of a Chevalley basis, for which[1]

$$[X_{\pm\alpha}, X_{\pm\beta}] \in \mathbb{Z}^\times \cdot X_{\pm(\alpha+\beta)}, \qquad \text{whenever } \alpha, \beta, \alpha+\beta \in \Delta^+ \tag{1.1}$$

This formula can be used to successively construct (up to constant multiples) the basis vectors $X_{\pm\alpha}$ starting from a henceforth fixed set of simple roots I.

[1]For any ring R, we shall use $R^\times$ to denote the set of all nonzero elements in R.

1.2. **Quantum groups.** A well-known q-deformation of (the universal enveloping algebra of) a finite type Lie algebra $\mathfrak{g}$ is the Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$. This is an associative $\mathbb{Q}(q)$-algebra, which also admits a triangular decomposition

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}^+) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^-)$$

Lusztig ([16]) constructed *root vectors*

$$\{f_\alpha\}_{\alpha \in \Delta^+} \in U_q(\mathfrak{n}^-) \tag{1.2}$$

which are q-deformations of $X_{-\alpha} \in \mathfrak{n}^-$. The input (respectively tools) for Lusztig's construction is that of a reduced decomposition of the longest word in the Weyl group associated to $\mathfrak{g}$ (respectively the action of the corresponding braid group on $U_q(\mathfrak{g})$). Such a choice is equivalent ([22]) to a total order of the set of positive roots

$$\Delta^+ = \{\dots < \alpha < \dots\} \tag{1.3}$$

which is *convex*, in the sense that for all pairs of positive roots $\alpha < \beta$ whose sum $\alpha + \beta$ is also a root, we have

$$\alpha < \alpha + \beta < \beta \tag{1.4}$$

The following commutation relations were proved by Levendorskii-Soibelman ([15]) for all positive roots $\alpha < \beta$ (the details of the proof can be found in [4, §9.3])

$$f_\alpha f_\beta - q^{-(\alpha,\beta)} f_\beta f_\alpha \in \bigoplus_{\beta > \gamma_1 \geq \dots \geq \gamma_t > \alpha}^{\gamma_1 + \dots + \gamma_t = \alpha + \beta} \mathbb{Z}[q, q^{-1}] \cdot f_{\gamma_1} \dots f_{\gamma_t} \tag{1.5}$$

In particular, if $\alpha < \beta$ form a *minimal pair* (i.e. $\alpha + \beta$ is a root and there do not exist roots α', β' such that $\alpha < \alpha' < \beta' < \beta$ and $\alpha' + \beta' = \alpha + \beta$) then we have

$$f_\alpha f_\beta - q^{-(\alpha,\beta)} f_\beta f_\alpha \in \mathbb{Z}[q, q^{-1}]^\times \cdot f_{\alpha+\beta} \tag{1.6}$$

This formula can be used to successively construct (up to constant multiples) the root vectors f_α starting from a set of simple roots I.

1.3. **Quantum loop groups.** The main object of our study is the quantum loop group $U_q(L\mathfrak{g})$, which is an associative algebra with a triangular decomposition

$$U_q(L\mathfrak{g}) = U_q(L\mathfrak{n}^+) \otimes U_q(L\mathfrak{h}) \otimes U_q(L\mathfrak{n}^-)$$

Whereas $U_q(\mathfrak{n}^-)$ was generated by the symbols $\{f_i\}_{i \in I}$, the negative half $U_q(L\mathfrak{n}^-)$ of the quantum loop group is generated by the coefficients of formal series

$$f_i(x) = \sum_{d \in \mathbb{Z}} \frac{f_{i,d}}{x^d}, \qquad \forall\, i \in I$$

(the explicit relations between these generators are well-known, and we recall them in Definition 2.2 below). By generalizing Lusztig's construction, Ding-Khoroshkin ([5]) defined so-called *fused currents* for every positive root

$$\widetilde{f}_\alpha(x) = \sum_{d \in \mathbb{Z}} \frac{\widetilde{f}_{\alpha,d}}{x^d}, \qquad \forall\, \alpha \in \Delta^+$$

whose coefficients are infinite sums which converge in certain representations of quantum loop groups. In Definition 2.17 below, we will define a completion of the negative half $U_q(L\mathfrak{n}^-)$ in which we conjecture that all the coefficients $\widetilde{f}_{\alpha,d}$ live, see Conjecture 2.30, thus providing a formal algebraic treatment of the main construction of [5].

Then it remains to ask whether the analogues of the properties (1.5) and (1.6) hold for fused currents. For instance, if $\alpha < \beta$ is a minimal pair of positive roots, we posit in Conjecture 2.31 that

$$\widetilde{f}_\alpha(x)\widetilde{f}_\beta(y) - \widetilde{f}_\beta(y)\widetilde{f}_\alpha(x) \cdot \frac{xq^v - yq^{(\alpha,\beta)}}{xq^{v+(\alpha,\beta)} - y} = c(q) \cdot \delta\left(\frac{xq^{v+(\alpha,\beta)}}{y}\right)\widetilde{f}_{\alpha+\beta}(xq^u) \tag{1.7}$$

for some $u, v \in \mathbb{Z}$, $c(q) \in \mathbb{Z}[q, q^{-1}]^\times$ that depend on α and β (let $\delta(x) = \sum_{d\in\mathbb{Z}} x^d$) [2].

1.4. **Specialization maps.** To understand fused currents explicitly, we will use the dual shuffle algebra picture (studied in the quantum loop group setting by Enriquez in [7], motivated by a construction of Feigin-Odesskii in [9]). We will recall the shuffle algebra in detail in Section 2, but in a nutshell, it is defined by

$$\mathcal{S} = \bigoplus_{\mathbf{k}=\sum_{i\in I} k_i \boldsymbol{\varsigma}^i \in \mathbb{N}^I} \mathcal{S}_{\mathbf{k}}$$

$$\mathcal{S}_{\mathbf{k}} = \Big\{\text{certain rational functions } R(z_{i1}, \dots, z_{ik_i})_{i\in I}\Big\}$$

In the present paper, $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\boldsymbol{\varsigma}^i \in \mathbb{N}^I$ is the I-tuple with a single 1 at position $i \in I$ and 0 everywhere else. The vector space $\mathcal{S}$ is made into an algebra via the shuffle product (2.8). The shuffle algebra $\mathcal{S}$ thus defined is relevant to us because it is isomorphic to the positive half of the quantum loop group

$$\mathcal{S} \simeq U_q(L\mathfrak{n}^+)$$

as well as dual to the negative half

$$\mathcal{S} \otimes U_q(L\mathfrak{n}^-) \xrightarrow{\langle\cdot,\cdot\rangle} \mathbb{Q}(q)$$

For every $\alpha \in \Delta^+$ we may describe the fused currents $\widetilde{f}_\alpha(x)$ by the dual linear functional, which we will call a *specialization map*

$$\mathcal{S}_\alpha \xrightarrow{\widetilde{\text{spec}}_\alpha^{(x)}} \mathbb{Q}(q)[x, x^{-1}], \qquad \widetilde{\text{spec}}_\alpha^{(x)}(R) = \Big\langle R, \widetilde{f}_\alpha(x)\Big\rangle$$

In Subsection 2.34, we conjecture that $\widetilde{\text{spec}}_\alpha^{(x)}(R)$ is equal (up to a scalar prefactor and a power of x) to a certain derivative of the rational function R when its variables are specialized according to (write $\alpha = \sum_{i\in I} k_i\boldsymbol{\varsigma}^i$ for suitable $k_i \in \mathbb{N}$)

$$\left\{z_{ib} = xq^{\sigma_{ib}}\right\}_{1\le b\le k_i}^{i\in I}$$

for certain $\sigma_{ib} \in \mathbb{Z}$. The specialization maps strongly depend on the total order (1.3), and we do not yet have a complete understanding of $\widetilde{\text{spec}}_\alpha^{(x)}$ in full generality.

[2] As $q \to 1$, the quantum loop group $U_q(L\mathfrak{g})$ converges to the universal enveloping algebra $U(L\mathfrak{g}) = U(\mathfrak{g}[z, z^{-1}])$, with the Lie bracket on the loop algebra $\mathfrak{g}[z, z^{-1}]$ defined via $[az^k, bz^l] = [a, b]z^{k+l}$ for $a, b \in \mathfrak{g}$, $k, l \in \mathbb{Z}$. One expects the limit of $\widetilde{f}_{\alpha,d}$ to be $X_{-\alpha}z^d$, so that (1.7) converges to

$$\left[X_{-\alpha}\delta\left(\frac{z}{x}\right), X_{-\beta}\delta\left(\frac{z}{y}\right)\right] = c(1)\cdot\delta\left(\frac{x}{y}\right)X_{-\alpha-\beta}\delta\left(\frac{z}{x}\right)$$

Extracting the coefficient of $x^{-k}y^{-l}$ from the formula above gives rise to (1.1).

1.5. **Quivers.** Beside laying out the general expectations of fused currents and specialization maps (as in the preceding paragraph), the main purpose of the present paper is to construct specialization maps in the particular case when the total order (1.3) is a refinement of the Auslander-Reiten partial order on Δ^+ developed in [25] (see Section 3 for an overview). Explicitly, the following is our main result.

Theorem 1.6. *Let Q be any orientation of a type ADE Dynkin diagram, and let us fix any total order (1.3) that refines the Auslander-Reiten partial order of the positive roots induced by Q. For every positive root $\alpha = \sum_{i \in I} k_i \varsigma^i \in \Delta^+$, define*

$$\mathcal{S}_\alpha \xrightarrow{\text{spec}_\alpha^{(x)}} \mathbb{Q}(q)[x, x^{-1}] \tag{1.8}$$

$$\text{spec}_\alpha^{(x)}(R) = \gamma_\alpha^{(x)} \cdot R(z_{i1}, \dots, z_{ik_i})\Big|_{z_{ib} \mapsto xq^{\tau(i)}, \forall i, b}$$

(see Definition 3.10 for the specific factor $\gamma_\alpha^{(x)} \in \mathbb{Q}(q^{1/2})^\times \cdot x^{\mathbb{Z}}$ that appears in the formula above) where $\tau \colon I \to \mathbb{Z}$ is a height function, i.e. satisfies $\tau(i) = \tau(j) + 1$ if there exists an edge in Q from i to j. If we let $f_\alpha(x)$ be the dual of the specialization maps

$$\text{spec}_\alpha^{(x)}(R) = \Big\langle R, f_\alpha(x) \Big\rangle$$

then for any minimal pair $\alpha < \beta$ of positive roots, we have

$$f_\alpha(x) f_\beta(y) - f_\beta(y) f_\alpha(x) \frac{xq - yq^{-1}}{x - y} = \delta\left(\frac{x}{y}\right) f_{\alpha+\beta}(x) \tag{1.9}$$

We conjecture that $f_\alpha(x)$ (respectively $\text{spec}_\alpha^{(x)}$) equals $\widetilde{f}_\alpha(x)$ (respectively $\widetilde{\text{spec}}_\alpha^{(x)}$) up to a constant multiple, which would establish formula (1.7) in the particular case of ADE types where the convex order on the set of positive roots is a refinement of the Auslander-Reiten partial order.

1.7. **Perspectives.** While the initial motivation for our work was the exploration of the fused currents $\widetilde{f}_\alpha(x)$ of [5], we propose that specialization maps can provide an altogether alternative definition of these objects. The latter viewpoint may be applied in principle to quantum loop groups of any Kac-Moody type, where the original definition of fused currents is unavailable due to the non-existence of the braid group action. Thus, the approach we propose is to define specialization maps

$$\mathcal{S}_\alpha \xrightarrow{\text{spec}_\alpha^{(x)}} \mathbb{Q}(q)[x, x^{-1}] \tag{1.10}$$

directly, and then redefine the fused currents $f_\alpha(x)$ as their duals

$$\text{spec}_\alpha^{(x)}(R) = \Big\langle R, f_\alpha(x) \Big\rangle \tag{1.11}$$

The specializations (1.10) should be defined so that one has a commutation relation

$$f_\alpha(x) f_\beta(y) - f_\beta(y) f_\alpha(x) \Big(\text{a rational function in } x, y\Big) = \Big(\text{a sum of } \delta \text{ functions and products of } f_\gamma\Big)$$

As the contents of the present paper show, the above construction features interesting combinatorics even when $\mathfrak{g}$ is of finite type, and it is bound to be challenging (yet doable in our opinion) for $\mathfrak{g}$ of affine type. For $\mathfrak{g}$ of general type, we expect the combinatorics of the specialization maps to be extremely difficult.

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2. Quantum loop groups and shuffle algebras

2.1. **Definitions.** Fix a Lie algebra $\mathfrak{g}$ of finite type and a decomposition $\Delta = \Delta^+ \sqcup \Delta^-$ of the corresponding root system into positive and negative roots. We also fix a set of simple roots $\{\alpha_i\}_{i \in I} \subset \Delta^+$ and consider the inner product $(\cdot, \cdot)$ on the root lattice. The Cartan matrix corresponding to $\mathfrak{g}$ is

$$(a_{ij})_{i,j \in I} \quad \text{with} \quad a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}$$

In what follows, let q be a formal variable, set $q_i = q^{\frac{(\alpha_i, \alpha_i)}{2}}$ for all $i \in I$, and consider the generating series

$$e_i(x) = \sum_{d \in \mathbb{Z}} \frac{e_{i,d}}{x^d}, \qquad f_i(x) = \sum_{d \in \mathbb{Z}} \frac{f_{i,d}}{x^d}, \qquad \varphi_i^{\pm}(x) = \sum_{d=0}^{\infty} \frac{\varphi_{i,d}^{\pm}}{x^{\pm d}}$$

and the formal delta function $\delta(x) = \sum_{d \in \mathbb{Z}} x^d$. For any $i, j \in I$, set

$$\zeta_{ij}\left(\frac{x}{y}\right) = \frac{x - yq^{-(\alpha_i, \alpha_j)}}{x - y} \tag{2.1}$$

We now recall the definition of the quantum loop group (new Drinfeld realization).

Definition 2.2. *The quantum loop group associated to* $\mathfrak{g}$ *is:*

$$U_q(L\mathfrak{g}) = \mathbb{Q}(q)\Big\langle e_{i,d}, f_{i,d}, \varphi_{i,d'}^{\pm} \Big\rangle_{i \in I}^{d \in \mathbb{Z}, d' \geq 0} \Big/ \text{relations (2.2)–(2.6)}$$

where we impose the following relations for all $i, j \in I$*:*

$$e_i(x)e_j(y)\zeta_{ji}\left(\frac{y}{x}\right) = e_j(y)e_i(x)\zeta_{ij}\left(\frac{x}{y}\right) \tag{2.2}$$

$$\sum_{\sigma \in S(1-a_{ij})} \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k}_{q_i} \cdot \tag{2.3}$$
$$e_i(x_{\sigma(1)}) \dots e_i(x_{\sigma(k)}) e_j(y) e_i(x_{\sigma(k+1)}) \dots e_i(x_{\sigma(1-a_{ij})}) = 0, \quad \text{if } i \neq j$$

$$\varphi_j^{\pm}(y)e_i(x)\zeta_{ij}\left(\frac{x}{y}\right) = e_i(x)\varphi_j^{\pm}(y)\zeta_{ji}\left(\frac{y}{x}\right) \tag{2.4}$$

$$\varphi_i^{\pm}(x)\varphi_j^{\pm'}(y) = \varphi_j^{\pm'}(y)\varphi_i^{\pm}(x), \quad \varphi_{i,0}^{+}\varphi_{i,0}^{-} = 1 \tag{2.5}$$

as well as the opposite relations with e's replaced by f's, and finally the relation:

$$[e_i(x), f_j(y)] = \frac{\delta_i^j \delta\left(\frac{x}{y}\right)}{q_i - q_i^{-1}} \cdot \left(\varphi_i^+(x) - \varphi_i^-(y)\right) \tag{2.6}$$

The algebra $U_q(L\mathfrak{g})$ is $\mathbb{Z}^I \times \mathbb{Z}$-graded via

$$\deg e_{i,d} = (\boldsymbol{\varsigma}^i, d), \qquad \deg \varphi^{\pm}_{i,d} = (0, \pm d), \qquad \deg f_{i,d} = (-\boldsymbol{\varsigma}^i, d)$$

where $\boldsymbol{\varsigma}^i = \underbrace{(0, \dots, 0, 1, 0, \dots, 0)}_{1 \text{ on the } i\text{-th position}}$. We have the triangular decomposition ([12])

$$U_q(L\mathfrak{g}) = U_q(L\mathfrak{n}^+) \otimes U_q(L\mathfrak{h}) \otimes U_q(L\mathfrak{n}^-)$$

into the subalgebras generated by $e_{i,d}$, $\varphi^{\pm}_{i,d}$, $f_{i,d}$, respectively. Note the isomorphism

$$U_q(L\mathfrak{n}^+) \xrightarrow{\sim} U_q(L\mathfrak{n}^-) \tag{2.7}$$

determined by sending $e_{i,d} \mapsto f_{i,-d}$ for all $i \in I$, $d \in \mathbb{Z}$ (here, we use that $U_q(L\mathfrak{n}^+)$ is generated by $e_{i,d}$ with the defining relations (2.2, 2.3), and similarly for $U_q(L\mathfrak{n}^-)$).

2.3. **The shuffle algebra.** We now recall the trigonometric degeneration ([7, 8]) of the Feigin-Odesskii shuffle algebra ([9]) of type $\mathfrak{g}$. Consider the vector space of rational functions

$$\mathcal{V} \ = \bigoplus_{\mathbf{k} = \sum_{i \in I} k_i \boldsymbol{\varsigma}^i \in \mathbb{N}^I} \mathbb{Q}(q)(\dots, z_{i1}, \dots, z_{ik_i}, \dots)^{\text{sym}}_{i \in I}$$

which are *color-symmetric*, meaning that they are symmetric in the variables $z_{i1}, \dots, z_{ik_i}$ for each $i \in I$ separately (the terminology is inspired by the fact that $i \in I$ is called the color of the variable z_{ib}, for any $i \in I$ and $b \geq 1$). We make the vector space $\mathcal{V}$ into a $\mathbb{Q}(q)$-algebra via the following shuffle product:

$$F(\dots, z_{i1}, \dots, z_{ik_i}, \dots) * G(\dots, z_{i1}, \dots, z_{il_i}, \dots) = \frac{1}{\mathbf{k}! \cdot \mathbf{l}!} \cdot \tag{2.8}$$

$$\text{Sym} \left[F(\dots, z_{i1}, \dots, z_{ik_i}, \dots) G(\dots, z_{i,k_i+1}, \dots, z_{i,k_i+l_i}, \dots) \prod_{i,j \in I} \prod_{b \leq k_i}^{c > k_j} \zeta_{ij}\left(\frac{z_{ib}}{z_{jc}}\right) \right]$$

In (2.8), Sym denotes symmetrization with respect to the:

$$(\mathbf{k} + \mathbf{l})! := \prod_{i \in I} (k_i + l_i)!$$

permutations of the variable sets $\{z_{i1}, \dots, z_{i,k_i+l_i}\}$ for each i independently.

Definition 2.4. *([7, 8], inspired by [9]) The shuffle algebra $\mathcal{S}$ is the subspace of $\mathcal{V}$ consisting of rational functions of the form:*

$$R(\dots, z_{i1}, \dots, z_{ik_i}, \dots) = \frac{r(\dots, z_{i1}, \dots, z_{ik_i}, \dots)}{\prod_{\{i \neq i'\} \subset I}^{\textit{unordered}} \prod_{1 \leq b \leq k_i}^{1 \leq b' \leq k_{i'}} (z_{ib} - z_{i'b'})} \tag{2.9}$$

where r is a color-symmetric Laurent polynomial that satisfies the wheel conditions:

$$r(\dots, z_{ib}, \dots)\Big|_{(z_{i1}, z_{i2}, z_{i3}, \dots, z_{i,1-a_{ij}}) \mapsto (w, wq_i^2, wq_i^4, \dots, wq_i^{-2a_{ij}}),\ z_{j1} \mapsto wq_i^{-a_{ij}}} = 0 \tag{2.10}$$

for any distinct $i, j \in I$.

It is elementary to show that $\mathcal{S}$ is closed under the shuffle product (2.8), and is thus an algebra. Because of (2.10), any r as in (2.9) is actually divisible by:

$$\prod_{\{i\neq i'\}\subset I: a_{ii'}=0}^{\text{unordered}} \prod_{1\leq b\leq k_i}^{1\leq b'\leq k_{i'}} (z_{ib} - z_{i'b'})$$

Therefore, rational functions R satisfying (2.9, 2.10) can only have simple poles on the diagonals $z_{ib} = z_{i'b'}$ with adjacent $i, i' \in I$, that is, such that $a_{ii'} < 0$.

2.5. **Shuffle algebras vs quantum loop groups.** The algebra $\mathcal{S}$ is graded by $\mathbf{k} = \sum_{i\in I} k_i \boldsymbol{\varsigma}^i \in \mathbb{N}^I$ that encodes the number of variables of each color, and by the total homogeneous degree $d \in \mathbb{Z}$. We write:

$$\deg R = (\mathbf{k}, d)$$

and say that $\mathcal{S}$ is $(\mathbb{N}^I \times \mathbb{Z})$-graded. We will denote the graded pieces by:

$$\mathcal{S} = \bigoplus_{\mathbf{k}\in\mathbb{N}^I} \mathcal{S}_{\mathbf{k}} \qquad \text{and} \qquad \mathcal{S}_{\mathbf{k}} = \bigoplus_{d\in\mathbb{Z}} \mathcal{S}_{\mathbf{k},d}$$

We now give the first connection between shuffle algebras and quantum loop groups.

Theorem 2.6. *([7] for the homomorphism, [21] for the isomorphism) The assignment $e_{i,d} \mapsto z_{i1}^d \in \mathcal{S}_{\boldsymbol{\varsigma}^i,d}$ gives rise to an algebra isomorphism*

$$\Upsilon\colon U_q(L\mathfrak{n}^+) \xrightarrow{\sim} \mathcal{S} \tag{2.11}$$

In particular, Theorem 2.6 implies that $\mathcal{S}$ is generated by the monomials $\{z_{i1}^d\}_{i\in I}^{d\in\mathbb{Z}}$ under the shuffle product (2.8). Besides being isomorphic to the positive half of quantum loop groups (as above), shuffle algebras are also dual to the negative half of quantum loop groups, as we recall next. In what follows, let $Dz = \frac{dz}{2\pi i z}$.

Theorem 2.7. *([7] for the construction of the pairing, [21] for its non-degeneracy) There is a non-degenerate pairing*

$$\mathcal{S} \otimes U_q(L\mathfrak{n}^-) \xrightarrow{\langle\cdot,\cdot\rangle} \mathbb{Q}(q) \tag{2.12}$$

given by

$$\Big\langle R, f_{i_1,-d_1} \dots f_{i_k,-d_k} \Big\rangle = \int_{|z_1|\ll\dots\ll|z_k|} \frac{R(z_1,\dots,z_k) z_1^{-d_1}\dots z_k^{-d_k}}{\prod_{1\leq a<b\leq k} \zeta_{i_a i_b}(z_a/z_b)} \prod_{a=1}^k Dz_a \tag{2.13}$$

for any $k \in \mathbb{N}$, $i_1, \dots, i_k \in I$, $d_1, \dots, d_k \in \mathbb{Z}$, $R \in \mathcal{S}_{\boldsymbol{\varsigma}_{i_1}+\dots+\boldsymbol{\varsigma}_{i_k}, d_1+\dots+d_k}$ (all pairings between elements of non-opposite degrees are set to be 0). In the right-hand side of (2.13), we plug each variable z_a into an argument of color i_a of the function R; since the latter is symmetric, the result is independent of any choices made. The symbol $\int$ in (2.13) refers to the contour integral over concentric circles centered at the origin of the complex plane (the notation $|z_a| \ll |z_b|$ means that the these circles are very far away from each other when compared to the poles of the integrand).

Note that the pairing (2.13) differs from that of [21, (5.16, 5.17)] by a scalar multiple, but we make the present choice to keep our formulas as clear as possible.

2.8. **The slope filtration.** We will now construct a filtration of $\mathcal{S}$ by a notion of slope $\mu \in \mathbb{Q}$, with the ultimate goal of defining a completion of the shuffle algebra. By Theorem 2.6, this will lead to a specific completion of the positive half of the quantum loop group, hence also of the negative half by the isomorphism (2.7). The coefficients of fused currents are expected to lie in this completion.

Given $\mathbf{k} = \sum_{i \in I} k_i \boldsymbol{\varsigma}^i$ and $\mathbf{l} = \sum_{i \in I} l_i \boldsymbol{\varsigma}^i$, we will write $\mathbf{l} \leq \mathbf{k}$ if $l_i \leq k_i$ for all $i \in I$. We also write $|\mathbf{l}| = \sum_{i \in I} l_i$. The following is a close relative of [19, Definition 3.3].

Definition 2.9. *For any $\mu \in \mathbb{Q}$, we say that $R \in \mathcal{S}_{\mathbf{k}}$ has slope $\leq \mu$ if*

$$\lim_{\xi \to \infty} \frac{R(\dots, \xi z_{i1}, \dots, \xi z_{il_i}, z_{i,l_i+1}, \dots, z_{ik_i}, \dots)}{\xi^{\mu|\mathbf{l}|}} \tag{2.14}$$

is finite for all $\mathbf{l} \leq \mathbf{k}$. Let $\mathcal{S}_{\mathbf{k}, \leq \mu} \subset \mathcal{S}_{\mathbf{k}}$ denote the set of such elements and set

$$\mathcal{S}_{\leq \mu} = \bigoplus_{\mathbf{k} \in \mathbb{N}^I} \mathcal{S}_{\mathbf{k}, \leq \mu}$$

One can show that $\mathcal{S}_{\leq \mu}$ is a subalgebra of $\mathcal{S}$ (cf. [17, Proposition 2.3] for the argument in a completely analogous setup). Moreover, the set

$$\mathcal{B}_\mu^+ = \bigoplus_{\substack{(\mathbf{k}, d) \in \mathbb{N}^I \times \mathbb{Z} \\ d = \mu|\mathbf{k}|}} \mathcal{B}_{\mathbf{k}, d} \quad \text{with} \quad \mathcal{B}_{\mathbf{k}, d} = \mathcal{S}_{\leq \frac{d}{|\mathbf{k}|}} \cap \mathcal{S}_{\mathbf{k}, d}$$

is also a subalgebra of $\mathcal{S}$, which we will refer to as a *slope subalgebra*. Note that each $\mathcal{B}_{\mathbf{k}, d}$ is a finite-dimensional $\mathbb{Q}(q)$-vector space (cf. the proof of Lemma 2.14). By analogy with [19, Theorem 1.1], one can show that the multiplication map

$$m \colon \bigotimes_{\mu \in \mathbb{Q}}^{\rightarrow} \mathcal{B}_\mu^+ \xrightarrow{\sim} \mathcal{S} \tag{2.15}$$

is an isomorphism of vector spaces (the arrow $\rightarrow$ refers to taking the product in increasing order of μ). More generally, the subalgebras $\mathcal{S}_{\leq \nu}$ of $\mathcal{S}$ admit factorizations as in (2.15), but with μ only running over the set $(-\infty, \nu]$.

Proposition 2.10. *For any $a_1 \in \mathcal{B}_{\nu_1}^+, \dots, a_t \in \mathcal{B}_{\nu_t}^+$, we have*

$$a_1 \dots a_t \ \in \bigotimes_{\mu \in [\min(\nu_s), \max(\nu_s)]}^{\rightarrow} \mathcal{B}_\mu^+ \tag{2.16}$$

Proof. Since $\mathcal{S}_{\leq \mu}$ is an algebra for any μ, the fact that $a_1, \dots, a_t$ lie in $\mathcal{S}_{\leq \max(\nu_s)}$ implies that $a_1 \dots a_t \in \mathcal{S}_{\leq \max(\nu_s)}$. By the sentence following (2.15), we have

$$a_1 \dots a_t \ \in \bigotimes_{\mu \in (-\infty, \max(\nu_s)]}^{\rightarrow} \mathcal{B}_\mu^+ \tag{2.17}$$

However, completely analogously to Definition 2.9, one can define the subalgebra $\mathcal{S}_{\geq \mu} \subset \mathcal{S}$ with $\mathcal{S}_{\geq \mu} \cap \mathcal{S}_{\mathbf{k}}$ consisting of rational functions such that

$$\lim_{\xi \to 0} \frac{R(\dots, \xi z_{i1}, \dots, \xi z_{il_i}, z_{i,l_i+1}, \dots, z_{ik_i}, \dots)}{\xi^{\mu|\mathbf{l}|}} \tag{2.18}$$

is finite for all $\mathbf{l} \leq \mathbf{k}$. Moreover, for any $(\mathbf{k}, d) \in \mathbb{N}^I \times \mathbb{Z}$, it is clear that

$$\mathcal{B}_{\mathbf{k},d} = \mathcal{S}_{\geq \frac{d}{|\mathbf{k}|}} \cap \mathcal{S}_{\mathbf{k},d}$$

because properties (2.14) and (2.18) are equivalent for a rational function of homogeneous degree $d = \mu|\mathbf{k}|$ in $\mathbf{k}$ variables. Hence, by analogy with (2.17), we obtain

$$a_1 \dots a_t \in \bigotimes_{\mu \in [\min(\nu_s), \infty)}^{\rightarrow} \mathcal{B}^+_\mu \tag{2.19}$$

Thus, combining (2.17, 2.19) with the fact that the map (2.15) is an isomorphism, we conclude (2.16). □

2.11. **Slopes and the pairing.** By abuse of notation, we will also refer to $\mathcal{B}^+_\mu$ as a subalgebra of $U_q(L\mathfrak{n}^+)$ via the isomorphism (2.11). Therefore, we obtain the following analogue of (2.15)

$$\bigotimes_{\mu \in \mathbb{Q}}^{\rightarrow} \mathcal{B}^+_\mu \xrightarrow{\sim} U_q(L\mathfrak{n}^+)$$

Using the isomorphism (2.7), we also have a factorization

$$\bigotimes_{\mu \in \mathbb{Q}}^{\rightarrow} \mathcal{B}^-_\mu \xrightarrow{\sim} U_q(L\mathfrak{n}^-) \tag{2.20}$$

where $\mathcal{B}^-_\mu$ refers to the image of $\mathcal{B}^+_\mu$ under (2.7) (we will denote the graded pieces of $\mathcal{B}^-_\mu$ by $\mathcal{B}_{-\mathbf{k},-d}$, with $\mu = \frac{d}{|\mathbf{k}|}$, in order to differentiate them from those of $\mathcal{B}^+_\mu$). The following result explains that the pairing (2.12) is determined by its values on the slope subalgebras (cf. [19, Proposition 3.12] for the argument in an analogous setup).

Proposition 2.12. *For any* $\{b^+_\mu \in \mathcal{B}^+_\mu, b^-_\mu \in \mathcal{B}^-_\mu\}_{\mu \in \mathbb{Q}}$ *(with almost all of the* b^+_μ, b^-_μ *being 1) we have the following formula for the pairing* (2.12)

$$\left\langle \prod_{\mu \in \mathbb{Q}}^{\rightarrow} b^+_\mu, \prod_{\mu \in \mathbb{Q}}^{\rightarrow} b^-_\mu \right\rangle = \prod_{\mu \in \mathbb{Q}} \langle b^+_\mu, b^-_\mu \rangle$$

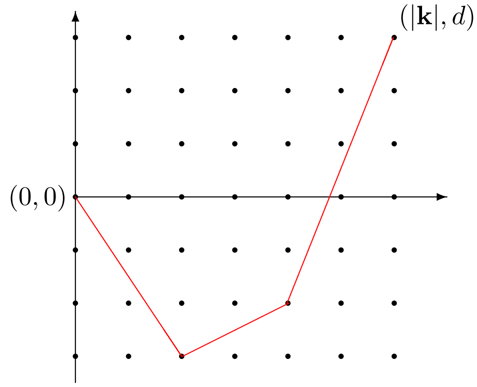
2.13. **Products and paths.** Let us consider a product

$$\pi = \prod_{\mu \in \mathbb{Q}}^{\rightarrow} b_\mu \in U_q(L\mathfrak{n}^-)_{-\mathbf{k},-d} \tag{2.21}$$

with $b_\mu \in \mathcal{B}^-_\mu$ for all $\mu \in \mathbb{Q}$, such that almost all of the b_μ are 1. We further assume that each b_μ is homogeneous of some degree $(-\mathbf{k}_\mu, -d_\mu) \in (-\mathbb{N}^I) \times \mathbb{Z}$, almost all of which will be $(0,0)$. To such a product (2.21) we associate the sequence of vectors

$$\Big(\dots, (|\mathbf{k}_\mu|, d_\mu), \dots \Big)_{\mu \in \mathbb{Q}} \subset \mathbb{N} \times \mathbb{Z}$$

Since almost all of the vectors in the sequence above are $(0,0)$, by placing these vectors head-to-tail (in increasing order of $\mu \in \mathbb{Q}$), we obtain a convex path $p = \text{path}(\pi)$ in the lattice plane from the origin $(0,0)$ to $(|\mathbf{k}|, d)$, see an example below. We call $(|\mathbf{k}|, d)$ the *size* of the path p and the vectors $(|\mathbf{k}_\mu|, d_\mu)$ the *legs* of the path.



Lemma 2.14. *For any convex path p, the vector space $U_q(L\mathfrak{n}^-)_p$ spanned by products (2.21) with $\mathrm{path}(\pi) = p$ is finite-dimensional.*

Proof. This is an immediate consequence of the fact that the graded pieces of the algebras $\mathcal{B}^-_\mu$ are finite-dimensional, itself a consequence of the fact that the space of Laurent polynomials in $|\mathbf{k}|$ variables of fixed total homogeneous degree, but with degree bounded from above in each variable, is finite-dimensional. □

Consider any collection of homogeneous elements

$$a_1, \dots, a_t \in U_q(L\mathfrak{n}^-)$$

By (2.20), each of these elements can be written as

$$a_s = \pi_s^1 + \pi_s^2 + \dots$$

where each $\pi_s^1, \pi_s^2, \dots$ is of the form (2.21). Then

$$P_s = \Big\{ \mathrm{path}(\pi_s^1), \mathrm{path}(\pi_s^2), \dots \Big\}$$

is a finite set of convex paths of size $(|\mathbf{k}_s|, d_s)$, and we will write P for the finite collection of concatenations of any path from P_1 with any path from P_2 … with any path from P_t.

Proposition 2.15. *With $a_1, \dots, a_t$ as above, the product $a_1 \dots a_t$ is a finite sum of products π of the form (2.21), such that $\mathrm{path}(\pi)$ lies below some path in P.*

The notion "lie below" in $\mathbb{Z}^2$ naturally refers to having smaller than or equal second coordinate: we say that a path p lies below a path p' if for any $x \in \mathbb{R}$, those y such that $(x, y) \in p$ are smaller than or equal to those y' such that $(x, y') \in p'$. In what follows, we only sketch the proof of Proposition 2.15, and refer the reader to [26, Section 5] for the full and original details of this kind of argument.

Proof. It suffices to assume that $a_s \in \mathcal{B}^-_{\mu_s}$ for various $\mu_s \in \mathbb{Q}$, so each P_s consists of a single leg and thus P consists of a single path p. If we have $\mu_s > \mu_{s+1}$ for some s (which corresponds to two consecutive legs in p which violate convexity), then we may use (the image under the isomorphism (2.7) of) Proposition 2.10 to write

$$a_s a_{s+1} = \sum (\pi \text{ as in } (2.21)) \tag{2.22}$$

The π's that appear in the right-hand side of (2.22) correspond to convex paths with the same endpoints as the concave path$(a_s a_{s+1})$. In particular, the aforementioned convex paths lie strictly below the aforementioned concave path and moreover the area between these convex and concave paths is a positive integer multiple of $\frac{1}{2}$. Therefore, if we plug the right-hand side of (2.22) in the middle of the product

$$a_1 \dots a_s a_{s+1} \dots a_t$$

then we are replacing a product corresponding to the path p by a sum of products corresponding to paths lying below p. However, due to Proposition 2.10, all of these paths still lie above the *convexification* $p^\sharp$ of the path p, which is the convex path obtained by reordering the segments of p in increasing order of slope. Repeating this algorithm will produce paths p' that are lower and lower down, but still bounded by $p^\sharp$ from below (since the convexification $p^\sharp$ lies below of any convexifications $(p')^\sharp$). The fact that the area between these paths and $p^\sharp$ decreases by a positive integer multiple of $\frac{1}{2}$ at every step means that the algorithm must terminate after finitely many steps. □

2.16. **The completion.** We are now ready to define our completion of $U_q(L\mathfrak{n}^-)$.

Definition 2.17. *Consider the vector space*

$$\widehat{U}_q(L\mathfrak{n}^-) = \bigoplus_{(\mathbf{k},d)\in\mathbb{N}^I\times\mathbb{Z}} \widehat{U}_q(L\mathfrak{n}^-)_{-\mathbf{k},-d} \tag{2.23}$$

where

$$\widehat{U}_q(L\mathfrak{n}^-)_{-\mathbf{k},-d} = \prod_{\textit{of size } (|\mathbf{k}|,d)}^{\textit{convex path } p} U_q(L\mathfrak{n}^-)_p$$

Proposition 2.18. *The algebra structure on $U_q(L\mathfrak{n}^-)$ extends uniquely to an algebra structure on $\widehat{U}_q(L\mathfrak{n}^-)$.*

Proof. We start by remarking that for any given path p of size $(|\mathbf{k}|, d)$, all but finitely many convex paths of the same size will lie below p. Let us argue by contradiction: suppose that infinitely many convex paths of size $(|\mathbf{k}|, d)$ failed to lie below p. Then each of these infinitely many paths would need to pass through at least one of the finitely many lattice points contained inside the trapezoid bounded by the line segment $(0, 0)$ to $(|\mathbf{k}|, d)$, the two vertical lines with x-coordinate 0 and $|\mathbf{k}|$, and the horizontal line corresponding to the smallest y-coordinate of any point on the path p. It is clearly impossible for infinitely many convex paths of fixed size to pass through a fixed lattice point (other than the endpoints of the path).

With this in mind, our task is to show the well-definedness of a product

$$\Big(\underbrace{\pi_1 + \pi_2 + \pi_3 + \dots}_{\text{elements of } U_q(L\mathfrak{n}^-)_{-\mathbf{k},-d}} \Big) \cdot \Big(\underbrace{\pi'_1 + \pi'_2 + \pi'_3 + \dots}_{\text{elements of } U_q(L\mathfrak{n}^-)_{-\mathbf{k}',-d'}} \Big) \tag{2.24}$$

where π_s (respectively π'_t) correspond to different convex paths p_s of size $(|\mathbf{k}|, d)$ (respectively p'_t of size $(|\mathbf{k}'|, d')$). We may evaluate the product (2.24) by foiling out the brackets and expressing each $\pi_s \pi'_t$ as a linear combination of products

$$\widetilde{\pi} \text{ as in (2.21)} \tag{2.25}$$

(cf. (2.16)). In order for such an infinite sum to be a well-defined element of $\widehat{U}_q(L\mathfrak{n}^-)$, one needs to show that each fixed $\widetilde{\pi}$ only appears for finitely many (s,t). Let us analyze the possible paths $\widetilde{p} = \text{path}(\widetilde{\pi})$ that correspond to elements (2.25), and it suffices to show that any such path appears only for finitely many pairs (s,t).

By Proposition 2.15, any path $\widetilde{p}$ corresponding to a product $\widetilde{\pi}$ in (2.25) must lie below the concatenation of p_s with p'_t. However, by the argument in the first paragraph of the proof, for any $N \in \mathbb{N}$ all but finitely many paths p_s (respectively all but finitely many paths p'_t) contain some point with vertical coordinate $< -N$. Therefore, the same is true for the concatenation of p_s and p'_t. By choosing N large enough compared to any given path $\widetilde{p}$, we can ensure that $\widetilde{p}$ does not lie below the concatenation of p_s with p'_t, except for finitely many pairs (s,t). $\square$

The proof of Proposition 2.18 also proves the following stronger statement.

Proposition 2.19. *For any $(\mathbf{k}, d) \in \mathbb{N}^I \times \mathbb{Z}$, any countable sum of products*

$$a_1 \dots a_t, \quad \textit{where } a_s \in \widehat{U}_q(L\mathfrak{n}^-)_{-\mathbf{k}_s, -d_s}, \forall s \in \{1, \dots, t\}, \sum_{s=1}^{t} \mathbf{k}_s = \mathbf{k}, \sum_{s=1}^{t} d_s = d$$

also lies in $\widehat{U}_q(L\mathfrak{n}^-)$ as long as all but finitely many such products have the property that the (not necessarily convex) path with legs $(|\mathbf{k}_1|, d_1), \dots, (|\mathbf{k}_t|, d_t)$ lies below any given convex path of size $(|\mathbf{k}|, d)$ in $\mathbb{N} \times \mathbb{Z}$.

2.20. **Quantum $\widehat{\mathfrak{g}}$.** We will now recall another completion of quantum affine algebras, that we will shortly connect with $\widehat{U}_q(L\mathfrak{n}^-)$ above. Let $U_q^{\text{ext}}(L\mathfrak{g})$ be the standard central extension of $U_q(L\mathfrak{g})$ with an extra central generator C such that

$$U_q(L\mathfrak{g}) \simeq U_q^{\text{ext}}(L\mathfrak{g})/(C-1)$$

We will not need the precise definition of the central extension, but observe that it does not affect the subalgebra $U_q(L\mathfrak{n}^-)$, or its completion (2.23). The algebra $U_q^{\text{ext}}(L\mathfrak{g})$ admits an automorphism ϖ defined on the generators via

$$\varpi\colon e_{i,d} \mapsto f_{i,-d}, \quad f_{i,d} \mapsto e_{i,-d}, \quad \varphi^{\pm}_{i,d'} \mapsto \varphi^{\mp}_{i,d'}, \quad C \mapsto C^{-1} \tag{2.26}$$

for all $i \in I, d \in \mathbb{Z}, d' \in \mathbb{N}$. Furthermore, let $U_q(\widehat{\mathfrak{g}})$ denote the Drinfeld-Jimbo quantum group associated to the affinization of the Lie algebra $\mathfrak{g}$, which is generated by

$$\{e_i, f_i, \varphi_i^{\pm 1}\}_{i \in \widehat{I}} \quad \text{with} \quad \widehat{I} = I \sqcup 0$$

A result of Drinfeld (proved by Beck, Damiani) establishes an algebra isomorphism

$$\Phi\colon U_q^{\text{ext}}(L\mathfrak{g}) \xrightarrow{\sim} U_q(\widehat{\mathfrak{g}}) \tag{2.27}$$

Let $U_q(\widehat{\mathfrak{n}}^+)$, $U_q(\widehat{\mathfrak{n}}^-)$, $U_q(\widehat{\mathfrak{h}})$ denote the subalgebras of $U_q(\widehat{\mathfrak{g}})$ generated by e_i, f_i, $\varphi_i^{\pm}$, respectively. Let $U_q(\widehat{\mathfrak{b}}^{\pm})$ denote the subalgebras generated by $U_q(\widehat{\mathfrak{n}}^{\pm})$ and $U_q(\widehat{\mathfrak{h}})$. According to [1] (respectively [14]), the subalgebras $U_q(\widehat{\mathfrak{n}}^{\pm})$ admit PBW bases in the root vectors $\{e_{\pm\gamma}\}_{\gamma \in \widehat{\Delta}^+}$[3] (termed *Cartan-Weyl basis* in [14]) constructed through Lusztig's braid group action (respectively, via iterated q-commutators in [14], for every normal ordering of $\widehat{\Delta}^+$) where $\widehat{\Delta}^+$ denotes the set of positive affine roots.

[3]For imaginary roots γ, we actually have $|I|$ root vectors $\{e_\gamma^{(i)}\}_{i \in I}$ instead of a single e_γ.

Claim 2.21. The root generators e_γ with $\gamma \in \widehat{\Delta}^+$ (respectively $-\gamma \in \widehat{\Delta}^+$) can be expressed as non-commutative polynomials in $\{e_i\}_{i\in\widehat{I}}$ (respectively $\{f_i\}_{i\in\widehat{I}}$) of degree $\mathrm{ht}(\gamma)$, with the height of an affine root defined by $\mathrm{ht}(\sum_{i\in\widehat{I}} r_i\alpha_i) = \sum_{i\in\widehat{I}} r_i$.

Consider the $\mathbb{Z}$-grading of $U_q(\widehat{\mathfrak{n}}^\pm)$ given by ht, i.e. setting $\deg e_i = 1$ and $\deg f_i = -1$ for all $i \in \widehat{I}$. With this in mind, we define the completion

$$\widehat{U}_q(\widehat{\mathfrak{g}}) \tag{2.28}$$

where a basis of neighborhoods of the identity are images under multiplication of

$$U_q(\widehat{\mathfrak{n}}^+)_{\geq N} \otimes U_q(\widehat{\mathfrak{h}}) \otimes U_q(\widehat{\mathfrak{n}}^-)_{\leq -N} \tag{2.29}$$

as N ranges over the natural numbers. Therefore, elements in the completion (2.28) are infinite sums of products, all but finitely many of which are of the form

$$\Big(\text{at least } N \text{ factors } e_i\Big)_{i\in\widehat{I}} \cdot \Big(\text{anything}\Big) \cdot \Big(\text{at least } N \text{ factors } f_i\Big)_{i\in\widehat{I}}$$

Remark 2.22. This completion is closely related to the one considered in [6], which was defined to consist of infinite sums

$$U_q(\widehat{\mathfrak{h}}) \cdot e_{-\gamma}^{n_\gamma} \dots e_{-\beta}^{n_\beta} e_{-\alpha}^{n_\alpha} e_{\alpha}^{m_\alpha} e_{\beta}^{m_\beta} \dots e_{\gamma}^{m_\gamma}$$

where $\alpha < \beta < \dots < \gamma$ with respect to the above normal ordering, such that for any weight λ and any $N \in \mathbb{Z}$, there are only finitely many terms of total weight λ which satisfy the condition $(n_\alpha + m_\alpha)\mathrm{ht}(\alpha) + (n_\beta + m_\beta)\mathrm{ht}(\beta) + \dots + (n_\gamma + m_\gamma)\mathrm{ht}(\gamma) \leq N$.

2.23. **The two completions.** We will now connect the completion $\widehat{U}_q(L\mathfrak{n}^-)$ of the subalgebra $U_q(L\mathfrak{n}^-)$ of the left-hand side of (2.27) with the completion (2.28) of the right-hand side. Our main result on this matter is the following.

Proposition 2.24. *The map* (2.27) *induces an injective algebra homomorphism*

$$\Phi \colon \widehat{U}_q(L\mathfrak{n}^-) \longrightarrow \widehat{U}_q(\widehat{\mathfrak{g}}) \tag{2.30}$$

Proof. We recall the fact (see [2, 3]) that the isomorphism (2.27) satisfies

$$\Phi(f_{i,d}) \in \begin{cases} U_q(\widehat{\mathfrak{n}}^-)_{\leq -1} & \text{if } d \leq 0 \\ U_q(\widehat{\mathfrak{b}}^+)_{\geq 1} & \text{if } d > 0 \end{cases}$$

In order to conclude the existence of a homomorphism (2.30), we need to show that infinite sums of (2.21) corresponding to convex paths lie in the completion (2.28). To this end, let us fix a degree $\mathbf{k} \in \mathbb{N}^I$, and consider the finitely many subspaces

$$\mathcal{B}_{-\mathbf{k}',-d'} \neq 0 \tag{2.31}$$

with $0 \leq d' < |\mathbf{k}'|$ and $\mathbf{k}' \leq \mathbf{k}$. All the subspaces (2.31) are finite-dimensional, so we may conclude that all their elements are written as sums of products of $f_{i,d}$'s with $d \in [-M, M]$ for some large enough natural number $M = M(\mathbf{k})$. Because of this, and the fact that the shift automorphism $\{f_{i,d} \mapsto f_{i,d-1}\}_{i\in I}^{d\in\mathbb{Z}}$ sends $\mathcal{B}_{-\mathbf{k}',-d'} \xrightarrow{\sim} \mathcal{B}_{-\mathbf{k}',-d'-|\mathbf{k}'|}$, we have

$$\mathcal{B}_{-\mathbf{k}',-d'} \subset \begin{cases} \Big\langle f_{i,0}, f_{i,-1}, \dots \Big\rangle_{i\in I} \subseteq U_q(\widehat{\mathfrak{n}}^-) & \text{if } d' \geq M|\mathbf{k}'| \\ \Big\langle f_{i,1}, f_{i,2}, \dots \Big\rangle_{i\in I} \subseteq U_q(\widehat{\mathfrak{b}}^+) & \text{if } d' < -M|\mathbf{k}'| \end{cases}$$

Since the degree by height of $\Phi(\mathcal{B}_{-\mathbf{k}',-d'}) \subset U_q(\widehat{\mathfrak{g}})$ is equal to $-|\mathbf{k}'| - d'\ell$, where $\ell \in \mathbb{Z}_{>0}$ denotes the height of the minimal imaginary root δ, and $\mathbf{k}' \leq \mathbf{k}$, we conclude that the subalgebras $\mathcal{B}_{-\mathbf{k}',-d'}$ correspond to elements of $U_q(\widehat{\mathfrak{g}})$ as follows

$$\begin{cases} \text{of degree} \leq O(-\mu) & \text{if } \mu \geq M \\ \text{of bounded degree} & \text{if } \mu \in [-M, M) \\ \text{of degree} \geq O(-\mu) & \text{if } \mu < -M \end{cases}$$

where $\mu = \frac{d'}{|\mathbf{k}'|}$ (above and below, "$O(\mu)$" refers to $a\mu+b$ for some $a > 0$, $b \in \mathbb{R}$). For any $N \in \mathbb{N}$, we may subdivide a convex path into "segments of slope $\leq O(-N)$", "segments of intermediate slope" and "segments of slope $\geq O(N)$". In doing so, we can ensure that any product (2.21) give rise to an element of the form (2.29) in $U_q(\widehat{\mathfrak{g}})$. This implies both that the map Φ is well-defined and that it is injective, because the slope (which measures the depth of the neighborhoods of 0 in the completion (2.23)) is linear in the number N (which measures the depth of the neighborhoods of zero in the completion (2.28)). □

Although it is inconsequential for the remainder of the present paper, we make the following conjecture, which will be proved in [20].

Conjecture 2.25. *The map Φ of (2.27) sends the slope subalgebras as follows*

$$\begin{aligned} \Phi(\mathcal{B}^+_\mu) &\subset U_q(\widehat{\mathfrak{n}}^+), && \textit{if } \mu \geq 0 \\ \Phi(\mathcal{B}^+_\mu) &\subset U_q(\widehat{\mathfrak{b}}^-), && \textit{if } \mu < 0 \\ \Phi(\mathcal{B}^-_\mu) &\subset U_q(\widehat{\mathfrak{n}}^-), && \textit{if } \mu \leq 0 \\ \Phi(\mathcal{B}^-_\mu) &\subset U_q(\widehat{\mathfrak{b}}^+), && \textit{if } \mu > 0 \end{aligned}$$

2.26. **The braid group action.** Although they do not use the language of completions (but point out the correct completion in their other paper [6], see Remark 2.22) and instead work in a certain class of admissible representations, [5] defined an action of the braid group of type $\mathfrak{g}$, generated by $\{T_i\}_{i\in I}$, on the completion (2.28). The authors of [5] work only with the simply-laced $\mathfrak{g}$, in which case the explicit formulas for T_i can be read off from [5, (1.15)–(1.21), (2.1)–(2.5)]:

- if $a_{ij} = 0$, then for any $d \in \mathbb{Z}, d' \in \mathbb{N}$

$$T_i(e_{j,d}) = e_{j,d}, \qquad T_i(f_{j,d}) = f_{j,d}, \qquad T_i(\varphi^\pm_{j,d'}) = \varphi^\pm_{j,d'} \tag{2.32}$$

- if $a_{ij} = -1$, then for any $d \in \mathbb{Z}, d' \in \mathbb{N}$

$$\begin{aligned} T_i(e_{j,d}) &= q^{2d}\left(-e_{j,d}e_{i,0} + q^{-1}e_{i,0}e_{j,d} - (q-q^{-1})\sum_{k\geq 1} q^{-k}e_{i,k}e_{j,d-k}\right), \\ T_i(f_{j,d}) &= q^{2d}\left(f_{i,0}f_{j,d} - qf_{j,d}f_{i,0} - (q-q^{-1})\sum_{k\geq 1} q^{k}f_{j,d+k}f_{i,-k}\right), \\ T_i(\varphi^\pm_{j,d'}) &= \sum_{k=0}^{d'} q^{\mp(k-2d')}\varphi^\pm_{i,k}\varphi^\pm_{j,d'-k} \end{aligned} \tag{2.33}$$

- if $j = i$, then for any $d \in \mathbb{Z}, d' \in \mathbb{N}$

$$
\begin{aligned}
T_i(e_{i,d}) &= \sum_{k\geq 0} \bar{\varphi}^+_{i,k} f_{i,d-k} C^{2d-k}, \\
T_i(f_{i,d}) &= \sum_{k\geq 0} e_{i,d+k} \bar{\varphi}^-_{i,k} C^{2d+k}, \\
T_i(\varphi^\pm_{i,d'}) &= \bar{\varphi}^\pm_{i,d'},
\end{aligned}
\tag{2.34}
$$

where $\{\bar{\varphi}^\pm_{i,k}\}^{k\geq 0}_{i\in I}$ are defined so that $\bar{\varphi}^\pm_i(z) = \sum_{k\geq 0} \bar{\varphi}^\pm_{i,k} z^{\mp k} = \big(\varphi^\pm_i(z)\big)^{-1}$.

where we applied the automorphism ϖ of (2.26), as our $e_{i,d}, f_{i,d}$ are $f_{i,-d}, e_{i,-d}$ of *loc. cit.* With this in mind, we propose the following reinterpretation of the main result of [5].

Proposition 2.27. *The maps T_i of (2.32)–(2.34) give rise to well-defined automorphisms*

$$
\widehat{U}_q(\widehat{\mathfrak{g}}) \xrightarrow{\sim} \widehat{U}_q(\widehat{\mathfrak{g}})
$$

which induce an action of the braid group on $\widehat{U}_q(\widehat{\mathfrak{g}})$.

2.28. **Fused currents.** Recall that any reduced decomposition of the longest element in the Weyl group associated to $\mathfrak{g}$ (with $s_i = s_{\alpha_i}$ being the simple reflections)

$$
w_0 = s_{i_1} \dots s_{i_n} \tag{2.35}
$$

yields a total (convex) order on the set Δ^+ of positive roots of $\mathfrak{g}$, by setting

$$
s_{i_n} \dots s_{i_2}(\alpha_{i_1}) > s_{i_n} \dots s_{i_3}(\alpha_{i_2}) > \dots > s_{i_n}(\alpha_{i_{n-1}}) > \alpha_{i_n} \tag{2.36}
$$

Definition 2.29. *([5]) For any reduced decomposition (2.35), any positive root $\alpha = s_{i_n} \dots s_{i_{k+1}}(\alpha_{i_k})$ (for some $k \in \{1, \dots, n\}$), and any $d \in \mathbb{Z}$, let*

$$
\widetilde{f}_{\alpha,d} = T^{-1}_{i_n} \dots T^{-1}_{i_{k+1}}(f_{i_k,d}) \in \widehat{U}_q(\widehat{\mathfrak{g}}) \tag{2.37}
$$

The fused current is defined by $\widetilde{f}_\alpha(x) = \sum_{d\in\mathbb{Z}} \frac{\widetilde{f}_{\alpha,d}}{x^d}$.

The original definition of [5] had the expressions (2.37) defined in an appropriate class of representations in which certain countable sums are well-defined. We posit that their definition is equivalent to the completion of Definition 2.17.

Conjecture 2.30. *For any $(\alpha, d) \in \Delta^+ \times \mathbb{Z}$, we have*

$$
\widetilde{f}_{\alpha,d} \in \mathrm{Im}\Big(\widehat{U}_q(L\mathfrak{n}^-)_{-\alpha,d} \longrightarrow \widehat{U}_q(\widehat{\mathfrak{g}})\Big)
$$

with respect to the algebra homomorphism (2.30) of Proposition 2.24.

We expect that the leading term of $\widetilde{f}_{\alpha,0}$ is precisely the root vector $f_\alpha \in U_q(\mathfrak{n}^-)$ of (1.2) under the natural embedding $U_q(\mathfrak{n}^-) \hookrightarrow U_q(L\mathfrak{n}^-)$ (that is, $f_{\alpha,0} = f_\alpha$ plus products (2.21) whose associated convex paths lie strictly below the horizontal line). Therefore, it is natural to develop the commutation relations for fused currents, akin to Levendorskii-Soibelman formulas (1.5) in the finite type quantum group. In particular, we will focus on the case of a minimal pair of positive roots $\alpha < \beta$, and seek an analogue of (1.6).

Conjecture 2.31. *For any minimal pair $\alpha < \beta$ w.r.t. the order (2.36), we have*

$$\widetilde{f}_\alpha(x)\widetilde{f}_\beta(y) - \widetilde{f}_\beta(y)\widetilde{f}_\alpha(x) \cdot \frac{xq^v - yq^{(\alpha,\beta)}}{xq^{v+(\alpha,\beta)} - y} = c(q) \cdot \delta\left(\frac{xq^{v+(\alpha,\beta)}}{y}\right) \widetilde{f}_{\alpha+\beta}(xq^u) \tag{2.38}$$

for some $u, v \in \mathbb{Z}$ and $c(q) \in \mathbb{Z}[q, q^{-1}]^\times$ depending on α, β, where $\delta(x) = \sum_{d\in\mathbb{Z}} x^d$. In the second term from the LHS of (2.38), the rational function is expanded in the region $|x| \gg |y|$.

Note that Conjecture 2.31 implies Conjecture 2.30, by the following inductive argument. Formula (2.38) allows us to write for all $(\alpha, d) \in \Delta^+ \times \mathbb{Z}$ (with $\mathrm{ht}(\alpha) > 1$)

$$\widetilde{f}_{\alpha,d} = \text{constant} \cdot \widetilde{f}_{\beta,d}\widetilde{f}_{\gamma,0} + \sum_{n=0}^{\infty} \text{constant} \cdot \widetilde{f}_{\gamma,n}\widetilde{f}_{\beta,d-n}$$

for some $\beta, \gamma \in \Delta^+$ with $\alpha = \beta + \gamma$. If all the $\widetilde{f}$'s in the right-hand side lie in $\widehat{U}_q(L\mathfrak{n}^-)$ by the induction hypothesis, then Proposition 2.19 implies that so does the entire right-hand side. Therefore, so does the left-hand side, which establishes the induction step.

2.32. **Pairing with elements in the completion.** We will now explain the importance of Conjecture 2.30, i.e. why is it important that $\widetilde{f}_{\alpha,d}$ should be interpreted as lying in the completion (2.23) rather than the completion (2.28). To us, this is relevant because the former completion interacts with the pairing (2.12) in the following natural way.

Proposition 2.33. *The pairing (2.12) naturally extends to a pairing*

$$\mathcal{S} \otimes \widehat{U}_q(L\mathfrak{n}^-) \xrightarrow{\langle \cdot,\cdot \rangle} \mathbb{Q}(q) \tag{2.39}$$

For any $(\mathbf{k}, d) \in \mathbb{N}^I \times \mathbb{Z}$, any linear functional $\mathcal{S}_{\mathbf{k},d} \xrightarrow{\lambda} \mathbb{Q}(q)$ can be written as

$$\lambda(R) = \langle R, f \rangle, \qquad \forall\, R \in \mathcal{S}_{\mathbf{k},d} \tag{2.40}$$

for a unique $f \in \widehat{U}_q(L\mathfrak{n}^-)_{-\mathbf{k},-d}$.

Proof. Let us recall the decomposition (2.15) for $\mathcal{S}$ and (2.20) for $U_q(L\mathfrak{n}^-)$. Any given $R \in \mathcal{S}$ decomposes in terms of the slope subalgebras $\mathcal{B}^+_\mu$ for μ in a finite subset $\Omega \subset \mathbb{Q}$. Proposition 2.12 says that the pairing (2.12) respects these decompositions, and so R pairs non-trivially only with products (2.21) whose corresponding convex path has legs with slopes in Ω. But only finitely many convex paths of any given size have legs with slopes in the finite subset $\Omega \subset \mathbb{Q}$, so we conclude that the pairing of R with any countable sum of products making up $\widehat{U}_q(L\mathfrak{n}^-)$ is well-defined.

Let us now prove the statement about λ. Recall that the restriction of (2.12) to

$$\mathcal{B}^+_\mu \otimes \mathcal{B}^-_\mu \xrightarrow{\langle \cdot,\cdot \rangle} \mathbb{Q}(q)$$

is a non-degenerate pairing of graded vector spaces which are finite-dimensional in every degree (and thus a perfect pairing). Therefore, so is

$$\left(\mathcal{B}_{\mathbf{k}_1,d_1} \otimes \cdots \otimes \mathcal{B}_{\mathbf{k}_t,d_t}\right) \otimes \left(\mathcal{B}_{-\mathbf{k}_1,-d_1} \otimes \cdots \otimes \mathcal{B}_{-\mathbf{k}_t,-d_t}\right) \xrightarrow{\langle \cdot,\cdot \rangle} \mathbb{Q}(q) \tag{2.41}$$

for any convex path with legs $(|\mathbf{k}_1|, d_1), \dots, (|\mathbf{k}_t|, d_t)$ in $\mathbb{N} \times \mathbb{Z}$ of size $(|\mathbf{k}|, d)$, in virtue of Proposition 2.12. For such a convex path, let us denote

$$\lambda_{(\mathbf{k}_1,d_1),\dots,(\mathbf{k}_t,d_t)} \colon \underbrace{\mathcal{B}_{\mathbf{k}_1,d_1} \otimes \dots \otimes \mathcal{B}_{\mathbf{k}_t,d_t}}_{\text{a direct summand of } \mathcal{S}} \to \mathbb{Q}(q)$$

the appropriate restriction of λ. Since the pairing (2.41) is perfect, there exists

$$f_{(\mathbf{k}_1,d_1),\dots,(\mathbf{k}_t,d_t)} \in \underbrace{\mathcal{B}_{-\mathbf{k}_1,-d_1} \otimes \dots \otimes \mathcal{B}_{-\mathbf{k}_t,-d_t}}_{\text{a direct summand of } U_q(L\mathfrak{n}^-)}$$

such that $\lambda_{(\mathbf{k}_1,d_1),\dots,(\mathbf{k}_t,d_t)}$ is given by pairing with $f_{(\mathbf{k}_1,d_1),\dots,(\mathbf{k}_t,d_t)}$. Then letting

$$f = \sum_{\substack{\text{convex paths of size } (|\mathbf{k}|,d) \\ \text{with legs } (|\mathbf{k}_1|,d_1),\dots,(|\mathbf{k}_t|,d_t)}} f_{(\mathbf{k}_1,d_1),\dots,(\mathbf{k}_t,d_t)}$$

yields the required element of $\widehat{U}_q(L\mathfrak{n}^-)$ in (2.40). The uniqueness of such f satisfying (2.40) is due to non-degeneracy of (2.12). ☐

2.34. **Specialization maps.** If the coefficients of the fused currents $\widetilde{f}_\alpha(x)$ lie in the completion $\widehat{U}_q(L\mathfrak{n}^-)$, as predicted by Conjecture 2.30, then Proposition 2.33 implies that they have a well-defined pairing with elements of the shuffle algebra.

Definition 2.35. *For any positive root α, define the specialization map*

$$\mathcal{S}_\alpha \xrightarrow{\widetilde{\text{spec}}_\alpha^{(x)}} \mathbb{Q}(q)[x, x^{-1}], \qquad \widetilde{\text{spec}}_\alpha^{(x)}(R) = \Big\langle R, \widetilde{f}_\alpha(x) \Big\rangle \tag{2.42}$$

Such specialization maps were studied in (super)type A in [27, 28], in types B_n and G_2 in [13], and in affine type A in [18], for a specific choice of the order on Δ^+. In all of these cases, the specialization maps were given by setting

$$\widetilde{\text{spec}}_\alpha^{(x)}(R) = \widetilde{\gamma}_\alpha^{(x)} \cdot R(\dots, z_{ib}, \dots)\Big|_{z_{ib} \mapsto xq^{\sigma_{ib}}, \forall i,b} \tag{2.43}$$

for some collection of integers $(\sigma_{ib})_{i \in I, b \geq 1}$ and some prefactor $\widetilde{\gamma}_\alpha^{(x)} \in \mathbb{Q}(q)^\times \cdot x^{\mathbb{Z}}$. In general, we expect the specialization maps to be given by a suitable derivative of R evaluated at a collection as in the right-hand side of (2.43), up to a prefactor.

It would be very interesting to obtain a complete description of the specialization maps (2.42) for any finite type root system and any reduced decomposition (2.35) of the longest word; in Section 3, we provide such a description in the particular case of ADE type quivers. However, we emphasize the fact that the specialization maps $\widetilde{\text{spec}}_\alpha^{(x)}$ should be considered in relation to the conjectural formula (2.38). Specifically, if $\alpha < \beta$ is a minimal pair and $R \in \mathcal{S}_{\alpha+\beta}$, then we have

$$\Big\langle R, \widetilde{f}_\alpha(x)\widetilde{f}_\beta(y) \Big\rangle = \frac{\widetilde{\text{spec}}_\alpha^{(x)} \otimes \widetilde{\text{spec}}_\beta^{(y)}(R)}{\prod_{i,b} \prod_{j,c} \zeta_{ij}\left(\frac{xq^{\sigma_{ib}}}{yq^{\tau_{jc}}}\right)}\Bigg|_{\text{expanded as } |x| \ll |y|} \tag{2.44}$$

(this formula comes from a topological coproduct on the Cartan-extended version of $\mathcal{S}$, cf. [19, (2.35)]), where (σ_{ib}) and (τ_{jc}) are the collections of integers associated to the specialization maps $\widetilde{\text{spec}}_\alpha^{(x)}$ and $\widetilde{\text{spec}}_\beta^{(y)}$, respectively. Here, $\widetilde{\text{spec}}_\alpha^{(x)} \otimes \widetilde{\text{spec}}_\beta^{(y)}(R)$ means that one divides the variables of R into two groups: the variables in one of the groups are specialized according to $\widetilde{\mathrm{spec}}_{\alpha}^{(x)}$, and the variables in the other group are specialized according to $\widetilde{\mathrm{spec}}_{\beta}^{(y)}$. Meanwhile, we postulate that

$$\prod_{i,b}\prod_{j,c}\frac{\zeta_{ji}\left(\frac{yq^{\tau_{jc}}}{xq^{\sigma_{ib}}}\right)}{\zeta_{ij}\left(\frac{xq^{\sigma_{ib}}}{yq^{\tau_{jc}}}\right)}=\frac{xq^{v}-yq^{(\alpha,\beta)}}{xq^{v+(\alpha,\beta)}-y} \tag{2.45}$$

with v as in (2.38), so that we have

$$\left\langle R,\widetilde{f}_{\beta}(y)\widetilde{f}_{\alpha}(x)\frac{xq^{v}-yq^{(\alpha,\beta)}}{xq^{v+(\alpha,\beta)}-y}\right\rangle=\frac{\widetilde{\mathrm{spec}}_{\alpha}^{(x)}\otimes\widetilde{\mathrm{spec}}_{\beta}^{(y)}(R)}{\prod_{i,b}\prod_{j,c}\zeta_{ij}\left(\frac{xq^{\sigma_{ib}}}{yq^{\tau_{jc}}}\right)}\Big|_{\text{expanded as }|x|\gg|y|} \tag{2.46}$$

Comparing the right-hand sides of the expansions (2.44) and (2.46), we see that we have the exact same rational function in x and y, first expanded as $|x|\ll|y|$ and then expanded as $|x|\gg|y|$. Therefore, formula (2.38) precisely entails the fact that said rational function has a single pole at $y=xq^{v+(\alpha,\beta)}$, and that the corresponding residue at this pole is none other than $c(q)\cdot\widetilde{\mathrm{spec}}_{\alpha+\beta}^{(xq^{u})}(R)$. We conclude that specialization maps give us a novel (and dual) way of thinking about the conjectural commutation relation (2.38) of fused currents.

3. Quivers of ADE type

3.1. Quivers and Hall algebras. We will henceforth assume that $\mathfrak{g}$ is a simply-laced finite type Lie algebra, and we choose an orientation Q of the Dynkin diagram of $\mathfrak{g}$. The inner product $(\cdot,\cdot)$ on the root lattice satisfies $(\beta,\alpha)\in\{-1,0,1\}$ for any $\alpha,\beta\in\Delta$ with $\beta\neq\pm\alpha$. Having made this choice, we may consider the pairing

$$\langle\cdot,\cdot\rangle\colon\mathbb{Z}^{I}\times\mathbb{Z}^{I}\longrightarrow\mathbb{Z},\qquad\langle\boldsymbol{v},\boldsymbol{w}\rangle=\sum_{i\in I}v_iw_i-\sum_{\overrightarrow{ij}}v_iw_j \tag{3.1}$$

and note that $(\boldsymbol{v},\boldsymbol{w})=\langle\boldsymbol{v},\boldsymbol{w}\rangle+\langle\boldsymbol{w},\boldsymbol{v}\rangle$. Let $\mathbb{F}_{q^2}$ be a finite field.

Definition 3.2. *Consider the category $\mathcal{C}$ of finite-dimensional representations of the quiver Q, i.e. collections of finite-dimensional vector spaces over $\mathbb{F}_{q^2}$ associated to the vertices of Q and linear maps associated to the edges of Q*

$$V=\left(V_i\xrightarrow{\phi_e}V_j\right)_{i,j\in I,e=\overrightarrow{ij}} \tag{3.2}$$

modulo change of basis of the vector spaces V_i. The Hall algebra of $\mathcal{C}$ is defined as

$$\mathcal{H}=\mathcal{H}(\mathcal{C})=\bigoplus_{[V]\in\mathrm{Ob}(\mathcal{C})/\sim}\mathbb{Q}\cdot[V]$$

endowed with the multiplication

$$[V]\cdot[W]=q^{\langle\boldsymbol{v},\boldsymbol{w}\rangle}\cdot\sum_{[X]\in\mathrm{Ob}(\mathcal{C})/\sim}[X]\cdot\#\big\{\textit{subreps } Y\subset X\textit{ s.t. }Y\simeq W,\ X/Y\simeq V\big\} \tag{3.3}$$

where $\boldsymbol{v}=(\dim V_i)_{i\in I}$ and $\boldsymbol{w}=(\dim W_i)_{i\in I}$.

It is well-known that the structure constants of the algebra $\mathcal{H}$ (i.e. the numbers that appear in the RHS of (3.3)) are Laurent polynomials in q with rational coefficients. Thus, one can think of q as a formal parameter, and of $\mathcal{H}$ as an algebra over $\mathbb{Q}(q)$. With this in mind, we have the following foundational result.

Theorem 3.3. *([11, 24]) There is an algebra isomorphism*

$$U_q(\mathfrak{n}^+) \xrightarrow{\sim} \mathcal{H} \tag{3.4}$$

determined by sending the generator e_i to the simple quiver representation with a one-dimensional vector space at the vertex i and 0 everywhere else.

We may grade $\mathcal{H}$ by associating to any quiver representation V its dimension vector $\boldsymbol{v} = (\dim V_i)_{i\in I} \in \mathbb{N}^I$, with respect to which (3.4) becomes an isomorphism of $\mathbb{N}^I$-graded algebras. For any $V, W \in \mathcal{C}$ with dimension vectors $\boldsymbol{v}$ and $\boldsymbol{w}$, respectively, the pairing $\langle \boldsymbol{v}, \boldsymbol{w}\rangle$ of (3.1) coincides with the Euler form (see [23]):

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle = \sum_{k\geq 0} (-1)^k \dim \mathrm{Ext}^k(V,W) = \dim \mathrm{Hom}(V,W) - \dim \mathrm{Ext}^1(V,W)$$

with the second equality based on the vanishing

$$\mathrm{Ext}^k(V,W) = 0, \qquad \forall\, k \geq 2, \quad \forall\, V, W \in \mathrm{Ob}(\mathcal{C})$$

3.4. **The Auslander-Reiten partial order.** It is well-known ([10]) that indecomposable representations of Q are in one-to-one correspondence with positive roots of $\mathfrak{g}$, i.e. up to isomorphism there is a single indecomposable representation V_α with dimension vector $\alpha \in \Delta^+$ (and there are no other indecomposables). For any two positive roots α and β, we have ([25, Section 4]) that

$$\text{either} \quad \mathrm{Hom}(V_\alpha, V_\beta) = 0 \quad \text{or} \quad \mathrm{Ext}^1(V_\alpha, V_\beta) = 0 \tag{3.5}$$

As shown in [25, Theorem 7], the isomorphism (3.4) sends

$$e_\alpha \mapsto q^{\kappa_\alpha} \cdot [V_\alpha]$$

for every positive root α, where the root vectors e_α are defined with respect to a certain reduced decomposition of the longest word w_0 (see [25, Section 13] for how to construct this reduced decomposition starting from an orientation of the Dynkin diagram of $\mathfrak{g}$) and $\kappa_\alpha \in \mathbb{Z}$ are not presently important to us (see [25, Section 3]). Very interestingly, the behavior and commutation relations of the e_α's do not depend on the total order on Δ^+ induced by the reduced decomposition, but rather only on the Auslander-Reiten partial order, which we will now recall.

Definition 3.5. *The Auslander-Reiten quiver associated to Q has the set Δ^+ of positive roots as vertices, and has an arrow $\alpha \to \beta$ if*

- $\alpha \neq \beta$,
- $\langle \alpha, \beta\rangle > 0$,
- *if $\gamma \in \Delta^+$ satisfies $\langle \alpha, \gamma\rangle > 0$ and $\langle \gamma, \beta\rangle > 0$, then $\gamma \in \{\alpha, \beta\}$.*

The Auslander-Reiten (AR for short) partial order on Δ^+ is defined by the property that $\alpha > \beta$ if and only if there is a path from α to β in the Auslander-Reiten quiver.

The AR partial order has the following properties for all positive roots $\alpha \neq \beta$:

- If α and β are incomparable, then $\langle \alpha, \beta\rangle = 0$,

- If $\alpha < \beta$ then $\langle \beta, \alpha \rangle \geq 0 \geq \langle \alpha, \beta \rangle$,
- If $\langle \alpha, \beta \rangle < 0$ or $\langle \beta, \alpha \rangle > 0$, then $\alpha < \beta$.

The first and the second properties follow from the third one, which the interested reader may find in [25, Section 4]. Moreover, the AR partial order is convex in the sense of (1.4) (this claim is implicit in [25, Section 13], who shows that any refinement of the AR partial order to a total order corresponds to a reduced decomposition of the longest word in the Weyl group; such total orders are known to be convex).

Lemma 3.6. *If $\alpha < \beta$ is a minimal pair adding up to a positive root $\alpha + \beta$, then*

$$\langle \alpha, \beta \rangle = -1 \quad \textit{and} \quad \langle \beta, \alpha \rangle = 0 \tag{3.6}$$

Proof. Let us assume that the positive roots α, β and $\alpha + \beta$ correspond to indecomposable quiver representations V, W and X, respectively. Then, combining $\langle \alpha, \beta \rangle \leq 0 \leq \langle \beta, \alpha \rangle$ with $-1 = (\alpha, \beta) = \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle$ and (3.5), we get:

$$\dim \mathrm{Ext}^1(V, W) = -\langle \alpha, \beta \rangle = t$$

and

$$\dim \mathrm{Hom}(W, V) = \langle \beta, \alpha \rangle = t - 1$$

for some $t \geq 1$. We will assume for the purpose of contradiction that $t \geq 2$, which implies that the space of extensions $\mathrm{Ext}^1(V, W)$ is at least 2-dimensional. For any non-trivial extension

$$0 \to W \to S_1 \oplus \cdots \oplus S_k \to V \to 0$$

(for various indecomposables $S_1, \ldots, S_k$) the fact that V and W are indecomposable implies that $\mathrm{Hom}(W, S_a)$ and $\mathrm{Hom}(S_a, V)$ are non-zero for all $1 \leq a \leq k$. Therefore, the dimension vectors γ_a of the S_a are contained between α and β, due to the assumption that the extension is non-trivial as well as the vanishing of Ext^1 in (3.5) and the third property of the AR partial order above. The following result will be proved at the end of the present proof.

Claim 3.7. If $\alpha < \beta$ is a minimal pair adding up to a positive root $\alpha + \beta$, there do not exist positive roots $\alpha < \gamma_1, \ldots, \gamma_k < \beta$ with $k > 1$ which add up to $\alpha + \beta$.

As a consequence of Claim 3.7, we conclude that all non-zero elements in $\mathrm{Ext}^1(V, W)$ are of the form

$$0 \to W \xrightarrow{f} X \xrightarrow{g} V \to 0 \tag{3.7}$$

Since we assumed that the space of extensions in question is at least 2-dimensional, let us consider another extension

$$0 \to W \xrightarrow{f'} X \xrightarrow{g'} V \to 0 \tag{3.8}$$

which is linearly independent from (3.7). The Baer sum of these two extensions

$$0 \to W \xrightarrow{f''} Q \xrightarrow{g''} V \to 0$$

which is defined by setting $Q = \mathrm{Ker}(g, g')/\mathrm{Im}(f, f')$ with

$$W \xrightarrow{(f,f')} X \oplus X \xrightarrow{(g,g')} V$$

is nonzero. By Claim 3.7 and the sentence immediately following it, we therefore have $Q \simeq X$. Consider the quiver representation $\mathrm{Ker}(g,g') = T_1 \oplus \cdots \oplus T_l$ (for various indecomposables T_a's) which has dimension $\alpha + 2\beta$. Invoking (3.5) and the third property of the AR partial order stated after Definition 3.5, we conclude that $\dim X \leq \dim T_a \leq \dim W$ for all $1 \leq a \leq l$, because $\mathrm{Hom}(T_a, X) \neq 0$ and either $\mathrm{Hom}(W, T_a) \neq 0$ or T_a is an indecomposable summand in $Q \simeq X$ so that $T_a \simeq X$. We can thus apply the following analogue of Claim 3.7, which will be proved several paragraphs down.

Claim 3.8. If $\alpha < \beta$ is a minimal pair adding up to a positive root $\alpha+\beta$, there do not exist positive roots $\alpha+\beta < \gamma_1, \dots, \gamma_k < \beta$ (respectively $\alpha < \gamma_1, \dots, \gamma_k < \alpha+\beta$) which add up to $\alpha + 2\beta$ (respectively $2\alpha + \beta$).

Therefore and with the convexity in mind, we have $\mathrm{Ker}(g,g') \simeq W \oplus X$, so we must have a short exact sequence

$$0 \to W \oplus X \to X \oplus X \xrightarrow{(g,g')} V \to 0$$

Since the only endomorphisms of X are scalars (this is true for all indecomposable representations, due to (3.5) and the equality $\langle \gamma, \gamma \rangle = \frac{1}{2}(\gamma,\gamma) = 1$ for any positive root γ), we conclude that g and g' are scalar multiples of each other. Similarly, one proves that f and f' are scalar multiples of each other. But this contradicts the fact that the short exact sequences (3.7) and (3.8) are linearly independent.

Let us now prove Claim 3.7. Assume for the purpose of contradiction that such $\gamma_1, \dots, \gamma_k$ existed, and choose a minimal $k \geq 2$ with said property. If $k = 2$, then this violates the minimality of the pair $\alpha < \beta$, hence $k \geq 3$. However, the fact that

$$2 = (\alpha+\beta, \alpha+\beta) = (\gamma_1 + \cdots + \gamma_k, \gamma_1 + \cdots + \gamma_k) = 2k + 2\sum_{a<b} (\gamma_a, \gamma_b)$$

implies that there must exist $a < b$ with $(\gamma_a, \gamma_b) = -1$. Therefore, $\gamma_a + \gamma_b$ is a positive root, which contradicts the minimality of k.

Let us now prove Claim 3.8. Assume for the purpose of contradiction that there exist positive roots

$$\alpha + \beta < \gamma_1, \dots, \gamma_k < \beta$$

which add up to $\alpha + 2\beta$ (the non-existence of $\alpha < \gamma_1, \dots, \gamma_k < \alpha + \beta$ which add up to $2\alpha + \beta$ is analogous, and is left as an exercise to the reader). Let us assume that k is minimal with this property. Since α, β and $\alpha + \beta$ are positive roots, $\alpha + 2\beta$ is not a positive root (as we cannot have all four of them having the same length), hence $k \geq 2$. If $k = 2$, then we have

$$(\gamma_1 + \gamma_2, \beta) = (\alpha + 2\beta, \beta) = 3$$

which is impossible because $(\gamma, \beta) \leq 1$ for every positive root $\gamma \neq \beta$. Therefore, $k \geq 3$ and we have

$$6 = (\alpha + 2\beta, \alpha + 2\beta) = (\gamma_1 + \cdots + \gamma_k, \gamma_1 + \cdots + \gamma_k) = 2k + 2\sum_{a<b} (\gamma_a, \gamma_b)$$

If $k > 3$, then there would exist $a < b$ with $(\gamma_a, \gamma_b) = -1$, hence $\gamma_a + \gamma_b$ is a positive root, which contradicts the minimality of k. Therefore, we must have $k = 3$. However,

$$(\gamma_1 + \gamma_2 + \gamma_3, \beta) = (\alpha + 2\beta, \beta) = 3$$

and

$$(\gamma_1 + \gamma_2 + \gamma_3, \alpha + \beta) = (\alpha + 2\beta, \alpha + \beta) = 3$$

implies that $(\gamma_a, \beta) = (\gamma_a, \alpha + \beta) = 1$ for all $a \in \{1, 2, 3\}$. Therefore, for every $a \in \{1, 2, 3\}$, each of $\beta - \gamma_a$ and $\alpha + \beta - \gamma_a$ is a (positive or negative) root. Let $\mathrm{ht}(\gamma) \in \mathbb{Z}_{>0}$ denote the height of a positive root $\gamma \in \Delta^+$. Because $\gamma_1 + \gamma_2 + \gamma_3 = \alpha + 2\beta$, we cannot have two or more of the γ_a's of height $\geq \mathrm{ht}(\alpha+\beta)$, and so either

- $\mathrm{ht}(\gamma_1), \mathrm{ht}(\gamma_2), \mathrm{ht}(\gamma_3) < \mathrm{ht}(\alpha+\beta)$. Thus, we have that $\alpha+\beta-\gamma_a = \delta_a$ is a positive root for all $a \in \{1, 2, 3\}$, and we must have $\delta_a < \alpha$ for all a due to the minimality of the pair $\alpha < \beta$ and the convexity. However, the fact that $\delta_1 + \delta_2 + \delta_3 = 2\alpha + \beta$ but $\delta_1, \delta_2, \delta_3 < \alpha < \alpha + \beta$ yields a contradiction.[4]
- Exactly one of the γ_a's has a height larger than $\mathrm{ht}(\alpha+\beta)$. Thus, up to relabeling, we may assume that $\alpha + \beta - \gamma_1 = \delta_1$, $\alpha + \beta - \gamma_2 = \delta_2$ but $\delta_3 = \gamma_3 - \alpha - \beta$ for positive roots $\delta_1, \delta_2, \delta_3$. As before, we must have

$$\delta_1, \delta_2 < \alpha < \alpha + \beta < \gamma_1, \gamma_2, \gamma_3 < \beta$$

  and $\delta_3 > \gamma_3$ by convexity. However, as explained before, $\varepsilon = \gamma_3 - \beta = \delta_3 + \alpha$ is also a positive root. Furthermore, we have $\varepsilon > \alpha$ and $\varepsilon < \gamma_3 < \beta$ by convexity. But then the equality

$$\gamma_1 + \gamma_2 + \varepsilon = \gamma_1 + \gamma_2 + \gamma_3 - \beta = \alpha + \beta$$

  yields a contradiction to Claim 3.7.

This completes our proof of Claim 3.8, and hence also of Lemma 3.6. □

3.9. **Specialization maps for the AR partial order.** Let $\tau\colon I \to \mathbb{Z}$ be any function with the property that $\tau(i) = \tau(j) + 1$ if there exists an arrow from i to j in the quiver Q. Such a map exists because finite type Dynkin diagrams do not have cycles, and it is unique up to a simultaneous translation. In Definition 2.4 (and the paragraph following it), we noticed that elements of the shuffle algebra are defined as a certain Laurent polynomial r divided by certain linear factors. More precisely, elements of the shuffle algebra $R \in \mathcal{S}_{\boldsymbol{v}}$ are of the form

$$R(\dots, z_{i1}, \dots, z_{iv_i}, \dots) = \frac{r(\dots, z_{i1}, \dots, z_{iv_i}, \dots)}{\prod_{\overrightarrow{ij}} \prod_{1 \leq b \leq v_i}^{1 \leq c \leq v_j} (z_{ib} - z_{jc})} \tag{3.9}$$

where r is a color-symmetric Laurent polynomial which satisfies the following three-variable wheel conditions whenever i and j are connected by an edge

$$r(\dots, z_{ib}, \dots)\Big|_{z_{i1} = qz_{j1}, z_{i2} = q^{-1}z_{j1}} = 0 \tag{3.10}$$

(indeed, (3.10) is just the particular case of (2.10) when $a_{ij} = -1$).

Definition 3.10. *For any $\boldsymbol{v} = (v_i)_{i \in I} \in \mathbb{N}^I$, define the specialization map*

$$\mathcal{S}_{\boldsymbol{v}} \xrightarrow{\mathrm{spec}^{(x)}_{\boldsymbol{v}}} \mathbb{Q}(q)[x, x^{-1}], \tag{3.11}$$

$$\mathrm{spec}^{(x)}_{\boldsymbol{v}}(R) = \gamma^{(x)}_{\boldsymbol{v}} \cdot r(\dots, z_{i1}, \dots, z_{iv_i}, \dots)\Big|_{z_{ib} \mapsto xq^{\tau(i)}, \forall i, b}$$

[4] A more general statement is true for convex orders $<$ on Δ^+: one cannot have $\alpha_1 < \dots < \alpha_k < \beta_1 < \dots < \beta_l$ such that $\alpha_1 + \dots + \alpha_k = \beta_1 + \dots + \beta_l$.

where r is the Laurent polynomial associated to $R \in \mathcal{S}_{\boldsymbol{v}}$ by formula (3.9) [5] *and*

$$\gamma^{(x)}_{\boldsymbol{v}} = q^{-\sum_{\overrightarrow{ij}} \tau(i) v_i v_j} \left[x q^{-\frac{1}{2}} (q - q^{-1}) \right]^{-\boldsymbol{v}\cdot\boldsymbol{v}} \tag{3.12}$$

Because of $\gamma^{(x)}_{\boldsymbol{v}}$, the map (3.11) actually takes values in $\mathbb{Q}(q^{\frac{1}{2}})[x, x^{-1}]$, but we will ignore this technicality, as it will produce no meaningful effects in the present paper.

3.11. **A key result.** In the present Subsection, we will prove some key results pertaining to the specialization maps (3.11). Let $\boldsymbol{v}\cdot\boldsymbol{w} = \sum_{i \in I} v_i w_i$ for all $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^I$.

Lemma 3.12. *For any $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^I$ and any $R \in \mathcal{S}_{\boldsymbol{v}+\boldsymbol{w}}$, we have*

$$\frac{\text{spec}^{(x)}_{\boldsymbol{v}} \otimes \text{spec}^{(y)}_{\boldsymbol{w}}(R)}{\prod_{i,j \in I} \zeta_{ij}\left(\frac{xq^{\tau(i)}}{yq^{\tau(j)}}\right)^{v_i w_j}} = \frac{\gamma^{(x)}_{\boldsymbol{v}} \gamma^{(y)}_{\boldsymbol{w}} \cdot r(\dots, xq^{\tau(i)}, \dots, yq^{\tau(j)}, \dots)}{(x-y)^{-\langle \boldsymbol{v}, \boldsymbol{w} \rangle} (yq^2 - x)^{\boldsymbol{v}\cdot\boldsymbol{w} - \langle \boldsymbol{w}, \boldsymbol{v} \rangle} (x - yq^{-2})^{\boldsymbol{v}\cdot\boldsymbol{w}} q^{\sum_{\overrightarrow{ij}} (\tau(i) v_i w_j + \tau(j) v_j w_i)}} \tag{3.13}$$

In the formula above, $\text{spec}^{(x)}_{\boldsymbol{v}} \otimes \text{spec}^{(y)}_{\boldsymbol{w}}(R)$ means that we split the variables of R into two sets, to which we separately apply the specialization maps $\text{spec}^{(x)}_{\boldsymbol{v}}$ and $\text{spec}^{(y)}_{\boldsymbol{w}}$.

Proof. By the definition of the specialization maps in (3.11), we have

$$\text{spec}^{(x)}_{\boldsymbol{v}} \otimes \text{spec}^{(y)}_{\boldsymbol{w}}(R) = \frac{\gamma^{(x)}_{\boldsymbol{v}} \gamma^{(y)}_{\boldsymbol{w}} \cdot r(\dots, xq^{\tau(i)}, \dots, yq^{\tau(j)}, \dots)}{\prod_{\overrightarrow{ij}} \left[(x - yq^{-1})^{v_i w_j} (yq - x)^{w_i v_j} q^{\tau(i) v_i w_j + \tau(j) v_j w_i} \right]}$$

Meanwhile, by the very definition of ζ_{ij} in (2.1), we have

$$\zeta_{ij}\left(\frac{xq^{\tau(i)}}{yq^{\tau(j)}}\right) = \begin{cases} \frac{x - yq^{-2}}{x-y} & \text{if } i = j \\ \frac{x-y}{x - yq^{-1}} & \text{if there is an arrow } \overrightarrow{ij} \\ \frac{yq^2 - x}{yq - x} & \text{if there is an arrow } \overrightarrow{ji} \\ 1 & \text{otherwise} \end{cases}$$

Dividing the formulas above yields (3.13). □

Because of the wheel conditions (3.10), the numerator of the RHS of (3.13) is also divisible by a number of linear factors of the form $(x - yq^{\pm 2})$. In the following result, we count these linear factors in the case when $\boldsymbol{v}$ and $\boldsymbol{w}$ are positive roots.

Proposition 3.13. *If $\boldsymbol{v}$ and $\boldsymbol{w}$ are positive roots, then for any $R \in \mathcal{S}_{\boldsymbol{v}+\boldsymbol{w}}$, the rational function (3.13) has a pole at $x = yq^2$ of order*

$$\leq \max\left(0, -\langle \boldsymbol{w}, \boldsymbol{v} \rangle\right)$$

and a pole at $x = yq^{-2}$ of order

$$\leq \max\left(0, \langle \boldsymbol{v}, \boldsymbol{w} \rangle\right)$$

If $\boldsymbol{v} < \boldsymbol{w}$, then the inequalities $\langle \boldsymbol{v}, \boldsymbol{w} \rangle \leq 0 \leq \langle \boldsymbol{w}, \boldsymbol{v} \rangle$ imply that the maxima above are both 0, hence, the rational function (3.13) does not have poles at $x = yq^{\pm 2}$.

---

[5] Compared with our general expectation in (2.43), the specialization map (3.11) sets all the variables z_{ib} for a given $i \in I$ to one and the same power of q (times x).

Proof. As a consequence of Theorem 2.6, any element of the shuffle algebra $\mathcal{S}_{\boldsymbol{v}+\boldsymbol{w}}$ is a linear combination of expressions of the form

$$R = z_{i_1 1}^{d_1} * \ldots * z_{i_k 1}^{d_k} \tag{3.14}$$

for any sequences $i_1, \ldots, i_k \in I$, $d_1, \ldots, d_k \in \mathbb{Z}$ such that $\boldsymbol{\varsigma}^{i_1} + \cdots + \boldsymbol{\varsigma}^{i_k} = \boldsymbol{v} + \boldsymbol{w}$. It therefore suffices to prove the required claims for R as in (3.14). In calculating the LHS of (3.13), the variables of R will be specialized to the multiset

$$M = \Big\{ \ldots, \underbrace{xq^{\tau(i)}, \ldots, xq^{\tau(i)}}_{v_i \text{ occurrences}}, \ldots, \underbrace{yq^{\tau(j)}, \ldots, yq^{\tau(j)}}_{w_j \text{ occurrences}}, \ldots \Big\} \tag{3.15}$$

By the very definition of the shuffle product, for R as in (3.14) we have

$$\text{LHS of (3.13)} = \sum_{\text{certain total orders } \succ \text{ on } M} \text{monomial} \prod_{xq^{\tau(i)} \succ yq^{\tau(j)}} \frac{\zeta_{ji}\left(\frac{yq^{\tau(j)}}{xq^{\tau(i)}}\right)}{\zeta_{ij}\left(\frac{xq^{\tau(i)}}{yq^{\tau(j)}}\right)} \tag{3.16}$$

Since

$$\frac{\zeta_{ji}\left(\frac{yq^{\tau(j)}}{xq^{\tau(i)}}\right)}{\zeta_{ij}\left(\frac{xq^{\tau(i)}}{yq^{\tau(j)}}\right)} = \begin{cases} \frac{x-yq^2}{xq^2-y} & \text{if } i = j \\ \frac{xq-yq^{-1}}{x-y} & \text{if there is an arrow } \overrightarrow{ij} \\ \frac{x-y}{xq^{-1}-yq} & \text{if there is an arrow } \overrightarrow{ji} \\ 1 & \text{otherwise} \end{cases} \tag{3.17}$$

the orders of the poles at $x = yq^2$ and $x = yq^{-2}$ in any given summand of (3.16) are

$$A = -\sum_{i \in I} \#\Big\{ \big(xq^{\tau(i)} \succ yq^{\tau(i)}\big) \Big\} + \sum_{\overrightarrow{ji}} \#\Big\{ \big(xq^{\tau(i)} \succ yq^{\tau(j)}\big) \Big\}$$

$$B = \sum_{i \in I} \#\Big\{ \big(xq^{\tau(i)} \succ yq^{\tau(i)}\big) \Big\} - \sum_{\overrightarrow{ij}} \#\Big\{ \big(xq^{\tau(i)} \succ yq^{\tau(j)}\big) \Big\}$$

respectively. It suffices to consider only those total orders $\succ$ on M of (3.15) for which $xq^{\tau(i)} \succ xq^{\tau(j)}$ and $yq^{\tau(i)} \succ yq^{\tau(j)}$ for all arrows $\overrightarrow{ij}$ (indeed, if one of such inequalities failed, then the corresponding summand in the RHS of (3.16) would include the factor $\zeta_{ij}(q) = 0$ for adjacent i, j). We will refer to such total orders as "allowable". Therefore, it just remains to show that

$$A \leq \max\big(0, -\langle \boldsymbol{w}, \boldsymbol{v} \rangle\big) \quad \text{and} \quad B \leq \max\big(0, \langle \boldsymbol{v}, \boldsymbol{w} \rangle\big) \tag{3.18}$$

for any allowable total order $\succ$ on M. We will only prove the inequality involving A, since it implies the inequality involving B due to the identity

$$B = \langle \boldsymbol{v}, \boldsymbol{w} \rangle + (A \text{ for the opposite quiver and the opposite order})$$

Fix an allowable total order $\succ$ on M, and we will call a variable $xq^{\tau(i)}$ (respectively $yq^{\tau(j)}$) "distinguished" if it is greater (respectively smaller) than any variable $yq^{\tau(i)}$ (respectively $xq^{\tau(j)}$) with respect to the total order $\succ$. The total number of distinguished variables $xq^{\tau(i)}$ (respectively $yq^{\tau(j)}$) will be encoded by the degree vectors $\boldsymbol{v'} \in [\boldsymbol{0}, \boldsymbol{v}]$ (respectively $\boldsymbol{w'} \in [\boldsymbol{0}, \boldsymbol{w}]$), meaning $0 \leq v'_i \leq v_i$ (respectively $0 \leq w'_i \leq w_i$) for any $i \in I$. As the order $\succ$ is allowable, if we have two variables

$$xq^{\tau(i)} \succ yq^{\tau(j)}$$

for some arrow $\overrightarrow{ji}$, then both variables $xq^{\tau(i)}$ and $yq^{\tau(j)}$ must be distinguished. We therefore conclude that

$$A \leq -\sum_{i\in I}(v'_i w_i + v_i w'_i - v'_i w'_i) + \sum_{\overrightarrow{ji}} v'_i w'_j$$

with the term $-(v'_i w_i + v_i w'_i - v'_i w'_i)$ counting those pairs of variables $xq^{\tau(i)} \succ yq^{\tau(i)}$ where at least one of the variables is distinguished. Therefore, (3.18) follows from the following combinatorial result.

Claim 3.14. If $\boldsymbol{v}, \boldsymbol{w}$ are positive roots, and $\boldsymbol{v}' \in [\mathbf{0}, \boldsymbol{v}], \boldsymbol{w}' \in [\mathbf{0}, \boldsymbol{w}]$ are arbitrary, then

$$-\sum_{i\in I}(v'_i w_i + v_i w'_i - v'_i w'_i) + \sum_{\overrightarrow{ji}} v'_i w'_j \leq \max\big(0, -\langle \boldsymbol{w}, \boldsymbol{v}\rangle\big) \tag{3.19}$$

It remains to prove the Claim above, and we will prove it in the case when $\boldsymbol{v}'$ and $\boldsymbol{w}'$ are allowed to have any real coordinates in the boxes $[\mathbf{0}, \boldsymbol{v}]$ and $[\mathbf{0}, \boldsymbol{w}]$, respectively. Because the aforementioned boxes are compact, the LHS of (3.19) must reach its maximum at a point in the box $[\mathbf{0}, \boldsymbol{v}] \times [\mathbf{0}, \boldsymbol{w}]$. However, because the LHS of (3.19) is linear in each coordinate v'_i, w'_i (and a linear function on an interval is maximized at one of the endpoints of the interval), it remains to prove (3.19) when each variable v'_i, w'_i is equal to either 0 or v_i, w_i, respectively. Thus, for any decompositions

$$I = I' \sqcup I'' = \tilde{I}' \sqcup \tilde{I}''$$

it remains to prove (3.19) when $v'_i = v_i$ for $i \in I'$, $v'_i = 0$ for $i \in I''$, $w'_i = w_i$ for $i \in \tilde{I}'$, $w'_i = 0$ for $i \in \tilde{I}''$. Thus, we need to prove the following inequality

$$-\sum_{i\in I\backslash(I''\cap\tilde{I}'')} v_i w_i \; + \sum_{\overrightarrow{ji},\, i\in I',\, j\in \tilde{I}'} v_i w_j \;\leq\; \max\big(0, -\langle \boldsymbol{w}, \boldsymbol{v}\rangle\big)$$

Clearly, the left-hand side of the equation above is maximized when $\tilde{I}' = I'$, $\tilde{I}'' = I''$. Therefore, it remains to prove that

$$-\langle \boldsymbol{w}_{I'}, \boldsymbol{v}_{I'}\rangle_{I'} \leq \max\big(0, -\langle \boldsymbol{w}, \boldsymbol{v}\rangle\big)$$

where $\boldsymbol{v}_{I'}, \boldsymbol{w}_{I'}$ denote the projections of the vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^I$ onto the coordinates indexed by I', while $\langle\cdot,\cdot\rangle_{I'}$ denotes the restriction of (3.1) to $\mathbb{Z}^{I'} \times \mathbb{Z}^{I'} \subset \mathbb{Z}^I \times \mathbb{Z}^I$.

From the point of view of quiver representations, the inequality above reads

$$\dim \mathrm{Ext}^1(W_{I'}, V_{I'}) - \dim \mathrm{Hom}(W_{I'}, V_{I'}) \leq \dim \mathrm{Ext}^1(W, V) \tag{3.20}$$

for any indecomposable representations V, W of the quiver Q, where $V_{I'}, W_{I'}$ denote their restrictions to the full subquiver corresponding to the vertex set $I' \subset I$ (here we used (3.5) again). In fact, inequality (3.20) follows from the stronger inequality

$$\dim \mathrm{Ext}^1(W_{I'}, V_{I'}) \leq \dim \mathrm{Ext}^1(W, V) \tag{3.21}$$

that holds for all finite-dimensional Q-representations V and W. In turn, inequality (3.21) follows immediately from the claim below.

Claim 3.15. For any finite-dimensional representations V, W of the quiver Q, let $V_{I'}, W_{I'}$ denote their restrictions to the full subquiver Q' corresponding to the vertex set $I' \subset I$. Then, the natural restriction map

$$\mathrm{Ext}^1(W,V) \twoheadrightarrow \mathrm{Ext}^1(W_{I'},V_{I'}) \tag{3.22}$$

is surjective.

It thus remains to prove Claim 3.15. Any element of $\mathrm{Ext}^1(W_{I'},V_{I'})$ can be represented by a collection of short exact sequences of vector spaces

$$\Big\{0 \to V_i \to X_i \to W_i \to 0\Big\}_{i \in I'} \tag{3.23}$$

which are compatible with the arrow maps of the subquiver Q' with the vertex set I'. Let us choose $\mathbb{F}_{q^2}$ vector space splittings $X_i \simeq V_i \oplus W_i$ for all $i \in I'$, with respect to which the short exact sequences above are

$$0 \to V_i \xrightarrow{(\mathrm{Id},0)} V_i \oplus W_i \xrightarrow{(0,\mathrm{Id})} W_i \to 0, \qquad \forall\, i \in I'$$

although we do not claim that the arrow maps in Q' respect these decompositions. To show that our given extension lies in the image of the map (3.22), we must extend it to an extension of the Q-representations V and W. To this end, we consider the split short exact sequences

$$0 \to V_i \xrightarrow{(\mathrm{Id},0)} V_i \oplus W_i \xrightarrow{(0,\mathrm{Id})} W_i \to 0, \qquad \forall\, i \in I \setminus I'$$

and extend $\{V_i \oplus W_i\}_{i \in I}$ to a Q-representation by defining the arrow maps to be

$$V_i \oplus W_i \xrightarrow{(\phi_{\overrightarrow{ij}},\psi_{\overrightarrow{ij}})} V_j \oplus W_j$$

whenever either i or j lie in $I \backslash I'$ (above, $\phi_{\overrightarrow{ij}} \colon V_i \to V_j$ and $\psi_{\overrightarrow{ij}} \colon W_i \to W_j$ denote the arrow maps in the quiver representations V and W, respectively, see (3.2)), while using the arrow maps $X_i \to X_j$ whenever $i, j \in I'$. The resulting Q-representation yields an element of $\mathrm{Ext}^1(W,V)$ which lifts our choice of extension (3.23). $\square$

3.16. **Fused currents for the AR partial order.** In the present Subsection, we invoke Proposition 2.33 to define for any $(\alpha, d) \in \Delta^+ \times \mathbb{Z}$

$$f_{\alpha,d} \in \widehat{U}_q(L\mathfrak{n}^-)_{-\alpha,d}$$

by the formula (we let $f_\alpha(x) = \sum_{d \in \mathbb{Z}} \frac{f_{\alpha,d}}{x^d}$ and call them the "fused currents")

$$\Big\langle R, f_\alpha(x) \Big\rangle = \mathrm{spec}^{(x)}_\alpha(R), \qquad \forall\, R \in \mathcal{S}_\alpha$$

Let us recall the fused currents $\widetilde{f}_\alpha(x)$ of [5] (see Subsections 2.28 and 2.34) in the particular case of a type ADE Dynkin diagram, defined with respect to any total convex order of the positive roots that refines the Auslander-Reiten partial order.

Conjecture 3.17. *For any orientation Q of a type ADE Dynkin diagram and any total convex order of the positive roots that refines the AR partial order, we have*

$$\widetilde{f}_\alpha(x) = c^{(x)}_\alpha \cdot f_\alpha(x) \quad \textit{or equivalently} \quad \widetilde{\mathrm{spec}}_\alpha(x) = c^{(x)}_\alpha \cdot \mathrm{spec}_\alpha(x)$$

for all $\alpha \in \Delta^+$, where $c^{(x)}_\alpha \in \mathbb{Q}(q,x)^\times$ is some scalar prefactor.

To motivate Conjecture 3.17, let us carry out the program from Subsection 2.34 for the objects $f_\alpha(x)$ and $\text{spec}^{(x)}_\alpha$ defined in the present Section.

Proof of Theorem 1.6, specifically equation (1.9). Combining together Lemma 3.6 and Proposition 3.13 implies that if $\alpha < \beta$ is a minimal pair such that $\alpha + \beta \in \Delta^+$, then

$$\frac{\text{spec}^{(x)}_\alpha \otimes \text{spec}^{(y)}_\beta (R)}{\prod_{i,j \in I} \zeta_{ij} \left(\frac{xq^{\tau(i)}}{yq^{\tau(j)}}\right)^{\alpha_i \beta_j}} = \frac{\text{Laurent polynomial}}{x-y} \tag{3.24}$$

for any $R \in \mathcal{S}_{\alpha+\beta}$. Meanwhile, the rational function (2.45) is easily computed to be

$$\prod_{i,j \in I} \left[\frac{\zeta_{ji}\left(\frac{yq^{\tau(j)}}{xq^{\tau(i)}}\right)}{\zeta_{ij}\left(\frac{xq^{\tau(i)}}{yq^{\tau(j)}}\right)} \right]^{\alpha_i\beta_j} \stackrel{(3.17)}{=} \left(\frac{x-y}{xq-yq^{-1}}\right)^{\langle \alpha,\beta\rangle} \left(\frac{xq^{-1}-yq}{x-y}\right)^{\langle \beta,\alpha\rangle} \stackrel{(3.6)}{=} \frac{xq-yq^{-1}}{x-y}$$

Thus, the difference of (2.44) and (2.46), specifically

$$\Big\langle R, f_\alpha(x) f_\beta(y) \Big\rangle \Big|_{|x| \ll |y|} - \Big\langle R, f_\beta(y) f_\alpha(x) \frac{xq-yq^{-1}}{x-y} \Big\rangle \Big|_{|x| \gg |y|}$$

is just the difference between the expansions at $|x| \ll |y|$ and $|x| \gg |y|$ of the rational function (3.24). Explicitly, the rational function in question is (3.13) for $\boldsymbol{v} = \alpha$ and $\boldsymbol{w} = \beta$. The aforementioned rational function has a single simple pole at $x = y$ with residue given by

$$\frac{\gamma^{(x)}_\alpha \gamma^{(x)}_\beta \cdot r(\dots, xq^{\tau(i)}, \dots)}{(xq^2 - x)^{\alpha\cdot\beta - \langle \beta,\alpha\rangle}(x - xq^{-2})^{\alpha\cdot\beta} q^{\sum_{\overrightarrow{ij}}(\tau(i)\alpha_i\beta_j + \tau(j)\alpha_j\beta_i)}} = \gamma^{(x)}_{\alpha+\beta} \cdot r(\dots, xq^{\tau(i)}, \dots)$$

(the latter equality is the reason for the specific formula of $\gamma^{(x)}_{\boldsymbol{v}}$ in (3.12); it is quite elementary, based on $\langle \beta, \alpha \rangle = 0$ of Lemma 3.6 and $\tau(i) - \tau(j) = 1$ for any arrow $\overrightarrow{ij}$ in Q, so we leave its proof as an exercise to the reader). Once we observe that the residue above is precisely $\text{spec}^{(x)}_{\alpha+\beta}(R)$, we conclude that

$$f_\alpha(x) f_\beta(y) - f_\beta(y) f_\alpha(x) \frac{xq-yq^{-1}}{x-y} = \delta\left(\frac{x}{y}\right) f_{\alpha+\beta}(x) \tag{3.25}$$

□

We remark that formula (3.25) allows one to inductively define the fused currents corresponding to the AR partial order from $f_{\alpha_i}(x) = f_i(x)$ for simple roots $\{\alpha_i\}_{i \in I}$.

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