AFFINE STANDARD LYNDON WORDS

CORBET ELKINS AND ALEXANDER TSYMBALIUK

Abstract. In this note, we establish the convexity and monotonicity for affine standard Lyndon words in all types, generalizing the A-type results of [AT]. We also derive partial results on the structure of imaginary standard Lyndon words and present a conjecture for their general form. Additionally, we provide computer code in Appendix which, in particular, allows to efficiently compute affine standard Lyndon words in exceptional types for all orders.

1. Introduction

1.1. **Summary.**

The free Lie algebras generated by a finite set $\{e_i\}_{i\in I}$ are known to have bases parametrized by Lyndon words (see Definition 2.2, Theorem 2.12) for each order on I. This was generalized to finitely generated Lie algebras $\mathfrak{a}$ in [LR]. Explicitly, if $\mathfrak{a}$ is generated by $\{e_i\}_{i\in I}$, then any order on I gives rise to the combinatorial basis parametrized by standard Lyndon words (see Definition 2.14, Theorem 2.16).

The key application of [LR] was to simple finite-dimensional $\mathfrak{g}$, or more precisely, to their maximal nilpotent subalgebras $\mathfrak{n}^+$. Evoking the root space decomposition:

$$\mathfrak{n}^+ = \bigoplus_{\alpha\in\Delta^+} \mathbb{C}\cdot e_\alpha\,, \qquad \Delta^+ = \big\{\text{positive roots}\big\},$$

it can be easily shown that there is a natural Lalonde-Ram bijection

$$\ell\colon \Delta^+ \xrightarrow{\ \sim\ } \big\{\text{standard Lyndon words}\big\}. \tag{1.1}$$

In this context, the bracketing $\mathsf{b}[\ell]$ (Definition 2.11) is on par with the general rule

$$[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha \in \mathbb{C}^\times \cdot e_{\alpha+\beta} \qquad \forall\, \alpha,\beta,\alpha+\beta\in\Delta^+.$$

A decade later, [Le] established an iterative Leclerc algorithm for (1.1). Moreover, the induced total order on Δ^+ is convex (see Proposition 2.21) by earlier work [R]. This played the key role in [Le], where it was shown that natural q-deformations of $\mathsf{b}[\ell]$ give rise to a basis of the corresponding positive half $U_q(\mathfrak{n}^+)$ of Drinfeld-Jimbo quantum group, recovering Lusztig's root generators, up to scalars.

In the recent work of Avdieiev and the second author [AT], the generalization to (untwisted) affine Lie algebras was initiated. Let $\widehat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$, whose Dynkin diagram is obtained by extending the Dynkin diagram of $\mathfrak{g}$ with a vertex 0. Thus, on the combinatorial side, we consider the alphabet $\widehat{I} = I\sqcup\{0\}$. The corresponding positive subalgebra $\widehat{\mathfrak{n}}^+\subset\widehat{\mathfrak{g}}$ still admits the root space decomposition $\widehat{\mathfrak{n}}^+ = \bigoplus_{\alpha\in\widehat{\Delta}^+}\widehat{\mathfrak{n}}^+_\alpha$ with $\widehat{\Delta}^+ = \{\text{positive affine roots}\}$. The key difference is that:

$$\dim\widehat{\mathfrak{n}}^+_\alpha = 1 \quad \forall\,\alpha\in\widehat{\Delta}^{+,\mathrm{re}}\,, \qquad \dim\widehat{\mathfrak{n}}^+_\alpha = |I| \quad \forall\,\alpha\in\widehat{\Delta}^{+,\mathrm{im}}.$$

Here, $\widehat{\Delta} = \widehat{\Delta}^{+,\mathrm{re}}\sqcup\widehat{\Delta}^{+,\mathrm{im}}$ is the decomposition into real and imaginary affine roots, with $\widehat{\Delta}^{+,\mathrm{im}} = \{k\delta|k\geq 1\}$. It is therefore natural to consider an extended set $\widehat{\Delta}^{+,\mathrm{ext}}$

of (4.2). Then, the degree reasoning as in [LR] provides a natural analogue of (1.1):

$$\mathrm{SL}\colon \widehat{\Delta}^{+,\mathrm{ext}} \xrightarrow{\sim} \big\{\text{affine standard Lyndon words}\big\}. \tag{1.2}$$

In [AT], we established a generalized Leclerc algorithm describing this bijection. As the key application, we then used it to inductively derive formulas for all affine standard Lyndon words in type A with any order on $\widehat{I}$.

The above explicit formulas in affine type A illustrated a stunning periodicity for affine standard Lyndon words, expressing all of them through $\mathrm{SL}(\alpha)$ with $|\alpha| < |\delta|$. Using the explicit formulas, we also established the pre-convexity:

$$\alpha < \alpha+\beta < \beta \quad \text{or} \quad \beta < \alpha+\beta < \alpha \qquad \forall\, \alpha,\beta,\alpha+\beta \in \widehat{\Delta}^{+,\mathrm{re}}, \tag{1.3}$$

as well as the monotonicity:

$$\alpha < \alpha+\delta < \alpha+2\delta < \cdots \quad \text{or} \quad \alpha > \alpha+\delta > \alpha+2\delta > \cdots \qquad \forall\, \alpha \in \widehat{\Delta}^{+,\mathrm{re}}. \tag{1.4}$$

The present note arose from an attempt to generalize the above results of [AT] to all types. In particular, our key results are the convexity (see Theorem 5.4 and Remark 5.8) and monotonicity (see Proposition 5.21), generalizing (1.3) and improving (1.4). We also propose a conjecture on the structure of all imaginary affine standard Lyndon words, proving it for all orders in any type with $0 \in \widehat{I}$ being the smallest letter (see Conjecture 6.26 and Theorem 6.14), thus rederiving A-type results. We note that out approach is completely opposite to that of [AT], as we establish convexity and monotonicity without having explicit formulas, and then use them to get information on the explicit form of affine standard Lyndon words.

1.2. **Outline.**

The structure of the present paper is the following:

- In Section 2, we recall classical results on Lyndon and standard Lyndon words, as well as the generalization to the case of affine root systems from [AT].
- In Section 3, we establish some basic properties of the standard, costandard, and other factorizations of standard Lyndon words, instrumental for this note.
- In Section 4, we investigate the behavior of the standard bracketing with respect to different splittings of words. The key results are Propositions 4.6 and 4.18, which also imply that the affine standard Lyndon words are the same whether using standard or costandard factorizations, see Remark 4.19. We also introduce the auxiliary sets $C(\alpha)$ for imaginary α and $O(\alpha)$ for real α, which are key for Section 5, and establish some basic properties of min/max elements of $O(\alpha)$.
- In Section 5, we prove the key results of this note. The convexity of Theorem 5.4 is stated using the above sets $C(\alpha)$ and $O(\alpha)$, see Definition 5.2, and generalizes the pre-convexity of (1.3), see Remark 5.8. Our second main result is Proposition 5.21 establishing (1.4) and specifying which monotonicity occurs.
- In Section 6, we explore the structure of imaginary affine standard Lyndon words and establish relations among those. We start by showing the compatibility of complete flags (4.1) in Proposition 6.2 and use it to deduce the monotonicity of Lemma 6.7. The main result, Conjecture 6.26, provides the structure of all $\mathrm{SL}_i(k\delta)$. In Theorem 6.14 we prove this result for the cases when the smallest simple root appears once in δ (which includes any order for any type with $0 \in \widehat{I}$ being the smallest element and recovers the result of [AT] for any order in affine type A), in Proposition 6.25 we prove it for $i = 1$ and any order, and finally

in Proposition 6.32 we reduce it to the $k = 2$ case. Using the latter result and computer code, we fully verified the Conjecture 6.26 for exceptional $\mathfrak{g}$.
- In Appendix A, we provide the computer code that we heavily used to find the correct patterns for affine standard Lyndon words as well as to verify Conjecture 6.26 for all orders on $\widehat{I}$ whenever $|I| \leq 8$.
- In Appendix B, we present the tables of all affine standard Lyndon words in affine type $G_2^{(1)}$ for all orders on $\widehat{I}$, which were computed using the above code.

1.3. Acknowledgement.

This note represents a part of the year-long REU project at Purdue University. We are grateful to Purdue University for support and for the opportunity to present these results at REU math conference in April 2025. C.E. is grateful for the Joel Spira Undergraduate Summer Research Award. A.T. is deeply indebted to Andrei Neguţ for numerous inspiring discussions over the years. A.T. is extremely grateful to IHES (Bures-sur-Yvette, France) for the hospitality and wonderful working conditions in the summer 2025, where the final version of this paper was prepared.

The work of both authors was partially supported by NSF Grant DMS-2302661.

2. Setup and Notations

In this section, we recall the classical results of [LR, Le] that provide a combinatorial construction of an important basis of finitely generated Lie algebras, with the main application to the maximal nilpotent subalgebras of simple Lie algebras. We further evoke the appropriate generalization of this to affine case, following [AT].

2.1. Lyndon words.

Let I be a finite ordered alphabet, and let I^* be the set of all finite length words in the alphabet I. For $u = [i_1 \dots i_k] \in I^*$, we define its length by $|u| = k$. Moreover, we consider the lexicographical order on I^* defined as follows:

$$[i_1 \dots i_k] < [j_1 \dots j_l] \quad \text{if} \begin{cases} i_1 = j_1, \dots, i_a = j_a, i_{a+1} < j_{a+1} \text{ for some } a \geq 0 \\ \text{ or} \\ i_1 = j_1, \dots, i_k = j_k \text{ and } k < l \end{cases} .$$

Definition 2.2. *A word $\ell = [i_1 \dots i_k]$ is called **Lyndon** if it is smaller than all of its cyclic permutations:*

$$[i_1 \dots i_{a-1} i_a \dots i_k] < [i_a \dots i_k i_1 \dots i_{a-1}] \qquad \forall\, a \in \{2, \dots, k\}.$$

For a word $w = [i_1 \dots i_k] \in I^*$, the subwords:

$$w_{a|} = [i_1 \dots i_a] \qquad \text{and} \qquad w_{|a} = [i_{k-a+1} \dots i_k]$$

with $0 \leq a \leq k$ are usually called a **prefix** and a **suffix** of w, respectively. We call such a prefix or a suffix proper if $0 < a < k$. It is well-known that Definition 2.2 is equivalent to the following one:

Definition 2.3. *A word w is Lyndon if it is smaller than all of its proper suffixes:*

$$w < w_{|a} \qquad \forall\, 0 < a < |w|.$$

As a corollary, we record the following basic property:

Lemma 2.4. *If $\ell_1 < \ell_2$ are Lyndon, then $\ell_1\ell_2$ is also Lyndon, and so $\ell_1\ell_2 < \ell_2\ell_1$.*

Let us now recall several basic facts from the theory of Lyndon words (cf. [Lo]).

Proposition 2.5. *Any Lyndon word ℓ with $|\ell| > 1$ has a factorization:*

$$\ell = \ell_1 \ell_2 \tag{2.1}$$

defined by the property that ℓ_2 is the longest proper suffix of ℓ which is also a Lyndon word. Then, ℓ_1 is also a Lyndon word.

Henceforth, we denote the longest proper Lyndon suffix of ℓ as ℓ^r and the remaining prefix as ℓ^l. The factorization $\ell = \ell^l \ell^r$ is called the **costandard factorization**.

Notation 2.6. *In the next section, we shall be using iterated superscripts of l, r that are chained left-to-right, e.g. $\ell^{lr} = (\ell^l)^r$, $\ell^{lrl} = ((\ell^l)^r)^l$, $\ell^{llrr} = (((\ell^l)^l)^r)^r$.*

We have an analogous factorization with the longest proper Lyndon prefix:

Proposition 2.7. *Any Lyndon word ℓ with $|\ell| > 1$ has a factorization:*

$$\ell = \ell_1 \ell_2 \tag{2.2}$$

defined by the property that ℓ_1 is the longest proper prefix of ℓ which is also a Lyndon word. Then, ℓ_2 is also a Lyndon word.

We shall denote such longest proper Lyndon prefix of ℓ by ℓ^{ls} and the remaining suffix by ℓ^{rs}. The factorization $\ell = \ell^{ls} \ell^{rs}$ is called the **standard factorization**.

Let L be the set of all Lyndon words. Any word can be canonically built from L:

Proposition 2.8. *Any word $w \in I^*$ has a unique factorization:*

$$w = \ell_1 \ldots \ell_k \quad \textit{with} \quad \ell_1 \geq \cdots \geq \ell_k \in \mathrm{L}. \tag{2.3}$$

The factorization (2.3) is called the **canonical factorization**. The following result is well-known (see [M]):

Lemma 2.9. *If ℓ is Lyndon and $w \in I^*$ has the canonical factorization (2.3), then*

$$\ell > w \Longleftrightarrow \ell > \ell_1.$$

Proof. The direction "$\Rightarrow$" is clear. Assume now that $\ell > \ell_1$. If ℓ_1 is not a prefix of ℓ, then $\ell > w$. If ℓ_1 is a prefix of ℓ, then $\ell = \ell_1 \ell^{(1)}$ with $\ell^{(1)} \neq \emptyset$. As ℓ is Lyndon, we get $\ell^{(1)} > \ell > \ell_1 \geq \ell_2$. This implies $\ell^{(1)} > \ell_2 \ldots \ell_k$ unless ℓ_2 is a prefix of $\ell^{(1)}$, i.e. $\ell^{(1)} = \ell_2 \ell^{(2)}$. We note that $\ell^{(2)} \neq \emptyset$ as $\ell = \ell_1 \ell_2$ would contradict the uniqueness of the canonical factorization. Repeating this argument k times we obtain $\ell > w$. $\square$

2.10. **Standard bracketing.**

Let $\mathfrak{a}$ be a Lie algebra generated by a finite set $\{e_i\}_{i \in I}$ labeled by the alphabet I.

Definition 2.11. *The **standard bracketing** of $\ell \in \mathrm{L}$ is given inductively by:*

- $\mathsf{b}[i] = e_i \in \mathfrak{a}$ *for* $i \in I$,
- $\mathsf{b}[\ell] = [\mathsf{b}[\ell^l], \mathsf{b}[\ell^r]] \in \mathfrak{a}$ *if* $|\ell| > 1$.

The major importance of this definition is due to the following result of Lyndon:

Theorem 2.12. *([Lo, Theorem 5.3.1]) If $\mathfrak{a}$ is a free Lie algebra in the generators $\{e_i\}_{i \in I}$, then the set $\{\mathsf{b}[\ell] \,|\, \ell \in \mathrm{L}\}$ provides a basis of $\mathfrak{a}$.*

2.13. **Standard Lyndon words.**
A generalization of Theorem 2.12 to Lie algebras $\mathfrak{a}$ generated by $\{e_i\}_{i\in I}$ was provided in [LR]. To state the result, define ${}_we \in U(\mathfrak{a})$ for any word $w \in I^*$:

$$ {}_{[i_1\dots i_k]}e = e_{i_1}\dots e_{i_k} \in U(\mathfrak{a}). \tag{2.4}$$

Consider the following new order on I^*:

$$ v \succeq w \quad \text{if} \begin{cases} |v|<|w| \\ \text{or} \\ |v|=|w| \text{ and } v\geq w \end{cases} . \tag{2.5}$$

The following definition is due to [LR]:

Definition 2.14. *(a) A word w is called **standard** if ${}_we \in U(\mathfrak{a})$ cannot be expressed as a linear combination of ${}_ve$ for various $v \succ w$, with ${}_we$ as in (2.4).*

*(b) A Lyndon word ℓ is called **standard Lyndon** if $\mathsf{b}[\ell] \in \mathfrak{a}$ cannot be expressed as a linear combination of $\mathsf{b}[v]$ for various Lyndon words $v \succ \ell$.*

The following result is nontrivial and justifies the above terminology:

Proposition 2.15. *([LR]) A Lyndon word is standard iff it is standard Lyndon.*

We shall use SL to denote the set of all standard Lyndon words. The major importance of this definition is due to the following result of Lalonde-Ram:

Theorem 2.16. *([LR, Theorem 2.1]) For any Lie algebra $\mathfrak{a}$ generated by a finite collection $\{e_i\}_{i\in I}$, the set $\big\{\mathsf{b}[\ell]\,|\,\ell\in \mathrm{SL}\big\}$ provides a basis of $\mathfrak{a}$.*

2.17. **Application to simple Lie algebras.**
Let $\mathfrak{g}$ be a simple Lie algebra with a root system $\Delta=\Delta^+\sqcup\Delta^-$ and simple roots $\{\alpha_i\}_{i\in I}$. We endow the root lattice $Q=\bigoplus_{i\in I}\mathbb{Z}\alpha_i$ with the symmetric pairing $(\cdot,\cdot)$ so that the Cartan matrix $(a_{ij})_{i,j\in I}$ of $\mathfrak{g}$ is given by $a_{ij}=2(\alpha_i,\alpha_j)/(\alpha_i,\alpha_i)$. The Lie algebra $\mathfrak{g}$ admits the standard **root space decomposition**:

$$ \mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha, \qquad \mathfrak{h}\subset\mathfrak{g} - \text{Cartan subalgebra}, \tag{2.6}$$

with $\dim(\mathfrak{g}_\alpha)=1$ for all $\alpha\in\Delta$. We pick root vectors $e_\alpha\in\mathfrak{g}_\alpha$ so that $\mathfrak{g}_\alpha=\mathbb{C}\cdot e_\alpha$.
Consider the *positive* Lie subalgebra $\mathfrak{n}^+=\bigoplus_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$ of $\mathfrak{g}$. Explicitly, $\mathfrak{n}^+$ is generated by $\{e_i\}_{i\in I}$ (where $e_i=e_{\alpha_i}$) subject to the classical Serre relations:

$$ \underbrace{[e_i,[e_i,\cdots,[e_i,e_j]\cdots]]}_{1-a_{ij}\text{ Lie brackets}}=0 \qquad \forall\ i\neq j. \tag{2.7}$$

Let $Q^+=\bigoplus_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$. The Lie algebra $\mathfrak{n}^+$ is naturally Q^+-graded via $\deg e_i=\alpha_i$.
Fix any order on the set I. According to Theorem 2.16, $\mathfrak{n}^+$ has a basis consisting of $\{e_\ell\,|\,\ell\in\mathrm{SL}\}$. Evoking the above Q^+-grading of $\mathfrak{n}^+$, it is natural to define the grading of words via $\deg[i_1\dots i_k]=\alpha_{i_1}+\cdots+\alpha_{i_k}\in Q^+$. Due to the decomposition (2.6) and the fact that the root vectors $\{e_\alpha\}_{\alpha\in\Delta^+}\subset\mathfrak{n}^+$ all live in distinct degrees $\alpha\in Q^+$, we conclude that there exists a **Lalonde-Ram bijection** [LR]:

$$ \ell\colon \Delta^+ \xrightarrow{\sim} \big\{\text{standard Lyndon words}\big\} \quad \text{with} \quad \deg\ell(\alpha)=\alpha. \tag{2.8}$$

For degree reasons, we also note that one can presently replace $v\succ w$ of (2.5) simply with $v>w$ subject to $\deg v=\deg w$ (so that $|v|=|w|$) in Definition 2.14.

2.18. **Results of Leclerc and Rosso.**

The Lalonde-Ram's bijection (2.8) was described explicitly in [Le]. We recall that for a root $\gamma=\sum_{i\in I} n_i\alpha_i \in \Delta^+$, its height is $|\gamma|=\mathrm{ht}(\gamma)=\sum_{i\in I} n_i \in \mathbb{Z}_{>0}$.

Proposition 2.19. *([Le, Proposition 25]) The bijection ℓ is inductively given by:*

- *for simple roots $\ell(\alpha_i)=[i]$,*
- *for other positive roots, we have the following **Leclerc algorithm**:*

$$\ell(\alpha)=\max\Big\{\ell(\gamma_1)\ell(\gamma_2)\,\Big|\,\alpha=\gamma_1+\gamma_2\,,\ \gamma_1,\gamma_2\in\Delta^+\,,\ \ell(\gamma_1)<\ell(\gamma_2)\Big\}. \tag{2.9}$$

The formula (2.9) recovers $\ell(\alpha)$ once we know $\ell(\gamma)$ for all $\{\gamma\in\Delta^+\,|\,\mathrm{ht}(\gamma)<\mathrm{ht}(\alpha)\}$.

Let us also recall another fundamental property of ℓ.

Definition 2.20. *A total order on the set Δ^+ of positive roots is called **convex** if:*

$$\alpha<\alpha+\beta<\beta \tag{2.10}$$

for all $\alpha<\beta\in\Delta^+$ such that $\alpha+\beta$ is also a root.

The following result is [Le, Proposition 26] (where it is attributed to Rosso [R]):

Proposition 2.21. *([Le, R]) Consider a total order $<$ on Δ^+ induced from the lexicographical order on standard Lyndon words:*

$$\alpha<\beta \quad\Longleftrightarrow\quad \ell(\alpha)<\ell(\beta)\ \text{ lexicographically}. \tag{2.11}$$

This order is convex.

2.22. **Affine Lie algebras.**

We now consider untwisted affine Kac-Moody algebras. Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra, $\{\alpha_i\}_{i\in I}$ be the simple roots, and $\theta\in\Delta^+$ be the highest root. We define $\widehat{I}=I\sqcup\{0\}$. Consider the affine root lattice $\widehat{Q}=Q\times\mathbb{Z}$ with the generators $\{(\alpha_i,0)\}_{i\in I}$ and $\alpha_0:=(-\theta,1)$. We endow $\widehat{Q}$ with the symmetric pairing

$$\big((\alpha,n),(\beta,m)\big)=(\alpha,\beta) \qquad \forall\ \alpha,\beta\in Q\,,\ n,m\in\mathbb{Z}.$$

This leads to the affine Cartan matrix $(a_{ij})_{i,j\in\widehat{I}}$ and the **affine Lie algebra** $\widehat{\mathfrak{g}}$. The associated affine root system $\widehat{\Delta}=\widehat{\Delta}^+\sqcup\widehat{\Delta}^-$ has the following explicit description:

$$\widehat{\Delta}^+=\big\{\Delta^+\times\mathbb{Z}_{\geq 0}\big\}\sqcup\big\{0\times\mathbb{Z}_{>0}\big\}\sqcup\big\{\Delta^-\times\mathbb{Z}_{>0}\big\}. \tag{2.12}$$

Here, $\delta=\alpha_0+\theta=(0,1)\in Q\times\mathbb{Z}$ is the **minimal imaginary root** of the affine root system $\widehat{\Delta}$. With this notation, we have the following root space decomposition:

$$\widehat{\mathfrak{g}}=\widehat{\mathfrak{h}}\oplus\bigoplus_{\alpha\in\widehat{\Delta}}\widehat{\mathfrak{g}}_\alpha, \qquad \widehat{\mathfrak{h}}\subset\widehat{\mathfrak{g}}-\text{Cartan subalgebra}. \tag{2.13}$$

Let us now recall another realization of $\widehat{\mathfrak{g}}$. To this end, consider the Lie algebra

$$\begin{gathered}\widetilde{\mathfrak{g}}=\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]\oplus\mathbb{C}\cdot\mathsf{c} \quad \text{with a Lie bracket given by}\\ [x\otimes t^n,y\otimes t^m]=[x,y]\otimes t^{n+m}+n\delta_{n,-m}(x,y)\cdot\mathsf{c} \quad\text{and}\quad [\mathsf{c},x\otimes t^n]=0,\end{gathered} \tag{2.14}$$

where $x,y\in\mathfrak{g}$, $m,n\in\mathbb{Z}$, and $(\cdot,\cdot)\colon\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ is a non-degenerate invariant pairing.

The rich theory of affine Lie algebras is mainly based on the following key result:

Claim 2.23. There exists a Lie algebra isomorphism:

$$\widehat{\mathfrak{g}} \xrightarrow{\sim} \widetilde{\mathfrak{g}}$$

determined on the generators by the following formulas:

$$e_i \mapsto e_i \otimes t^0 \qquad f_i \mapsto f_i \otimes t^0 \qquad h_i \mapsto h_i \otimes t^0 \qquad \forall\, i \in I$$
$$e_0 \mapsto e_{-\theta} \otimes t^1 \qquad f_0 \mapsto e_\theta \otimes t^{-1} \qquad h_0 \mapsto [e_{-\theta}, e_\theta] \otimes t^0 + (e_{-\theta}, e_\theta)\mathsf{c}.$$

In view of this result, we can explicitly describe the root subspaces from (2.13):

$$\widehat{\mathfrak{g}}_{(\alpha,k)} = \mathfrak{g}_\alpha \otimes t^k \quad \forall\, (\alpha,k) \in \widehat{\Delta}^{+,\mathrm{re}} := \big\{\Delta^+ \times \mathbb{Z}_{\geq 0}\big\} \sqcup \big\{\Delta^- \times \mathbb{Z}_{>0}\big\},$$
$$\widehat{\mathfrak{g}}_{k\delta} = \mathfrak{h} \otimes t^k \quad \text{for } k\delta \in \widehat{\Delta}^{+,\mathrm{im}} := \big\{0 \times \mathbb{Z}_{>0}\big\}.$$

As $\dim(\mathfrak{g}_\alpha) = 1$ for any $\alpha \in \Delta$ and $\dim(\mathfrak{h}) = \mathrm{rank}(\mathfrak{g}) = |I|$, we thus obtain:

$$\dim(\widehat{\mathfrak{g}}_\alpha) = 1 \quad \forall\, \alpha \in \widehat{\Delta}^{+,\mathrm{re}}, \qquad \dim(\widehat{\mathfrak{g}}_\alpha) = |I| \quad \forall\, \alpha \in \widehat{\Delta}^{+,\mathrm{im}}. \tag{2.15}$$

In what follows, we shall always write xt^n instead of $x \otimes t^n$.

2.24. Affine standard Lyndon words.

Replacing $\mathfrak{g}$ of Subsection 2.17 with $\widehat{\mathfrak{g}}$ of Subsection 2.22, we likewise consider only the positive subalgebra $\widehat{\mathfrak{n}}^+ = \bigoplus_{\alpha \in \widehat{\Delta}^+} \widehat{\mathfrak{g}}_\alpha$, which is generated by $\{e_i\}_{i \in \widehat{I}}$ subject to the Serre relations (2.7) for $i \neq j \in \widehat{I}$. Endowing $\widehat{I}$ with any order allows us to introduce Lyndon and standard Lyndon words (with respect to $\widehat{\mathfrak{n}}^+$). Henceforth, we shall often use the term **affine standard Lyndon words** in the present setup.

The key difference is that some root subspaces are higher dimensional, see (2.15). Thus, we no longer have a bijection (2.8). However, a similar degree reasoning implies that there is a unique affine standard Lyndon word in each real degree $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, denoted by $\mathrm{SL}(\alpha)$, and $|I|$ affine standard Lyndon words in each imaginary degree $\alpha \in \widehat{\Delta}^{+,\mathrm{im}}$, denoted by $\mathrm{SL}_1(\alpha) > \cdots > \mathrm{SL}_{|I|}(\alpha)$. These words can be computed through the following **generalized Leclerc algorithm**:

Proposition 2.25. *([AT, Proposition 3.4]) The affine standard Lyndon words (with respect to $\widehat{\mathfrak{n}}^+$) are determined inductively by the following rules:*

(a) For simple roots, we have $\mathrm{SL}(\alpha_i) = [i]$. For other real $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, we have:

$$\mathrm{SL}(\alpha) = \max\left\{ \mathrm{SL}_*(\gamma_1)\mathrm{SL}_*(\gamma_2) \,\middle|\, \substack{\alpha=\gamma_1+\gamma_2,\ \gamma_1,\gamma_2 \in \widehat{\Delta}^+ \\ \mathrm{SL}_*(\gamma_1) < \mathrm{SL}_*(\gamma_2) \\ [\mathsf{b}[\mathrm{SL}_*(\gamma_1)], \mathsf{b}[\mathrm{SL}_*(\gamma_2)]] \neq 0} \right\}, \tag{2.16}$$

where $\mathrm{SL}_*(\gamma)$ denotes $\mathrm{SL}(\gamma)$ for $\gamma \in \widehat{\Delta}^{+,\mathrm{re}}$ and any of $\{\mathrm{SL}_k(\gamma)\}_{k=1}^{|I|}$ for $\gamma \in \widehat{\Delta}^{+,\mathrm{im}}$.

(b) For imaginary $\alpha \in \widehat{\Delta}^{+,\mathrm{im}}$, the corresponding $|I|$ affine standard Lyndon words $\{\mathrm{SL}_k(\alpha)\}_{k=1}^{|I|}$ are the $|I|$ lexicographically largest words from the list as in the right-hand side of (2.16) whose standard bracketings are linearly independent.

We shall call $u = \mathrm{SL}_*(\gamma)$ real (resp. imaginary) if $\gamma \in \widehat{\Delta}^{+,\mathrm{re}}$ (resp. $\gamma \in \widehat{\Delta}^{+,\mathrm{im}}$).

Remark 2.26. We note that the condition $[\mathsf{b}[\mathrm{SL}_*(\gamma_1)], \mathsf{b}[\mathrm{SL}_*(\gamma_2)]] \neq 0$ implies that γ_1, γ_2 must be real in (b). Moreover, this condition always holds if $\alpha, \gamma_1, \gamma_2$ are real.

We conclude this section with the notation that will be used through this note:

Notation 2.27. *For any $\alpha = (\alpha', k) \in \widehat{\Delta}^{+,\mathrm{re}}$, we set $h_\alpha := [e_{\alpha'}, e_{-\alpha'}] \in \mathfrak{h}$. For any $w \in \widehat{I}^*$ with $\deg w = \alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, we set $h_w = h_\alpha$. We note that h_α, h_w depend on the choice of root vectors and thus are defined up to nonzero constants. To address this ambiguity, we shall write $x \sim y$ if $x = cy$ for some $c \in \mathbb{C}\backslash\{0\}$.*

3. Properties of factorizations

In this section, we establish some properties of the standard, costandard, and other factorizations of standard Lyndon words. The following result is well-known:

Lemma 3.1. *For a Lyndon word w, the smallest proper suffix is w^r.*

Proof. Denote the smallest proper suffix of w as u, that is, $u = \min\{w_{|a}\}_{0<a<|w|}$. First, we note that u is Lyndon as any proper suffix of u is lexicographically larger than u. Thus u is a suffix of w^r. But if u was a proper suffix of w^r, then $w^r < u$ as w^r is Lyndon, yielding a contradiction with the minimality of u. Hence $u = w^r$. □

Lemma 3.2. *For any Lyndon word ℓ with $|\ell|, |\ell^l| > 1$, we have $\ell^{lr} \geq \ell^r$.*

Proof. If $\ell^{lr} < \ell^r$, then $\ell^{lr}\ell^r$ would be Lyndon by Lemma 2.4. The latter implies $\ell^{lr}\ell^r < \ell^r$, a contradiction with Lemma 3.1. This proves $\ell^{lr} \geq \ell^r$. □

We note that the above lemmas admit the following "prefix" counterpart:

Lemma 3.3. *(a) For any Lyndon word ℓ with $|\ell|, |\ell^{rs}| > 1$, we have $(\ell^{rs})^{ls} \leq \ell^{ls}$.*

(b) For a Lyndon word w with $|w| > 1$, the biggest proper Lyndon prefix is w^{ls}.

Proof. Part (b) is obvious from the definition of the standard factorization. As per part (a), if $\ell^{rs,ls} := (\ell^{rs})^{ls} > \ell^{ls}$, then $\ell^{ls}\ell^{rs,ls}$ would be Lyndon by Lemma 2.4, thus contradicting part (b) as $\ell^{ls} < \ell^{ls}\ell^{rs,ls}$. □

Our next several results relate any factorization to the costandard one:

Lemma 3.4. *Consider any factorization of a Lyndon word $\ell = \ell_1\ell_2$ with $\ell_1, \ell_2 \in \mathrm{L}$. Then, ℓ^r belongs to the set $P := \{\ell_2, \ell_1^r\ell_2, \ell_1^{lr}\ell_1^r\ell_2, \ell_1^{llr}\ell_1^{lr}\ell_1^r\ell_2, \dots\}$.*

Proof. Suppose that $\ell^r \notin P$ and let $u \in P$ be the longest suffix of ℓ^r (such u exists as ℓ_2 is a suffix of ℓ^r), so that $\ell^r = vu$. But there exists a Lyndon word w of the form $\ell^{l\dots lr}$ that has v as a proper suffix and $wu \in P$. Thus, we have $w < v$ and so $wu < vu = \ell^r$, which contradicts Lemma 3.1. Therefore, $\ell^r \in P$. □

Lemma 3.5. *For any factorization of a Lyndon word $\ell = \ell_1\ell_2$ with $\ell_1, \ell_2 \in \mathrm{L}$, every element of the set $\bar{P} = \{\ell_2, \ell_1^r\ell_2, \ell_1^{lr}\ell_1^r\ell_2, \ell_1^{llr}\ell_1^{lr}\ell_1^r\ell_2, \dots, \ell^r\}$ is Lyndon.*

Proof. We prove this by induction on the length. The base case ℓ_2 is obvious.

For the step of induction, fix $u \in \bar{P}$ and assume that the induction hypothesis holds for any $v \in \bar{P}$ with $|v| < |u|$. Suppose that u is not Lyndon. Split u into $u = u_1u_2$ with $u_2 \in \bar{P}$ and $u_1 = \ell_1^{l\cdots lr}$, and let k denote the number of l's in the above superscript. Similarly, we split $\ell^r = v_1v_2$ with $v_2 \in \bar{P}$ and $v_1 = \ell_1^{l\cdots lr}$, and let p be the number of l's in the above superscript, so that $p > k$ (the case $p = k$ is obvious). As u_1 is Lyndon and u_2 is Lyndon by the induction hypothesis, we note that $u_1 \geq u_2$, as otherwise u_1u_2 would be Lyndon by Lemma 2.4, a contradiction. Also $u_1 \leq v_1$, due to a repeated application of Lemma 3.2. Hence, we obtain:

$$u_2 \leq u_1 \leq v_1 < v_1v_2 = \ell^r.$$

Thus, ℓ^r is larger than its proper suffix u_2, a contradiction with ℓ^r being Lyndon. Therefore u is a Lyndon word, which completes the step of induction. □

Corollary 3.6. *For any factorization of a Lyndon word $\ell = \ell_1\ell_2$ with $\ell_1, \ell_2 \in \mathrm{L}$ and $\ell_2 \neq \ell^r$, there exists a factorization $\ell_1 = uv$ with $u, v \in \mathrm{L}$ such that $v\ell_2 \in \mathrm{L}$. Moreover, one can choose the costandard factorization $u = \ell_1^l, v = \ell_1^r$.*

Proof. Since $\ell^r \neq \ell_2$, we have $\ell_1^r \ell_2 \in \bar{P}$ and hence it is a Lyndon word by Lemma 3.5. Therefore, Lyndon words $u = \ell_1^l$ and $v = \ell_1^r$ satisfy both conditions. □

Corollary 3.7. *For any factorization of a Lyndon word $\ell = \ell_1 \ell_2$ with $\ell_1, \ell_2 \in \mathrm{L}$ and $\ell_2 \neq \ell^r$, there exists another factorization $\ell = uv\ell_2$ such that $u\ell_2 v, vu\ell_2 > \ell$ and $u\ell_2$ is Lyndon. Moreover, one can choose the costandard factorization $u = \ell_1^l, v = \ell_1^r$.*

Proof. By Corollary 3.6, there is a splitting $\ell = uv\ell_2$ with $u, v, uv, v\ell_2 \in \mathrm{L}$. Thus $u < v < \ell_2$, and so $u\ell_2$ is Lyndon by Lemma 2.4. As $v\ell_2 \in \mathrm{L}$, we have $v\ell_2 < \ell_2 v$, hence, $\ell = uv\ell_2 < u\ell_2 v$. As $uv \in \mathrm{L}$, we have $uv < vu$, so that $\ell = uv\ell_2 < vu\ell_2$. □

As another interesting application of Lemma 3.5, we have:

Corollary 3.8. *For a factorization of a Lyndon word $\ell = \ell_1 \ell_2$ with $\ell_1, \ell_2 \in \mathrm{L}$:*

$$\ell_2 = \ell^r \iff \ell_1^r \geq \ell_2.$$

Proof. The "$\Rightarrow$" direction follows from Lemma 3.2. For the "$\Leftarrow$" direction, if $\ell_1^r \geq \ell_2$, then $\ell_1^r \ell_2 \notin \mathrm{L}$. Evoking Lemma 3.5, we get $\bar{P} = \{\ell_2\}$ so that $\ell_2 = \ell^r$. □

We also note that Lemmas 3.4–3.5 admit "prefix" counterparts:

Lemma 3.9. *Consider any factorization of a Lyndon word $\ell = \ell_1 \ell_2$ with $\ell_1, \ell_2 \in \mathrm{L}$. Define u_i and v_i inductively via $u_1 = \ell_1, v_1 = \ell_2$ and $u_i = u_{i-1} v_{i-1}^{ls}, v_i = v_{i-1}^{rs}$, as long as $|v_{i-1}| > 1$. Then, every element in $\{u_1, u_2, \ldots, u_n\}$ is Lyndon, where n is the smallest integer such that $v_n^{ls} \leq u_n$ or $|v_n| = 1$.*

Proof. We prove that u_k is Lyndon for $1 \leq k \leq n$ by induction on k. The base case $k = 1$ is obvious. For the inductive step, if u_{k-1} is Lyndon and $u_k = u_{k-1} v_{k-1}^{ls}$ with $v_{k-1}^{ls} > u_{k-1}$, then u_k is also Lyndon by Lemma 2.4. □

Lemma 3.10. *Consider any factorization $\ell = \ell_1 \ell_2$ with $\ell, \ell_1, \ell_2 \in \mathrm{L}$. Then, ℓ^{ls} is equal to u_n as defined in Lemma 3.9.*

Proof. Assume the contrary: $\ell^{ls} \neq u_n$. Then u_n must be a prefix of ℓ^{ls}, so that $\ell^{ls} = u_n w$ with $w \neq \emptyset$. Consider the canonical factorization $w = w_1 \ldots w_k$, cf. (2.3). Due to Lemma 3.12, the word $u_n w_1$ is Lyndon, so that $u_n < w_1$. On the other hand, we have $w_1 \leq v_n^{ls}$ by Lemma 3.3, which contradicts to our choice of n. □

The next two results will be needed later:

Lemma 3.11. *For $u \in \mathrm{L}$, consider any splitting $u = vw$ with $v \in \mathrm{L}$, and let $w = w_1 w_2 \ldots w_N$ be the canonical factorization. Then $w_1 \geq w_2 \geq \cdots \geq w_N > v$.*

Proof. As $u = vw$ is Lyndon, we have $w_N > vw$, which implies $w_N > v$. □

Lemma 3.12. *For $u \in \mathrm{L}$, consider any splitting $u = vw$ with $v \in \mathrm{L}$, and let $w = w_1 w_2 \ldots w_N$ be the canonical factorization. Then $vw_1, vw_1 w_2, \ldots, vw_1 \ldots w_N \in \mathrm{L}$.*

Proof. We prove that $vw_1 \ldots w_n$ is Lyndon for all $1 \leq n \leq N$ by induction on n. For the base case $n = 1$, it suffices to show that $v < w_1$, due to Lemma 2.4. If not, then $w_N \leq w_1 \leq v < vw = u$, which contradicts the condition that u is Lyndon. For the step of induction, assume that $vw_1 \ldots w_{n-1} \in \mathrm{L}$. Then applying the base case to $v' = vw_1 \ldots w_{n-1}$, $w' = w_n w_{n+1} \ldots w_N$, we get $vw_1 \ldots w_n = v' w_n \in \mathrm{L}$. □

The above result admits a natural "prefix" counterpart:

Lemma 3.13. *For $u \in \mathrm{L}$, consider any splitting $u = vw$ with $w \in \mathrm{L}$ and the canonical factorization $v = v_1 v_2 \dots v_N$. Then $v_N w, v_{N-1}v_N w, \dots, v_1 \dots v_N w \in \mathrm{L}$.*

Proof. We prove that $v_{N-n+1} \dots v_N w \in \mathrm{L}$ for $1 \le n \le N$ by induction on n. For the base case $n = 1$, it suffices to show that $v_N < w$, due to Lemma 2.4. If not, then $v_1 \ge \dots \ge v_N \ge w$, hence a contradiction with $u \in \mathrm{L}$, due to the uniqueness of the canonical factorization. For the step of induction, assume that $v_{N-n+2} \dots v_N w \in \mathrm{L}$. Applying the base case to $v' = v_1 v_2 \dots v_{N-n+1}$ and $w' = v_{N-n+2} \dots v_N w$, we get $v_{N-n+1} \dots v_N w = v_{N-n+1} w' \in \mathrm{L}$. □

We conclude this section with the following two results on cyclic permutations:

Lemma 3.14. *For any word $w \in I^*$, consider its canonical factorization (2.3): $w = w_1 w_2 \dots w_n$. If there is a cyclic permutation of w that is Lyndon, then it must be of the form $w_i w_{i+1} \dots w_n w_1 \dots w_{i-1}$ for some $i \in \{1, 2, \dots, n\}$.*

Proof. Let ℓ be a cyclic permutation of w that is Lyndon (unique if exists). If ℓ is not of the stated form, then there is some $w_i = uv$ such that v is a proper prefix of ℓ and u is a proper suffix. But then $\ell > v > w_i > u$, which contradicts to $\ell \in \mathrm{L}$. □

Corollary 3.15. *For any word $w \in I^*$, consider its canonical factorization (2.3): $w = w_1 w_2 \dots w_n$. If $\ell \in \mathrm{L}$ is a cyclic permutation of w, then there is i such that*

$$\underbrace{w}_{k \text{ times}} = w_1 w_2 \dots w_{i-1} \underbrace{\ell}_{k-1 \text{ times}} w_i w_{i+1} \dots w_n \qquad \forall\, k \in \mathbb{Z}_{>0}.$$

4. Full flags and auxiliary sets

In this section, we investigate the behavior of the standard bracketing with respect to different splittings of words. For imaginary roots, it will be crucially important to consider not individual standard bracketings $\mathsf{b}[\mathrm{SL}_i(k\delta)]$ but rather the induced complete flags:

$$\begin{gathered} 0 = \mathcal{S}^k_0 \subset \mathcal{S}^k_1 \subset \dots \subset \mathcal{S}^k_{|I|} = \mathfrak{h}t^k \qquad \forall\, k \in \mathbb{Z}_{>0}, \\ \text{with} \quad \mathcal{S}^k_i := \mathrm{span}\left\{\mathsf{b}[\mathrm{SL}_1(k\delta)], \dots, \mathsf{b}[\mathrm{SL}_i(k\delta)]\right\}. \end{gathered} \tag{4.1}$$

4.1. Extended set of roots.

Following [AT, (5.1)], let us consider the following upgrade of (2.12):

$$\widehat{\Delta}^{+,\mathrm{ext}} = \widehat{\Delta}^{+,\mathrm{re}} \cup \widehat{\Delta}^{+,\mathrm{imx}} \quad \text{with} \quad \widehat{\Delta}^{+,\mathrm{imx}} = \left\{(k\delta, r) \,|\, k \ge 1, 1 \le r \le |I|\right\}, \tag{4.2}$$

counting imaginary roots with multiplicities. We can thus naturally generalize (2.8):

$$\mathrm{SL}\colon \widehat{\Delta}^{+,\mathrm{ext}} \xrightarrow{\sim} \left\{\text{affine standard Lyndon words}\right\}, \quad \mathrm{SL}((k\delta, r)) = \mathrm{SL}_r(k\delta). \tag{4.3}$$

We also consider the induced order on $\widehat{\Delta}^{+,\mathrm{ext}}$, in analogy with (2.11):

$$\alpha < \beta \iff \mathrm{SL}(\alpha) < \mathrm{SL}(\beta) \qquad \forall\, \alpha, \beta \in \widehat{\Delta}^{+,\mathrm{ext}}. \tag{4.4}$$

Lemma 4.2. *Let w be a Lyndon word and uv be the standard factorization of a Lyndon word, with $|w| < |uv|$. Then $w > u \iff w > uv$.*

Proof. We prove the "⇒" direction by induction on the length of $|uv|$. The base case $|uv| = 2$ is clear. As per the step of induction, let us assume the contrary, that is $u < w < uv$. Then $w = uy$ with $y = y_1 \dots y_N$ being a canonical factorization. Due to Lemma 3.11, the Lyndon word y_1 is $> u$ and has length less than $|v|$. Let $v = zt$ be the standard factorization of v. We claim that $z \le u$. If not, then

uz would be Lyndon by Lemma 2.4, contradicting the fact that u is the longest Lyndon prefix. Thus, $z \leq u < y_1 \leq y < v = zt$, which cannot occur due to the induction assumption (applied to y_1 in place of w and zt in place of uv). This yields a contradiction, thus establishing the step of induction.

The "$\Leftarrow$" direction is a consequence of the inequalities $w \geq uv > u$. $\square$

4.3. ***W*-set and pseudo-bracketing.**

The key difficulty in extending the convexity (2.10) to affine root systems $\widehat{\Delta}^+$ lies in the treatment of imaginary roots. For example, while (2.16) guarantees that

$$\alpha + \beta > \min\{\alpha, \beta\}$$

if $\alpha, \beta, \alpha + \beta \in \widehat{\Delta}^{+,\mathrm{re}}$ (cf. Remark 2.26), the generalization of this to the case when some roots are imaginary is not obvious, and will be established in Corollary 4.9, cf. Remark 4.12. To this end, we start with the following definition:

Definition 4.4. *(a) Define the set W as follows:*

$$W = \big\{ w = (u,v) \,\big|\, u, v \in \mathrm{SL}, u < v \big\},$$

whereas we often write $w_1 = u, w_2 = v$. We endow W with the following ordering:

$$(u,v) < (u',v') \quad \textit{if} \begin{cases} |uv| < |u'v'| \\ \quad \textit{or} \\ |uv| = |u'v'| \ \textit{and} \ uv > u'v' \\ \quad \textit{or} \\ uv = u'v' \ \textit{and} \ u < u' \end{cases} . \tag{4.5}$$

*Finally, for any $w \in W$, we define its **pseudo-bracketing** $\bar{\mathsf{b}}[w] \in \mathfrak{a}$ via:*

$$\bar{\mathsf{b}}[w] = [\mathsf{b}[w_1], \mathsf{b}[w_2]].$$

(b) For any $\alpha \in \widehat{\Delta}^+$, define the subset W_α of W via:

$$W_\alpha = \big\{ (u,v) \,\big|\, u, v \in \mathrm{SL}, u < v, \deg(u) + \deg(v) = \alpha \big\}.$$

(c) Define the subset $\overline{W}$ of W as follows:

$$\overline{W} = \big\{ (u,v) \,\big|\, u, v \in \mathrm{SL}, u < v, uv \in \mathrm{SL} \big\}.$$

(d) For any $\alpha \in \widehat{\Delta}^{+,\mathrm{ext}}$, we define the subset $\overline{W}_\alpha$ of $\overline{W}$ via:

$$\overline{W}_\alpha = \big\{ (u,v) \,\big|\, u, v \in \mathrm{SL}, u < v, uv = \mathrm{SL}(\alpha) \big\}.$$

Remark 4.5. For any standard Lyndon word w, the set $\overline{W}$ contains the costandard factorization (w^l, w^r), the standard factorization (w^{ls}, w^{rs}), as well as possibly some more (ℓ_1, ℓ_2) arising from factorizations $w = \ell_1 \ell_2$ into two standard Lyndon words. Moreover, (w^l, w^r) is the smallest and (w^{ls}, w^{rs}) is the biggest among all of those.

Proposition 4.6. *Let $w_1, \ldots, w_N$ be the elements of $W_{k\delta}$ listed in increasing order. Then for any $1 \leq n \leq N$, we have:*

$$\mathrm{span}\Big\{ \bar{\mathsf{b}}[w_1], \bar{\mathsf{b}}[w_2], \ldots, \bar{\mathsf{b}}[w_n] \Big\} = \mathcal{S}^k_m, \tag{4.6}$$

where $m = \max\{i \,|\, (w_n)_1 (w_n)_2 \leq \mathrm{SL}_i(k\delta)\}$ and $m = 0$ if $(w_n)_1 (w_n)_2 > \mathrm{SL}_1(k\delta)$.

Proof. We prove (4.6) by induction on n.

In the base case $n=1$, either $w_1=(\mathrm{SL}_1^l(k\delta),\mathrm{SL}_1^r(k\delta))$ or $(w_1)_1(w_1)_2>\mathrm{SL}_1(k\delta)$. In the first case, we have $\bar{\mathsf{b}}[w_1]=\mathsf{b}[\mathrm{SL}_1(k\delta)]$ and $m=1$. In the second case, $m=0$ while $\bar{\mathsf{b}}[w_1]=0$ as w_1 represents the costandard factorization of a non-standard Lyndon word (cf. Remark 4.5). Thus, the equality (4.6) holds for $n=1$.

For the inductive step, we shall assume that (4.6) holds for $n'=n-1$ with the right-hand side $\mathcal{S}^k_{m'}$. We shall now consider three cases:

• If $w_n=(\mathrm{SL}_i^l(k\delta),\mathrm{SL}_i^r(k\delta))$, then $m=i$ and $\bar{\mathsf{b}}[w_n]=\mathsf{b}[\mathrm{SL}_i(k\delta)]$, while the inductive hypothesis yields $\mathrm{span}\{\bar{\mathsf{b}}[w_1],\bar{\mathsf{b}}[w_2],\ldots,\bar{\mathsf{b}}[w_{n-1}]\}=\mathcal{S}^k_{i-1}$. This implies that $\mathrm{span}\{\bar{\mathsf{b}}[w_1],\bar{\mathsf{b}}[w_2],\ldots,\bar{\mathsf{b}}[w_n]\}=\mathcal{S}^k_i$, as claimed.

• If w_n represents the costandard factorization of a non-standard Lyndon word, then $\bar{\mathsf{b}}[w_n]=\mathsf{b}[(w_n)_1(w_n)_2]\in\mathrm{span}\{\bar{\mathsf{b}}[w_1],\bar{\mathsf{b}}[w_2],\ldots,\bar{\mathsf{b}}[w_{n-1}]\}$. It thus remains to show that $m=m'$, that is, $\mathrm{SL}_{m'+1}(k\delta)<(w_n)_1(w_n)_2<\mathrm{SL}_{m'}(k\delta)$. But if not, then we would actually have $\mathrm{SL}_{m'+1}(k\delta)=(w_n)_1(w_n)_2$, due to the ordering (4.5) and $\mathrm{SL}_{m'+1}(k\delta)<(w_{n-1})_1(w_{n-1})_2\leq\mathrm{SL}_{m'}(k\delta)$, thus contradicting $(w_n)_1(w_n)_2\notin\mathrm{SL}$.

• If w_n does not represent the costandard factorization of any Lyndon word, then $(w_n)_1(w_n)_2=(w_{n-1})_1(w_{n-1})_2$ (as the costandard factorization of $(w_n)_1(w_n)_2$ is among $\{w_j\}_{j=1}^{n-1}$). Therefore, we have $m=m'$. It thus remains to show:

$$\bar{\mathsf{b}}[w_n]\in\mathrm{span}\left\{\bar{\mathsf{b}}[w_1],\bar{\mathsf{b}}[w_2],\ldots,\bar{\mathsf{b}}[w_{n-1}]\right\}. \tag{4.7}$$

Using Corollary 3.6, let us split $(w_n)_1(w_n)_2=uv(w_n)_2$ with $u=(w_n)_1^l, v=(w_n)_1^r$, so that $u,v,v(w_n)_2\in\mathrm{L}$. We note that $\mathsf{b}[(w_n)_1]=[\mathsf{b}[u],\mathsf{b}[v]]$. For the notation simplicity, let $y=(w_n)_2$, so that $u<v<y$. By the Jacobi identity, we have:

$$[\mathsf{b}[u],[\mathsf{b}[v],\mathsf{b}[y]]]+[\mathsf{b}[v],[\mathsf{b}[y],\mathsf{b}[u]]]+[\mathsf{b}[y],[\mathsf{b}[u],\mathsf{b}[v]]]=0.$$

We shall assume that y is not imaginary, as otherwise $\bar{\mathsf{b}}[w_n]=0$, implying (4.7). Hence, it suffices to show:

$$[\mathsf{b}[u],[\mathsf{b}[v],\mathsf{b}[y]]],[\mathsf{b}[v],[\mathsf{b}[y],\mathsf{b}[u]]]\in\mathrm{span}\{\bar{\mathsf{b}}[w_1],\bar{\mathsf{b}}[w_2],\ldots,\bar{\mathsf{b}}[w_{n-1}]\}.$$

If u is imaginary or $[\mathsf{b}[v],\mathsf{b}[y]]=0$, then $[\mathsf{b}[u],[\mathsf{b}[v],\mathsf{b}[y]]]=0$. Otherwise, $[\mathsf{b}[u],[\mathsf{b}[v],\mathsf{b}[y]]]$ is a multiple of $[\mathsf{b}[u],\mathsf{b}[\mathrm{SL}(\deg(v)+\deg(y))]]$ and $u\neq\mathrm{SL}(\deg(v)+\deg(y))$. We note that $\mathrm{SL}(\deg(v)+\deg(y))\geq vy$ due to (2.16), as $[\mathsf{b}[v],\mathsf{b}[y]]\neq0$. As $u<v<vy\leq\mathrm{SL}(\deg(v)+\deg(y))$, we see that $(u,\mathrm{SL}(\deg(v)+\deg(y)))$ is in $W_{k\delta}$ and is smaller than w_n, that is, belongs to $\{w_1,\ldots,w_{n-1}\}$. This implies that $[\mathsf{b}[u],[\mathsf{b}[v],\mathsf{b}[y]]]\in\mathrm{span}\{\bar{\mathsf{b}}[w_1],\bar{\mathsf{b}}[w_2],\ldots,\bar{\mathsf{b}}[w_{n-1}]\}$.

If v is imaginary or $[\mathsf{b}[y],\mathsf{b}[u]]=0$, then $[\mathsf{b}[v],[\mathsf{b}[y],\mathsf{b}[u]]]=0$. Otherwise, $[\mathsf{b}[v],[\mathsf{b}[y],\mathsf{b}[u]]]$ is a multiple of $[\mathsf{b}[v],\mathsf{b}[\mathrm{SL}(\deg(y)+\deg(u))]]$ and also $v\neq\mathrm{SL}(\deg(y)+\deg(u))$. Let $\ell\in\mathrm{L}$ denote the appropriate concatenation of v and $\mathrm{SL}(\deg(y)+\deg(u))$. As $[\mathsf{b}[y],\mathsf{b}[u]]\neq0$ and $u<y$, we get $\mathrm{SL}(\deg(u)+\deg(y))\geq uy$ due to (2.16). Combining this with Corollary 3.7, we obtain $\ell>uvy$. Hence, either $(v,\mathrm{SL}(\deg(y)+\deg(u))$ or $(\mathrm{SL}(\deg(y)+\deg(u),v)$ is in $W_{k\delta}$ and is smaller than w_n, that is, belongs to $\{w_1,\ldots,w_{n-1}\}$. This implies that $[\mathsf{b}[v],[\mathsf{b}[y],\mathsf{b}[u]]]\in\mathrm{span}\{\bar{\mathsf{b}}[w_1],\bar{\mathsf{b}}[w_2],\ldots,\bar{\mathsf{b}}[w_{n-1}]\}$.

This completes the proof of the inductive step, and hence of the proposition. □

Let us illustrate this proposition with a couple of examples (cf. Notation 2.27):

Example 4.7. Consider affine type $F_4^{(1)}$ with the order $3<4<0<2<1$. Using the code (see Listing 6 of Appendix A), the set W_δ, written in increasing order, with the pairs corresponding to the costandard factorization of SL-words highlighted in bold, is as follows:

$$(\mathbf{3432104321}, \mathbf{32}), (\mathbf{3432104}, \mathbf{32321}), (343210432, 321), (34321043232, 1),$$
$$(\mathbf{343210321}, \mathbf{324}), (34321032132, 4), (343210, 324321), (34321032, 3214),$$
$$(\mathbf{343214}, \mathbf{323210}), (34321432, 3210), (34321432321, 0), (3432143232, 01),$$
$$(34321, 3243210), (3432132, 32104), (34321343210, 2), (34, 3243210321),$$
$$(34324, 3213210), (3432, 32143210), (343234321, 012), (3432343210, 21),$$
$$(33210, 3243214), (3321, 32432104), (3, 32432104321), (332, 321432104).$$

Then, $\overline{\mathsf{b}}[(343210432, 321)] \sim h_{321}t$ and $\overline{\mathsf{b}}[(34321043232, 1)] \sim h_1t$ are in the span of $\mathsf{b}[\mathrm{SL}_1(\delta)] = \mathsf{b}[343210432132] \sim h_{32}t$ and $\mathsf{b}[\mathrm{SL}_2(\delta)] = \mathsf{b}[343210432321] \sim h_{32321}t$. Likewise, for non-highlighted elements in the second line: $\overline{\mathsf{b}}[(34321032132, 4)] \sim h_4t$, $\overline{\mathsf{b}}[(343210, 324321)] \sim h_{324321}t$, $\overline{\mathsf{b}}[(34321032, 3214)] \sim h_{3214}t$, all of which are in the span of above $\mathsf{b}[\mathrm{SL}_1(\delta)]$, $\mathsf{b}[\mathrm{SL}_2(\delta)]$, and $\mathsf{b}[\mathrm{SL}_3(\delta)] = \mathsf{b}[343210321324] \sim h_{324}t$.

Example 4.8. Consider affine type $E_6^{(1)}$ with the order $3 < 0 < 1 < 5 < 4 < 6 < 2$. Using the code (see Listing 6 of Appendix A), the set W_δ, written in increasing order, with the pairs corresponding to the costandard factorization of SL-words highlighted in bold, is as follows:

$$(\mathbf{3645032641}, \mathbf{32}), (\mathbf{364503264}, \mathbf{321}), (36450326432, 1), (\mathbf{364503261}, \mathbf{324}),$$
$$(36450326132, 4), (36450326, 3241), (\mathbf{36450}, \mathbf{3241326}), (364503241, 326),$$
$$(36450324132, 6), (36450324, 3261), (36450321, 3264), (3645032, 32641),$$
$$(\mathbf{3645}, \mathbf{32413260}), (36453241, 3260), (36453241326, 0), (3645324132, 06),$$
$$(3645324, 32610), (3645321, 32640), (364532, 326410), (\mathbf{36403261}, \mathbf{3245}),$$
$$(36403261324, 5), (3640326132, 54), (3640326, 32451), (3640, 32451326),$$
$$(3640321, 32645), (36403213645, 2), (364032, 326451), (3640323645, 12),$$
$$(364, 324513260), (364321, 326450), (36432, 3264510), (360, 324513264),$$
$$(36, 3245132640), (345, 326103264), (34, 3261032645), (3, 32641032645).$$

Then, for example, the pseudo-bracketing of all elements in the third line are $\overline{\mathsf{b}}[(36450324132, 6)] \sim h_6t$, $\overline{\mathsf{b}}[(36450324, 3261)] \sim h_{3261}t$, $\overline{\mathsf{b}}[(36450321, 3264)] \sim h_{3264}t$, $\overline{\mathsf{b}}[(3645032, 32641)] \sim h_{32641}t$, which are in the linear span of $\mathsf{b}[\mathrm{SL}_1(\delta)] = \mathsf{b}[364503264132] \sim h_{32}t$, $\mathsf{b}[\mathrm{SL}_2(\delta)] = \mathsf{b}[364503264321] \sim h_{321}t$, $\mathsf{b}[\mathrm{SL}_3(\delta)] = \mathsf{b}[364503261324] \sim h_{324}t$, and $\mathsf{b}[\mathrm{SL}_4(\delta)] = \mathsf{b}[364503241326] \sim h_{3241326}t$.

As an important corollary of Proposition 4.6, we obtain:

Corollary 4.9. *Consider two standard Lyndon words u, v such that $u < v$ and $\deg(u) + \deg(v) = k\delta$. Then $\mathrm{SL}_i(k\delta) < uv \implies [\mathsf{b}[u], \mathsf{b}[v]] \in \mathcal{S}^k_{i-1}$.*

Proof. As $(u, v) \in W_{k\delta}$, we get $[\mathsf{b}[u], \mathsf{b}[v]] \in \mathcal{S}^k_m$ with $m = \max\{j \mid uv \le \mathrm{SL}_j(k\delta)\}$ by Proposition 4.6. As $\mathrm{SL}_i(k\delta) < uv$, we see that $m \le i - 1$, implying the result. ☐

4.10. **O-sets and their properties.**

We can now introduce a set of roots that will be key to our notion of convexity:

Definition 4.11. *For any $\alpha = (k\delta, i) \in \widehat{\Delta}^{+,\mathrm{imx}}$, define the following set:*

$$C(\alpha) = \left\{ \beta \,\middle|\, \begin{matrix} \beta\in\widehat{\Delta}^{+,\mathrm{re}}, \beta=\beta'+p\delta \text{ with } \beta'\in\widehat{\Delta}^{+,\mathrm{re}}, |\beta'|<|\delta|, p\in\mathbb{Z}_{\geq 0} \\ [\mathsf{b}[\mathrm{SL}(\beta')], \mathsf{b}[\mathrm{SL}(k\delta-\beta')]] \notin \mathcal{S}^k_{i-1} \\ \exists\, j\leq i \text{ s.t. } [\mathsf{b}[\mathrm{SL}(\beta')], \mathsf{b}[\mathrm{SL}_j(k\delta)]]\neq 0 \end{matrix} \right\}.$$

Remark 4.12. The importance of the second line in the right-hand side is that for $u, v \in C(\alpha)$ with $\deg(u)+\deg(v) = k\delta$, we have $\mathrm{SL}(\alpha) > \min\{u, v\}$ by Corollary 4.9.

Definition 4.13. *For any $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, define the following set:*

$$O(\alpha) = \Big\{ \beta \,\big|\, \beta \in \widehat{\Delta}^{+,\mathrm{imx}}, \alpha \in C(\beta) \Big\}.$$

The max and min of $O(\alpha) \cap \{(k\delta, \bullet)\}$ are of special interest:

Definition 4.14. *For any $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$ and $k \in \mathbb{Z}_{>0}$, we define*

$$\begin{aligned} M_k(\alpha) &= \max\big\{\beta \in O(\alpha) \,\big|\, |\beta| = |k\delta|\big\}, \\ m_k(\alpha) &= \min\big\{\beta \in O(\alpha) \,\big|\, |\beta| = |k\delta|\big\}. \end{aligned} \tag{4.8}$$

In what follows, we shall often use the following segmental property of $O(\alpha)$:

$$\Big\{\beta \in O(\alpha) \,\Big|\, |\beta| = |k\delta|\Big\} = \Big\{\beta \in \widehat{\Delta}^{+,\mathrm{imx}} \,\Big|\, |\beta| = |k\delta|, m_k(\alpha) \leq \beta \leq M_k(\alpha)\Big\},$$

which follows from the fact that

$$[\mathsf{b}[\mathrm{SL}(\beta')], \mathsf{b}[\mathrm{SL}(k\delta - \beta')]] \notin \mathcal{S}^k_{i-1}$$

implies the same for any $i' < i$, while

$$\exists\, j \leq i \text{ such that } [\mathsf{b}[\mathrm{SL}(\beta')], \mathsf{b}[\mathrm{SL}_j(k\delta)]] \neq 0$$

implies the same for any $i' > i$. This also provides more explicit description of (4.8):

Lemma 4.15. *For any $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$ and $k \in \mathbb{Z}_{>0}$, we have:*

$$M_k(\alpha) = \max\big\{\beta \in \widehat{\Delta}^{+,\mathrm{imx}} \,\big|\, |\beta| = |k\delta|,\ [\mathsf{b}[\mathrm{SL}(\beta)], \mathsf{b}[\mathrm{SL}(\alpha)]] \neq 0\big\}.$$

Proof. Let $M'_k(\alpha) = (k\delta, i)$ denote the right-hand side above. First, we claim that $M'_k(\alpha) \in O(\alpha)$. Indeed, $h_\alpha t^k \notin \mathcal{S}^k_{i-1}$ follows from $h_\alpha t^k$ being orthogonal to the spanning set of $\mathcal{S}^k_{i-1}$. Thus $M_k(\alpha) \geq M'_k(\alpha)$. But $M_k(\alpha) > M'_k(\alpha)$ would contradict the definition of $M'_k(\alpha)$. This establishes the result: $M_k(\alpha) = M'_k(\alpha)$. □

Corollary 4.16. *If $M_k(\alpha) = (k\delta, i)$ for $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, then $[h, \mathrm{SL}(\alpha)] = 0\ \forall\, h \in \mathcal{S}^k_{i-1}$.*

Lemma 4.17. *For any $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$ and $k \in \mathbb{Z}_{>0}$, we have:*

$$m_k(\alpha) = (k\delta, i) \quad \text{with} \quad h_\alpha t^k \in \mathcal{S}^k_i \backslash \mathcal{S}^k_{i-1}.$$

Proof. Pick i such that $h_\alpha t^k \in \mathcal{S}^k_i \backslash \mathcal{S}^k_{i-1}$. First, we claim that $(k\delta, i) \in O(\alpha)$. To do so, it suffices to prove that there exists a $j \leq i$ with $[\mathsf{b}[\mathrm{SL}_j(k\delta)], \mathsf{b}[\mathrm{SL}(\alpha)]] \neq 0$. The latter follows immediately by noting that $h_\alpha t^k$ can not be orthogonal to the spanning set of $\mathcal{S}^k_i$ which contains $h_\alpha t^k$. Thus $m_k(\alpha) \leq (k\delta, i)$. But $m_k(\alpha) < (k\delta, i)$ would contradict $\alpha \in C(m_k(\alpha))$. This completes the proof. □

Proposition 4.18. *For any factorization $u = u_1 u_2$ with $u, u_1, u_2 \in \mathrm{SL}$, we have:*

(a) $[\mathsf{b}[u_1], \mathsf{b}[u_2]] \neq 0$*;*

(b) if u is imaginary and $u = \mathrm{SL}_i(k\delta)$, then $[\mathsf{b}[u_1], \mathsf{b}[u_2]] \in \mathcal{S}^k_i \backslash \mathcal{S}^k_{i-1}$.

Remark 4.19. This result immediately implies that if we were to use the standard factorization instead of the costandard one in Definition 2.11, we would still get exactly the same affine standard Lyndon words as well as the same flags $\mathcal{S}^k_\bullet$.

Proof of Proposition 4.18. Let $w_1, w_2, \dots, w_n, \dots$ be the elements of $\overline{W}$ ordered in increasing order. Then, $[\mathsf{b}[u_1], \mathsf{b}[u_2]] = \bar{\mathsf{b}}[w_n]$ if $w_n = (u_1, u_2)$. We shall prove both parts by induction on n. The base case $n = 1$ is clear, since w_1 represents the costandard factorization of an affine standard Lyndon word. For the inductive step, we assume that the result holds for all $\{w_m\}_{m<n}$.

First, we consider the case when $(w_n)_1(w_n)_2$ is real. Part (a) is clear if w_n represents the costandard factorization. Let us now assume that w_n does not represent the costandard factorization. Evoking Corollary 3.6, consider the costandard factorization $(w_n)_1 = uv$, so that $u, v, vy \in \mathrm{L}$, where we use $y = (w_n)_2$. Being factors of a standard Lyndon word, we actually have $u, v, vy \in \mathrm{SL}$. By the Jacobi identity:

$$[\mathsf{b}[u], [\mathsf{b}[v], \mathsf{b}[y]]] + [\mathsf{b}[v], [\mathsf{b}[y], \mathsf{b}[u]]] + [\mathsf{b}[y], [\mathsf{b}[u], \mathsf{b}[v]]] = 0. \tag{4.9}$$

Therefore, it suffices to show that $[\mathsf{b}[u], [\mathsf{b}[v], \mathsf{b}[y]]] \neq 0$ and $[\mathsf{b}[v], [\mathsf{b}[y], \mathsf{b}[u]]] = 0$.

First, let us show that $[\mathsf{b}[u], [\mathsf{b}[v], \mathsf{b}[y]]] \neq 0$. This is clear if $\deg(vy)$ is real, since we have $[\mathsf{b}[v], \mathsf{b}[y]] \neq 0$ (by the inductive hypothesis applied to (v, y)) and so $[\mathsf{b}[u], [\mathsf{b}[v], \mathsf{b}[y]]]$ is a nonzero multiple of $[\mathsf{b}[u], \mathsf{b}[vy]]$ which is nonzero (by the inductive hypothesis applied to (u, vy)). If $\deg(vy)$ is imaginary, then we claim that actually $vy = \mathrm{SL}(M_k(\deg(u)))$ for $k = |vy|/|\delta|$. Indeed, if $vy > \mathrm{SL}(M_k(\deg(u)))$, then $[\mathsf{b}[u], \mathsf{b}[vy]] = 0$ by Corollary 4.16, which contradicts the inductive hypothesis applied to (u, vy). If $vy < \mathrm{SL}(M_k(\deg(u)))$, then we would get a contradiction with the generalized Leclerc algorithm (2.16), since $uvy < u\mathrm{SL}(M_k(\deg(u)))$ and $[\mathsf{b}[u], \mathsf{b}[\mathrm{SL}(M_k(\deg(u)))]] \neq 0$ by Lemma 4.15. This proves the claimed equality $vy = \mathrm{SL}(M_k(\deg(u)))$. Applying the induction hypothesis (part (b)) to the pair (v, y), we then have $[\mathsf{b}[v], \mathsf{b}[y]] \in \mathcal{S}^k_i \backslash \mathcal{S}^k_{i-1}$ whereas $M_k(\deg(u)) = (k\delta, i)$. Hence $[\mathsf{b}[u], [\mathsf{b}[v], \mathsf{b}[y]]] \neq 0$ by Lemma 4.15.

Next, let us prove that $[\mathsf{b}[v], [\mathsf{b}[y], \mathsf{b}[u]]] = 0$. Assuming the contradiction, we see that $[\mathsf{b}[y], \mathsf{b}[u]] \neq 0$, and hence $\deg(y) + \deg(u) \in \widehat{\Delta}^+$. If $\deg(y) + \deg(u) \in \widehat{\Delta}^{+,\mathrm{re}}$, then $uy \leq \mathrm{SL}(\deg(u) + \deg(y)) =: z$ by the generalized Leclerc algorithm (2.16). If $\deg(y) + \deg(u) = k\delta$, then we have $uy \leq \mathrm{SL}(M_k(\deg(v))) =: z$ by Corollary 4.16. Evoking Corollary 3.7, in both cases we see that the appropriate concatenation of v and z is bigger than uvy and $[\mathsf{b}[v], \mathsf{b}[z]] \neq 0$. This implies that uvy is not standard Lyndon, due to the generalized Leclerc algorithm, a contradiction. Therefore, indeed we have $[\mathsf{b}[v], [\mathsf{b}[y], \mathsf{b}[u]]] = 0$.

Let us now consider the case when $(w_n)_1(w_n)_2$ is imaginary, that is, equal to $\mathrm{SL}_i(k\delta)$ for some i, k. It suffices to prove part (b) only. The result is clear if w_n represents the costandard factorization of $\mathrm{SL}_i(k\delta)$. Let us now assume that w_n does not represent the costandard factorization of $\mathrm{SL}_i(k\delta)$. Using the same notations u, v, y as above, and evoking the equality (4.9), it thus suffices to prove:

$$[\mathsf{b}[u], [\mathsf{b}[v], \mathsf{b}[y]]] \in \mathcal{S}^k_i \backslash \mathcal{S}^k_{i-1}, \qquad [\mathsf{b}[v], [\mathsf{b}[y], \mathsf{b}[u]]] \in \mathcal{S}^k_{i-1}.$$

By the inductive hypothesis applied to (u, vy), we see that $\deg(u), \deg(vy) \in \widehat{\Delta}^{+,\mathrm{re}}$, and furthermore $[\mathsf{b}[u], \mathsf{b}[vy]] \in \mathcal{S}^k_i \backslash \mathcal{S}^k_{i-1}$. On the other hand, by the inductive hypothesis applied to (v, y), we know that $[\mathsf{b}[v], \mathsf{b}[y]]$ is a nonzero multiple of $\mathsf{b}[vy]$. This implies the first inclusion above: $[\mathsf{b}[u], [\mathsf{b}[v], \mathsf{b}[y]]] \in \mathcal{S}^k_i \backslash \mathcal{S}^k_{i-1}$.

If $[\mathsf{b}[v],[\mathsf{b}[y],\mathsf{b}[u]]]\neq 0$, then $\deg(v),\deg(u)+\deg(y)\in\widehat{\Delta}^{+,\mathrm{re}}$. As $u<v<y$, we then have $uy\leq \mathrm{SL}(\deg(u)+\deg(y))$ by (2.16). Evoking Corollary 3.7 once again, we see that the appropriate concatenation of v and $\mathrm{SL}(\deg(u)+\deg(y))$ is bigger than $uvy=\mathrm{SL}_i(k\delta)$. Therefore, $[\mathsf{b}[v],\mathsf{b}[\mathrm{SL}(\deg(u)+\deg(y))]]\in\mathcal{S}^k_{i-1}$ by Corollary 4.9. Since $[\mathsf{b}[u],\mathsf{b}[y]]$ is a multiple of $\mathsf{b}[\mathrm{SL}(\deg(u)+\deg(y))]$, we thus get the second inclusion above: $[\mathsf{b}[v],[\mathsf{b}[y],\mathsf{b}[u]]]\in\mathcal{S}^k_{i-1}$. □

Corollary 4.20. *For any $1\leq i\leq |I|$ and $k\in\mathbb{Z}_{>0}$, we cannot have $\mathrm{SL}_i(k\delta)=uv$ with $u,v\in\mathrm{SL}$ being imaginary words.*

Proof. If u,v were imaginary, we would have $[\mathsf{b}[u],\mathsf{b}[v]]=0$ as $[\mathfrak{h}t^a,\mathfrak{h}t^b]=0$ for all $a,b>0$, a contradiction with Proposition 4.18(a). □

Corollary 4.21. *For any $\alpha\in\widehat{\Delta}^{+,\mathrm{re}}$ and $k\in\mathbb{Z}_{>0}$, consider any factorization $\mathrm{SL}(M_k(\alpha))=u_1u_2$ with $u_1,u_2\in\mathrm{SL}$. Then $[[\mathsf{b}[u_1],\mathsf{b}[u_2]],\mathsf{b}[\mathrm{SL}(\alpha)]]\neq 0$.*

Proof. Let $M_k(\alpha)=(k\delta,i)$. Due to Proposition 4.18(b), we have $[\mathsf{b}[u_1],\mathsf{b}[u_2]]=\sum_{j=1}^{i} a_j\mathsf{b}[\mathrm{SL}_j(k\delta)]$ with $a_j\in\mathbb{C}$ and $a_i\neq 0$. But $[\mathsf{b}[\mathrm{SL}_j(k\delta)],\mathsf{b}[\mathrm{SL}(\alpha)]]$ vanishes for $j<i$ and is nonzero for $j=i$ by Lemma 4.15. The result follows. □

Corollary 4.22. *For any $1\leq i\leq |I|$, $k\in\mathbb{Z}_{>0}$, and any splitting $\mathrm{SL}_i(k\delta)=uv$ with $u,v\in\mathrm{SL}$, we have $m_k(\deg(u))=(k\delta,i)=m_k(\deg(v))$.*

Proof. By Corollary 4.20 we know that $\deg(u)\in\widehat{\Delta}^{+,\mathrm{re}}$ and so $h_{\deg(u)}t^k\in\mathcal{S}^k_i\backslash\mathcal{S}^k_{i-1}$ by Proposition 4.18(b). Therefore, $m_k(\deg(u))=(k\delta,i)$ by Lemma 4.17. The proof of $m_k(\deg(v))=(k\delta,i)$ is analogous. □

Corollary 4.23. *For any $\alpha\in\widehat{\Delta}^{+,\mathrm{re}}$ consider any splitting $\mathrm{SL}(\alpha)=uv$ with $u,v\in\mathrm{SL}$. If $\deg(u)=k\delta$, then $u=\mathrm{SL}(M_k(\alpha))$. If $\deg(v)=k\delta$, then $v=\mathrm{SL}(M_k(\alpha))$.*

Proof. First, let us assume that $\deg(u)=k\delta$. If $u<\mathrm{SL}(M_k(\alpha))$, then we would get a contradiction to the generalized Leclerc algorithm (2.16). Indeed, for $\mathrm{SL}(M_k(\alpha))<v$ we have $\mathrm{SL}(\alpha)=uv<\mathrm{SL}(M_k(\alpha))v$, while for $\mathrm{SL}(M_k(\alpha))>v$ we have $\mathrm{SL}(\alpha)=uv<vu<v\mathrm{SL}(M_k(\alpha))$, as well as $[\mathsf{b}[\mathrm{SL}(M_k(\alpha))],\mathsf{b}[v]]\neq 0$. On the other hand, if $u>\mathrm{SL}(M_k(\alpha))=\mathrm{SL}(M_k(\alpha-k\delta))$, then $[\mathsf{b}[u],\mathsf{b}[v]]=0$ by Corollary 4.16, a contradiction to Proposition 4.18(a). Therefore, $u=\mathrm{SL}(M_k(\alpha))$.

The proof of the claim for the case $\deg(v)=k\delta$ is analogous. □

5. Convexity and Monotonicity

In this section, we prove two key results of the present note, which generalize the results of [AT, §5] to all types and do not rely on the explicit formulas of SL-words.

5.1. Statement of convexity.

For any $\alpha\in\widehat{\Delta}^{+,\mathrm{ext}}$, let us introduce sets $\mathcal{L}_\alpha,\mathcal{R}_\alpha$ that are key for the convexity:

Definition 5.2. *(a) For $\alpha\in\widehat{\Delta}^{+,\mathrm{re}}$, we define:*

$$\mathcal{L}_\alpha=\left\{\beta\,\middle|\,\substack{\beta\in\widehat{\Delta}^{+,\mathrm{re}}\\ \gamma:=\alpha-\beta\in\widehat{\Delta}^{+,\mathrm{re}}\\ \beta<\gamma}\right\}\cup\left\{\beta\,\middle|\,\substack{\beta\in O(\alpha),|\beta|=|k\delta|<|\alpha|\\ M_k(\alpha)<\alpha-k\delta}\right\}\cup\left\{\alpha-k\delta\,\middle|\,\substack{0<|k\delta|<|\alpha|\\ M_k(\alpha)>\alpha-k\delta}\right\},$$

$$\mathcal{R}_\alpha=\left\{\gamma\,\middle|\,\substack{\gamma\in\widehat{\Delta}^{+,\mathrm{re}}\\ \beta:=\alpha-\gamma\in\widehat{\Delta}^{+,\mathrm{re}}\\ \beta<\gamma}\right\}\cup\left\{\beta\,\middle|\,\substack{\beta\in O(\alpha),|\beta|=|k\delta|<|\alpha|\\ M_k(\alpha)>\alpha-k\delta}\right\}\cup\left\{\alpha-k\delta\,\middle|\,\substack{0<|k\delta|<|\alpha|\\ M_k(\alpha)<\alpha-k\delta}\right\}.$$

(b) For $\alpha \in \widehat{\Delta}^{+,\mathrm{imx}}$, we define the corresponding sets $\mathcal{L}_\alpha, \mathcal{R}_\alpha \subset C(\alpha)$ via:

$$\mathcal{L}_\alpha = \big\{\beta \,\big|\, \beta \in C(\alpha),\ |\beta| < |\alpha|, \text{ and } \beta < \alpha - \beta\big\},$$
$$\mathcal{R}_\alpha = \big\{\alpha - \beta \,\big|\, \beta \in \mathcal{L}_\alpha\big\}.$$

Remark 5.3. While the above usage of $M_k(\alpha)$ in (a) may look strange, we will see in Lemma 5.7 that $m_k(\alpha) < \alpha - k\delta \iff M_k(\alpha) < \alpha - k\delta$ and $m_k(\alpha) > \alpha - k\delta \iff M_k(\alpha) > \alpha - k\delta$. Thus, one can use any $\beta \in O(\alpha)$, $|\beta| = |k\delta|$, instead of $M_k(\alpha)$.

The first main result of this section is the following convexity:

Theorem 5.4. *For any $\alpha \in \widehat{\Delta}^{+,\mathrm{ext}}$, we have:*

$$\beta < \alpha < \gamma \qquad \forall\, \beta \in \mathcal{L}_\alpha, \gamma \in \mathcal{R}_\alpha.$$

Equivalently:

$$\max(\mathcal{L}_\alpha) < \alpha < \min(\mathcal{R}_\alpha) \qquad \forall\, \alpha \in \widehat{\Delta}^{+,\mathrm{ext}}. \tag{5.1}$$

Similarly to the convexity for finite type of Proposition 2.21, the inequality

$$\max(\mathcal{L}_\alpha) < \alpha$$

is easy, as it follows from Lemma 5.9. Thus, the key is to prove $\alpha < \min(\mathcal{R}_\alpha)$.

Remark 5.5. We note that (5.1) implies the pre-convexity (1.3), see Remark 5.8.

5.6. **Auxiliary results.**

In preparation for the general monotonicity of Proposition 5.21, let us establish:

Lemma 5.7. *Fix $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$ and $k \in \mathbb{Z}_{>0}$. If Theorem 5.4 holds for all roots of height $\leq |\alpha + k\delta|$, then the following properties are equivalent:*

(1) $\alpha < \alpha+\delta < \cdots < \alpha+k\delta < M_1(\alpha)$ (resp. $\alpha > \alpha+\delta > \cdots > \alpha+k\delta > M_1(\alpha)$),
(2) $\alpha < M_1(\alpha)$ (resp. $\alpha > M_1(\alpha)$),
(3) α is less than (resp. greater than) all elements in $O(\alpha)$ of height $< |\alpha+k\delta|$.

Proof. First, let us prove the "(2) ⇒ (1)" implication. If $\alpha < M_1(\alpha)$, then $\alpha \in \mathcal{L}_{\alpha+\delta}$ and $M_1(\alpha) \in \mathcal{R}_{\alpha+\delta}$, so that $\alpha < \alpha + \delta < M_1(\alpha)$ by the convexity (5.1). But $M_1(\alpha + p\delta) = M_1(\alpha)$ for any $p \in \mathbb{Z}_{>0}$, as $\mathsf{b}[\mathrm{SL}(\alpha + p\delta)] \sim \mathsf{b}[\mathrm{SL}(\alpha)]t^p$. Thus, repeating the above argument with $\alpha + \delta$ instead of α, then with $\alpha + 2\delta$ and so on, we eventually get the desired chain $\alpha < \alpha + \delta < \alpha + 2\delta < \cdots < \alpha + k\delta < M_1(\alpha)$. Similarly, if $M_1(\alpha) < \alpha$, then $M_1(\alpha) \in \mathcal{L}_{\alpha+\delta}$ and $\alpha \in \mathcal{R}_{\alpha+\delta}$, so that $M_1(\alpha) < \alpha + \delta < \alpha$ by the convexity (5.1). Then, following the same logic as above, we obtain $\alpha > \alpha + \delta > \alpha + 2\delta > \cdots > \alpha + k\delta > M_1(\alpha)$.

Let us prove the "(1) ⇒ (3)" implication for $\alpha < \alpha + \delta < \cdots < \alpha + k\delta < M_1(\alpha)$. Pick any $\beta \in O(\alpha)$ with $|\beta| = |p\delta| < |\alpha + k\delta|$. We consider two cases:

- If $|\beta| < |\alpha|$, then set $\alpha' := \alpha - p\delta$. Considering α' and $M_p(\alpha)$, we see that either $\alpha' \in \mathcal{L}_\alpha, M_p(\alpha) \in \mathcal{R}_\alpha$ or $\alpha' \in \mathcal{R}_\alpha, M_p(\alpha) \in \mathcal{L}_\alpha$. However, it must be the first of these options, due to Theorem 5.4 and the assumption $\alpha' < \alpha$. Hence, $\beta \in \mathcal{R}_\alpha$, and we obtain $\alpha < \beta$ by applying Theorem 5.4 once again.
- If $|\beta| > |\alpha|$, let us choose $s \in \mathbb{Z}_{\geq 0}$ so that $\overline{\alpha} = \alpha - s\delta \in \widehat{\Delta}^{+,\mathrm{re}}$ satisfies $0 < |\overline{\alpha}| < |\delta|$. As $\alpha < M_1(\alpha)$, we have $\alpha - \delta < \alpha < M_1(\alpha)$ by Theorem 5.4. Repeating this argument, we get $\overline{\alpha} = \alpha - s\delta < \alpha < M_1(\alpha)$. Define $\overline{\alpha}' := \delta - \overline{\alpha}$. Applying Theorem 5.4 to $\delta = \overline{\alpha} + \overline{\alpha}'$, we obtain $\overline{\alpha} < M_1(\alpha) < \overline{\alpha}'$. Pick $t \in \mathbb{Z}_{\geq 0}$ so that $|\alpha + t\delta| < |p\delta| < |\alpha + (t+1)\delta|$. Combining the assumption with $M_1(\alpha) < \overline{\alpha}'$ proved above, we get $\alpha + t\delta < M_1(\alpha) < \overline{\alpha}'$. But then we have $\alpha + t\delta \in \mathcal{L}_\beta$ and

$\overline{\alpha}' \in \mathcal{R}_\beta$, so that $\alpha + t\delta < \beta$ by Theorem 5.4. Combining this with $\alpha \leq \alpha + t\delta$, we ultimately get $\alpha < \beta$.

The proof of "(1) $\Rightarrow$ (3)" implication for $\alpha > \alpha + \delta > \cdots > \alpha + k\delta > M_1(\alpha)$ is similar. Pick $\beta \in O(\alpha)$ with $\beta = |p\delta| < |\alpha + k\delta|$. We consider two cases:

- If $|\beta| < |\alpha|$, then consider $\alpha' = \alpha - p\delta$, so that $\alpha' > \alpha$ by the assumption. Considering α' and $M_p(\alpha)$, we likewise deduce from Theorem 5.4 that $\alpha' \in \mathcal{R}_\alpha$, $M_p(\alpha) \in \mathcal{L}_\alpha$. Therefore, we have $\beta \in \mathcal{L}_\alpha$ and so $\beta < \alpha$ by Theorem 5.4.
- If $|\beta| > |\alpha|$, then consider $\overline{\alpha}, \overline{\alpha}'$ defined as before. As $M_1(\alpha) < \alpha$, we have $M_1(\alpha) < \alpha < \alpha - \delta$ by Theorem 5.4. Repeating this argument, we get $M_1(\alpha) < \alpha < \alpha - s\delta = \overline{\alpha}$. Applying Theorem 5.4 to $\delta = \overline{\alpha} + \overline{\alpha}'$, we then obtain $\overline{\alpha}' < M_1(\alpha) < \overline{\alpha}$. Pick t as above, and note that $\overline{\alpha}' < M_1(\alpha) < \alpha + t\delta$. Thus $\alpha + t\delta \in \mathcal{R}_\beta$ and $\overline{\alpha}' \in \mathcal{L}_\beta$, and so $\beta < \alpha + t\delta$ by Theorem 5.4. Combining this with $\alpha + t\delta \leq \alpha$, we get $\beta < \alpha$.

Finally, the implication "(3) $\Rightarrow$ (2)" is obvious. This completes the proof. □

Remark 5.8. The above result provides a more natural version of Theorem 5.4:

- if $\alpha < \beta$ with $\alpha, \beta, \alpha + \beta \in \widehat{\Delta}^{+,\mathrm{re}}$, then $\alpha < \alpha + \beta < \beta$;
- if $\alpha < \beta$ with $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}, \beta \in O(\alpha)$, then $\alpha < \alpha + \beta < \beta$;
- if $\alpha < \beta$ with $\beta \in \widehat{\Delta}^{+,\mathrm{re}}, \alpha \in O(\beta)$, then $\alpha < \alpha + \beta < \beta$;
- if $\alpha < \beta$ with $\alpha, \beta \in \widehat{\Delta}^{+,\mathrm{re}}$ and $\alpha + \beta = k\delta$, then $\alpha < \gamma < \beta$ for all $\gamma \in O(\alpha)$ satisfying $|\gamma| = |k\delta|$.

The next result indicates the importance of the set $\mathcal{L}_\alpha$:

Lemma 5.9. *For $\alpha \in \widehat{\Delta}^{+,\mathrm{ext}}$, we have:*

$$\max(\mathcal{L}_\alpha) = \deg(\mathrm{SL}^{ls}(\alpha)).$$

Proof. First, note $\mathrm{SL}^{ls}(\alpha) < \mathrm{SL}(\alpha) < \mathrm{SL}^{rs}(\alpha)$. We claim that $\deg(\mathrm{SL}^{ls}(\alpha)) \in \mathcal{L}_\alpha$. This is clear if $\deg(\mathrm{SL}^{ls}(\alpha)), \deg(\mathrm{SL}^{rs}(\alpha)) \in \widehat{\Delta}^{+,\mathrm{re}}$. If $\deg(\mathrm{SL}^{rs}(\alpha)) \in \widehat{\Delta}^{+,\mathrm{imx}}$, then $\deg(\mathrm{SL}^{rs}(\alpha)) = M_k(\alpha)$ for some k by Corollary 4.23, and the claim follows. Likewise, if $\deg(\mathrm{SL}^{ls}(\alpha)) \in \widehat{\Delta}^{+,\mathrm{imx}}$, then $\deg(\mathrm{SL}^{ls}(\alpha)) = M_k(\alpha)$ for some k.

Let us now assume the contrary, that is, $\max(\mathcal{L}_\alpha) > \deg(\mathrm{SL}^{ls}(\alpha))$. This implies $\max(\mathcal{L}_\alpha) > \alpha$, due to Lemma 4.2. We shall now consider two cases:

(1) For $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, we claim that we get a contradiction with the generalized Leclerc algorithm (2.16). This is clear if $\beta = \max(\mathcal{L}_\alpha)$ is real. On the other hand, if $\beta \in \widehat{\Delta}^{+,\mathrm{imx}}$, then we must have $\beta = M_k(\alpha)$ for some k by Corollary 4.23, and so $M_k(\alpha) < \alpha - k\delta$. Combining this with $[\mathsf{b}[\mathrm{SL}(M_k(\alpha))], \mathsf{b}[\mathrm{SL}(\alpha - k\delta)]] \neq 0$, we get $\mathrm{SL}(M_k(\alpha))\mathrm{SL}(\alpha - k\delta) > \mathrm{SL}(\alpha)$, a contradiction with the generalized Leclerc algorithm (2.16).

(2) For $\alpha = (k\delta, i) \in \widehat{\Delta}^{+,\mathrm{imx}}$, let $\beta = \max(\mathcal{L}_\alpha)$. Then $\gamma := \alpha - \beta$ satisfies $\gamma \in \mathcal{R}_\alpha \subset C(\alpha)$ and $\alpha < \beta < \gamma$. This contradicts Remark 4.12.

This completes the proof, as we get contradiction in both cases. □

We conclude this subsection with several simple results:

Lemma 5.10. *For $\alpha, \beta, \alpha + \beta \in \widehat{\Delta}^{+,\mathrm{re}}$, $\gamma \in \widehat{\Delta}^{+,\mathrm{imx}}$ if $[\mathsf{b}[\mathrm{SL}(\gamma)], \mathsf{b}[\mathrm{SL}(\alpha + \beta)]] \neq 0$, then $\mathsf{b}[\mathrm{SL}(\gamma)]$ commutes with at most one of $\mathsf{b}[\mathrm{SL}(\alpha)]$, $\mathsf{b}[\mathrm{SL}(\beta)]$.*

Proof. According to the Jacobi identity, we have:

$$[\mathsf{b}[\mathrm{SL}(\gamma)],[\mathsf{b}[\mathrm{SL}(\alpha)],\mathsf{b}[\mathrm{SL}(\beta)]]]+[\mathsf{b}[\mathrm{SL}(\alpha)],[\mathsf{b}[\mathrm{SL}(\beta)],\mathsf{b}[\mathrm{SL}(\gamma)]]]+ \\ [\mathsf{b}[\mathrm{SL}(\beta)],[\mathsf{b}[\mathrm{SL}(\gamma)],\mathsf{b}[\mathrm{SL}(\alpha)]]]=0.$$

Thus, if $\mathsf{b}[\mathrm{SL}(\gamma)]$ commuted with both $\mathsf{b}[\mathrm{SL}(\alpha)]$ and $\mathsf{b}[\mathrm{SL}(\beta)]$, then we would have

$$[\mathsf{b}[\mathrm{SL}(\gamma)],[\mathsf{b}[\mathrm{SL}(\alpha)],\mathsf{b}[\mathrm{SL}(\beta)]]]=0.$$

The latter contradicts the assumption, since $[\mathsf{b}[\mathrm{SL}(\alpha)],\mathsf{b}[\mathrm{SL}(\beta)]]\sim\mathsf{b}[\mathrm{SL}(\alpha+\beta)]$. □

Corollary 5.11. *Let $\alpha,\beta\in\widehat{\Delta}^{+,\mathrm{re}}$ satisfy $\alpha+\beta\in\widehat{\Delta}^{+,\mathrm{re}}$. If $M_k(\alpha)>M_k(\beta)$ then $M_k(\alpha)=M_k(\alpha+\beta)$, and if $M_k(\alpha)=M_k(\beta)$ then $M_k(\alpha+\beta)\leq M_k(\alpha)$.*

Proof. If $M_k(\alpha)>M_k(\beta)$, then $\mathsf{b}[\mathrm{SL}(M_k(\alpha))]$ commutes with $\mathsf{b}[\mathrm{SL}(\beta)]$. Combining this with $[\mathsf{b}[\mathrm{SL}(\alpha+k\delta)],\mathsf{b}[\mathrm{SL}(\beta)]]\neq 0$ (due to $\alpha+\beta\in\widehat{\Delta}^{+,\mathrm{re}}$) and the fact that $[\mathsf{b}[\mathrm{SL}(M_k(\alpha))],\mathsf{b}[\mathrm{SL}(\alpha)]]$ is a nonzero multiple of $\mathsf{b}[\mathrm{SL}(\alpha)]t^k$, we obtain $[\mathsf{b}[\mathrm{SL}(M_k(\alpha))],\mathsf{b}[\mathrm{SL}(\alpha+\beta)]]\neq 0$ by the Jacobi identity. Hence, $M_k(\alpha+\beta)\geq M_k(\alpha)$ by Lemma 4.15. But the strict inequality $M_k(\alpha+\beta)>M_k(\alpha)$ would contradict Lemma 5.10. Therefore, we have $M_k(\alpha+\beta)=M_k(\alpha)$.

Likewise, if $M_k(\alpha)=M_k(\beta)$, then $M_k(\alpha+\beta)\leq M_k(\alpha)$ by Lemma 5.10. □

Lemma 5.12. *Let $\alpha,\beta\in\widehat{\Delta}^{+,\mathrm{re}}$ satisfy $\alpha+\beta\in\widehat{\Delta}^{+,\mathrm{re}}$. If $m_k(\alpha)<m_k(\beta)$ then $m_k(\alpha+\beta)=m_k(\alpha)$, and if $m_k(\alpha)=m_k(\beta)$ then $m_k(\alpha+\beta)\geq m_k(\alpha)$.*

Proof. Both results follow immediately from Lemma 4.17, since $h_{\alpha+\beta}=h_\alpha+h_\beta$. □

The following result is standard, cf. [NT, Claim 2.35]:

Lemma 5.13. *Assume that roots $\alpha,\beta,\alpha',\beta'\in\widehat{\Delta}^{+,\mathrm{re}}$ satisfy $\alpha+\beta=\alpha'+\beta'$ and*

(a) $\alpha+\beta\notin\widehat{\Delta}^{+,\mathrm{im}}$

or

(b) $\alpha+\beta\in\widehat{\Delta}^{+,\mathrm{im}}$, *but* $(\alpha,\beta')\neq 0$.

Then one of the following four cases must hold:

$$\alpha+\gamma=\alpha' \quad \textit{and} \quad \beta-\gamma=\beta'$$

or

$$\alpha-\gamma=\alpha' \quad \textit{and} \quad \beta+\gamma=\beta'$$

or

$$\alpha+\gamma=\beta' \quad \textit{and} \quad \beta-\gamma=\alpha'$$

or

$$\alpha-\gamma=\beta' \quad \textit{and} \quad \beta+\gamma=\alpha'$$

with $\gamma\in\widehat{\Delta}^+\cup\{0\}$.

Proof. Assume first that $\alpha+\beta\notin\widehat{\Delta}^{+,\mathrm{im}}$. If $(\alpha,\alpha')>0$, then the reflection $s_\alpha(\alpha')=\alpha'-k\alpha$ is a root, for some $k\in\mathbb{Z}_{>0}$. This implies that $\alpha'-\alpha\in\Delta\sqcup\{0\}$, proving the claim. The same argument applies if $(\alpha,\beta')>0$, $(\beta,\alpha')>0$, or $(\beta,\beta')>0$. However, one of these inequalities must hold as $(\alpha+\beta,\alpha'+\beta')=(\alpha+\beta,\alpha+\beta)>0$.

In the case (b), we have $(\alpha+\beta,\alpha'+\beta')=(\alpha+\beta,\alpha+\beta)=0$, but $(\alpha,\beta')\neq 0$. Hence, the above argument can be applied without changes. □

5.14. **Proof of convexity.**

In this subsection, we establish the convexity (5.1).

Proof of Theorem 5.4. As noted before, Lemma 5.9 implies that $\max(\mathcal{L}_\mu) < \mu$ for any $\mu \in \widehat{\Delta}^{+,\mathrm{ext}}$. Thus it remains to show that $\mu < \min(\mathcal{R}_\mu)$. Evoking Lemma 4.2 and Lemma 5.9, the latter is equivalent to

$$\max(\mathcal{L}_\mu) < \min(\mathcal{R}_\mu). \tag{5.2}$$

We show this by induction on the height of μ (with the conventions $|(k\delta, i)| = |k\delta|$). The base case $|\mu| = 2$ is obvious. For the inductive step, let $\mathrm{SL}(\mu) = \mathrm{SL}(\alpha)\mathrm{SL}(\beta)$ be the standard factorization, so that $\alpha = \max(\mathcal{L}_\mu)$ by Lemma 5.9. Pick any $\alpha' \in \mathcal{L}_\mu, \beta' \in \mathcal{R}_\mu$ with $\alpha' + \beta' = \alpha + \beta$. We can assume that $\alpha' < \alpha$ and also that $\beta' < \beta$ as otherwise $\alpha < \mu < \beta \leq \beta'$. Let us assume the contrary to (5.2):

$$\beta' < \alpha.$$

We start with several general results:

Claim 5.15. For any $\alpha, \beta, \alpha + \beta \in \widehat{\Delta}^{+,\mathrm{re}}$ and $k \in \mathbb{Z}_{>0}$ such that $\alpha' = \alpha + k\delta$ and $\beta' = \beta - k\delta$ are also affine positive roots, if Theorem 5.4 holds for all roots of height $< |\alpha + \beta|$, then we cannot have $\alpha', \beta' < \alpha, \beta$.

Proof. Assume the contrary: $\alpha', \beta' < \alpha, \beta$. According to Lemma 5.7, we have:

$$m_p(\alpha) \leq M_p(\alpha) < \alpha', \beta' < \alpha, \beta < m_p(\beta) \leq M_p(\beta) \qquad \forall\, 0 < p < |\alpha + \beta|/|\delta|.$$

Hence, we have $m_p(\alpha+\beta) = m_p(\alpha)$ and $M_p(\alpha+\beta) = M_p(\beta)$, due to Lemma 5.12 and Corollary 5.11. Applying Theorem 5.4 to $\alpha, \beta', \alpha + \beta'$, we obtain $m_p(\alpha + \beta) < \beta' < \alpha + \beta' < \alpha < M_p(\alpha + \beta)$. As $m_p(\alpha + \beta') = m_p(\alpha + \beta), M_p(\alpha + \beta') = M_p(\alpha + \beta)$, the resulting inequalities $m_p(\alpha+\beta') < \alpha+\beta' < M_p(\alpha+\beta')$ contradict to Lemma 5.7. □

Claim 5.16. If $\alpha + \beta \in \widehat{\Delta}^{+,\mathrm{im}}$, $\alpha' = \alpha + k\delta$, $\beta' = \beta - k\delta$ with $\alpha, \beta, \alpha + k\delta, \beta - k\delta \in \widehat{\Delta}^{+,\mathrm{re}}$, $k \neq 0$, and Theorem 5.4 holds for all roots of height $< |\alpha + \beta|$, then we cannot have $\alpha', \beta' < \alpha, \beta$.

Proof. Assume the contrary: $\alpha', \beta' < \alpha, \beta$. If $k > 0$, then according to Lemma 5.7, we have $M_1(\alpha) < \alpha', \beta' < \alpha, \beta < M_1(\beta)$, a contradiction with $M_1(\alpha) = M_1(\beta)$ due to $\alpha + \beta \in \widehat{\Delta}^{+,\mathrm{im}}$. If $k < 0$, then we likewise have $M_1(\beta) < \alpha', \beta' < \alpha, \beta < M_1(\alpha)$, which again contradicts the equality $M_1(\alpha) = M_1(\beta)$. □

The next two claims cover the cases when exactly one of $\alpha, \alpha', \beta, \beta'$ is in $\widehat{\Delta}^{+,\mathrm{imx}}$.

Claim 5.17. If $\alpha, \beta, \alpha + \beta \in \widehat{\Delta}^{+,\mathrm{re}}$ and Theorem 5.4 holds for all roots of height $< |\alpha+\beta|$, then we cannot have $\alpha+\beta-k\delta, m_k(\alpha+\beta) < \alpha, \beta$ whenever $|k\delta| < |\alpha+\beta|$.

Proof. Assume the contrary, that is, $\alpha + \beta - k\delta, m_k(\alpha + \beta) < \alpha, \beta$ for some k with $|k\delta| < |\alpha+\beta|$. According to Lemma 5.12, we have $m_k(\alpha+\beta) \geq \min\{m_k(\alpha), m_k(\beta)\}$. We can assume without loss of generality that $m_k(\alpha) \leq m_k(\beta)$, so that $m_k(\alpha) < \alpha$.

Let us first consider the case $|k\delta| > |\alpha|$. Applying Theorem 5.4 to $\bar{\alpha} = k\delta - \alpha$ and α, we get $\bar{\alpha} < m_k(\alpha) < \alpha$, and so $\bar{\alpha} < m_k(\alpha) \leq m_k(\alpha + \beta) < \beta$. But then $(\alpha + \beta - k\delta) + \bar{\alpha} = \beta$ while $(\alpha + \beta - k\delta), \bar{\alpha} < \beta$, which contradicts Theorem 5.4.

Let us now consider the case $|k\delta| < |\alpha|$. Applying Theorem 5.4 to $\alpha - k\delta, \beta$ and using $\alpha + \beta - k\delta < \beta$, we get $\alpha - k\delta < \alpha + \beta - k\delta$. Thus, $\alpha - k\delta < \alpha + \beta - k\delta < \alpha$, and we get $\alpha - k\delta < \alpha < m_k(\alpha)$ by Lemma 5.7. But then $m_k(\alpha) > \alpha > m_k(\alpha + \beta)$, a contradiction with $m_k(\alpha + \beta) \geq \min\{m_k(\alpha), m_k(\beta)\} = m_k(\alpha)$. □

Claim 5.18. If $\alpha, \beta, \alpha+\beta \in \widehat{\Delta}^{+,\mathrm{re}}$ and Theorem 5.4 holds for all roots of height $< |\alpha+\beta|$, then we cannot have $\alpha, \beta < M_k(\alpha+\beta), \alpha+\beta-k\delta$ whenever $|k\delta| < |\alpha+\beta|$.

Proof. Assume the contrary: $\alpha, \beta < M_k(\alpha+\beta), \alpha+\beta-k\delta$ for some k with $|k\delta| < |\alpha+\beta|$. According to Corollary 5.11, we have $M_k(\alpha+\beta) \leq \max\{M_k(\alpha), M_k(\beta)\}$. We can assume without loss of generality that $M_k(\alpha) \geq M_k(\beta)$, so that $\alpha < M_k(\alpha)$.

Let us first consider the case $|k\delta| > |\alpha|$. Applying Theorem 5.4 to $\bar{\alpha} = k\delta - \alpha$ and α, we get $\alpha < M_k(\alpha) < \bar{\alpha}$, and so $\alpha < M_k(\alpha+\beta) \leq M_k(\alpha) < \bar{\alpha}$. But then $(\alpha+\beta-k\delta)+\bar{\alpha} = \beta$, while $(\alpha+\beta-k\delta), \bar{\alpha} > \beta$, which contradicts Theorem 5.4.

Let us now consider the case $|k\delta| < |\alpha|$. Applying Theorem 5.4 to $\alpha - k\delta, \beta$ and using $\alpha+\beta-k\delta > \beta$, we get $\alpha - k\delta > \alpha+\beta-k\delta$. Thus, $\alpha-k\delta > \alpha+\beta-k\delta > \alpha$, and we get $\alpha - k\delta > \alpha > M_k(\alpha)$ by Lemma 5.7. But then $M_k(\alpha) < \alpha < M_k(\alpha+\beta)$, a contradiction with $M_k(\alpha+\beta) \leq \max\{M_k(\alpha), M_k(\beta)\} = M_k(\alpha)$. □

We cannot have more than two of $\alpha, \alpha', \beta, \beta'$ in $\widehat{\Delta}^{+,\mathrm{imx}}$. If two of these roots are imaginary, then we claim that $\alpha < \beta'$. To this end, we consider four cases:

- $\alpha, \alpha' \in \widehat{\Delta}^{+,\mathrm{re}}$, $\beta, \beta' \in \widehat{\Delta}^{+,\mathrm{imx}}$.
  Since $\beta, \beta' \in O(\alpha) = O(\alpha')$ and $\alpha < \beta$, we then have $\alpha < \beta'$ by Lemma 5.7.
- $\alpha, \alpha' \in \widehat{\Delta}^{+,\mathrm{imx}}$, $\beta, \beta' \in \widehat{\Delta}^{+,\mathrm{re}}$.
  As $\alpha, \alpha' \in O(\beta') = O(\beta)$ and $\alpha' < \beta'$, we must have $\alpha < \beta'$ by Lemma 5.7.
- $\alpha, \beta' \in \widehat{\Delta}^{+,\mathrm{re}}$, $\beta, \alpha' \in \widehat{\Delta}^{+,\mathrm{imx}}$.
  If $\beta' < \alpha$, then we would have $\alpha' < \beta' < \alpha < \beta$, which contradicts Lemma 5.7 as $\alpha', \beta \in O(\beta') = O(\alpha)$. Therefore, we must have $\beta' > \alpha$.
- $\alpha', \beta \in \widehat{\Delta}^{+,\mathrm{re}}$, $\beta', \alpha \in \widehat{\Delta}^{+,\mathrm{imx}}$.
  If $\beta' < \alpha$, then we would have $\alpha' < \beta' < \alpha < \beta$, which contradicts to Lemma 5.7 as $\alpha, \beta' \in O(\beta) = O(\alpha')$. Therefore, we must have $\beta' > \alpha$.

Returning to the proof of Theorem 5.4, if $\alpha+\beta = \alpha'+\beta'$ is real, then we can apply Lemma 5.13(a). On the other hand, if $\alpha'+\beta' = k\delta$ with $k > 0$ and $\alpha' < \beta'$ are real roots, then it suffices to show that $\mu = M_k(\beta')$ is $< \beta'$. In this case $[[\mathsf{b}[\mathrm{SL}(\alpha)], \mathsf{b}[\mathrm{SL}(\beta)]], \mathsf{b}[\mathrm{SL}(\beta')]] \neq 0$ by Corollary 4.21. Therefore, $[h_\alpha t^k, e_{\beta'}] \neq 0$, which implies that $(\alpha, \beta') \neq 0$, hence, we meet the requirements of Lemma 5.13(b).

Applying Lemma 5.13, we get four different cases to consider. If $\gamma = k\delta$ with $k > 0$, then we cannot have $\alpha', \beta' < \alpha, \beta$ in all cases, due to Claim 5.15 if $\alpha+\beta \in \widehat{\Delta}^{+,\mathrm{re}}$ or Claim 5.16 if $\alpha+\beta \in \widehat{\Delta}^{+,\mathrm{im}}$. If $\gamma \in \widehat{\Delta}^{+,\mathrm{re}}$, then evoking Claims 5.17–5.18 and the above four bullets, we can further assume that $\alpha, \beta, \alpha', \beta' \in \widehat{\Delta}^{+,\mathrm{re}}$. The analysis in these cases is analogous to that of [NT, Proof of Proposition 2.34]:

- $\alpha+\gamma = \alpha'$, $\beta-\gamma = \beta'$.
  Applying the inductive hypothesis to $\alpha' = \gamma+\alpha$ and $\beta = \gamma+\beta'$ and using $\alpha' < \alpha$, $\beta' < \beta$, we get $\gamma < \alpha' < \alpha$ and $\beta' < \beta < \gamma$, a contradiction with $\alpha < \beta$.
- $\alpha-\gamma = \alpha'$, $\beta+\gamma = \beta'$.
  Applying the inductive hypothesis to $\alpha = \gamma+\alpha'$ and $\beta' = \gamma+\beta$ and using $\alpha' < \alpha$, $\beta' < \beta$, we get $\alpha' < \alpha < \gamma$ and $\gamma < \beta' < \beta$, a contradiction with $\beta' < \alpha$.
- $\alpha+\gamma = \beta'$, $\beta-\gamma = \alpha'$.
  Applying the inductive hypothesis to $\beta' = \gamma+\alpha$ and $\beta = \gamma+\alpha'$ and using $\beta' < \alpha$, $\alpha' < \beta$, we get $\gamma < \beta' < \alpha$ and $\alpha' < \beta < \gamma$, a contradiction with $\alpha < \beta$.
- $\alpha-\gamma = \beta'$, $\beta+\gamma = \alpha'$.
  Applying the inductive hypothesis to $\alpha = \gamma+\beta'$ and $\alpha' = \gamma+\beta$ and using $\beta' < \alpha$, $\alpha' < \beta$, we get $\beta' < \alpha < \gamma$ and $\gamma < \alpha' < \beta$, a contradiction with $\alpha' < \beta'$.

Thus, in all cases, we get a contradiction with the assumed inequality $\beta' < \alpha$. $\square$

5.19. **Monotonicity.**
This subsection generalizes [AT, §5.3].

Definition 5.20. *For $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, consider the decomposition $\alpha = \alpha' + k\delta$ with $\alpha' \in \widehat{\Delta}^{+,\mathrm{re}}, |\alpha'| < |\delta|, k \in \mathbb{Z}_{\geq 0}$ (i.e. $\alpha' \in \Delta^+ \cup (\delta - \Delta^+)$). Define the* ***chain*** *of α as:*

$$\mathrm{ch}(\alpha) = (\alpha', \alpha' + \delta, \alpha' + 2\delta, \ldots).$$

Combining Lemma 5.7 with Theorem 5.4, we get our second key result:

Proposition 5.21. *For any $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, the following properties are equivalent:*

(1) $\mathrm{ch}(\alpha)$ is monotone increasing (resp. decreasing),
(2) $\alpha < M_1(\alpha)$ (resp. $\alpha > M_1(\alpha)$),
(3) α is less than (resp. greater than) all elements in $O(\alpha)$.

In particular, each chain $\mathrm{ch}(\alpha)$ is monotonous.

Remark 5.22. This monotonicity heavily relies on Lemma 5.7, which was established at the same time as the convexity of Theorem 5.4 by induction on the height.

Corollary 5.23. *For $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$ with $|\alpha| < |\delta|$, the chain $\mathrm{ch}(\alpha)$ is monotone increasing (resp. decreasing) iff $\mathrm{ch}(\delta - \alpha)$ is monotone decreasing (resp. increasing).*

Proof. If $\mathrm{ch}(\alpha)$ is monotone increasing, then $\alpha < M_1(\alpha)$ by Proposition 5.21. Since $\alpha, \delta - \alpha \in C(M_1(\alpha))$, we have $\alpha < M_1(\alpha) < \delta - \alpha$ by Theorem 5.4, and so $\mathrm{ch}(\delta - \alpha)$ is monotone decreasing due to Proposition 5.21. Similarly, if $\mathrm{ch}(\alpha)$ is monotone decreasing, then $M_1(\alpha) < \alpha$, so that $\delta - \alpha < M_1(\alpha) < \alpha$ by Theorem 5.4, which implies that the chain $\mathrm{ch}(\delta - \alpha)$ is monotone increasing, due to Proposition 5.21. $\square$

Remark 5.24. We note that Proposition 5.21 and Corollary 5.23 are natural generalizations of the A-type results from [AT, Proposition 5.4, Remark 5.5], where they were rather derived from the explicit formulas for affine standard Lyndon words.

Lemma 5.25. *If $\alpha, \beta, \alpha + \beta \in \widehat{\Delta}^{+,\mathrm{re}}$, and both chains $\mathrm{ch}(\alpha), \mathrm{ch}(\beta)$ are increasing (resp. decreasing), then the chain $\mathrm{ch}(\alpha + \beta)$ is also increasing (resp. decreasing).*

Proof. Let us first assume that $\mathrm{ch}(\alpha), \mathrm{ch}(\beta)$ are both increasing. If $M_1(\alpha) \neq M_1(\beta)$, then we have $\alpha, \beta < \max\{M_1(\alpha), M_1(\beta)\}$ by Proposition 5.21. Assume without loss of generality that $\alpha < \beta$. Then, combining Theorem 5.4 and Corollary 5.11, we get:

$$\alpha < \alpha + \beta < \beta < \max\{M_1(\alpha), M_1(\beta)\} = M_1(\alpha + \beta).$$

Hence, $\mathrm{ch}(\alpha + \beta)$ is increasing by Proposition 5.21.

Let us now consider the case $M_1(\alpha) = M_1(\beta)$. Let $\alpha = (\bar{\alpha}, p)$, $\beta = (\bar{\beta}, s)$ with $|\bar{\alpha}|, |\bar{\beta}| < |\delta|$ and $p, s \in \mathbb{Z}_{\geq 0}$. Assume without loss of generality that $\delta - \bar{\alpha} < \delta - \bar{\beta}$. We note that $M_1(\alpha) = M_1(\bar{\alpha}) = M_1(\delta - \bar{\alpha})$ and $M_1(\beta) = M_1(\bar{\beta}) = M_1(\delta - \bar{\beta})$. Then both chains $\mathrm{ch}(\delta - \bar{\alpha}), \mathrm{ch}(\delta - \bar{\beta})$ are decreasing by Corollary 5.23. Combining Proposition 5.21, Theorem 5.4, and Corollary 5.11, we thus obtain:

$$M_1(2\delta - (\bar{\alpha} + \bar{\beta})) \leq M_1(\alpha) = M_1(\beta) < \delta - \bar{\alpha} < 2\delta - (\bar{\alpha} + \bar{\beta}) < \delta - \bar{\beta}.$$

Then, the chain $\mathrm{ch}(2\delta - (\bar{\alpha} + \bar{\beta}))$ is decreasing by Proposition 5.21, and hence the chain $\mathrm{ch}(\bar{\alpha} + \bar{\beta}) = \mathrm{ch}(\alpha + \beta)$ is increasing by evoking Corollary 5.23 once again.

The case when $\mathrm{ch}(\alpha), \mathrm{ch}(\beta)$ are both decreasing follows from above. Indeed, if $\alpha = (\bar{\alpha}, p), \beta = (\bar{\beta}, s)$ with $|\bar{\alpha}|, |\bar{\beta}| < |\delta|$ and $p, s \in \mathbb{Z}_{\geq 0}$, then the chains $\mathrm{ch}(\delta -$

$\bar{\alpha}), \mathrm{ch}(\delta - \bar{\beta})$ are both increasing by Corollary 5.23. Hence, $\mathrm{ch}(2\delta - (\bar{\alpha} + \bar{\beta}))$ is increasing by above, and so $\mathrm{ch}(\bar{\alpha}+\bar{\beta}) = \mathrm{ch}(\alpha+\beta)$ is decreasing by Corollary 5.23. □

For $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, the set $O(\alpha)$ provides an upper or lower bound on $\mathrm{ch}(\alpha)$, due to Remark 5.8. However, the following result often yields better bounds on $\mathrm{ch}(\alpha)$:

Lemma 5.26. *(a) If $\alpha, \beta, \alpha + \beta \in \widehat{\Delta}^{+,\mathrm{re}}$, with both $\mathrm{ch}(\alpha), \mathrm{ch}(\beta)$ increasing, then:*

$$\mathrm{ch}(\alpha + \beta) < \min\{m_k(\alpha), m_k(\beta)\} \qquad \forall\, k \in \mathbb{Z}_{>0}.$$

(b) If $\alpha, \beta, \alpha + \beta \in \widehat{\Delta}^{+,\mathrm{re}}$, with both $\mathrm{ch}(\alpha), \mathrm{ch}(\beta)$ decreasing, then:

$$\mathrm{ch}(\alpha + \beta) > \max\{M_k(\alpha), M_k(\beta)\} \qquad \forall\, k \in \mathbb{Z}_{>0}.$$

Proof. (a) According to Lemma 5.25, the chain $\mathrm{ch}(\alpha + \beta)$ is increasing, hence, $\mathrm{ch}(\alpha + \beta) < m_k(\alpha + \beta)$ by Proposition 5.21. If $m_k(\alpha) \neq m_k(\beta)$, then $m_k(\alpha + \beta) = \min\{m_k(\alpha), m_k(\beta)\}$, and the result follows. If $m_k(\alpha) = m_k(\beta)$, then $\mathrm{ch}(\alpha), \mathrm{ch}(\beta) < m_k(\alpha)$ by Proposition 5.21. Pick any $\gamma \in \mathrm{ch}(\alpha + \beta)$ with $|\gamma| \geq |\alpha + \beta|$, so that $\gamma = \alpha+\beta+p\delta$ for some $p \in \mathbb{Z}_{\geq 0}$. Assuming without loss of generality that $\alpha < \beta$, we then have $\alpha < \beta \leq \beta+p\delta < m_k(\alpha)$ by Proposition 5.21, so that $\gamma < \beta+p\delta < m_k(\alpha)$ by Theorem 5.4. The claim follows as $\mathrm{ch}(\alpha + \beta)$ is increasing

(b) The chain $\mathrm{ch}(\alpha+\beta)$ is decreasing by Lemma 5.25, hence, $M_k(\alpha+\beta) < \mathrm{ch}(\alpha+\beta)$ by Proposition 5.21. If $M_k(\alpha) \neq M_k(\beta)$, then $M_k(\alpha+\beta) = \max\{M_k(\alpha), M_k(\beta)\}$ by Corollary 5.11, and the result follows. If $M_k(\alpha) = M_k(\beta)$, then $M_k(\alpha) < \mathrm{ch}(\beta), \mathrm{ch}(\alpha)$ by Proposition 5.21. Pick any $\gamma \in \mathrm{ch}(\alpha+\beta)$ with $|\gamma| \geq |\alpha+\beta|$, so that $\gamma = \alpha+\beta+p\delta$ for some $p \in \mathbb{Z}_{\geq 0}$. Assuming without loss of generality that $\alpha < \beta$, we then have $M_k(\alpha) < \alpha+p\delta \leq \alpha < \beta$ by Proposition 5.21, so that $M_k(\alpha) < \alpha+p\delta < \gamma$ by Theorem 5.4. The claim follows as $\mathrm{ch}(\alpha + \beta)$ is decreasing. □

The following result is crucial for the next section:

Lemma 5.27. *For any $i \in \{1, 2, \ldots, |I| - 1\}$ and $k \in \mathbb{Z}_{>0}$, we have:*

$$\mathrm{SL}_{i+1}(k\delta) < \mathrm{SL}_i^{ls}(k\delta). \tag{5.3}$$

Proof. As $|\mathrm{SL}_i^{ls}(k\delta)| < |k\delta|$, the inequality (5.3) is equivalent to $\mathrm{SL}_{i+1}^{ls}(k\delta) < \mathrm{SL}_i^{ls}(k\delta)$ by Lemma 4.2. Assuming the contrary to the latter, we get $\mathrm{SL}_{i+1}^{ls}(k\delta) > \mathrm{SL}_i^{ls}(k\delta)$, since $\mathrm{SL}_{i+1}^{ls}(k\delta) \neq \mathrm{SL}_i^{ls}(k\delta)$. Evoking Lemma 4.2 once again, we then get $\mathrm{SL}_i(k\delta) < \mathrm{SL}_{i+1}^{ls}(k\delta) < \mathrm{SL}_{i+1}(k\delta)$, a contradiction. This establishes (5.3). □

We conclude this section with a couple of interesting observations:

Lemma 5.28. *If $\alpha, \beta, \gamma \in \widehat{\Delta}^{+,\mathrm{re}}$, $\mathrm{SL}(\alpha) = \mathrm{SL}(\gamma)\mathrm{SL}(\beta)$, and $\mathrm{ch}(\beta)$ is increasing, then $\mathrm{ch}(\alpha)$ is increasing.*

Proof. We have the following chain of inequalities $\mathrm{SL}(\alpha + \delta) \geq \mathrm{SL}(\gamma)\mathrm{SL}(\beta + \delta) > \mathrm{SL}(\gamma)\mathrm{SL}(\beta) = \mathrm{SL}(\alpha)$ and $\mathrm{SL}(\gamma) < \mathrm{SL}(\beta) < \mathrm{SL}(\beta + \delta)$. Hence $\mathrm{ch}(\alpha)$ increases. □

Lemma 5.29. *For any $1 \leq i \leq |I|$ and $k \in \mathbb{Z}_{>0}$, let $\alpha = \deg(\mathrm{SL}_i^{ls}(k\delta))$ and pick any decomposition $\alpha = \beta + \gamma$ with $\beta, \gamma \in \widehat{\Delta}^{+,\mathrm{re}}$. We cannot have both $\mathrm{ch}(\beta), \mathrm{ch}(\gamma)$ decreasing. Moreover, if $\mathrm{ch}(\beta), \mathrm{ch}(\gamma)$ are both increasing, then exactly one of $m_k(\beta), m_k(\gamma)$ is equal to $m_k(\alpha)$, and the other must be larger than $m_k(\alpha)$.*

Proof. If both chains $\mathrm{ch}(\beta), \mathrm{ch}(\gamma)$ are decreasing, then so is $\mathrm{ch}(\alpha)$ by Lemma 5.25. On the other hand, $\alpha < (k\delta, i) = m_k(\alpha)$ (with the equality due to Corollary 4.22), which implies that $\mathrm{ch}(\alpha)$ is increasing by Proposition 5.21, a contradiction.

Assume now that both chains $\mathrm{ch}(\beta), \mathrm{ch}(\gamma)$ are increasing. Then we have $\mathrm{ch}(\alpha) \leq \min\{m_k(\beta), m_k(\gamma)\}$ by Lemma 5.26. If $\min\{m_k(\beta), m_k(\gamma)\} < (k\delta, i)$, then the above would yield $\alpha < (k\delta, i+1)$, a contradiction with Lemma 5.27. According to Lemma 5.12 and Corollary 4.22, we have $(k\delta, i) = m_k(\alpha) \geq \min\{m_k(\beta), m_k(\gamma)\}$. Therefore, it suffices to show that $m_k(\gamma) = m_k(\beta) = (k\delta, i)$ is not possible. Assume the contrary, and without loss of generality we can also assume that $\beta < \gamma$. Due to Theorem 5.4, we then get $\gamma > \alpha = \deg(\mathrm{SL}_i^{ls}(k\delta))$, which implies $\gamma > (k\delta, i)$ by Lemma 4.2. But then $\mathrm{ch}(\gamma)$ is decreasing by Proposition 5.21, a contradiction. □

6. Imaginary affine standard Lyndon words

In this section, we investigate the imaginary affine standard Lyndon words and relations among those. Our analysis is crucially based on the results of Section 5.

6.1. Compatibility of flags and basic inequalities.

We start with the important compatibility of flags $\{\mathcal{S}^k_\bullet\}_{k\geq 1}$ from (4.1):

Proposition 6.2. *For any $i \in \{0, 1, \ldots, |I|\}$ and $k \in \mathbb{Z}_{>0}$, we have:*

$$\mathcal{S}_i^{k+1} = \mathcal{S}_i^k t.$$

Additionally, if $i > 0$ and $w = \mathrm{SL}_i^{ls}((k+1)\delta)$, then $m_k(\deg(w)) = (k\delta, i)$.

Proof. The proof proceeds by induction on i. The base case $i = 0$ is obvious. For the step of induction, let us assume that both claims hold for all $j < i$. Let us first verify that $m_k(\deg(w)) = (k\delta, i)$. If $m_k(\deg(w)) > (k\delta, i)$, then we actually have $[\mathsf{b}[w], \mathsf{b}[\mathrm{SL}((k+1)\delta - \deg(w))]] \sim h_w t^{k+1} \in \mathcal{S}_{i-1}^k t$ by Lemma 4.17. As $\mathcal{S}_{i-1}^k t = \mathcal{S}_{i-1}^{k+1}$ by the inductive hypothesis, we get a contradiction with Proposition 4.18(b). If $m_k(\deg(w)) < (k\delta, i)$, then $w < \mathrm{SL}(m_k(\deg(w))) \leq \mathrm{SL}_{i+1}(k\delta) < \mathrm{SL}_i^{ls}(k\delta)$ by Proposition 5.21 and Lemma 5.27. Let $\alpha = \deg(\mathrm{SL}_i^{ls}(k\delta))$, so that $\mathrm{SL}(\alpha) = \mathrm{SL}_i^{ls}(k\delta) > \mathrm{SL}_i((k+1)\delta)$ by Lemma 4.2. As $|\alpha| < |(k+1)\delta|$, we thus have $\mathrm{SL}(\alpha)\mathrm{SL}((k+1)\delta - \alpha) > \mathrm{SL}_i((k+1)\delta)$. We also have $\alpha < (k\delta, i) < (k+1)\delta - \alpha$ by Proposition 5.21. Then $[\mathsf{b}[\mathrm{SL}(\alpha)], \mathsf{b}[\mathrm{SL}((k+1)\delta - \alpha)]] \sim h_\alpha t^{k+1} \in \mathcal{S}_{i-1}^{k+1}$ by Corollary 4.9. But this contradicts the inductive hypothesis as $h_\alpha t^k \in \mathcal{S}_i^k \backslash \mathcal{S}_{i-1}^k$ by Corollary 4.22 and Lemma 4.17. This establishes $m_k(\deg(w)) = (k\delta, i)$.

We thus get $[\mathsf{b}[w], \mathsf{b}[\mathrm{SL}((k+1)\delta - \deg(w))]] \in \mathcal{S}_i^k t \backslash \mathcal{S}_{i-1}^k t$, due to Lemma 4.17. On the other hand, $[\mathsf{b}[w], \mathsf{b}[\mathrm{SL}((k+1)\delta - \deg(w))]] \in \mathcal{S}_i^{k+1} \backslash \mathcal{S}_{i-1}^{k+1}$ by Proposition 4.18. As $\mathcal{S}_{i-1}^{k+1} = \mathcal{S}_{i-1}^k t$ by the inductive hypothesis, and quotients $\mathcal{S}_i^k t / \mathcal{S}_{i-1}^k t, \mathcal{S}_i^{k+1} / \mathcal{S}_{i-1}^{k+1}$ are one-dimensional, we obtain $\mathcal{S}_i^{k+1} = \mathcal{S}_i^k t$. This completes the inductive step. □

Remark 6.3. In A-type, we rather used explicit formulas for $\mathsf{b}[\mathrm{SL}_i(k\delta)]$ of [AT, (4.67, 4.73)], which had a similar periodicity for $k \geq 2$, but not for $k = 1$. Thus, the above result simplifies several arguments from the proof of [AT, Theorem 4.7].

Corollary 6.4. *For any $k, p \in \mathbb{Z}_{>0}$ and $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$, we have:*

$$M_k(\alpha) = (k\delta, i) \iff M_p(\alpha) = (p\delta, i),$$
$$m_k(\alpha) = (k\delta, i) \iff m_p(\alpha) = (p\delta, i).$$

Proof. This follows directly from repeated applications of the above result. □

Combining this with Corollary 4.22, we obtain:

Corollary 6.5. *For any* $k, p \in \mathbb{Z}_{>0}$*,* $i \in \{1, 2, \dots, |I|\}$*, and any splitting* $\mathrm{SL}_i(k\delta) = uv$ *with* $u, v \in \mathrm{SL}$*, we have:* $m_p(\deg(u)) = (p\delta, i) = m_p(\deg(v))$*.*

Lemma 6.6. *For any* $i \in \{1, 2, \dots, |I|\}$ *and* $k \in \mathbb{Z}_{>0}$*, let* $\alpha = \deg(\mathrm{SL}_i^{ls}(k\delta))$*. Then:*

$$\mathrm{SL}(\alpha + \delta) \leq \mathrm{SL}_i^{ls}((k+1)\delta).$$

Proof. Assume by contradiction that $\mathrm{SL}(\alpha+\delta) > \mathrm{SL}_i^{ls}((k+1)\delta)$. Then $\mathrm{SL}(\alpha+\delta) > \mathrm{SL}_i((k+1)\delta)$ by Lemma 4.2. Combining Proposition 5.21 and Corollary 5.23, we get $\alpha+\delta < M_1(\alpha) < k\delta - \alpha$, so that $\mathrm{SL}(\alpha+\delta)\mathrm{SL}(k\delta-\alpha)$ is Lyndon by Lemma 2.4. We also note that $[\mathsf{b}[\mathrm{SL}(\alpha+\delta)], \mathsf{b}[\mathrm{SL}(k\delta-\alpha)]] \in \mathcal{S}_i^{k+1} \backslash \mathcal{S}_{i-1}^{k+1}$ by Lemma 4.17 and Proposition 6.2. But this contradicts Corollary 4.9, as $\mathrm{SL}_i((k+1)\delta) < \mathrm{SL}(\alpha+\delta) < \mathrm{SL}(\alpha+\delta)\mathrm{SL}(k\delta-\alpha)$. This completes the proof. □

The following result generalizes [AT, Remark 5.6] to all types:

Lemma 6.7. *For any* $i \in \{1, 2, \dots, |I|\}$*, we have:*

$$\mathrm{SL}_i(\delta) > \mathrm{SL}_i(2\delta) > \mathrm{SL}_i(3\delta) > \dots$$

Proof. Pick any $k \in \mathbb{Z}_{>0}$. We have $m_k(\deg(\mathrm{SL}_i^{ls}((k+1)\delta))) = (k\delta, i)$ by Corollary 6.5. As $\mathrm{SL}_i^{ls}((k+1)\delta) < \mathrm{SL}_i((k+1)\delta) = \mathrm{SL}(m_{k+1}(\deg(\mathrm{SL}_i^{ls}((k+1)\delta))))$, we have $\mathrm{SL}_i^{ls}((k+1)\delta) < \mathrm{SL}_i(k\delta) = \mathrm{SL}(m_k(\deg(\mathrm{SL}_i^{ls}((k+1)\delta))))$ due to Proposition 5.21. We thus conclude that $\mathrm{SL}_i((k+1)\delta) < \mathrm{SL}_i(k\delta)$ due to Lemma 4.2. □

The following result is important for the rest of this section:

Lemma 6.8. *For any* $i \in \{1, 2, \dots, |I|\}$ *and* $k \in \mathbb{Z}_{>0}$*, let* $\beta = \deg(\mathrm{SL}_i^{ls}(k\delta))$*. Then* $|\beta| > |(k-1)\delta|$*.*

Proof. It suffices to consider the case $k > 1$. Let $\alpha = \deg(\mathrm{SL}_i^{ls}((k-1)\delta))$, so that $\mathrm{SL}(\alpha+\delta) \leq \mathrm{SL}(\beta)$ by Lemma 6.6. We also note that $\mathrm{SL}(\beta) < \mathrm{SL}_i((k-1)\delta)$ due to Proposition 5.21, since $\mathrm{SL}(\beta) < \mathrm{SL}_i(k\delta) = \mathrm{SL}(m_k(\beta))$ and $\mathrm{SL}_i((k-1)\delta) = \mathrm{SL}(m_{k-1}(\beta))$ by Corollary 6.4. Evoking Proposition 5.21 again, we get $\alpha < \alpha + \delta$. Combining all the above, we obtain:

$$\mathrm{SL}_i^{ls}((k-1)\delta) = \mathrm{SL}(\alpha) < \mathrm{SL}(\alpha+\delta) \leq \mathrm{SL}(\beta) < \mathrm{SL}_i((k-1)\delta).$$

This implies $|\beta| \geq |(k-1)\delta|$ by Lemma 4.2, and $|\beta| \neq |(k-1)\delta|$ by Corollary 4.20. □

We conclude this subsection with the following interesting observation:

Corollary 6.9. *For any standard Lyndon word* w *and any splitting* $w = uv$*, we cannot have* $u = \mathrm{SL}_i(k\delta)$ *or* $v = \mathrm{SL}_i(k\delta)$ *for some* i *and* $k > 1$*.*

Proof. First, we claim that it suffices to assume that both $u, v \in \mathrm{SL}$. Indeed, if $u = \mathrm{SL}_i(k\delta)$ but $v \notin \mathrm{L}$, then there is a prefix $w' = uv'$ of uv with $w', v' \in \mathrm{L}$ by Lemma 3.12. Similarly, if $v = \mathrm{SL}_i(k\delta)$ but $u \notin \mathrm{L}$, then there is a suffix $w' = u'v$ of uv with $w', u' \in \mathrm{L}$ by Lemma 3.13. In fact, $w', u', v' \in \mathrm{SL}$ as subwords of $w \in \mathrm{SL}$.

The result follows from Corollary 4.20 if $\deg(w) \in \widehat{\Delta}^{+,\mathrm{imx}}$. Let us now assume that $\deg(w) \in \widehat{\Delta}^{+,\mathrm{re}}$. We have $M_k(\alpha) < M_1(\alpha)$ for any $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$ and $k > 1$, due to Corollary 6.4 and Lemma 6.7. Assume first that $v = \mathrm{SL}_i(k\delta)$ for some i and $k > 1$. Then $v = \mathrm{SL}(M_k(\deg(u)))$ by Corollary 4.23. Thus $\mathrm{ch}(\deg(u)) < M_1(\deg(u))$ and so $\mathrm{SL}(\deg(w) - (k-1)\delta) \geq u\mathrm{SL}(M_1(\deg(u)))$ by the generalized Leclerc algorithm. Then $\mathrm{SL}(\deg(w) - (k-1)\delta) \geq u\mathrm{SL}(M_1(\deg(u))) > u\mathrm{SL}(M_k(\deg(u))) =$

w, which contradicts to Proposition 5.21 as $u < w < v = \mathrm{SL}(M_k(\deg(u))) = \mathrm{SL}(M_k(\deg(w)))$ by Theorem 5.4. The case $u = \mathrm{SL}_i(k\delta)$ for some i and $k > 1$ is treated similarly. First, we note that $u = \mathrm{SL}(M_k(\deg(v)))$ by Corollary 4.23. Then, we have $\mathrm{SL}(\deg(uv) - \delta) > uv > \mathrm{SL}(M_1(\deg(v))) = \mathrm{SL}(M_1(\deg(uv)))$, in accordance with Proposition 5.21. As $[\mathsf{b}[\mathrm{SL}(M_1(\deg(v)))], \mathsf{b}[\mathrm{SL}(\deg(uv) - \delta)]] \neq 0$, we have $\mathrm{SL}(M_1(\deg(v)))\mathrm{SL}(\deg(uv) - \delta) \leq uv$ by the generalized Leclerc algorithm. This contradicts the above inequality $u = \mathrm{SL}(M_k(\deg(v))) < \mathrm{SL}(M_1(\deg(v)))$. □

6.10. **Special orders.**

In this subsection, we obtain explicit formulas for all imaginary words $\mathrm{SL}_i(k\delta)$ when δ contains only one instance of the smallest simple root, denoted by α_ε.

Remark 6.11. This applies to any order in type A, as well as to an arbitrary type and the orders on the alphabet $\widehat{I}$ with $0 \in \widehat{I}$ being the smallest letter.

We start with the following simple observation:

Lemma 6.12. *If α_ε occurs once in δ, then for any $i \in \{1, 2, \ldots, |I|\}$, we have $|\mathrm{SL}_i^{rs}(\delta)| = 1$ and all $\mathrm{SL}_i(\delta)$ end with different letters.*

We also have a simple criteria for chains to be increasing rather than decreasing:

Lemma 6.13. *If α_ε occurs once in δ, then for any $\alpha \in \widehat{\Delta}^{+,\mathrm{re}}$ with $|\alpha| < |\delta|$:*

$$\mathrm{ch}(\alpha) \text{ increases} \iff \alpha \text{ contains } \alpha_\varepsilon.$$

Proof. To prove the "⇒" direction, it suffices to show that if α does not contain α_ε then $\mathrm{ch}(\alpha)$ decreases. Comparing the first letters, we get $M_1(\alpha) \leq \mathrm{SL}_1(\delta) < \mathrm{SL}(\alpha)$, and so $\mathrm{ch}(\alpha)$ decreases by Proposition 5.21. The "⇐" direction follows from above and Corollary 5.23, since if α contains α_ε then $\delta - \alpha \in \widehat{\Delta}^{+,\mathrm{re}}$ does not. □

We are now ready to establish the structure of all $\mathrm{SL}_i(k\delta)$ in the present setup:

Theorem 6.14. *If α_ε occurs once in δ, then we have:*

$$\mathrm{SL}_i(k\delta) = \mathrm{SL}_i^{ls}(\delta) \underbrace{\mathrm{SL}(M_1(\gamma_i))}_{k-1 \text{ times}} \mathrm{SL}_i^{rs}(\delta), \qquad \mathrm{SL}_i^{ls}(k\delta) = \mathrm{SL}_i^{ls}(\delta) \underbrace{\mathrm{SL}(M_1(\gamma_i))}_{k-1 \text{ times}},$$

for any $k \in \mathbb{Z}_{>0}$, $i \in \{1, 2, \ldots, |I|\}$, where $\gamma_i := \deg(\mathrm{SL}_i^{ls}(\delta))$.

Proof. The proof proceeds by induction on k, the base case $k = 1$ is obvious. As per the inductive step, let us assume the validity for $k-1$. Let $\alpha = \deg(\mathrm{SL}_i^{ls}((k-1)\delta))$, so that $b = \mathrm{SL}_i^{rs}((k-1)\delta) = \mathrm{SL}_i^{rs}(\delta)$, which is a single letter by Lemma 6.12.

Let $\beta = \deg(\mathrm{SL}_i^{ls}(k\delta))$, so that $\alpha + \delta \leq \beta$ by Lemma 6.6. By Corollary 6.5, we have $m_{k-1}(\alpha) = m_{k-1}(\beta) = ((k-1)\delta, i)$. Since $\mathrm{ch}(\alpha), \mathrm{ch}(\beta)$ are increasing by Theorem 5.4 and Proposition 5.21, we have $\alpha < \beta < ((k-1)\delta, i)$. Therefore, we have $\mathrm{SL}(\beta) = \mathrm{SL}(\alpha)w$ for some nonempty word w satisfying $w < \mathrm{SL}_i^{rs}((k-1)\delta)$.

Let $w = w_1 \ldots w_n$ be the canonical factorization. We claim that w_1 contains the smallest letter ε. If not, then the first letter ι of w_1 is bigger than ε, so that $\mathrm{SL}(\alpha)\iota$ is (standard) Lyndon by Lemma 2.4, but then $(k-1)\delta - \alpha_b + \alpha_\iota = \deg(\mathrm{SL}(\alpha)\iota) \in \widehat{\Delta}^+$ implies $\iota = b$ and so $\mathrm{SL}_i((k-1)\delta) = \mathrm{SL}(\alpha)b$ is a prefix of $\mathrm{SL}_i(k\delta)$, a contradiction with Corollary 6.7. Thus w_1 contains ε, and so $\iota = \varepsilon$ as $w_1 \in \mathrm{L}$. Since w contains at most one ε, we get $n = 1$, as otherwise we have a contradiction with $w_1 \geq w_2$.

Thus, $n = 1$ so that w is Lyndon. Note that $|\deg(w)| \leq |\delta|$ as $|\alpha| = |(k-1)\delta| - 1$. If $|\deg(w)| < |\delta|$, then since w contains the smallest letter ε, the word $\mathrm{SL}(\delta - \deg(w))$

does not and so $\mathrm{SL}(\alpha)w\mathrm{SL}(\delta-\deg(w))$ is Lyndon by Lemma 2.4. This implies $\mathrm{SL}(\alpha+\delta) \geq \mathrm{SL}(\alpha)w\mathrm{SL}(\delta-\deg(w)) > \mathrm{SL}(\alpha)w = \mathrm{SL}(\beta)$, a contradiction with Lemma 6.6. Therefore, $\deg(w)=\delta$. Then $\mathrm{SL}_i^{ls}(k\delta) = \mathrm{SL}(\alpha+\delta) = \mathrm{SL}(\alpha)\mathrm{SL}(M_1(\alpha))$ by Corollary 4.23. Combining this with the inductive hypothesis and the equality $M_1(\alpha)=M_1(\gamma_i)$ completes the induction. □

The following result pertains to a slight generalization of the present setup:

Lemma 6.15. *Fix $i \in \{1,2,\ldots,|I|\}$ and let $\alpha = \deg(\mathrm{SL}_i^{ls}(\delta))$. If*

$$\mathrm{SL}_i(k\delta) = \mathrm{SL}(\alpha)\underbrace{\mathrm{SL}(M_1(\alpha))}_{k-1 \text{ times}}\mathrm{SL}(\delta-\alpha) \qquad \forall\, k \in \mathbb{Z}_{>0},$$

then

$$\mathrm{SL}((\delta-\alpha)+p\delta) = \underbrace{\mathrm{SL}(M_1(\alpha))}_{p \text{ times}}\mathrm{SL}(\delta-\alpha) \qquad \forall\, p \in \mathbb{Z}_{\geq 0}.$$

Proof. We prove the result by induction on $p>0$ (as the $p=0$ case is obvious). For the base case $p=1$, we have $\mathrm{SL}(M_1(\alpha)) < \mathrm{SL}(\delta-\alpha)$ by Theorem 5.4, cf. Remark 5.8. Thus $\mathrm{SL}(M_1(\alpha))\mathrm{SL}(\delta-\alpha)$ is a Lyndon factor (see Lemma 2.4) of a standard Lyndon word $\mathrm{SL}_i(2\delta)$, so that $\mathrm{SL}(M_1(\alpha))\mathrm{SL}(\delta-\alpha) = \mathrm{SL}((\delta-\alpha)+\delta)$.

For the inductive step, let us assume that the result holds for $p-1$, so that $\mathrm{SL}(M_1(\alpha)) < \mathrm{SL}((\delta-\alpha)+(p-1)\delta)$. Then $\mathrm{SL}(M_1(\alpha))\mathrm{SL}((\delta-\alpha)+(p-1)\delta)$ is a Lyndon factor of a standard Lyndon word $\mathrm{SL}_i((p+1)\delta)$. Hence, similarly to the base case, we conclude that $\mathrm{SL}((\delta-\alpha)+p\delta) = \mathrm{SL}(M_1(\alpha))\mathrm{SL}((\delta-\alpha)+(p-1)\delta)$. Combining this with the inductive hypothesis completes the step of induction. □

The above result implies the following corollary in the present setup:

Corollary 6.16. *If α_ε occurs once in δ, then for any simple root α_i with $i \neq \varepsilon$:*

$$\mathrm{SL}(\alpha_i+k\delta) = \underbrace{\mathrm{SL}(M_1(\alpha_i))}_{k \text{ times}}\, i \qquad \forall\, k \in \mathbb{Z}_{\geq 0}.$$

Proof. There are $|\widehat{I}| = |I|+1$ simple roots. According to Lemma 6.12, all of them besides α_ε appear as $\deg(\mathrm{SL}_j^{rs}(\delta))$ for some j. Thus, the result follows from Lemma 6.15, which can be applied due to Theorem 6.14. □

6.17. **General orders.**

We shall now discuss the case of general orders on $\widehat{I}$.

Lemma 6.18. *For any $i \in \{1,2,\ldots,|I|\}$ and $k>1$, we have:*

$$\mathrm{SL}_i^{ls}(k\delta) = \mathrm{SL}_i^{ls}((k-1)\delta)w \quad \textit{for some word} \quad w \neq \emptyset.$$

Proof. According to Lemma 6.6, we have $\mathrm{SL}(\deg(\mathrm{SL}_i^{ls}((k-1)\delta))+\delta) \leq \mathrm{SL}_i^{ls}(k\delta)$. Combining Corollary 6.5 and Proposition 5.21, we see that $\mathrm{ch}(\deg(\mathrm{SL}_i^{ls}((k-1)\delta)))$ is increasing, so that $\mathrm{SL}_i^{ls}((k-1)\delta) < \mathrm{SL}(\deg(\mathrm{SL}_i^{ls}((k-1)\delta))+\delta) \leq \mathrm{SL}_i^{ls}(k\delta)$. Additionally, we have $\mathrm{SL}_i^{ls}(k\delta) < \mathrm{SL}_i(k\delta) < \mathrm{SL}_i((k-1)\delta)$ by Corollary 6.7. Hence, we have $\mathrm{SL}_i^{ls}(k\delta) = \mathrm{SL}_i^{ls}((k-1)\delta)w$ for some nonempty word $w < \mathrm{SL}_i^{rs}((k-1)\delta)$. □

We now prove several technical results that will ultimately yield Proposition 6.25.

Lemma 6.19. *Fix any $i \in \{1,2,\ldots,|I|\}$, $k \geq 1$, $\alpha,\beta \in \widehat{\Delta}^{+,\mathrm{re}}$ with $\alpha+\beta \in \widehat{\Delta}^{+,\mathrm{re}}$, $\alpha \geq \deg(\mathrm{SL}_i^{ls}(k\delta))$, $m_k(\alpha) = (k\delta,i)$, $|\beta| < |2\delta|$, $|\alpha+\beta| < |(k+1)\delta|$, $\mathrm{ch}(\alpha)$ increasing, $\mathrm{ch}(\alpha+\beta)$ increasing, and $\alpha+\beta < (k\delta,i)$. Then $\alpha+\delta > \alpha+\beta$.*

Proof. Suppose to the contrary that $\alpha+\beta > \alpha+\delta$. As $\alpha+\delta > \alpha$ we get $\alpha+\beta > \alpha$, and so $\beta > \alpha$ by Theorem 5.4. Hence, $|\alpha+\beta| \geq |k\delta|$ as otherwise we get a contradiction with Lemma 4.2.

First, assume that $\mathrm{ch}(\beta)$ is increasing. If $|\beta| > |\delta|$, then $\beta-\delta < \beta$. Considering the splitting $\alpha+\beta = (\beta-\delta)+(\alpha+\delta)$, we get $\alpha+\delta < \alpha+\beta < \beta-\delta$ by Theorem 5.4. Combining $\alpha < \alpha+\beta < \beta-\delta$ with $|\beta-\delta| < |\delta|$, we obtain $\beta-\delta > (k\delta, i)$ by Lemma 4.2. As $\mathrm{ch}(\beta)$ is increasing, we must have $m_k(\beta-\delta) > (k\delta, i)$ by Proposition 5.21. As $m_k(\alpha) = (k\delta, i)$, we get $m_k(\alpha+(\beta-\delta)) = (k\delta, i)$ by Lemma 5.12. As $\mathrm{ch}(\alpha+\beta)$ is increasing, we obtain $\alpha+(\beta-\delta) < (k\delta, i)$. On the other hand, $\alpha < \beta-\delta$ implies $\alpha < \alpha+(\beta-\delta)$ by Theorem 5.4, which together with $|\alpha+(\beta-\delta)| < |k\delta|$ yields $\alpha+(\beta-\delta) > (k\delta, i)$ due to Lemma 4.2, a contradiction. If $|\beta| < |\delta|$, we then have $\beta > (k\delta, i)$ by Lemma 4.2, as $\beta > \alpha$. Since $\mathrm{ch}(\beta)$ is increasing, we thus get $m_k(\beta) > (k\delta, i)$ by Proposition 5.21. According to Corollary 5.23, the chain $\mathrm{ch}(\delta-\beta)$ is decreasing and is greater than $(k\delta, i)$, so that $\delta-\beta > (k\delta, i) > \alpha+\delta$. Applying Theorem 5.4 to the decomposition $(\alpha+\beta)+(\delta-\beta) = \alpha+\delta$, we finally obtain $\alpha+\delta > \alpha+\beta$, a contradiction.

Let us now assume that $\mathrm{ch}(\beta)$ is decreasing. If $|\beta| > |\delta|$, then $\beta-\delta > \beta$. We must have $\beta-\delta > \alpha+\beta-\delta > \alpha$ by Theorem 5.4. Since $\alpha+\beta < (k\delta, i)$ and $\mathrm{ch}(\alpha+\beta)$ increases, we have $\alpha+\beta-\delta < (k\delta, i)$. As $|\alpha+\beta-\delta| < |k\delta|$, we then obtain $\alpha+\beta-\delta < \alpha$ by Lemma 4.2. This contradicts above $\beta-\delta > \alpha+\beta-\delta > \alpha$. If $|\beta| < |\delta|$, then as $|\alpha+\beta-\delta| < |k\delta|$, we again conclude that $\alpha+\beta-\delta < \alpha$ by Lemma 4.2. As $\alpha = (\alpha+\beta-\delta)+(\delta-\beta)$, we obtain $\alpha < \delta-\beta$ by Theorem 5.4. Applying Lemma 4.2 once again, we get $(k\delta, i) < \delta-\beta$. On the other hand, $(k\delta, i) > \alpha$ and so $(k\delta, i) > \alpha+\delta$ by Proposition 5.21. Thus $\delta-\beta > \alpha+\delta$ and we get a contradiction with Theorem 5.4 applying it to $\alpha+\delta = (\alpha+\beta)+(\delta-\beta)$. ∎

Corollary 6.20. *Assume that all the conditions of Lemma 6.19 hold, as well as* $\beta > \alpha$ *and* $\mathrm{SL}(\alpha+\beta)$ *is a prefix of* $\mathrm{SL}_i^{ls}((k+1)\delta)$. *Then* $|\beta| < |\delta|$ *and* $m_k(\beta) > (k\delta, i)$.

Proof. Let us first prove that $|\beta| < |\delta|$. If not, then $|\beta| > |\delta|$ as β is real. Since $\mathrm{SL}(\alpha+\delta) > \mathrm{SL}(\alpha+\beta)$ by the previous lemma, $|\mathrm{SL}(\alpha+\delta)| < |\mathrm{SL}(\alpha+\beta)|$, and $\mathrm{SL}(\alpha+\beta)$ is a prefix of $\mathrm{SL}_i^{ls}((k+1)\delta)$, we obtain $\mathrm{SL}(\alpha+\delta) > \mathrm{SL}_i^{ls}((k+1)\delta)$. Therefore, $\alpha+\delta > ((k+1)\delta, i)$ by Lemma 4.2. As $m_k(\alpha) = (k\delta, i)$ and $\mathrm{ch}(\alpha)$ increases, we have $\alpha+\delta < m_{k+1}(\alpha+\delta) = ((k+1)\delta, i)$ by Proposition 5.21 and Corollary 6.4, a contradiction with above. Thus, indeed we have $|\beta| < |\delta|$.

Assume that $m_k(\beta) < (k\delta, i)$. Then $m_k(\alpha+\beta) < (k\delta, i)$ by Lemma 5.12. We then get $\mathrm{SL}_i^{ls}(\delta) > \mathrm{SL}_{i+1}(\delta) \geq \mathrm{SL}_{i+1}(k\delta) \geq \mathrm{SL}(m_k(\alpha+\beta))$, due to Lemma 5.27 and Corollary 6.7. Since $\mathrm{SL}(\alpha+\beta)$ is a prefix of $\mathrm{SL}_i^{ls}((k+1)\delta)$, $\mathrm{SL}_i^{ls}(\delta)$ must be a prefix of $\mathrm{SL}(\alpha+\beta)$ by repeated applications of Lemma 6.18, which implies $\mathrm{SL}(m_k(\alpha+\beta)) < \mathrm{SL}_i^{ls}(\delta) < \mathrm{SL}(\alpha+\beta)$. On the other hand, $\alpha+\beta < m_k(\alpha+\beta) < (k\delta, i)$ as $\mathrm{ch}(\alpha+\beta)$ is increasing, a contradiction with the above. Thus $m_k(\beta) \geq (k\delta, i)$.

If $m_k(\beta) = (k\delta, i)$, then $\mathrm{ch}(\beta)$ is decreasing, as otherwise $\beta < (k\delta, i)$ by Proposition 5.21, which together with $|\beta| < |\delta|$ and Lemma 4.2 implies $\beta < \deg(\mathrm{SL}_i^{ls}(k\delta)) < \alpha$, a contradiction. Then, we have $\delta-\beta < (k\delta, i) < \beta$ by Proposition 5.21 and Corollary 5.23. Thus $\delta-\beta < \deg(\mathrm{SL}_i^{ls}(k\delta)) < \alpha < \alpha+\delta$, where the first inequality follows from Lemma 4.2 while the last one holds as $\mathrm{ch}(\alpha)$ increases. Applying Theorem 5.4 to the decomposition $(\alpha+\beta)+(\delta-\beta) = \alpha+\delta$, we get $\alpha+\delta < \alpha+\beta$, a contradiction with Lemma 6.19. Therefore, $m_k(\beta) > (k\delta, i)$. ∎

In the next few results, we investigate prefixes of $\mathrm{SL}_i(k\delta)$.

Lemma 6.21. *For any $i \in \{1, 2, \ldots, |I|\}$ and $k \in \mathbb{Z}_{>0}$, no proper prefix of $\mathrm{SL}_i(k\delta)$ can be an imaginary standard Lyndon word.*

Proof. Assume the contradiction, that is, $\mathrm{SL}_j(p\delta)$ is a prefix of $\mathrm{SL}_i(k\delta)$ for $0 < p < k$ and some i, j. As $\mathrm{SL}_j(p\delta) > \mathrm{SL}_j(k\delta)$ by Corollary 6.7, we obtain $j > i$. On the other hand, $\mathrm{SL}_i^{ls}(\delta)$ is a prefix of $\mathrm{SL}_i(k\delta)$ by repeated application of Lemma 6.18. But then $\mathrm{SL}_i^{ls}(\delta) > \mathrm{SL}_j(\delta) > \mathrm{SL}_j(p\delta)$, due to Lemma 5.27 and Corollary 6.7. This contradicts to $\mathrm{SL}_i^{ls}(\delta)$ being a shorter prefix of $\mathrm{SL}_i(k\delta)$ than $\mathrm{SL}_j(p\delta)$. □

Lemma 6.22. *For any $i \in \{1, 2, \ldots, |I|\}$, $k \in \mathbb{Z}_{>0}$, and any splitting $\mathrm{SL}_i(k\delta) = \ell w$ with $\ell \in \mathrm{SL}$ and $|\ell| \geq |\mathrm{SL}_i^{ls}(\delta)|$, the chain $\mathrm{ch}(\deg(\ell))$ is increasing. Moreover, we have $m_k(\deg(\ell)) = (k\delta, i)$.*

Proof. Fix i, k. By Lemma 6.21 there are no imaginary standard Lyndon proper prefixes of $\mathrm{SL}_i(k\delta)$. We will now perform two rounds of induction: the first will show that $\mathrm{ch}(\deg(\ell))$ is increasing and $m_k(\deg(\ell)) \geq (k\delta, i)$, while the second will then show that $m_k(\deg(\ell)) = (k\delta, i)$.

The first induction is on the decreasing length of ℓ. The base case $\ell = \mathrm{SL}_i^{ls}(k\delta)$ is clear, due to Corollary 4.22 and Proposition 5.21. Note that by repeated application of Lemma 6.18, we have $\mathrm{SL}_i^{ls}(\delta)$ is a prefix of all ℓ since $|\ell| \geq |\mathrm{SL}_i^{ls}(\delta)|$. As per the induction step, we assume the induction hypothesis holds for all Lyndon prefixes u with $|u| > |\ell|$. Pick such shortest u, so that $u = \ell v$ with $\ell = u^{ls}, v = u^{rs}$. We have $v > \mathrm{SL}_i(k\delta)$ since $v > u$ (as u is Lyndon), $|v| < |u|$, and u is a prefix of $\mathrm{SL}_i(k\delta)$. This implies that either $\mathrm{ch}(\deg(v))$ is decreasing, or $m_k(\deg(v)) > (k\delta, i)$, or v is imaginary, due to Proposition 5.21. Let us consider each of these cases separately:

- If $\mathrm{ch}(\deg(v))$ is decreasing, then the chain $\mathrm{ch}(\deg(\ell))$ is increasing by Lemma 5.25 and the induction hypothesis. On the other hand, as $\mathrm{SL}_i^{ls}(\delta) \leq \ell$, we get $\ell > \mathrm{SL}_{i+1}(\delta)$ by Lemma 5.27. If we had $m_k(\deg(\ell)) < (k\delta, i)$, then $m_1(\deg(\ell)) < (\delta, i)$ by Corollary 6.4, which then contradicts to Proposition 5.21, as we get $\mathrm{SL}_{i+1}(\delta) < \ell < \mathrm{SL}(m_1(\deg(\ell))) \leq \mathrm{SL}_{i+1}(\delta)$. Thus, $m_k(\deg(\ell)) \geq (k\delta, i)$.
- If $m_k(\deg(v)) > (k\delta, i)$ then $m_k(\deg(\ell)) \geq (k\delta, i)$ by Lemma 5.12 and the induction assumption. Thus $\mathrm{ch}(\deg(\ell))$ increases by Proposition 5.21 as $\ell < \mathrm{SL}_i(k\delta)$.
- If v is imaginary, then $m_k(\deg(\ell)) = m_k(\deg(u))$ and $\mathrm{ch}(\deg(\ell)) = \mathrm{ch}(\deg(u))$. Hence, the results for ℓ follow immediately from the inductive hypothesis for u.

The second round of induction proceeds by increasing length of ℓ. The base case $\ell = \mathrm{SL}_i^{ls}(\delta)$ (by repeated applications of Lemma 6.18) follows from Corollary 6.5. For the step of induction, let us assume that $m_k(\deg(u)) = (k\delta, i)$ for all Lyndon prefixes u satisfying $|\mathrm{SL}_i^{ls}(\delta)| \leq |u| < |\ell|$. Given ℓ, set $u = \ell^{ls}$ and $v = \ell^{rs}$, so that the result holds for u by the induction assumption. Let $\alpha = \deg(u)$ and $\beta = \deg(v)$.

If β is imaginary, then the result follows from the inductive hypothesis for u. Therefore, we shall assume that $\beta \in \widehat{\Delta}^{+,\mathrm{re}}$. By the inductive hypothesis, we have $m_k(\alpha) = (k\delta, i)$ and $\mathrm{ch}(\alpha)$ is increasing. Let $p \in \mathbb{Z}_{>0}$ be the largest such that $\mathrm{SL}_i^{ls}(p\delta)$ is a prefix of u. We note that $|\alpha + \beta| \leq |\mathrm{SL}_i^{ls}((p+1)\delta)| < |(p+1)\delta|$, since $\mathrm{SL}_i^{ls}((p+1)\delta)$ is a Lyndon prefix by Lemma 6.18. On the other hand, we have $|u| = |\alpha| \geq |\mathrm{SL}_i^{ls}(p\delta)| > |(p-1)\delta|$ by Lemma 6.8. Therefore, $|\beta| < |2\delta|$. As shown in the first round of induction, the chain $\mathrm{ch}(\alpha + \beta)$ is increasing. We also have $\beta > \alpha$ and $\alpha + \beta < (k\delta, i) < (p\delta, i)$ by Corollary 6.7. We can thus apply Corollary 6.20 to deduce $m_p(\beta) > (p\delta, i)$, so that $m_k(\beta) > (k\delta, i)$ by Corollary 6.4. Then, $m_k(\alpha + \beta) = (k\delta, i)$ by Lemma 5.12 and the inductive hypothesis. □

Corollary 6.23. *For any $i \in \{1,2,\ldots,|I|\}$ and $k \geq 1$, we have:*

$$\mathrm{SL}_i^{ls}(k\delta) = \mathrm{SL}_i^{ls}((k-1)\delta)w \quad \textit{for some word} \quad w \neq \emptyset.$$

Consider the canonical factorization $w = w_1 \ldots w_n$. Then $|w_j| < |\delta|$ for $j > 1$. We also have $n = 1$ iff $\deg(w_1) = \delta$. Finally, for $n > 1$, we have $|w_1| < |\delta|$, $\mathrm{ch}(\deg(w_1))$ is decreasing, and $m_k(\deg(w_1)) > (k\delta, i)$.

Proof. The first claim follows from Lemma 6.18.

By Lemma 3.12, $\mathrm{SL}_i^{ls}((k-1)\delta)w_1$ is a Lyndon prefix of $\mathrm{SL}_i^{ls}(k\delta)$, and is thus equal to $\mathrm{SL}(\alpha+\beta)$ where $\alpha = \deg(\mathrm{SL}_i^{ls}((k-1)\delta))$ and $\beta = \deg(w_1)$. We have $\mathrm{SL}_i^{ls}((k-1)\delta)w_1 < \mathrm{SL}_i(k\delta) = \mathrm{SL}(m_k(\deg(\mathrm{SL}_i^{ls}((k-1)\delta)w_1)))$ by Corollary 4.22. Then, $\mathrm{SL}_i^{ls}((k-1)\delta)w_1 < \mathrm{SL}_i((k-1)\delta)$ as $m_{k-1}(\deg(\mathrm{SL}_i^{ls}((k-1)\delta)w_1)) = ((k-1)\delta, i)$ by Corollary 6.5. We note that $|\mathrm{SL}_i^{ls}((k-1)\delta)w_1| \geq |(k-1)\delta|$, as otherwise $\mathrm{SL}_i^{ls}((k-1)\delta) < \mathrm{SL}_i^{ls}((k-1)\delta)w_1 < \mathrm{SL}_i((k-1)\delta)$ contradicts Lemma 4.2. Hence

$$|w_j| < |\delta| \qquad \forall\, j > 1.$$

According to Lemma 6.22, $m_k(\alpha+\beta) = (k\delta, i)$ and $\mathrm{ch}(\alpha+\beta)$ is increasing. Additionally $|\beta| < |2\delta|$ since $|\alpha| > |(k-2)\delta|$ by Lemma 6.8. If $n = 1$ and $\beta \in \widehat{\Delta}^{+,\mathrm{re}}$, then by Lemma 6.19, $\alpha+\beta < \alpha+\delta$, a contradiction with Lemma 6.6. This proves that $\deg(w_1) = \delta$ if $n = 1$. If $\beta = \delta$ and $n > 1$, then applying Lemma 6.19 to α and $\delta + \deg(w_2)$ we get $\alpha + \delta + \deg(w_2) < \alpha + \delta$, which contradicts Corollary 6.24 below as $\mathrm{SL}(\alpha+\delta) = \mathrm{SL}(\alpha)w_1 < \mathrm{SL}(\alpha)w_1w_2 = \mathrm{SL}(\alpha+\delta+\deg(w_2))$. This proves

$$n = 1 \iff \beta = \delta.$$

If $n > 1$, then $\beta \in \widehat{\Delta}^{+,\mathrm{re}}$, and so $|\beta| < |\delta|$ and $m_{k-1}(\beta) > ((k-1)\delta, i)$ by Corollary 6.20. The latter implies $m_k(\beta) > (k\delta, i)$ by Corollary 6.4. Suppose that $\mathrm{ch}(\beta)$ is increasing. We then have $\alpha + \beta < (k\delta, i) < (k\delta, i-1) < \delta - \beta$ by Proposition 5.21, so that $\alpha+\beta < \delta-\beta$. Thus we get $\mathrm{SL}(\alpha+\delta) \geq \mathrm{SL}(\alpha+\beta)\mathrm{SL}(\delta-\beta) = \mathrm{SL}(\alpha)\mathrm{SL}(\beta)\mathrm{SL}(\delta-\beta)$. But since $\mathrm{SL}(\delta-\beta) > \mathrm{SL}(m_1(\beta)) > \mathrm{SL}(\beta) = w_1 \geq w_2$, we get $\mathrm{SL}(\delta-\beta) > w_2w_3\ldots w_n$ by Lemma 2.9. But then $\mathrm{SL}(\alpha+\delta) > \mathrm{SL}(\alpha)w_1w_2\ldots w_n$, contradicting Lemma 6.6. Hence $\mathrm{ch}(\beta)$ must be decreasing. □

Corollary 6.24. *Using notations of Corollary 6.23, set $\gamma_j = \deg(\mathrm{SL}_i^{ls}((k-1)\delta)) + \sum_{p=1}^{j} \deg(w_p)$ for any $1 \leq j \leq n$. Then $\mathrm{ch}(\gamma_j)$ is increasing and $m_k(\gamma_j) = (k\delta, i)$ for any j. If $\deg(w_1) \neq \delta$, then $m_k(\deg(w_j)) > (k\delta, i)$ for all j.*

Proof. According to Lemma 3.12, each $\mathrm{SL}_i^{ls}((k-1)\delta)w_1\ldots w_j$ is a Lyndon prefix of $\mathrm{SL}_i^{ls}(k\delta)$, and is thus equal to $\mathrm{SL}(\gamma_j)$. Note that each γ_j is real by Lemma 6.21. Moreover, $\mathrm{ch}(\gamma_j)$ is increasing and $m_k(\gamma_j) = (k\delta, i)$ by Lemma 6.22.

If $\deg(w_1) \neq \delta$, let us now show that $m_k(\deg(w_j)) > (k\delta, i)$ by induction on j. The base case $j = 1$ follows from Corollary 6.23. As per the step of induction, assume that $m_k(\deg(w_p)) > (k\delta, p)$ for all $p < j$. Then γ_{j-1} and $\deg(w_j)$ satisfy the requirements for Corollary 6.20, hence we have $m_{k-1}(\deg(w_j)) > ((k-1)\delta, i)$, and so $m_k(\deg(w_j)) > (k\delta, i)$ by Corollary 6.4. □

We can now describe the biggest imaginary affine standard Lyndon words:

Proposition 6.25. *For any $k \in \mathbb{Z}_{>0}$, we have:*

$$\mathrm{SL}_1(k\delta) = \mathrm{SL}_1^{ls}(\delta)\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1\ \textit{times}}\ \mathrm{SL}_1^{rs}(\delta), \qquad \mathrm{SL}_1^{ls}(k\delta) = \mathrm{SL}_1^{ls}(\delta)\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1\ \textit{times}}\ .$$

Proof. The proof is by induction on k, the base case $k = 1$ being trivial. For the inductive step, assume the above equalities hold for $k-1$. Suppose first that $n > 1$ as used in Corollary 6.23. The latter implies $m_k(\deg(w_1)) > (k\delta, 1)$ which is impossible, a contradiction. Thus $n = 1$ and $\deg(w_1) = \delta$ by Corollary 6.23. By Corollary 4.23 and $M_1(\deg(\mathrm{SL}_1^{ls}((k-1)\delta))) = M_1(\deg(\mathrm{SL}_1^{ls}(\delta))) = (\delta, 1)$, we get:

$$\mathrm{SL}_1^{ls}(k\delta) = \mathrm{SL}_i^{ls}((k-1)\delta)\mathrm{SL}_1(\delta).$$

Combining this with the inductive hypothesis completes the inductive step. □

We now propose the structure of all imaginary affine standard Lyndon words:

Conjecture 6.26. *For all $i \in \{1, 2, \dots, |I|\}$ and $k \in \mathbb{Z}_{>0}$, we have:*

$$\mathrm{SL}_i(k\delta) = \mathrm{SL}_i^{ls}(\delta) \underbrace{w}_{k-1 \text{ times}} \mathrm{SL}_i^{rs}(\delta), \qquad \mathrm{SL}_i^{ls}(k\delta) = \mathrm{SL}_i^{ls}(\delta) \underbrace{w}_{k-1 \text{ times}}, \tag{6.1}$$

where w is a cyclic permutation of some $\mathrm{SL}_j(\delta)$ for $j \le i$.

Remark 6.27. Taking $w = \mathrm{SL}_1(\delta)$ if $i = 1$ or $w = \mathrm{SL}(M_1(\deg(\mathrm{SL}_i^{ls}(\delta))))$ if the smallest simple root occurs once in δ, we obtain Proposition 6.25 and Theorem 6.14.

Remark 6.28. The first part of the conjecture implies the form of w in the second part. Indeed, we must have $\deg(w) = \delta$, and since α_0 occurs once in δ, there must be a cyclic permutation ℓ of w which is Lyndon. Then ℓ is a subword of $\mathrm{SL}_i(3\delta)$ by Corollary 3.15, and so $\ell \in \mathrm{SL}$. This implies that $\ell = \mathrm{SL}_j(\delta)$ for some j. If $j > i$, then $\mathrm{SL}_i(3\delta)$ has a suffix of the form $\mathrm{SL}_j(\delta)w$ for some w. Combining Lemma 5.27 and Lemma 6.18, we get $\mathrm{SL}_j(\delta) \le \mathrm{SL}_{i+1}(\delta) < \mathrm{SL}_i^{ls}(\delta) < \mathrm{SL}_i^{ls}(3\delta) < \mathrm{SL}_i(3\delta)$. As $\mathrm{SL}_j(\delta)$ cannot be a prefix of $\mathrm{SL}_i(3\delta)$ by Lemma 6.21, we then get $\mathrm{SL}_j(\delta)w < \mathrm{SL}_i(3\delta)$, a contradiction with $\mathrm{SL}_i(3\delta)$ being Lyndon. This shows that $j \le i$.

Remark 6.29. Using the computer code (see Appendix A) we verified the validity of (6.1) for all $1 \le i \le |I|$ and $k = 2$ in all exceptional types and orders on $\widehat{I}$. This confirms Conjecture 6.26 for all orders in exceptional types, due to Proposition 6.32.

We note that w being a nontrivial cyclic permutation of some $\mathrm{SL}_j(\delta)$ is a new phenomena in comparison to [AT], which we illustrate with a couple of examples:

Example 6.30. Consider the affine type $C_3^{(1)}$ with the order $1 < 3 < 0 < 2$. Using the code, we find the following structure of imaginary standard Lyndon words:

$$\mathrm{SL}_1(\delta) = 123120, \quad \mathrm{SL}_2(\delta) = 121203, \quad \mathrm{SL}_3(\delta) = 101232,$$
$$\mathrm{SL}_1(k\delta) = 123\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 120 \qquad \text{for } k > 1,$$
$$\mathrm{SL}_2(k\delta) = 12120\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 3 \qquad \text{for } k > 1,$$
$$\mathrm{SL}_3(k\delta) = 10123\ \underbrace{120123}_{k-1 \text{ times}}\ 2 \qquad \text{for } k > 1.$$

Here, we note that 120123 is a cyclic permutation of $\mathrm{SL}_1(\delta) = 123120$.

Example 6.31. Consider the affine type $F_4^{(1)}$ with the order $4 < 2 < 0 < 3 < 1$. Using the code, we find the following structure of imaginary standard Lyndon words:

$$\mathrm{SL}_1(\delta) = 432132432130, \qquad \mathrm{SL}_2(\delta) = 432134321302,$$
$$\mathrm{SL}_3(\delta) = 432104321323, \qquad \mathrm{SL}_4(\delta) = 432343213021,$$
$$\mathrm{SL}_1(k\delta) = 432132\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 432130 \qquad \text{for } k>1,$$
$$\mathrm{SL}_2(k\delta) = 43213432130\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 2 \qquad \text{for } k>1,$$
$$\mathrm{SL}_3(k\delta) = 43210432132\,\underbrace{432130432132}_{k-1 \text{ times}}\,3 \qquad \text{for } k>1,$$
$$\mathrm{SL}_4(k\delta) = 43234321302\ \underbrace{\mathrm{SL}_2(\delta)}_{k-1 \text{ times}}\ 1 \qquad \text{for } k>1.$$

Here, we note that 432130432132 is a cyclic permutation of $\mathrm{SL}_1(\delta) = 432132432130$.

Proposition 6.32. *For any i, if (6.1) holds for $k=2$, then (6.1) holds for all k.*

Proof. We prove this by induction on k. The base cases $k=1$ and $k=2$ are clear. For the induction step, assume that (6.1) holds for $k-1$. By Lemma 6.18, we have $\mathrm{SL}_i^{ls}(k\delta) = \mathrm{SL}_i^{ls}((k-1)\delta)w'$ for some w'. It thus suffices to show that $w'=w$, where we use the assumption to write $\mathrm{SL}_i^{ls}(2\delta) = \mathrm{SL}_i^{ls}(\delta)w$. Let $\beta = \deg(\mathrm{SL}_i^{ls}(\delta))$.

First, let us assume that $w' < w$. Consider the canonical factorizations $w' = w'_1w'_2\dots w'_m$ and $w = w_1w_2\dots w_n$. Choose p so that $w_p \neq w'_p$ and $w_k = w'_k$ for $k<p$. By the result of [M] (a generalization of Lemma 2.9), we have $w'_p < w_p$. We note that $\mathrm{SL}_i^{ls}((k-1)\delta)w_1\dots w_{p-1} = \mathrm{SL}_i^{ls}((k-1)\delta)w'_1\dots w'_{p-1}$ is Lyndon, due to Lemma 3.12. Set $\alpha := \deg(\mathrm{SL}_i^{ls}((k-1)\delta)w_1w_2\dots w_{p-1}) + \deg(w_p)$, which by the inductive hypothesis can be written as $\alpha = \beta + \sum_{j=1}^p \deg(w_j) + (k-2)\delta$. As $\mathrm{SL}_i^{ls}(\delta)w_1\dots w_p$ is a Lyndon prefix of $\mathrm{SL}_i(2\delta)$, we see that $\beta + \sum_{j=1}^p \deg(w_j) \in \widehat{\Delta}^{+,\mathrm{re}}$ and evoking Lemma 6.22 we get $m_2(\alpha) = m_2(\beta + \sum_{j=1}^p \deg(w_j)) = (2\delta, i)$ and $\mathrm{ch}(\alpha)$ increases. Recall that $\mathrm{SL}_i^{ls}(\delta) < w_p$ by Lemma 3.11. According to Corollary 6.23, we have $|w_p| \leq |\delta|$ with the equality iff $p=n=1$. Then $\mathrm{SL}_i(\delta) < w_p$ by Lemma 4.2 unless $p=n=1$, while in the latter case we have $w_p \geq \mathrm{SL}_i(\delta)$ by Corollary 4.23. As $\mathrm{SL}_i^{ls}((k-1)\delta)w_1\dots w_{p-1} < \mathrm{SL}_i(k\delta) < \mathrm{SL}_i(\delta) \leq w_p$, we see that $\mathrm{SL}(\alpha) \geq \mathrm{SL}_i^{ls}((k-1)\delta)w_1\dots w_p > \mathrm{SL}_i^{ls}(k\delta)$ by the generalized Leclerc algorithm and Lemma 2.9. As $|\alpha| = |\beta + (k-2)\delta + \sum_{j=1}^n \deg(w_j)| < |k\delta|$, we get $\mathrm{SL}_i(k\delta) < \mathrm{SL}(\alpha)$ by Lemma 4.2. On the other hand, as $\mathrm{ch}(\alpha)$ increases and $m_k(\alpha) = (k\delta, i)$ by Corollary 6.4, we obtain $\mathrm{SL}_i(k\delta) > \mathrm{SL}(\alpha)$, a contradiction.

Assume now that $w' > w$. Consider the canonical factorizations of w', w, and choose p as in the previous case, so that $w'_p > w_p$ now. Evoking Lemma 6.22, we again see that $\tilde{\alpha} := \deg(\mathrm{SL}_i^{ls}((k-1)\delta)w'_1w'_2\dots w'_p) \in \widehat{\Delta}^{+,\mathrm{re}}$, $m_k(\tilde{\alpha}) = (k\delta, i)$, and $\mathrm{ch}(\tilde{\alpha})$ increases. Combining the inductive hypothesis with Lemma 2.9, we get $\mathrm{SL}_i^{ls}((k-1)\delta) = \mathrm{SL}_i^{ls}((k-2)\delta)w < \mathrm{SL}_i^{ls}((k-2)\delta)w'_1w'_2\dots w'_p$. Arguing as in the previous case, we also have $w'_p > \mathrm{SL}_i((k-1)\delta)$. Hence $\mathrm{SL}_i^{ls}((k-2)\delta)w'_1w'_2\dots w'_{p-1} < \mathrm{SL}_i^{ls}((k-1)\delta) < w'_p$. We thus have $\mathrm{SL}(\tilde{\alpha}-\delta) \geq \mathrm{SL}_i^{ls}((k-2)\delta)w'_1w'_2\dots w'_p > \mathrm{SL}_i^{ls}((k-1)\delta)$, and so $\mathrm{SL}_i((k-1)\delta) < \mathrm{SL}(\tilde{\alpha}-\delta)$ by Lemma 4.2. But since $m_{k-1}(\tilde{\alpha}) = ((k-1)\delta, i)$ and $\mathrm{ch}(\tilde{\alpha})$ increases, we get $\mathrm{SL}_i((k-1)\delta) > \mathrm{SL}(\tilde{\alpha}-\delta)$, a contradiction.

Therefore, we must have $w'=w$, which completes the step of induction. □

Appendix A. Code

In this section, we present the source code (written using Python):
https://github.com/corbyte/AffineStandardLyndonWords.
Almost everything in the code is done through a rootSystem object and example of the initialization can be seen below:

Listing 1. RootSystem Initialization

```
"""
rootSystem(ordering,type:str):
Initialization of root system

ordering -- list of ordering for the rootsystem with ordering[0] < ordering[1] < and so on
type -- type of the rootsystem
"""

G2 = rootSystem([2,1,0],'G')

G2.delta
#[1,2,3]
G2.baseRoots
# Will return all roots in the root system with height \leq \delta
```

In addition to the rootSystem class another important class is the word class: with this class you can do comparison and concatenation between words. The word class acts as a wrapper around a list of elements from the letter class:

Listing 2. Word Class

```
#'b'<'a'<'c'
u #abc
v #bc
u < v
# False
u + v
# abcbc
print(u)
#a,b,c
u.no_commas()
#abc
```

Getting a standard Lyndon word for a given rootSystem and ordering is very quick, additionally one can quickly get chains of standard Lyndon words:

Listing 3. Standard Lyndon Words

```
root_system # any rootSystem object
l = root_system.SL(degree) #Where degree is an element of the root system


#l will be an array of word objects, if degree is real there will only be one, but if the degree is imaginary there will be several

chain = root_system.chain(degree)
```

```

#chain is a list of all currenly generated standard Lyndon words with degree, degree + k\delta

root_system.periodicity(degree)

#Returns the periodicity of ch(degree)
```

The notation to use for degree is that the $(i+1)$-th element of the degree list you want corresponds to the multiplicity of α_i in that degree. Additionally, words can be quickly parsed into the "block format":

Listing 4. Block Format

```
G2 = rootSystem([1,2,0],'G')

G2.SL(G2.delta*8 + [1,0,0])[0].no_commas()
#'1222101222102122210101222101222102122210101222102'

print(G2.parse_to_block_format(G2.SL(G2.delta*8 + [1,0,0])[0]))
#[im,1,2] 2 [im,1,1] 10 [im,1,2] 2 [im,1,1] 10 [im,1,1] 2
#[im,i,j] means that there is an \SL_i(\delta) j times in that spot

#There is an additional parameter which will have the code look for rotated imaginary words

C3 = rootSystem([1,3,0,2],'C')
C3.SL(C3.delta*3)[2].no_commas()
#'101231201231201232'
C3.SL(C3.delta)[0].no_commas()
#'123120'
print(C3.parse_to_block_format(C3.SL(C3.delta*3)[2]))
#1 [1,5,2] 01232

# Where [i,j,k] means that \SL_i(\delta) is rotated j-1 letters and repeated k times
```

There are some additional functions which will give useful information about standard Lyndon words and degrees:

Listing 5. Additional Functions

```
G2 = rootSystem([1,2,0],'G')
#get_monotonicity returns 1 if the chain is increasing and -1 if it is decreasing
G2.get_monotonicity([0,1,0])
#1
#rootSystem.M_k(degree) returns i where M_k(degree) = (k\delta,i)
G2.M_k([0,1,0])
#1
#rootSystem.m_k(degree) returns i where m_k(degree) = (k\delta,i)
G2.m_k([0,1,0])
#2

#mod_delta(\alpha+k\delta) will return (\alpha,k\delta)

```

```
G2.mod_delta(G2.delta*5 + [0,1,0])
#(array([0, 1, 0]), 5)

#generate_up_to_delta(k) will generate all standard Lyndon words upto
    height n\delta, results will be cached

G2.generate_up_to_delta(5)

#get_decompositions(\alpha) will return all possible \beta,\gamma \in
    \wDelta^{+} such that \beta + \gamma = \alpha
G2.get_decompositions(G2.delta)

#You can also get the standard and costandard factorization of words

l = G2.SL(G2.delta)[1]

print(*[i.no_commas()for i in G2.costfac(l)],sep=',')
#2,21210
print(*[i.no_commas()for i in G2.standfac(l)],sep=',')
#22121,0
```

Listing 6. W-set

```
    F4 = rootSystem([3,4,0,2,1],"F")
    F4.text_W_set(1)
    #Prints the W_{\delta} set for F4 with ordering 3<4<0<2<1
    print(*[i.no_commas() for i in F4.SL(F4.delta)],sep="\n")
    #Prints imaginary SL words of height delta
    E6 = rootSystem([3,0,1,5,4,6,2],"E")
    E6.text_W_set(1)
    #Prints the W_{\delta} set for E6 with ordering 3<0<1<5<4<6<2
    print(*[i.no_commas() for i in E6.SL(E6.delta)],sep="\n")
    #Prints imaginary SL words of height delta
```

Appendix B. Explicit formulas for $G_2^{(1)}$

In this appendix, we present the list of all affine standard Lyndon words in affine type $G_2^{(1)}$. These were derived using the code of Appendix A. We use the conventions that α_1 is a long root and α_2 is a short root of G_2. We note that having these formulas at hand, one can directly verify them by induction on the height of a root using the generalized Leclerc algorithm. While similar to type $A_n^{(1)}$ of [AT], the structure is more compelling as we get up to 5 “chunks” with $\mathrm{SL}_1(\delta)$ for real roots.

B.1. **Order** $0 < 1 < 2$.

	$\mathrm{SL}(\cdot)$
$(\delta, 1)$	012221
$(\delta, 2)$	012212

$\alpha_0 + k\delta$	
$k = 0$	0
$k = 1$	0120122
$k = 2$	0122012201221
$k \equiv 0 \mod 3$, $k \geq 3$	$01221 \underbrace{\mathrm{SL}_1(\delta)}_{k/3-2 \text{ times}} 0122201221 \underbrace{\mathrm{SL}_1(\delta)}_{k/3-1 \text{ times}} 01221 \underbrace{\mathrm{SL}_1(\delta)}_{k/3-1 \text{ times}} 01222$
$k \equiv 1 \mod 3$, $k \geq 3$	$01221 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1 \text{ times}} 01221 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1 \text{ times}} 0122201221 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1 \text{ times}} 01222$
$k \equiv 2 \mod 3$, $k \geq 3$	$01221 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1 \text{ times}} 0122201221 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1 \text{ times}} 0122201221 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil-1 \text{ times}}$

$\alpha_1 + k\delta$	
k	$\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}} 1$

$\alpha_2 + k\delta$	
k	$\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}} 2$

$\alpha_1 + \alpha_2 + k\delta$	
$k \equiv 0 \mod 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/2 \text{ times}} 1 \underbrace{\mathrm{SL}_1(\delta)}_{k/2 \text{ times}} 2$
$k \equiv 1 \mod 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/2\rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2\rfloor \text{ times}} 1$

$\alpha_0 + \alpha_1 + k\delta$	
$k = 0$	01
$k = 1$	01201221
$k = 2$	01220122101221
$k \equiv 0 \mod 3$, $k \geq 3$	$01221 \underbrace{\mathrm{SL}_1(\delta)}_{k/3-1 \text{ times}} 01221 \underbrace{\mathrm{SL}_1(\delta)}_{k/3-1 \text{ times}} 01221 \underbrace{\mathrm{SL}_1(\delta)}_{k/3-1 \text{ times}} 01222$
$k \equiv 1 \mod 3$, $k \geq 3$	$01221 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1 \text{ times}} 01221 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1 \text{ times}} 0122201221 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil-1 \text{ times}}$
$k \equiv 2 \mod 3$, $k \geq 3$	$01221 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1 \text{ times}} 0122201221 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil-1 \text{ times}} 01221 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil-1 \text{ times}}$

$\alpha_1 + 2\alpha_2 + k\delta$	
$k \equiv 0 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}} 1 \underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}} 2$
$k \equiv 1 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor \text{ times}} 1 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor \text{ times}} 2$
$k \equiv 2 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor \text{ times}} 1$

$\alpha_0+\alpha_1+\alpha_2+k\delta$	
$k=0$	012
$k=1$	012201221
$k \equiv 0 \mod 2$ $k \geq 2$	$01221 \underbrace{\mathrm{SL}_1(\delta)}_{k/2-1 \text{ times}} 01221 \underbrace{\mathrm{SL}_1(\delta)}_{k/2-1 \text{ times}} 01222$
$k \equiv 1 \mod 2$ $k \geq 2$	$01221 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2 \rfloor-1 \text{ times}} 0122201221 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/2 \rceil-1 \text{ times}}$

$\alpha_1+3\alpha_2+k\delta$	
$k \equiv 0 \mod 4$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}} 1 \underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}} 2$
$k \equiv 1 \mod 4$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}} 1 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}} 2$
$k \equiv 2 \mod 4$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}} 1 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}} 2$
$k \equiv 3 \mod 4$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}} 1$

$\alpha_0+\alpha_1+2\alpha_2+k\delta$	
$k=0$	0122
$k \geq 1$	$01221 \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}} 01222$

$2\alpha_1+3\alpha_2+k\delta$	
$k \equiv 0 \mod 5$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}} 1 \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}} 1 \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}} 2$
$k \equiv 1 \mod 5$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}} 1 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}} 1$
$k \equiv 2 \mod 5$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}} 1 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}} 1 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}} 2$
$k \equiv 3 \mod 5$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}} 1 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}} 1$
$k \equiv 4 \mod 5$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}} 1 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}} 2 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}} 1$

$\alpha_0+\alpha_1+3\alpha_2+k\delta$	
k	$01222 \underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}$

$\alpha_0+2\alpha_1+2\alpha_2+k\delta$	
k	$01221 \underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}$

$(k\delta, 1)$	
$k>1$	$01222 \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}} 1$

$(k\delta, 2)$	
$k>1$	$01221 \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}} 2$

B.2. **Order** $0 < 2 < 1$**.**

	$\mathrm{SL}(\cdot)$
$(\delta, 1)$	012212
$(\delta, 2)$	012221

$\alpha_0 + k\delta$	
$k = 0$	0
$k = 1$	0120122
$k = 2$	0122012201221
$k = 3$	0122012210122101222
$k = 4$	0122201221012220122101221
$k \equiv 0 \mod 2$, $k \geq 5$	$01222 \underbrace{\mathrm{SL}_1(\delta)}_{k/2-2 \text{ times}} 0122101222 \underbrace{\mathrm{SL}_1(\delta)}_{k/2-2 \text{ times}} 0122101221$
$k \equiv 1 \mod 2$, $k \geq 5$	$01222 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2 \rfloor-2 \text{ times}} 01221012210122101222 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/2 \rceil-2 \text{ times}}$

$\alpha_1 + k\delta$	
k	$\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}} 1$

$\alpha_2 + k\delta$	
k	$\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}} 2$

$\alpha_1 + \alpha_2 + k\delta$	
$k \equiv 0 \mod 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/2 \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/2 \text{ times}}\ 1$
$k \equiv 1 \mod 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/2 \rceil \text{ times}}\ 1\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2 \rfloor \text{ times}}\ 2$

$\alpha_0 + \alpha_1 + k\delta$	
$k = 0$	01
$k = 1$	01201221
$k = 2$	01220122101221
$k \geq 3$	$01222 \underbrace{\mathrm{SL}_1(\delta)}_{k-3 \text{ times}} 012210122101221$

$\alpha_1 + 2\alpha_2 + k\delta$	
$k \equiv 0 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}}\ 1$
$k \equiv 1 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3 \rceil \text{ times}}\ 1\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3 \rfloor \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3 \rfloor \text{ times}}\ 2$
$k \equiv 2 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3 \rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3 \rceil \text{ times}}\ 1\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3 \rfloor \text{ times}}\ 2$

$\alpha_0 + \alpha_1 + \alpha_2 + k\delta$	
$k = 0$	012
$k = 1$	012201221
$k \geq 2$	$01222 \underbrace{\mathrm{SL}_1(\delta)}_{k-2 \text{ times}} 0122101221$

$\alpha_1 + 3\alpha_2 + k\delta$	
$k \equiv 0 \mod 4$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}}\ 1$
$k \equiv 1 \mod 4$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}}\ 1\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 2$
$k \equiv 2 \mod 4$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}}\ 1\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 2$
$k \equiv 3 \mod 4$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}}\ 1\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 2$

$\alpha_0 + \alpha_1 + 2\alpha_2 + k\delta$	
$k = 0$	0122
$k \geq 1$	$01222\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 01221$

$2\alpha_1 + 3\alpha_2 + k\delta$	
$k \equiv 0 \mod 5$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}}\ 1\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}}\ 1$
$k \equiv 1 \mod 5$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 1\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 1$
$k \equiv 2 \mod 5$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 1\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 1\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 2$
$k \equiv 3 \mod 5$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 1\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 1$
$k \equiv 4 \mod 5$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 1\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 1\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 2$

$\alpha_0 + \alpha_1 + 3\alpha_2 + k\delta$	
k	$01222\,\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}$

$\alpha_0 + 2\alpha_1 + 2\alpha_2 + k\delta$	
k	$01221\,\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}$

$(k\delta, 1)$	
$k > 1$	$01221\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 2$

$(k\delta, 2)$	
$k > 1$	$01222\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 1$

B.3. **Order** $1 < 0 < 2$.

	$\mathrm{SL}(\cdot)$
$(\delta, 1)$	120122
$(\delta, 2)$	121220

$\alpha_1 + k\delta$	
$k=0$	1
$k=1$	1212120
$k=2$	1212012012122
$k \equiv 0 \mod 2$, $k \geq 3$	$12122 \underbrace{\mathrm{SL}_1(\delta)}_{k/2-2\text{ times}} 12012012012122 \underbrace{\mathrm{SL}_1(\delta)}_{k/2-1\text{ times}}$
$k \equiv 1 \mod 2$, $k \geq 3$	$12122 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2\rfloor-1\text{ times}} 12012122 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2\rfloor-1\text{ times}} 120120$

$\alpha_2 + k\delta$	
$k=0$	2
$k \equiv 0 \mod 3$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/3\text{ times}} 122 \underbrace{\mathrm{SL}_1(\delta)}_{k/3\text{ times}} 0 \underbrace{\mathrm{SL}_1(\delta)}_{k/3-1\text{ times}} 122$
$k \equiv 1 \mod 3$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor\text{ times}} 122 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor\text{ times}} 122 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor\text{ times}} 0$
$k \equiv 2 \mod 3$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil\text{ times}} 0 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor\text{ times}} 122 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor\text{ times}} 122$

$\alpha_0 + k\delta$	
k	$\underbrace{\mathrm{SL}_1(\delta)}_{k\text{ times}} 0$

$\alpha_1 + \alpha_2 + k\delta$	
$k=0$	12
$k \geq 1$	$12122 \underbrace{\mathrm{SL}_1(\delta)}_{k-1\text{ times}} 120$

$\alpha_0 + \alpha_1 + k\delta$	
$k=0$	10
$k=1$	12120120
$k \geq 2$	$12122 \underbrace{\mathrm{SL}_1(\delta)}_{k-2\text{ times}} 120120120$

$\alpha_1 + 2\alpha_2 + k\delta$	
k	$\underbrace{\mathrm{SL}_1(\delta)}_{k\text{ times}} 122$

$\alpha_0 + \alpha_1 + \alpha_2 + k\delta$	
k	$120 \underbrace{\mathrm{SL}_1(\delta)}_{k\text{ times}}$

$\alpha_1 + 3\alpha_2 + k\delta$	
$k=1$	1222
$k \equiv 0 \mod 4$, $k \geq 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/4\text{ times}} 122 \underbrace{\mathrm{SL}_1(\delta)}_{k/4\text{ times}} 122 \underbrace{\mathrm{SL}_1(\delta)}_{k/4\text{ times}} 0 \underbrace{\mathrm{SL}_1(\delta)}_{k/4-1\text{ times}} 122$
$k \equiv 1 \mod 4$, $k \geq 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}} 122 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}} 122 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}} 122 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}} 0$
$k \equiv 2 \mod 4$, $k \geq 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4\rceil\text{ times}} 0 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}} 122 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}} 122 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}} 122$
$k \equiv 3 \mod 4$, $k \geq 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4\rceil\text{ times}} 122 \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4\rceil\text{ times}} 0 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}} 122 \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}} 122$

$\alpha_0+\alpha_1+2\alpha_2+k\delta$	
$k\equiv 0 \mod 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/2 \text{ times}}\ 122\ \underbrace{\mathrm{SL}_1(\delta)}_{k/2 \text{ times}}\ 0$
$k\equiv 1 \mod 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/2\rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2\rfloor \text{ times}}\ 122$

$2\alpha_1+3\alpha_2+k\delta$	
k	$12122\,\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}$

$\alpha_0+\alpha_1+3\alpha_2+k\delta$	
$k=0$	12220
$k\equiv 0 \mod 5$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}}\ 122\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}}\ 122\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5-1 \text{ times}}\ 122$
$k\equiv 1 \mod 5$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 122\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 122\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 122\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 0$
$k\equiv 2 \mod 5$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 122\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 122\ \underbrace{\mathrm{SL}_{b1}(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 122\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 0$
$k\equiv 3 \mod 5$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 122\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 122\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 122$
$k\equiv 4 \mod 5$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil \text{ times}}\ 122\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 122\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor \text{ times}}\ 122\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil \text{ times}}\ 0$

$\alpha_0+2\alpha_1+2\alpha_2+k\delta$	
$k=0$	12120
$k\geq 1$	$12122\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 120120$

$(k\delta,1)$	
$k>1$	$120\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 122$

$(k\delta,2)$	
$k>1$	$12122\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 0$

B.4. **Order** $1<2<0$**.**

	$\mathrm{SL}(\cdot)$
$(\delta,1)$	122210
$(\delta,2)$	122102

$\alpha_0 + k\delta$	
$k = 0$	0
$k \equiv 0 \mod 5$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/5\text{ times}}\ 10\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5\text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5\text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5\text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5-1\text{ times}}\ 10$
$k \equiv 1 \mod 5$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 10\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 10\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 2$
$k \equiv 2 \mod 5$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil\text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 10\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 10$
$k \equiv 3 \mod 5$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil\text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 10\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil\text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 10\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 2$
$k \equiv 4 \mod 5$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil\text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil\text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 10\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil\text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 10$

$\alpha_1 + k\delta$	
$k = 0$	1
$k = 1$	1212210
$k = 2$	1221221012210
$k \equiv 0 \mod 3$, $k \geq 3$	$12210\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3-1\text{ times}}\ 12210\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3-1\text{ times}}\ 12210\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3-1\text{ times}}\ 1222$
$k \equiv 1 \mod 3$, $k \geq 3$	$12210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1\text{ times}}\ 12210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1\text{ times}}\ 122212210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil-1\text{ times}}$
$k \equiv 2 \mod 3$, $k \geq 3$	$12210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1\text{ times}}\ 122212210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil-1\text{ times}}\ 12210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil-1\text{ times}}$

$\alpha_2 + k\delta$	
k	$\underbrace{\mathrm{SL}_1(\delta)}_{k\text{ times}}\,2$

$\alpha_1 + \alpha_2 + k\delta$	
$k = 0$	12
$k = 1$	12212210
$k \equiv 0 \mod 2$, $k \geq 2$	$12210\ \underbrace{\mathrm{SL}_1(\delta)}_{k/2-1\text{ times}}\ 12210\ \underbrace{\mathrm{SL}_1(\delta)}_{k/2-1\text{ times}}\ 1222$
$k \equiv 1 \mod 2$, $k \geq 2$	$12210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2\rfloor-1\text{ times}}\ 122212210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/2\rceil-1\text{ times}}$

$\alpha_0 + \alpha_1 + k\delta$	
k	$\underbrace{\mathrm{SL}_1(\delta)}_{k\text{ times}}\,10$

$\alpha_1 + 2\alpha_2 + k\delta$	
$k = 1$	122
$k \geq 2$	$12210\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1\text{ times}}\ 1222$

$\alpha_0+\alpha_1+\alpha_2+k\delta$	
$k \equiv 0 \mod 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/2 \text{ times}}\ 10\ \underbrace{\mathrm{SL}_1(\delta)}_{k/2 \text{ times}}\ 2$
$k \equiv 1 \mod 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/2\rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2\rfloor \text{ times}}\ 10$

$\alpha_1+3\alpha_2+k\delta$	
k	$1222\,\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}$

$\alpha_0+\alpha_1+2\alpha_2+k\delta$	
$k \equiv 0 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}}\ 10\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}}\ 2$
$k \equiv 1 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor \text{ times}}\ 10\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor \text{ times}}\ 2$
$k \equiv 2 \mod 3$	$[\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor \text{ times}}\ 10$

$2\alpha_1+3\alpha_2+k\delta$	
$k=0$	12122
$k=1$	12212212210
$k=2$	12212210122101222
$k \equiv 0 \mod 3$ $k \geq 3$	$12210\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3-1 \text{ times}}\ 12210\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3-1 \text{ times}}\ 122212210\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3-1 \text{ times}}\ 1222$
$k \equiv 1 \mod 3$	$12210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1 \text{ times}}\ 122212210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1 \text{ times}}\ 122212210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil-1 \text{ times}}$
$k \equiv 2 \mod 3$	$12210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor-1 \text{ times}}\ 122212210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil-1 \text{ times}}\ 12210\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil-1 \text{ times}}\ 1222$

$\alpha_0+\alpha_1+3\alpha_2+k\delta$	
$k \equiv 0 \mod 4$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}}\ 10\ \underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}}\ 2$
$k \equiv 1 \mod 4$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4\rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor \text{ times}}\ 10\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor \text{ times}}\ 2$
$k \equiv 2 \mod 4$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4\rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4\rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor \text{ times}}\ 10\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor \text{ times}}\ 2$
$k \equiv 3 \mod 4$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4\rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4\rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4\rceil \text{ times}}\ 2\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor \text{ times}}\ 10$

$\alpha_0+2\alpha_1+2\alpha_2+k\delta$	
k	$12210\,\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}$

$(k\delta,1)$	
$k>1$	$1222\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 10$

$(k\delta,2)$	
$k>1$	$12210\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 2$

B.5. **Order** $2<0<1$**.**

	$\mathrm{SL}(\cdot)$
$(\delta,1)$	221021
$(\delta,2)$	221210

$\alpha_0+k\delta$	
k	$\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}\,0$

$\alpha_1 + k\delta$	
$k = 0$	1
$k \equiv 0 \mod 4$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/4 \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{k/4-1 \text{ times}}\ 21$
$k \equiv 1 \mod 4$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 0$
$k \equiv 2 \mod 4$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 21$
$k \equiv 3 \mod 4$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4 \rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4 \rfloor \text{ times}}\ 21$

$\alpha_2 + k\delta$	
$k = 0$	2
$k = 1$	2212210
$k \geq 2$	$22121\ \underbrace{\mathrm{SL}_1(\delta)}_{k-2 \text{ times}}\ 22102210$

$\alpha_1 + \alpha_2 + k\delta$	
k	$\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}\ 21$

$\alpha_0 + \alpha_1 + k\delta$	
$k = 0$	01
$k \equiv 0 \mod 5$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5 \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5-1 \text{ times}}\ 21$
$k \equiv 1 \mod 5$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 0$
$k \equiv 2 \mod 5$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 0$
$k \equiv 3 \mod 5$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 21$
$k \equiv 4 \mod 5$, $k \geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5 \rceil \text{ times}}\ 0$

$\alpha_1 + 2\alpha_2 + k\delta$	
$k = 0$	221
$k \geq 1$	$22121\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 2210$

$\alpha_0 + \alpha_1 + \alpha_2 + k\delta$	
$k \equiv 0 \mod 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/2 \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/2 \text{ times}}\ 0$
$k \equiv 1 \mod 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/2 \rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2 \rfloor \text{ times}}\ 21$

$\alpha_1 + 3\alpha_2 + k\delta$	
$k = 0$	2221
$k = 1$	2212212210
$k = 2$	2212210221022121
$k \equiv 0 \mod 2$, $k \geq 3$	$22121\ \underbrace{\mathrm{SL}_1(\delta)}_{k/2-2 \text{ times}}\ 22102210221022121\ \underbrace{\mathrm{SL}_1(\delta)}_{k/2-1 \text{ times}}$
$k \equiv 1 \mod 2$, $k \geq 3$	$22121\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2 \rfloor-1 \text{ times}}\ 221022121\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2 \rfloor-1 \text{ times}}\ 22102210$

$\alpha_0 + \alpha_1 + 2\alpha_2 + k\delta$	
k	$2210\,\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}$

$2\alpha_1 + 3\alpha_2 + k\delta$	
k	$22121\,\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}$

$\alpha_0 + \alpha_1 + 3\alpha_2 + k\delta$	
$k = 0$	22210
$k = 1$	22122102210
$k \geq 2$	$22121\ \underbrace{\mathrm{SL}_1(\delta)}_{k-2 \text{ times}}\ 221022102210$

$\alpha_0 + 2\alpha_1 + 2\alpha_2 + k\delta$	
$k \equiv 0 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}}\ 0$
$k \equiv 1 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3 \rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3 \rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3 \rfloor \text{ times}}\ 21$
$k \equiv 2 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3 \rceil \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3 \rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3 \rfloor \text{ times}}\ 21$

$(k\delta, 1)$	
$k > 1$	$2210\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 21$

$(k\delta, 2)$	
$k > 1$	$22121\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 0$

B.6. **Order** $2 < 1 < 0$.

	$\mathrm{SL}(\cdot)$
$(\delta, 1)$	221021
$(\delta, 2)$	221210

$\alpha_0 + k\delta$	
k	$\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}\,0$

$\alpha_1+k\delta$	
$k=0$	1
$k\equiv 0 \mod 4$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/4\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/4\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/4\text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{k/4-1\text{ times}}\ 21$
$k\equiv 1 \mod 4$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}}\ 0$
$k\equiv 2 \mod 4$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4\rceil\text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}}\ 21$
$k\equiv 3 \mod 4$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4\rceil\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/4\rceil\text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/4\rfloor\text{ times}}\ 21$

$\alpha_2+k\delta$	
$k=0$	2
$k=1$	2212210
$k\geq 2$	$22121\ \underbrace{\mathrm{SL}_1(\delta)}_{k-2\text{ times}}\ 22102210$

$\alpha_1+\alpha_2+k\delta$	
k	$\underbrace{\mathrm{SL}_1(\delta)\,21}_{k\text{ times}}$

$\alpha_0+\alpha_1+k\delta$	
$k=0$	10
$k\equiv 0 \mod 5$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/5\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5\text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5\text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{k/5-1\text{ times}}\ 21$
$k\equiv 1 \mod 5$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 0$
$k\equiv 2 \mod 5$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil\text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 0$
$k\equiv 3 \mod 5$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil\text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil\text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 21$
$k\equiv 4 \mod 5$, $k\geq 1$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil\text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/5\rfloor\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/5\rceil\text{ times}}\ 0$

$\alpha_1+2\alpha_2+k\delta$	
$k=0$	221
$k\geq 1$	$22121\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1\text{ times}}\ 2210$

$\alpha_0+\alpha_1+\alpha_2+k\delta$	
$k\equiv 0 \mod 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/2\text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/2\text{ times}}\ 0$
$k\equiv 1 \mod 2$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/2\rceil\text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2\rfloor\text{ times}}\ 21$

$\alpha_1 + 3\alpha_2 + k\delta$	
$k=0$	2221
$k=1$	2212212210
$k=2$	2212210221022121
$k \equiv 0 \mod 2$, $k \geq 3$	$22121\ \underbrace{\mathrm{SL}_1(\delta)}_{k/2-2 \text{ times}}\ 22102210221022121\ \underbrace{\mathrm{SL}_1(\delta)}_{k/2-1 \text{ times}}$
$k \equiv 1 \mod 2$, $k \geq 3$	$22121\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2\rfloor-1 \text{ times}}\ 221022121\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/2\rfloor-1 \text{ times}}\ 22102210$

$\alpha_0 + \alpha_1 + 2\alpha_2 + k\delta$	
k	$2210\,\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}$

$2\alpha_1 + 3\alpha_2 + k\delta$	
k	$22121\,\underbrace{\mathrm{SL}_1(\delta)}_{k \text{ times}}$

$\alpha_0 + \alpha_1 + 3\alpha_2 + k\delta$	
$k=0$	22210
$k=1$	22122102210
$k \geq 2$	$22121\ \underbrace{\mathrm{SL}_1(\delta)}_{k-2 \text{ times}}\ 221022102210$

$\alpha_0 + 2\alpha_1 + 2\alpha_2 + k\delta$	
$k \equiv 0 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{k/3 \text{ times}}\ 0$
$k \equiv 1 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor \text{ times}}\ 21$
$k \equiv 2 \mod 3$	$\underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil \text{ times}}\ 21\ \underbrace{\mathrm{SL}_1(\delta)}_{\lceil k/3\rceil \text{ times}}\ 0\ \underbrace{\mathrm{SL}_1(\delta)}_{\lfloor k/3\rfloor \text{ times}}\ 21$

$(k\delta, 1)$	
$k > 1$	$2210\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 21$

$(k\delta, 2)$	
$k > 1$	$22121\ \underbrace{\mathrm{SL}_1(\delta)}_{k-1 \text{ times}}\ 0$

References

[AT] Y. Avdieiev, A. Tsymbaliuk, *Affine standard Lyndon words: A-type*, Int. Math. Res. Not. IMRN (2024), no. 21, 13488–13524.

[LR] P. Lalonde, A. Ram, *Standard Lyndon bases of Lie algebras and enveloping algebras*, Trans. Amer. Math. Soc. 347 (1995), no. 5, 1821–1830.

[Le] B. Leclerc, *Dual canonical bases, quantum shuffles and q-characters*, Math. Z. 246 (2004), no. 4, 691–732.

[Lo] M. Lothaire, *Combinatorics of words*, Cambridge University Press, Cambridge (1997), xviii+238pp.

[M] G. Melançon, *Combinatorics of Hall trees and Hall words*, J. Combinat. Theory A 59 (1992), no. 2, 285–308.

[NT] A. Neguţ, A. Tsymbaliuk, *Quantum loop groups and shuffle algebras via Lyndon words*, Adv. Math. 439 (2024), Paper No. 109482, 69pp.

[R] M. Rosso, *Lyndon bases and the multiplicative formula for R-matrices*, unpublished preprint (2002).

C.E.: Purdue University, Department of Mathematics, West Lafayette, IN, USA
Email address: `cdelkins@purdue.edu`

A.T.: Purdue University, Department of Mathematics, West Lafayette, IN, USA
Email address: `sashikts@gmail.com`