Multiplicative Slices, Relativistic Toda and Shifted Quantum Affine Algebras

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To Tony Joseph on his 75th birthday, with admiration

Abstract We introduce the shifted quantum affine algebras. They map homomorphically into the quantized K-theoretic Coulomb branches of $3d$ $\mathcal{N} = 4$ SUSY quiver gauge theories. In type A, they are endowed with a coproduct, and they act on the equivariant K-theory of parabolic Laumon spaces. In type A_1, they are closely related to the type A open relativistic quantum Toda system.

Mathematics Subject Classification: 17B37, 81R10, 81T13

1 Introduction

1.1 Summary

The goal of this paper is to initiate the study of *shifted quantum affine algebras*[1] and *shifted* $\mathfrak{v}$*-Yangians*. They arise as a tool to write down via generators and

[1] They were introduced by B. Feigin in 2010.

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M. Gorelik et al. (eds.), *Representations and Nilpotent Orbits of Lie Algebraic Systems*, Progress in Mathematics 330,
https://doi.org/10.1007/978-3-030-23531-4_6

relations the quantized K-theoretic Coulomb branches of $3d$ $\mathcal{N} = 4$ SUSY quiver gauge theories (see [10, Remark 3.9(2)]), similarly to the appearance of shifted Yangians in the study of the quantized Coulomb branches of $3d$ $\mathcal{N} = 4$ SUSY quiver gauge theories [10].[2] Similarly to [24], the shifted quantum affine algebras carry a coproduct, see Sect. 10 for partial results in this direction. The multiplicative analogue of the construction [4] equips the equivariant K-theory of parabolic Laumon spaces with an action of the quantized K-theoretic Coulomb branch for a type A quiver, and hence with an action of a shifted quantum affine algebra of type A. Similarly to [24], the unframed case of type A_1 quiver is closely related to the open relativistic quantum Toda system of type A.

1.2 Outline of the Paper

- In Sect. 2, we give a construction of the completed phase space of the (quasiclassical) relativistic open Toda system for arbitrary simply-connected semisimple algebraic group G via quasihamiltonian and Poisson reductions. It is a direct multiplicative analogue of the Kazhdan–Kostant construction of the (nonrelativistic) open Toda integrable system. We want to stress right away that it depends on a choice of a pair of Coxeter elements in the Weyl group W of G, via a choice of Steinberg's cross-section.[3] In the case when the two Coxeter elements coincide, the resulting completed phase space is isomorphic to the universal centralizer $\mathfrak{Z}^G_G$, see Sect. 2.3. In the case $G = SL(n)$, the universal centralizer is isomorphic to a natural n-fold cover of the moduli space of centered periodic $SU(2)$-monopoles of charge n, see Corollary 2.6.
- The conjectural quantization of the above construction of the completed phase space of the relativistic open Toda is described in Sect. 3.12. We conjecture that it is isomorphic to the corresponding spherical symmetric nil-DAHA which is realized as an equivariant K-theory of a twisted affine Grassmannian, i.e. as a sort of twisted quantized Coulomb branch (the twist is necessary in the case of non-simply-laced G). The bulk of Sect. 3 is occupied by the review of Cherednik's definition of symmetric nil-DAHA, its residue construction, and its realization as the equivariant K-theory of a twisted affine flag variety. In the simply-laced case no twist is required, and the spherical nil-DAHA in question is isomorphic to the convolution algebra $K^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_G)$ up to some finite extension. This convolution algebra is defined for arbitrary reductive G. In case $G = GL(n)$, this convolution algebra is likely to have a presentation via generators and relations (as a truncated shifted quantum affine algebra of type A_1), see Sect. 9. From this presentation and Proposition 11.21 we obtain a homomorphism $K^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_G) \to K^{L(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_L)$ for any Levi subgroup

[2] We must admit right away that we were not able to prove the desired presentation of the quantized Coulomb branch for a single quiver.

[3] The appearance of Coxeter elements in the construction of relativistic Toda lattice goes back at least to [60].

$L \subset G = GL(n)$. We conjecture an existence of such a homomorphism for arbitrary Levi subgroup L in arbitrary reductive group G, but we have no clue as to a geometric construction of such a homomorphism. It would be important for a study of equivariant quantum K-theory of the flag variety $\mathcal{B}$ of G. Its analogue for the equivariant Borel-Moore homology convolution algebra $H_\bullet^{G(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_G) \to H_\bullet^{L(\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_L)$ is constructed in [24]. However, the construction is not geometric; it uses an isomorphism with the quantum open (nonrelativistic) Toda lattice.

- Recall that for an arbitrary $3d$ $\mathcal{N} = 4$ SUSY quiver gauge theory of type ADE, the non-quantized K-theoretic Coulomb branch is identified with a multiplicative generalized slice in the corresponding affine Grassmannian [10, Remarks 3.9(2), 3.17]. These multiplicative slices are studied in detail in Sect. 4 (in the unframed case, they were studied in detail in [25]). In particular, they embed into the loop group $G(z)$, and it is likely that the image coincides with the space of scattering matrices of singular periodic monopoles [14]. The multiplication in the loop group gives rise to the multiplication of slices, which is conjecturally quantized by the coproduct of the corresponding shifted quantum affine algebras.
- In Sect. 5, we introduce the *shifted quantum affine algebras* $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$ and $\mathcal{U}^{\mathrm{ad}}_{\mu^+,\mu^-}$ (*simply-connected* and *adjoint* versions, respectively) for any simple Lie algebra $\mathfrak{g}$ and its two coweights μ^+, μ^- (these algebras depend only on $\mu = \mu^+ + \mu^-$ up to an isomorphism). For $\mu^+ = \mu^- = 0$, they are central extensions of the standard quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{g})$ and its adjoint version $U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{g})$. These algebras can be viewed as trigonometric versions of the shifted Yangians $\mathbf{Y}_\mu$, see [10, 24, 45].

  An alternative (but equivalent) definition of $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$ was suggested to us by B. Feigin in Spring 2010 in an attempt to generalize the results of [7] to the K-theoretic setting (which is the subject of Sect. 12 of the present paper). In this approach, we consider an algebra with the same generators and defining relations as $U_{\boldsymbol{v}}(L\mathfrak{g})$ in the new Drinfeld realization with just one modification: the relation $[e_i(z), f_j(w)] = \frac{\delta_{ij}\delta(z/w)}{\boldsymbol{v}_i-\boldsymbol{v}_i^{-1}}\left(\psi_i^+(z) - \psi_i^-(z)\right)$ is replaced by $p_i(z)[e_i(z), f_j(w)] = \frac{\delta_{ij}\delta(z/w)}{\boldsymbol{v}_i-\boldsymbol{v}_i^{-1}}\left(\psi_i^+(z) - \psi_i^-(z)\right)$ for any collection of rational functions $\{p_i(z)\}_{i\in I}$ (here I parametrizes the set of vertices of the Dynkin diagram of $\mathfrak{g}$). For $\mathfrak{g} = \mathfrak{sl}_2$ and $\mu^+ = \mu^- \in -\mathbb{N}$, the algebra $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$ appeared in [18, § 5.2].

  We also provide an alternative presentation of the antidominantly shifted quantum affine algebras with a finite number of generators and defining relations, see Theorem 5.5 and Appendix A for its proof. We note that this result (and its proof) also holds for any affine Lie algebra, except for type $A_1^{(1)}$. In the unshifted case, more precisely for $U_{\boldsymbol{v}}(L\mathfrak{g})$, it can be viewed as a $\boldsymbol{v}$-version of the famous Levendorskii presentation of the Yangian $Y(\mathfrak{g})$, see [47]. Motivated by Guay et al. [33], we also provide a slight modification of this presentation in Theorem A.3.
- In Sect. 6, we introduce other generators of $\mathcal{U}^{\mathrm{ad}}_{\mu^+,\mu^-}$, which can be encoded by the generating series $\{A_i^\pm(z), B_i^\pm(z), C_i^\pm(z), D_i^\pm(z)\}_{i\in I}$. We provide a complete list

of the defining relations between these generators for antidominant $\mu^+, \mu^- \in \Lambda^-$ (we use Λ^- to denote the submonoid of the coweight lattice Λ spanned by antidominant coweights), see Theorem 6.6 and Appendix B for its proof. This should be viewed as a shifted $\boldsymbol{v}$-version of the corresponding construction for Yangians of [30]. We note that while some of the relations were established (without a proof) in loc. cit., the authors did not aim at providing a complete list of the defining relations. However, a rational analogue of Theorem 6.6 provides such a list.

We would like to point out that this is one of the few places where it is essential to work with the *adjoint* version. In the simplest case, that is of $U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{sl}_2)$, these generating series coincide with the entries of the matrices $T^{\pm}(z)$ from the RTT realization of $U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{sl}_2)$, see [17] and our discussion in Sect. 11.4.

- In Sect. 7, we construct homomorphisms

$$\widetilde{\Phi}^{\underline{\lambda}}_{\mu}\colon \mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_N^{\pm 1}] \longrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_N^{\pm 1}]$$

from the adjoint version of shifted quantum affine algebras to the $\mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_N^{\pm 1}]$-algebras $\widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_N^{\pm 1}]$ of difference operators on multidimensional tori, see Theorem 7.1 and Appendix C for its proof. Here $\underline{\lambda} = (\omega_{i_1},\ldots,\omega_{i_N})$ is a sequence of fundamental coweights, such that $\lambda - \mu$ is a sum of simple coroots with coefficients in $\mathbb{N}$, where $\lambda := \sum_{s=1}^{N}\omega_{i_s}$. This result can be viewed as a $\boldsymbol{v}$-version of the corresponding construction for shifted Yangians of [10, Theorem B.15], while the *unshifted case* of it, more precisely the case of $U_{\boldsymbol{v}}(L\mathfrak{g})$, appeared (without a proof) in [31]. For $\mathfrak{g}=\mathfrak{sl}_2,\ N=0$ and antidominant shift, the above homomorphism made its first appearance in [18, Section 6].
- In Sect. 8, we consider the quantized K-theoretic Coulomb branch $\mathcal{A}^{\boldsymbol{v}}$ in the particular case of quiver gauge theories of ADE type (a straightforward generalization of the constructions of [9, 10], with the equivariant Borel-Moore homology replaced by the equivariant K-theory). There is a natural embedding $\mathbf{z}^*(\iota_*)^{-1}\colon \mathcal{A}^{\boldsymbol{v}} \hookrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_N^{\pm 1}]$. In Theorem 8.1, we show that our homomorphism $\widetilde{\Phi}^{\underline{\lambda}}_{\mu}$ of Sect. 7 factors through the above embedding (with $\mathbb{C}[\boldsymbol{v}^{\pm 1}]$ extended to $\mathbb{C}(\boldsymbol{v})$), giving rise to a homomorphism

$$\overline{\Phi}^{\underline{\lambda}}_{\mu}\colon \mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_N^{\pm 1}] \longrightarrow \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}.$$

This is a $\boldsymbol{v}$-version of the corresponding result for shifted Yangians of [10, Theorem B.18].

In Sect. 8.3, we add certain truncation relations to the relations defining $\mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_N^{\pm 1}]$ to obtain the *truncated shifted quantum affine algebras* $\mathcal{U}^{\underline{\lambda}}_{\mu}$ such that the homomorphism $\overline{\Phi}^{\underline{\lambda}}_{\mu}$ factors through the projection and the same named homomorphism $\mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_N^{\pm 1}] \twoheadrightarrow \mathcal{U}^{\underline{\lambda}}_{\mu} \xrightarrow{\overline{\Phi}^{\underline{\lambda}}_{\mu}} \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$. We expect that $\overline{\Phi}^{\underline{\lambda}}_{\mu}\colon \mathcal{U}^{\underline{\lambda}}_{\mu} \to \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$ is an isomorphism, see Conjecture 8.9.

In Sect. 8.4, we define the *shifted $\boldsymbol{v}$-Yangians* ${}^{\imath}\mathcal{Y}^{\boldsymbol{v}}_{\mu}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}] \subset \mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \dots, \mathsf{z}_N^{\pm 1}]$ and their truncated quotients ${}^{\imath}\mathcal{Y}^{\lambda}_{\mu} \subset \mathcal{U}^{\lambda}_{\mu}$. We conjecture that $\overline{\Phi}^{\lambda}_{\mu} : {}^{\imath}\mathcal{Y}^{\lambda}_{\mu} \to \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$ is an isomorphism, see Conjecture 8.13.

One of our biggest failures is the failure to define the integral forms ${}^{\imath}\mathfrak{Y}^{\lambda}_{\mu} \subset {}^{\imath}\mathcal{Y}^{\lambda}_{\mu}$ and $\mathfrak{U}^{\lambda}_{\mu} \subset \mathcal{U}^{\lambda}_{\mu}$ over $\mathbb{C}[\boldsymbol{v}^{\pm 1}] \subset \mathbb{C}(\boldsymbol{v})$ that would (at least conjecturally) map isomorphically onto $\mathcal{A}^{\boldsymbol{v}} \subset \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$. Only in the case of $\mathfrak{g} = \mathfrak{sl}_2$, making use of the $ABCD$-generators of Sect. 6, we are able to introduce the desired integral form in Sect. 9.1 (see also [29] for the integral forms for $\mathfrak{g} = \mathfrak{sl}_n$). It is worth noting that for arbitrary simply-laced $\mathfrak{g}$ and any $i \in I$, the images under $\overline{\Phi}^{\lambda}_{\mu}$ of the generators $B^{+}_{i,r}$ and $e_{i,r}$ (resp. $C^{+}_{i,r}$ and $f_{i,r}$) are the classes of dual exceptional collections of vector bundles on the corresponding minuscule Schubert varieties in the affine Grassmannian, see Remark 8.4.

The desired integral forms ${}^{\imath}\mathfrak{Y}^{\lambda}_{\mu}$ and $\mathfrak{U}^{\lambda}_{\mu}$ are expected to be quantizations of a certain cover ${}^{\dagger}\hat{\mathcal{W}}^{\lambda^*}_{\mu^*}$ of a multiplicative slice introduced in Sect. 4.6, see Conjecture 8.14. Here $*$ stands for the involution $\mu \mapsto -w_0\mu$ of the coweight lattice Λ.

- In Sect. 9, we prove the surjectivity of the homomorphism $\overline{\Phi}^{0}_{-n\alpha}$ in the simplest case of $\mathfrak{g} = \mathfrak{sl}_2$ and antidominant shifts, see Theorem 9.2. This identifies the slightly localized and extended quantized K-theoretic Coulomb branch $K^{\widetilde{GL}(n,\mathcal{O}) \rtimes \widetilde{\mathbb{C}}^{\times}}_{\mathrm{loc}}(\mathrm{Gr}_{GL(n)})$ with a quotient of the localized version of the truncated shifted quantum affine algebra $\mathfrak{U}^{0}_{-n\alpha,\mathrm{loc}}$ (where $\widetilde{GL}(n)$ and $\widetilde{\mathbb{C}}^{\times}$ stand for the two-fold covers of $GL(n), \mathbb{C}^{\times}$; while the localization is obtained by inverting $1 - \boldsymbol{v}^{2m}$, $1 \leq m \leq n$). We reduce the proof of the isomorphism $\mathfrak{U}^{0}_{-n\alpha,\mathrm{loc}} \xrightarrow{\sim} K^{\widetilde{GL}(n,\mathcal{O}) \rtimes \widetilde{\mathbb{C}}^{\times}}_{\mathrm{loc}}(\mathrm{Gr}_{GL(n)})$ to a verification of an identity with quantum resultants in $\mathcal{U}^{0}_{-n\alpha}$, see Remarks 9.6, and 9.12. It would be interesting to describe explicitly a basis of $\mathfrak{U}^{0}_{-n\alpha,\mathrm{loc}}$ projecting to the "canonical" basis of $K^{\widetilde{GL}(n,\mathcal{O}) \rtimes \widetilde{\mathbb{C}}^{\times}}_{\mathrm{loc}}(\mathrm{Gr}_{GL(n)})$ formed by the classes of irreducible equivariant perverse coherent sheaves [8].
- In Sect. 10, we discuss generalizations of the classical coproducts on $U_{\boldsymbol{v}}(L\mathfrak{g})$ to the shifted setting. We start by considering the simplest case $\mathfrak{g} = \mathfrak{sl}_2$. We will denote $\mathcal{U}^{\mathrm{sc}}_{0,b\alpha/2}$ simply by $\mathcal{U}^{\mathrm{sc}}_{0,b}$ (here $b \in \mathbb{Z}$ and α is the simple positive coroot). We construct homomorphisms

$$\Delta_{b_1,b_2} : \mathcal{U}^{\mathrm{sc}}_{0,b} \longrightarrow \mathcal{U}^{\mathrm{sc}}_{0,b_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2}$$

for any $b_1, b_2 \in \mathbb{Z}$, which recover the classical Drinfeld-Jimbo coproduct for $b_1 = b_2 = 0$. Our construction is parallel to the one for shifted Yangians of [24] and proceeds in two steps. First, we define such homomorphisms in the *antidominant* case $b_1, b_2 \in \mathbb{Z}_{\leq 0}$, see Theorem 10.5 and Appendix D for its proof. The proof is crucially based on the aforementioned alternative presentation of the antidominantly shifted quantum affine algebras with a finite number of generators and defining relations of Theorem 5.5. Second, we use the algebra embeddings

$\iota_{n,m_1,m_2}\colon \mathcal{U}^{\mathrm{sc}}_{0,n} \hookrightarrow \mathcal{U}^{\mathrm{sc}}_{0,n+m_1+m_2}$ (here $m_1, m_2 \leq 0$) to reduce the general case to the antidominant one, see Theorem 10.10 and Appendix F for its proof. We note that our proof of injectivity of the *shift homomorphisms* ι_{n,m_1,m_2} is based on the PBW property of the shifted quantum affine algebras of $\mathfrak{sl}_2$, see Lemma 10.9 and Theorem E.2 of Appendix E.

In Sects. 10.6 and 10.7, we generalize the aforementioned case of $\mathfrak{sl}_2$ to the case of $\mathfrak{sl}_n$ ($n \geq 2$). The idea is again to treat first the case of antidominant shifts and then deduce the general case. To achieve the former goal, it is essential to have explicit formulas for the action of the Drinfeld-Jimbo coproduct on the generators $\{e_{i,-1}, f_{i,1}, h_{i,\pm 1}\}_{i\in I}$ of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. This is the key technical result, stated in Theorem 10.13 and proved in Appendix G. Once this is established, it is easy to guess the formulas for the homomorphism $\Delta_{\mu_1,\mu_2}\colon \mathcal{U}^{\mathrm{sc}}_{0,\mu_1+\mu_2} \to \mathcal{U}^{\mathrm{sc}}_{0,\mu_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$ in the case $\mu_1, \mu_2 \in \Lambda^-$ (antidominant), see Theorem 10.16 and its proof in Appendix H. In Theorem 10.20 we derive the construction of Δ_{μ_1,μ_2} for general $\mu_1, \mu_2 \in \Lambda$ by utilizing the algebra embeddings $\iota_{\mu,\nu_1,\nu_2}\colon \mathcal{U}^{\mathrm{sc}}_{0,\mu} \hookrightarrow \mathcal{U}^{\mathrm{sc}}_{0,\mu+\nu_1+\nu_2}$ for $\mu \in \Lambda, \nu_1, \nu_2 \in \Lambda^-$, see Theorem 10.19 and its proof in Appendix I (the latter is based on the shuffle realization of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ of [53, 63]).

Motivated by Finkelberg et al. [24], we expect that our construction of homomorphisms Δ_{μ_1,μ_2} can be generalized to any simply-laced $\mathfrak{g}$ and its two coweights $\mu_1, \mu_2 \in \Lambda$. However, we failed to achieve this due to a lack of explicit formulas for the Drinfeld-Jimbo coproduct of the generators $\{e_{i,-1}, f_{i,1}, h_{i,\pm 1}\}_{i\in I}$ of $U_{\boldsymbol{v}}(L\mathfrak{g})$ (even for $\mathfrak{g} = \mathfrak{sl}_n$, the formulas of Theorem 10.13 seem to be new, to our surprise).

Moreover, we expect that this coproduct extends to

$$\Delta^{\mathrm{ad}}_{\mu_1,\mu_2}\colon \mathcal{U}^{\mathrm{ad}}_{0,\mu+\mu_2}[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_{N_1+N_2}^{\pm 1}] \longrightarrow \mathcal{U}^{\mathrm{ad}}_{0,\mu_1}[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_{N_1}^{\pm 1}] \otimes \mathcal{U}^{\mathrm{ad}}_{0,\mu_2}[\mathsf{z}_{N_1+1}^{\pm 1},\ldots,\mathsf{z}_{N_1+N_2}^{\pm 1}],$$

which descends to the same named homomorphism $\Delta^{\mathrm{ad}}_{\mu_1,\mu_2}\colon \mathcal{U}^{\underline{\lambda}}_{\mu_1+\mu_2} \to \mathcal{U}^{\underline{\lambda}^{(1)}}_{\mu_1} \otimes \mathcal{U}^{\underline{\lambda}^{(2)}}_{\mu_2}$ between truncated algebras, see Conjecture 11.22. We check a particular case of this conjecture for $\mathfrak{g} = \mathfrak{sl}_2$ in Proposition 11.21, using the RTT realization of $\mathcal{U}^{\mathrm{ad}}_{0,2b}$ of Theorem 11.11.

- In Sect. 11, we discuss relativistic/trigonometric Lax matrices, the shifted RTT algebras of $\mathfrak{sl}_2$ and their relation to the shifted quantum affine algebras of $\mathfrak{sl}_2$. This yields a link between two seemingly different appearances of the RTT relations (both trigonometric and rational).

  In Sect. 11.2, we recall the Kuznetsov-Tsyganov [43] local relativistic Lax matrix $L_i^{\boldsymbol{v},0}(z)$ satisfying the trigonometric RTT-relation. The complete monodromy matrix $T_n^{\boldsymbol{v},0}(z) = L_n^{\boldsymbol{v},0}(z)\cdots L_1^{\boldsymbol{v},0}(z)$ also satisfies the same relation, and its matrix coefficient $T_n^{\boldsymbol{v},0}(z)_{11}$ encodes all the hamiltonians of the q-difference quantum open Toda lattice for $GL(n)$ [19, 56].

  We introduce two more local Lax matrices $L_i^{\boldsymbol{v},\pm 1}(z)$ satisfying the same trigonometric RTT-relation. They give rise to the plethora of 3^n complete monodromy matrices $T^{\boldsymbol{v}}_{\vec{k}}(z), \vec{k} \in \{-1,0,1\}^n$, given by the length n products of the three local Lax matrices in arbitrary order. The matrix coefficient $T^{\boldsymbol{v}}_{\vec{k}}(z)_{11}$ encodes the hamiltonians of the corresponding modified quantum difference Toda lattice; the quadratic hamiltonians are given by the formula (11.8). At the quasiclassical level, these integrable systems go back to [21]. We show that among these 3^n integrable systems there are no more than 3^{n-2} nonequivalent, see Lemma 11.6. It is shown in [35] that they are all obtained by the construction of [56] using arbitrary pairs of orientations of the A_{n-1} Dynkin diagram, see Remark 11.7.

In Sect. 11.4, we introduce the *shifted RTT algebras of* $\mathfrak{sl}_2$, denoted by $\mathcal{U}^{\mathrm{rtt}}_{0,-2n}$, and construct isomorphisms $\Upsilon_{0,-2n}\colon \mathcal{U}^{\mathrm{ad}}_{0,-2n} \xrightarrow{\sim} \mathcal{U}^{\mathrm{rtt}}_{0,-2n}$ for any $n \in \mathbb{N}$, see Theorem 11.8 and Theorem 11.11. For $n = 0$, this recovers the isomorphism of the *new Drinfeld* and the *RTT* realizations of the quantum loop algebra $U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{sl}_2)$, due to [17]. We also identify the $ABCD$ generators of $\mathcal{U}^{\mathrm{ad}}_{0,-2n}$ of Sect. 6 with the generators of $\mathcal{U}^{\mathrm{rtt}}_{0,-2n}$, see Corollary 11.10.

Viewing the Lax matrix $L^{\boldsymbol{v},-1}_1(z)$ as a homomorphism from $\mathcal{U}^{\mathrm{rtt}}_{0,-2}$ to the algebra of difference operators on $\mathbb{C}^\times$ and composing it with $\Upsilon_{0,-2}$, we recover the homomorphism $\widetilde{\Phi}^0_{-2}$ of Sect. 7. More generally, among all pairwise isomorphic shifted algebras $\{\mathcal{U}^{\mathrm{ad}}_{b,-2-b} | b \in \mathbb{Z}\}$ only those with $b, -2-b \leq 0$ admit an RTT realization, i.e., there are analogous isomorphisms $\Upsilon_{b,-2-b}\colon \mathcal{U}^{\mathrm{ad}}_{b,-2-b} \xrightarrow{\sim} \mathcal{U}^{\mathrm{rtt}}_{b,-2-b}$. Moreover, recasting the homomorphisms $\widetilde{\Phi}_{b,-2-b}$ (generalizations of $\widetilde{\Phi}^0_{-2}$ for $b = 0$) as the homomorphisms $\mathcal{U}^{\mathrm{rtt}}_{b,-2-b} \to \hat{\mathcal{A}}^{\boldsymbol{v}}_1$, we recover the other two Lax matrices $L^{\boldsymbol{v},0}_1(z)$ (for $b = -1$) and $L^{\boldsymbol{v},1}_1(z)$ (for $b = -2$).

Finally, we use the RTT presentation of $U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{sl}_2)$ to derive explicit formulas for the action of the Drinfeld-Jimbo coproduct on the Drinfeld *half-currents*, see Proposition 11.18 and Appendix J for its proof. We also show that the same formulas hold in the antidominantly shifted setting for the homomorphisms Δ_{b_1,b_2}, see Proposition 11.19. As a consequence of the latter, the homomorphism $\Delta^{\mathrm{ad}}_{2b_1,2b_2}$ is intertwined with the RTT coproduct $\Delta^{\mathrm{rtt}}_{2b_1,2b_2}$, see Corollary 11.20, which is used to prove the aforementioned Proposition 11.21 on the descent of $\Delta^{\mathrm{ad}}_{2b_1,2b_2}$ to the truncated versions.

- In Sect. 12, we provide yet another geometric realization of the shifted quantum affine algebras (resp. shifted Yangians) of $\mathfrak{sl}_n$ via the parabolic Laumon spaces. Roughly speaking, this arises by combining our homomorphism $\overline{\Phi}^{\lambda}_{\mu}$ of Sect. 8 (resp. $\overline{\Phi}^{\lambda}_{\mu}$ of [10, Theorem B.18]) with an action of the quantized K-theoretic (resp. cohomological) Coulomb branch $\mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$ on the localized equivariant K-theory (resp. cohomology) of parabolic Laumon spaces, constructed in [4], see Remark 12.3(c).

  For any $\pi = (p_1, \ldots, p_n) \in \mathbb{Z}^n_{>0}$, we construct an action of $\mathcal{U}^{\mathrm{sc}}_{0,\mu}$, the simply-connected shifted quantum affine algebra of $\mathfrak{sl}_n$ with the shift $\mu = \sum_{j=1}^{n-1}(p_{j+1} - p_j)\omega_j$, on $M(\pi)$: the direct sum of localized equivariant K-theory of $\mathfrak{Q}_{\underline{d}}$, see Theorem 12.2. Here $\mathfrak{Q}_{\underline{d}}$ is the type π Laumon based parabolic quasiflags' space, which we recall in Sect. 12.1. In Theorem 12.6, we slightly generalize this by constructing an action of the shifted quantum affine algebra of $\mathfrak{gl}_n$ (defined in Sect. 12.7) on $M(\pi)$. In Theorem 12.4, we establish an isomorphism $M(\pi') \otimes M(\pi'') \xrightarrow{\sim} M(\pi)$ (here $\pi = \pi' + \pi''$) of $\mathcal{U}^{\mathrm{sc}}_{0,\mu}$-modules, where the action on the source arises from the formal coproduct $\widetilde{\Delta}\colon \mathcal{U}^{\mathrm{sc}}_{0,\mu} \to \mathcal{U}^{\mathrm{sc}}_{0,\mu'} \widehat{\otimes} \mathcal{U}^{\mathrm{sc}}_{0,\mu''}$, constructed in Sect. 10.1 (an analogue of the Drinfeld formal coproduct on $U_{\boldsymbol{v}}(L\mathfrak{g})$).

The *rational* counterpart of these results is established in Theorem 12.7, where we construct an action of $\mathcal{Y}^{\hbar}_{\mu}$ (the shifted Yangian of $\mathfrak{sl}_n$ with scalars extended to $\mathbb{C}(\hbar)$) on $V(\pi)$: the sum of localized equivariant cohomology of $\mathfrak{Q}_{\underline{d}}$. The dominant case ($p_1 \leq \ldots \leq p_n$) of this result was treated in [7], where the proof was deduced from the Gelfand-Tsetlin formulas of [27]. In contrast, our straightforward proof is valid for any π and, thus, gives an alternative proof of the above Gelfand-Tsetlin formulas. We also propose a $\boldsymbol{v}$-analogue of the Gelfand-Tsetlin formulas of [27], see Proposition 12.8.

Our construction can be also naturally generalized to provide the actions of the shifted quantum toroidal (resp. affine Yangian) algebras of $\mathfrak{sl}_n$ on the sum of localized equivariant K-theory (resp. cohomology) of the parabolic affine Laumon spaces, see Sect. 12.9.

In Sect. 12.10, we introduce the *Whittaker* vectors in the completions of $M(\pi)$ and $V(\pi)$:

$$\mathfrak{m} := \sum_{\underline{d}} [\mathcal{O}_{\mathfrak{Q}_{\underline{d}}}] \in M(\pi)^{\wedge} \text{ and } \mathfrak{v} := \sum_{\underline{d}} [\mathfrak{Q}_{\underline{d}}] \in V(\pi)^{\wedge}.$$

This name is motivated by their *eigenvector properties* of Proposition 12.11, Remark 12.12(c).

Motivated by the work of Brundan-Kleshchev, see [12], we expect that the truncated shifted quantum affine algebras $\mathcal{U}^{N\omega_{n-1}}_{\mu}$ of $\mathfrak{sl}_n$ should be $\boldsymbol{v}$-analogues of the finite W-algebras $W(\mathfrak{sl}_N, e_\pi)$, see [57], where $N := \sum p_i$ and $e_\pi \in \mathfrak{sl}_N$ is a nilpotent element of Jordan type π.

2 Relativistic Open Toda Lattice

2.1 Quasihamiltonian Reduction

Let $G \supset B \supset T$ be a reductive group with a Borel and Cartan subgroups. Let $T \subset B_- \subset G$ be the opposite Borel subgroup; let U (resp. U_-) be the unipotent radical of B (resp. B_-). We consider the *double* $D(G) = G \times G$ (see, e.g., [2, § 3.2]) equipped with an action of $G \times G$: $(u_1, u_2) \cdot (g_1, g_2) = (u_1 g_1 u_2^{-1}, u_2 g_2 u_2^{-1})$, and with a moment map $\mu = (\mu_1, \mu_2)\colon D(G) \to G \times G,\ \mu(g_1, g_2) = (g_1 g_2 g_1^{-1}, g_2^{-1})$ (see [2, Remark 3.2]). The double $D(G)$ carries a (non-closed) 2-form $\omega_D =$

$\frac{1}{2}(\mathrm{Ad}_{g_2} g_1^*\theta, g_1^*\theta) + \frac{1}{2}(g_1^*\theta, g_2^*\theta + g_2^*\overline{\theta})$ where $(\cdot,\cdot)$ is a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, and θ (resp. $\overline{\theta}$) is the left- (resp. right-) invariant Maurer–Cartan form on G.

We choose a pair of Coxeter elements $c, c' \in W = N_G(T)/T$, and their representatives $\dot{c}, \dot{c}' \in N_G(T)$. Steinberg's cross-section $\Sigma_G^{\dot{c}} \subset G$ is defined as $Z^0(G) \cdot (U_-\dot{c} \cap \dot{c}U)$. If G is semisimple simply-connected, then the composed morphism $\Sigma_G^{\dot{c}} \hookrightarrow G \to G/\!\!/\mathrm{Ad}_G = T/W$ is an isomorphism [58, Theorem 1.4]. For arbitrary G, the composed morphism $\varrho\colon \Sigma_G^{\dot{c}} \to T/W$ is a ramified Galois cover with Galois group $\pi_1(G/Z^0(G))$. Furthermore, we consider $\Xi_G^{\dot{c}} := Z^0(G) \cdot U_-\dot{c}U_- \supset \Sigma_G^{\dot{c}}$. According to [58, § 8.9] (for a proof, see, e.g., [39]), $\Sigma_G^{\dot{c}}$ meets any U_--orbit (with respect to the conjugation action) on $\Xi_G^{\dot{c}}$ in exactly one point, and the conjugation action of U_- on $\Xi_G^{\dot{c}}$ is free, so that $\Xi_G^{\dot{c}}/\mathrm{Ad}_{U_-} \simeq \Sigma_G^{\dot{c}}$.

For example, according to [58, Example 7.4b)], for an appropriate choice of $\dot{c}$, the Steinberg cross-section $\Sigma_{SL(n)}^{\dot{c}}$ consists of the matrices with 1's just above the main diagonal, $(-1)^{n-1}$ in the bottom left corner, arbitrary entries elsewhere in the first column, and zeros everywhere else (in our conventions, B (resp. B_-) is the subgroup of upper triangular (resp. lower triangular) matrices in $SL(n)$). Hence $\Xi_{SL(n)}^{\dot{c}}$ consists of matrices with 1's just above the main diagonal, and zeros everywhere above that.

Following [26], we define the phase space of the open relativistic Toda lattice as the quasihamiltonian reduction ${}^\dagger\mathfrak{Z}^{c',c}(G) := \mu^{-1}(\Xi_G^{\dot{c}'} \times \mathrm{inv}(\Xi_G^{\dot{c}}))/U_- \times U_-$ where $\mathrm{inv}\colon G \to G$ is the inversion $g \mapsto g^{-1}$. The composed projection

$$\mu^{-1}(\Xi_G^{\dot{c}'} \times \mathrm{inv}(\Xi_G^{\dot{c}})) \to \mathrm{inv}(\Xi_G^{\dot{c}}) \hookrightarrow G \twoheadrightarrow G/\!\!/\mathrm{Ad}_G = T/W$$

gives rise to an integrable system $\varpi\colon {}^\dagger\mathfrak{Z}^{c',c}(G) \to T/W$ which factors through ${}^\dagger\mathfrak{Z}^{c',c}(G) \xrightarrow{\tilde{\varpi}} \Sigma_G^{\dot{c}} \xrightarrow{\varrho} T/W$.

Lemma 2.1 *If G is semisimple simply-connected, then ${}^\dagger\mathfrak{Z}^{c',c}(G)$ is smooth, and ω_D gives rise to a symplectic form on ${}^\dagger\mathfrak{Z}^{c',c}(G)$.*

Proof The morphism $\Xi_G^{\dot{c}} \to \Sigma_G^{\dot{c}} = T/W$ is smooth by [58, Theorem 1.5], so the fibered product $\Xi_G^{\dot{c}'} \times_{T/W} \Xi_G^{\dot{c}} \subset \Xi_G^{\dot{c}'} \times \Xi_G^{\dot{c}}$ is smooth. But

$$\mu\colon D(G) \supset \mu^{-1}(\Xi_G^{\dot{c}'} \times \mathrm{inv}(\Xi_G^{\dot{c}})) \to \Xi_G^{\dot{c}'} \times \mathrm{inv}(\Xi_G^{\dot{c}}) \simeq \Xi_G^{\dot{c}'} \times \Xi_G^{\dot{c}}$$

is a submersion onto $\Xi_G^{\dot{c}'} \times_{T/W} \Xi_G^{\dot{c}}$, hence $M := \mu^{-1}(\Xi_G^{\dot{c}'} \times \mathrm{inv}(\Xi_G^{\dot{c}}))$ is smooth, and its quotient modulo the free action of $U_- \times U_-$ is smooth as well.

The restriction of ω_D to M is $U_- \times U_-$-invariant, so it descends to a 2-form ω on ${}^\dagger\mathfrak{Z}^{c',c}(G)$. This 2-form is closed since the differential $d\omega_D = -\mu^*(\chi_1 + \chi_2)$ (see [2, Definition 2.2(B1)]) where $\chi = \frac{1}{12}(\theta, [\theta, \theta])$ is the canonical closed biinvariant 3-form on G, and χ_1 (resp. χ_2) is its pull-back from the first (resp. second) copy of G. But the restriction $\chi|_{\Xi_G^{\dot{c}}}$ vanishes identically since $(\mathfrak{b}_-, [\mathfrak{b}_-, \mathfrak{b}_-]) = 0$.

It remains to check the nondegeneracy of ω, that is given $(g_1, g_2) \in M$ to check that $\operatorname{Ker} \omega_D|_M(g_1, g_2)$ is contained in the span $v(\mathfrak{n}_- \oplus \mathfrak{n}_-)$ of tangent vectors at (g_1, g_2) arising from the action of $U_- \times U_-$. The argument in the proof of [2, Theorem 5.1] shows that $\operatorname{Ker} \omega_D|_M(g_1, g_2) \subset v(\mathfrak{g} \oplus \mathfrak{g})$. However, it is clear that $T_{(g_1,g_2)}M \cap v(\mathfrak{g} \oplus \mathfrak{g}) = v(\mathfrak{n}_- \oplus \mathfrak{n}_-)$.

The lemma is proved. □

2.2 Poisson Reduction

Note that $T \cdot \Xi_G^{\dot{c}} = \Xi_G^{\dot{c}} \cdot T = \operatorname{Ad}_T(\Xi_G^{\dot{c}}) = B_- \cdot \dot{c} \cdot B_- =: C_c$ (a Coxeter Bruhat cell). One can check that the natural morphism

$$^{\dagger}\mathfrak{Z}^{c',c}(G) = \mu^{-1}(\Xi_G^{\dot{c}'} \times \operatorname{inv}(\Xi_G^{\dot{c}}))/U_- \times U_- \to \mu^{-1}(C_{c'} \times \operatorname{inv}(C_c))/B_- \times B_-$$

is an isomorphism. Moreover, the action of $B_- \times B_-$ on $\mu^{-1}(C_{c'} \times \operatorname{inv}(C_c))$ factors through the free action of $(B_- \times B_-)/\Delta_{Z(G)}$: the quotient modulo the diagonal copy of the center of G.

The double $D(G) = G \times G$ carries the Semenov-Tian-Shansky Poisson structure [59, Section 2]. Following loc. cit., $G \times G$ with this Poisson structure is denoted by $(D_+(G), \{,\}_+)$, the *Heisenberg double*. Another Poisson structure on $G \times G$ denoted $\{,\}_-$ in *loc. cit.* is the *Drinfeld double* $D_-(G)$. The diagonal embedding $G \hookrightarrow D_-(G)$ is Poisson with respect to the standard Poisson structure on G denoted π_G in [20, § 2.1]. The dual (Semenov-Tian-Shansky) Poisson structure on G is denoted π in [20, § 2.2].

The Heisenberg double $D_+(G)$ is equipped with two commuting (left and right) dressing Poisson actions of the Drinfeld double $D_-(G)$. Restricting to the diagonal $G \hookrightarrow D_-(G)$ we obtain two commuting Poisson actions of (G, π_G) on $D_+(G)$. The multiplicative moment map of this action is nothing but $\mu\colon D_+(G) \to (G, \pi) \times (G, \pi)$ of Sect. 2.1 (a Poisson morphism). Now $C_c \subset G$ is a coisotropic subvariety [20, § 6.2] of (G, π), and $\mu^{-1}(C_{c'} \times \operatorname{inv}(C_c)) \hookrightarrow D(G)$ is a coisotropic subvariety of $(D_+(G), \{,\}_+)$. The action of $G \times G$ on $(D_+(G), \{,\}_+)$ is Poisson if $G \times G$ is equipped with the direct product of the standard Poisson-Lie structures denoted π_G in [20, § 2.1]. Note that $B_- \times B_- \subset G \times G$ is a Poisson-Lie subgroup; its Poisson structure will be denoted $\pi_{B_-} \times \pi_{B_-}$.

The characteristic distribution [20, § 6.2] of the coisotropic subvariety $\mu^{-1}(C_{c'} \times \operatorname{inv}(C_c)) \subset (D_+(G), \{,\}_+)$ coincides with the distribution defined by the tangent spaces to the $B_- \times B_-$-orbits in $\mu^{-1}(C_{c'} \times \operatorname{inv}(C_c))$. By [20, Proposition 6.7] we obtain a Poisson structure on $\mu^{-1}(C_{c'} \times \operatorname{inv}(C_c))/(B_- \times B_-) \simeq {}^{\dagger}\mathfrak{Z}^{c',c}(G)$. This Poisson structure coincides with the one arising from the symplectic form ω on ${}^{\dagger}\mathfrak{Z}^{c',c}(G)$.

2.3 The Universal Centralizer

Recall that the universal centralizer [49, Section 8] $\mathfrak{Z}^G_G \subset G \times \Sigma^{\dot{c}}_G$ is defined as $\mathfrak{Z}^G_G = \{(g, x) : gxg^{-1} = x\}$. In case $c = c'$ and $\dot{c} = \dot{c}'$, we have an evident embedding $\mathfrak{Z}^G_G \hookrightarrow \mu^{-1}(\Xi^{\dot{c}}_G \times \mathrm{inv}(\Xi^{\dot{c}}_G))$, and the composed morphism $\eta\colon \mathfrak{Z}^G_G \hookrightarrow \mu^{-1}(\Xi^{\dot{c}}_G \times \mathrm{inv}(\Xi^{\dot{c}}_G)) \twoheadrightarrow {}^{\dagger}\mathfrak{Z}^{c,c}(G)$. Clearly, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{Z}^G_G & \xrightarrow{\eta} & {}^{\dagger}\mathfrak{Z}^{c,c}(G) \\ \big\downarrow \mathrm{pr} & & \tilde{\varpi} \big\downarrow \\ \Sigma^{\dot{c}}_G & = & \Sigma^{\dot{c}}_G \end{array}$$

Proposition 2.2 *For semisimple simply-connected G, the morphism $\eta\colon \mathfrak{Z}^G_G \to {}^{\dagger}\mathfrak{Z}^{c,c}(G)$ is an isomorphism.*

Proof First we prove the surjectivity of η. We use the equality $U_- \times U_- = (U_- \times \{e\}) \times \Delta_{U_-}$. Given $(g_1, g_2) \in \mu^{-1}(\Xi^{\dot{c}}_G \times \mathrm{inv}(\Xi^{\dot{c}}_G))$ we first act by $(u_2, u_2) \in \Delta_{U_-}\colon (g_1, g_2) \mapsto (u_2g_1u_2^{-1}, u_2g_2u_2^{-1})$. We can find a unique u_2 such that $u_2g_2u_2^{-1} \in \Sigma^{\dot{c}}_G$. Let us denote the resulting $(u_2g_1u_2^{-1}, u_2g_2u_2^{-1})$ by (h_1, h_2) for brevity. Now we act by the left shift $h_1 \mapsto u_1h_1$ which takes $h_1h_2h_1^{-1}$ to $u_1h_1h_2h_1^{-1}u_1^{-1}$. We can find a unique u_1 such that $u_1h_1h_2h_1^{-1}u_1^{-1} \in \Sigma^{\dot{c}}_G$. Now both $h_2 = u_2g_2u_2^{-1}$ and $u_1h_1h_2h_1^{-1}u_1^{-1}$ are in $\Sigma^{\dot{c}}_G$. Being conjugate they must coincide, hence $(u_1h_1, h_2) \in \mathfrak{Z}^G_G$.

Now if $\eta(g, x) = \eta(g', x')$, then there is $u_2 \in U_-$ such that $u_2xu_2^{-1} = x'$, hence $x = x'$ and $u_2 = e$. Then $g' = u_1g$ for some $u_1 \in U_-$, and both g and g' commute with x, hence $u_1xu_1^{-1} = x$, hence $u_1 = e$, so that $g = g'$.

So η is bijective at the level of $\mathbb{C}$-points. But ${}^{\dagger}\mathfrak{Z}^{c,c}(G)$ is smooth, hence η is an isomorphism. □

Remark 2.3 For arbitrary reductive G the morphism η is an affine embedding, but it fails to be surjective already for $G = PGL(2)$ where the class of (g_1, g_2) such that $g_2 = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ and $g_1g_2g_1^{-1} = \begin{pmatrix} -a & -1 \\ 1 & 0 \end{pmatrix}$ does not lie in the image of η when $a \neq 0$. Similarly, for $G = GL(2)$, the class of (g_1, g_2) such that $g_2 = \begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}$ and $g_1g_2g_1^{-1} = \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}$ does not lie in the image of η. It also follows that the natural projection ${}^{\dagger}\mathfrak{Z}^{c,c}(SL(2)) \to {}^{\dagger}\mathfrak{Z}^{c,c}(PGL(2))$ is *not* surjective.

Remark 2.4 For G semisimple simply-connected, the reduction

$$(\mathbf{D}(G), \omega_{\mathbf{D}(G)})/\!\!/\mathrm{diag}(G)$$

[2, Example 6.1, Remark 6.2] inherits a symplectic structure on its nonsingular locus. We have a natural morphism $\mathfrak{Z}^G_G \to (\mathbf{D}(G), \omega_{\mathbf{D}(G)})/\!\!/\mathrm{diag}(G)$ which is a birational isomorphism (but not an isomorphism: e.g., it contracts the centralizer of a regular unipotent element). Thus an open subvariety of $\mathfrak{Z}^G_G$ is equipped with a symplectic form pulled back from $(\mathbf{D}(G), \omega_{\mathbf{D}(G)})/\!\!/\mathrm{diag}(G)$. This form extends to a symplectic form on the entire $\mathfrak{Z}^G_G$ [8, § 2.4]. The isomorphism $\eta\colon \mathfrak{Z}^G_G \xrightarrow{\sim} {}^\dagger\mathfrak{Z}^{c,c}(G)$ is a symplectomorphism.

2.4 Comparison with the Coxeter–Toda Lattice

We compare ${}^\dagger\mathfrak{Z}^{c',c}(G)$ with the construction of [38]. Throughout this section we assume G to be semisimple simply-connected. The left action of the center $Z(G)$ on $D(G)$, $\xi \cdot (g_1, g_2) = (\xi g_1, g_2)$ gives rise to the action of $Z(G)$ on ${}^\dagger\mathfrak{Z}^{c',c}(G) = M/U_- \times U_-$ where $M = \mu^{-1}(\Xi^{\dot c'}_G \times \mathrm{inv}(\Xi^{\dot c}_G)) \subset D(G) = G \times G$. We consider an open subset $M \supset \overset{\bullet}{M} := (U_- \cdot T \cdot \dot w_0 \cdot U_- \times G) \cap M$ given by the condition that g_1 lies in the big Bruhat cell $C_{w_0} \subset G$. Clearly, $\overset{\bullet}{M} \subset M$ is $U_- \times U_-$-invariant, and we define ${}^\dagger\overset{\bullet}{\mathfrak{Z}}{}^{c',c}(G) := \overset{\bullet}{M}/U_- \times U_-$, an open subvariety of ${}^\dagger\mathfrak{Z}^{c',c}(G)$. Let $S \subset \overset{\bullet}{M}$ be given by the condition $g_1 \in T \cdot \dot w_0$. Then the composed projection $S \hookrightarrow \overset{\bullet}{M} \twoheadrightarrow {}^\dagger\overset{\bullet}{\mathfrak{Z}}{}^{c',c}(G)$ is an isomorphism. Moreover, the projection $\mathrm{pr}_2\colon S \to G$ is a $Z(G)$-torsor over its image $\Xi^{\dot c}_G \cap \mathrm{Ad}_T(\dot w_0 \Xi^{\dot c'}_G \dot w_0^{-1}) = \Xi^{\dot c}_G \cap \mathrm{Ad}_T(U \dot w_0 \dot c' \dot w_0^{-1} U)$. Finally, note that the composed projection

$$\Xi^{\dot c}_G \cap \mathrm{Ad}_T(U \dot w_0 \dot c' \dot w_0^{-1} U) \hookrightarrow T \cdot U_- \cdot \dot c \cdot U_- \cdot T \cap T \cdot U \cdot \dot w_0 \dot c' \dot w_0^{-1} \cdot U \cdot T \twoheadrightarrow$$
$$\twoheadrightarrow (T \cdot U_- \cdot \dot c \cdot U_- \cdot T \cap T \cdot U \cdot \dot w_0 \dot c' \dot w_0^{-1} \cdot U \cdot T)/\mathrm{Ad}_T =: G^{\dot c, \dot w_0 \dot c' \dot w_0^{-1}}/\mathrm{Ad}_T$$

is an isomorphism. But according to [38] (see also [34]), $G^{\dot c, \dot w_0 \dot c' \dot w_0^{-1}}/\mathrm{Ad}_T$ is the phase space of the Coxeter–Toda lattice. All in all, we obtain an isomorphism (respecting the symplectic structures) $Z(G)\backslash {}^\dagger\overset{\bullet}{\mathfrak{Z}}{}^{c',c} \xrightarrow{\sim} G^{\dot c, \dot w_0 \dot c' \dot w_0^{-1}}/\mathrm{Ad}_T$.

For example, for an appropriate choice of $\dot c, \dot c' \in SL(n)$, the slice S is formed by all the tridiagonal matrices of determinant 1 with 1's just above the main diagonal, and with the invertible entries just below the main diagonal (see [34, Introduction]).

We also define an open subset $S \supset \overset{\circ}{S} := \{(g_1, g_2) \in M : g_1 \in T \cdot \dot w_0,\ g_2 \in U_- \cdot T \cdot U\}$. It is equipped with a projection $\mathrm{pr}_1\colon \overset{\circ}{S} \to T \cdot \dot w_0 \xrightarrow{\sim} T$, and with another projection $\mathrm{pr}_2\colon \overset{\circ}{S} \to U_- \cdot T \cdot U \twoheadrightarrow T$. One can check that $(\mathrm{pr}_1, \mathrm{pr}_2)\colon \overset{\circ}{S} \xrightarrow{\sim} T \times T$. We define an open subvariety ${}^\dagger\mathfrak{Z}^{c',c}(G) \supset {}^\dagger\overset{\bullet}{\mathfrak{Z}}{}^{c',c}(G) \supset {}^\dagger\overset{\circ}{\mathfrak{Z}}{}^{c',c}(G)$ as the isomorphic image of $\overset{\circ}{S}$. Thus ${}^\dagger\overset{\circ}{\mathfrak{Z}}{}^{c',c}(G) \simeq T \times T$.

2.5 Trigonometric Zastava for $SL(2)$

Recall the degree n trigonometric open zastava ${}^\dagger\overset{\circ}{Z}{}^n$ for the group $SL(2)$ (see [25]). This is the moduli space of pairs of relatively prime polynomials ($Q = z^n + q_1 z^{n-1} + \ldots + q_n,\ R = r_1 z^{n-1} + r_2 z^{n-2} + \ldots + r_n$) such that $q_n \neq 0$. We have a morphism $\zeta\colon \mathfrak{Z}^{GL(n)}_{GL(n)} \to {}^\dagger\mathring{Z}^n$ taking a pair $(g, x) \in \mathfrak{Z}^{GL(n)}_{GL(n)}$ to (Q, R) where Q is the characteristic polynomial of x, and R is a unique polynomial of degree less than n such that $R(x) = g$. We denote by $\mathrm{pr}\colon {}^\dagger\mathring{Z}^n \to (\mathbb{C}^\times)^{(n)}$ the morphism taking (Q, R) to the set of roots of Q.

Recall that $\Sigma^{\dot{c}}_{GL(n)} = Z^0(GL(n)) \cdot \Sigma^{\dot{c}}_{SL(n)} \simeq Z(GL(n)) \times \Sigma^{\dot{c}}_{SL(n)} = \mathbb{C}^\times \times \Sigma^{\dot{c}}_{SL(n)}$. We denote by $\mathsf{p}\colon \mathfrak{Z}^{GL(n)}_{GL(n)} \to \mathbb{C}^\times$ the composed projection $\mathfrak{Z}^{GL(n)}_{GL(n)} \to \Sigma^{\dot{c}}_{GL(n)} \to \mathbb{C}^\times$.

Proposition 2.5 *The following square is Cartesian:*

$$\begin{array}{ccc} \mathfrak{Z}^{GL(n)}_{GL(n)} & \xrightarrow{\ \zeta\ } & {}^\dagger\mathring{Z}^n \\ \big\downarrow{\scriptstyle \mathsf{p}} & & {\scriptstyle q_n}\big\downarrow \\ \mathbb{C}^\times & \xrightarrow{\ n\ } & \mathbb{C}^\times \end{array}$$

Thus $\mathfrak{Z}^{GL(n)}_{GL(n)}$ *is an unramified* $\mathbb{Z}/n\mathbb{Z}$*-cover of* ${}^\dagger\mathring{Z}^n$.

Proof Clear from the above discussion. □

Following [1, end of chapter 2], we consider the subvariety ${}^\dagger\tilde{Z}^n_1 \hookrightarrow {}^\dagger\mathring{Z}^n$ formed by the pairs (Q, R) such that $q_n = 1$ and the resultant of Q and R, denoted Result(Q, R), equals 1. Note that we have an evident embedding $\mathfrak{Z}^{SL(n)}_{SL(n)} \hookrightarrow \mathfrak{Z}^{GL(n)}_{GL(n)}$.

Corollary 2.6 *The restriction of the morphism* ζ *to* $\mathfrak{Z}^{SL(n)}_{SL(n)} \subset \mathfrak{Z}^{GL(n)}_{GL(n)}$ *gives rise to an isomorphism* $\zeta\colon \mathfrak{Z}^{SL(n)}_{SL(n)} \xrightarrow{\sim} {}^\dagger\tilde{Z}^n_1$.

Proof For $(g, x) \in \mathfrak{Z}^{GL(n)}_{GL(n)}$, the inclusion $x \in SL(n)$ is equivalent to $q_n = 1$, while we claim that the inclusion $g \in SL(n)$ is equivalent to $\mathrm{Result}(Q, R) = 1$. The latter follows by combining the equalities $gx = xg$ and $g = R(x)$ with the standard equality $\mathrm{Result}(Q, R) = \prod_{i=1}^n R(\xi_i)$, where $\{\xi_i\}_{i=1}^n$ are the roots of Q. Since $\{\xi_i\}_{i=1}^n$ are the generalized eigenvalues (taken with corresponding multiplicities) of x, it is easy to see that $\{R(\xi_i)\}_{i=1}^n$ are the generalized eigenvalues of g, hence, $\det(g) = \prod_{i=1}^n R(\xi_i)$. □

For a future use we define an unramified $\mathbb{Z}/2\mathbb{Z}$-cover ${}^\dagger\hat{Z}^n \to {}^\dagger\mathring{Z}^n$ where ${}^\dagger\hat{Z}^n$ is the moduli space of pairs of relatively prime polynomials $(Q = q_0 z^n + q_1 z^{n-1} + \ldots + q_n,\ R = r_1 z^{n-1} + r_2 z^{n-2} + \ldots + r_n)$ such that $q_n \cdot q_0 = (-1)^n$. The projection ${}^\dagger\hat{Z}^n \to {}^\dagger\mathring{Z}^n$ takes (Q, R) to $(q_0^{-1} Q, R)$.

Finally, there are important embeddings $\Psi\colon {}^\dagger\mathring{Z}^n, {}^\dagger\hat{Z}^n \hookrightarrow SL(2, \mathbb{C}[z])$ taking (Q, R) to a unique matrix $\begin{pmatrix} Q & \tilde{R} \\ R & \tilde{Q} \end{pmatrix}$ such that $\deg \tilde{R} \le n > \deg \tilde{Q}$, and $\tilde{R}(0) = 0$, that is $\tilde{R} = \tilde{r}_0 z^n + \tilde{r}_1 z^{n-1} + \ldots + \tilde{r}_{n-1} z$. Identifying ${}^\dagger\hat{Z}^n$ and ${}^\dagger\mathring{Z}^n$ with their images inside $SL(2, \mathbb{C}[z])$, the matrix multiplication gives rise to the multiplication morphisms ${}^\dagger\hat{Z}^k \times {}^\dagger\hat{Z}^l \to {}^\dagger\hat{Z}^{k+l}$, ${}^\dagger\mathring{Z}^k \times {}^\dagger\mathring{Z}^l \to {}^\dagger\mathring{Z}^{k+l}$.

3 Quantum Relativistic Open Toda and Nil-DAHA

Throughout this section (with the exception of Sect. 3.11 dealing with $G = GL(n)$) G is an almost simple simply-connected complex algebraic group.

3.1 Root Systems and Foldings

Let $G^\vee$ be the Langlands dual (adjoint) group with a Cartan torus $T^\vee$. We choose a Borel subgroup $B^\vee \supset T^\vee$. It defines the set of simple positive roots $\{\alpha_i,\ i \in I\}$. Let $\mathfrak{g}^\vee$ be the Lie algebra of $G^\vee$. We realize $\mathfrak{g}^\vee$ as a *folding* of a simple simply-laced Lie algebra $\mathfrak{g}'^\vee$, i.e. as invariants of an outer automorphism σ of $\mathfrak{g}'^\vee$ preserving a Cartan subalgebra $\mathfrak{t}'^\vee \subset \mathfrak{g}'^\vee$ and acting on the root system of $(\mathfrak{g}'^\vee, \mathfrak{t}'^\vee)$. In particular, σ gives rise to the same named automorphism of the Langlands dual Lie algebras $\mathfrak{g}' \supset \mathfrak{t}'$ (note that say, if $\mathfrak{g}$ is of type B_n, then $\mathfrak{g}'$ is of type A_{2n-1}, while if $\mathfrak{g}$ is of type C_n, then $\mathfrak{g}'$ is of type D_{n+1}; in particular, $\mathfrak{g} \not\subset \mathfrak{g}'$). We choose a σ-invariant Borel subalgebra $\mathfrak{t}' \subset \mathfrak{b}' \subset \mathfrak{g}'$ such that $\mathfrak{b} = (\mathfrak{b}')^\sigma$. The corresponding set of simple roots is denoted by I'. We denote by Ξ the finite cyclic group generated by σ, and $d := |\Xi|$. Let $G' \supset T'$ denote the corresponding simply-connected Lie group and its Cartan torus. The *coinvariants* $X_*(T')_\sigma$ of σ on the coroot lattice $X_*(T')$ of $(\mathfrak{g}', \mathfrak{t}')$ coincide with the root lattice of $\mathfrak{g}^\vee$. We have an injective map $a:\ X_*(T')_\sigma \to X_*(T')^\sigma$ from coinvariants to invariants defined as follows: given a coinvariant α with a representative $\tilde{\alpha} \in X_*(T')$ we set $a(\alpha) := \sum_{\xi\in\Xi} \xi(\tilde{\alpha})$.

To compare with the notations of [36, § 4.4, Remark 4.5], we are in the symmetric case with $Q'_0 = Y := X^*(T^\vee) = X_*(T) = X_*(T')_\sigma$, and $Q_0 \subset X := X^*(T')_\sigma$ generated by the classes of simple roots of $T' \subset B' \subset G'$. Note that Q'_0 is generated by the classes of simple coroots of $T' \subset B' \subset G'$, and we have a canonical identification $Q_0 = Q'_0$ sending a coroot $\tilde{\alpha}$ to the corresponding root $\tilde{\alpha}^\vee$. The Weyl group W of $G \supset T$ coincides with the invariants $(W')^\sigma$ of σ on the Weyl group W' of $G' \supset T'$ (our W is denoted W_0 in [36]). The W-invariant pairing $X \times Y \to \mathbb{Q}$ defined in [36, § 4.4] is actually integer valued: $X \times Y \to \mathbb{Z}$, so that $m = 1$ (notations of loc. cit.). To compare with notations of [13, Section 1], $P := X$, $Q := Q_0$, and the natural pairing $P \times P \to \mathbb{Q}$ gives rise to the embedding $Q = Y \hookrightarrow P$. We will also need an extended lattice $Y_{\mathrm{ad}} := X_*(T_{\mathrm{ad}}) = X_*(T'_{\mathrm{ad}})_\sigma \supset Y$. Note that $\Pi := Y_{\mathrm{ad}}/Y = (X_*(T'_{\mathrm{ad}})/X_*(T'))_\sigma$. Also note that the above W-invariant identification $Q_0 = Q'_0$ extends to the W-invariant identification $Q_0 \subset X = Y_{\mathrm{ad}} \supset Q'_0$. The extended pairing $X \times Y_{\mathrm{ad}} \to \mathbb{Q}$ is no more integer valued in general, and we denote by m_{ad} the maximal denominator appearing in the values of this pairing. Finally, $R \subset X$ stands for the set of roots.

3.2 Affine Flags

We fix a primitive root of unity ζ of order $d = \mathrm{ord}(\sigma)$. We set $\mathcal{K} = \mathbb{C}((\mathsf{t})) \supset \mathcal{O} = \mathbb{C}[[\mathsf{t}]]$. The group ind-scheme $G'(\mathcal{K})$ is equipped with an automorphism ς defined as the composition of two automorphisms: a) σ on G'; b) $\mathsf{t} \mapsto \zeta\mathsf{t}$. This automorphism preserves the Iwahori subgroup $\mathbf{I}' \subset G'(\mathcal{K})$. We denote by $\mathcal{F}\ell$ the twisted affine flag space $G'(\mathcal{K})^\varsigma/(\mathbf{I}')^\varsigma$: an ind-proper ind-scheme of ind-finite type, see [55]. We denote by $\mathfrak{u} \subset \mathrm{Lie}(\mathbf{I}')^\varsigma$ its pronilpotent radical. The trivial (Tate) bundle $\underline{\mathfrak{g}'(\mathcal{K})^\varsigma}$ with the fiber $\mathfrak{g}'(\mathcal{K})^\varsigma$ over $\mathcal{F}\ell$ has a structure of an ind-scheme. It contains a profinite dimensional vector subbundle $\underline{\mathfrak{u}}$ whose fiber over a point $b \in \mathcal{F}\ell$ represented by a compact subalgebra in $\mathfrak{g}'(\mathcal{K})^\varsigma$ is the pronilpotent radical of this subalgebra. The trivial vector bundle $\underline{\mathfrak{g}'(\mathcal{K})^\varsigma}$ also contains a trivial vector subbundle $\mathfrak{u} \times \mathcal{F}\ell$. We will call $\underline{\mathfrak{u}}$ the cotangent bundle of $\mathcal{F}\ell$, and we will call the intersection $\mathbf{\Lambda} := \underline{\mathfrak{u}} \cap (\mathfrak{u} \times \mathcal{F}\ell)$ the affine Steinberg variety.

To simplify the notations we will write $\mathbf{I}$ for $(\mathbf{I}')^\varsigma$, and $\mathbf{K}$ for $G'(\mathcal{O})^\varsigma$. The convolution product on the complexified equivariant coherent K-theory $K^{\mathbb{C}^\times \times \mathbf{I} \rtimes \mathbb{C}^\times}(\mathbf{\Lambda})$ is defined as in [9, Remark 3.9(3)] (cf. [8, § 7.1] and [64, § 2.2, 2.3]). Here the first copy of $\mathbb{C}^\times$ acts by dilations in fibers of $\underline{\mathfrak{u}}$, while the second one acts by loop rotations, and $K_{\mathbb{C}^\times \times \mathbb{C}^\times}(\mathrm{pt}) = \mathbb{C}[t^{\pm 1}, q^{\pm 1}]$.

3.3 DAHA, Symmetric Case

Following [36, § 4.6], we set $\widetilde{X} := X \oplus \mathbb{Z}\delta = X^*(T')_\sigma \oplus \mathbb{Z}\delta$. This is the group of characters of $\mathbf{I} \rtimes \mathbb{C}^\times$. Note that the Picard group $\mathrm{Pic}(\mathcal{F}\ell)$ is canonically isomorphic to $X \oplus \mathbb{Z}\omega_0$. The $\mathbf{I}$-orbits on $\mathcal{F}\ell$ are parametrized by the affine Weyl group $W_a \simeq Y \rtimes W = X_*(T')_\sigma \rtimes W$. We denote by $\mathbf{\Lambda}_e \simeq \mathfrak{u}$ the closed subscheme of $\mathbf{\Lambda}$: the preimage of the one-point $\mathbf{I}$-orbit $\mathcal{F}\ell_e \subset \mathcal{F}\ell$. For $\tilde{\lambda} = (\check{\lambda}, k) \in \widetilde{X}$ we denote by $\mathcal{O}_{\mathbf{\Lambda}_e}\langle\tilde{\lambda}\rangle \in K^{\mathbb{C}^\times \times \mathbf{I} \rtimes \mathbb{C}^\times}(\mathbf{\Lambda})$ the (class of the) direct image of the structure sheaf of $\mathbf{\Lambda}_e$ twisted by the character $\tilde{\lambda}$ of $\mathbf{I} \rtimes \mathbb{C}^\times$. Let $\widetilde{I} \subset W_a$ be the set of one-dimensional $\mathbf{I}$-orbits on $\mathcal{F}\ell$. For $i \in \widetilde{I}$ we denote by $\mathcal{F}\ell_i$ the corresponding orbit, and by $\overline{\mathcal{F}\ell_i}$ its closure, isomorphic to a projective line. We denote by $\mathbf{\Lambda}_i \subset \mathbf{\Lambda}$ the closed subscheme of $\mathbf{\Lambda}$: the closure of the preimage of $\mathcal{F}\ell_i$. We denote by $\omega_{\mathbf{\Lambda}_i}$ the (class of the) direct image (wrt the closed embedding $\mathbf{\Lambda}_i \hookrightarrow \mathbf{\Lambda}$) of the inverse image (wrt the smooth projection $\mathbf{\Lambda}_i \to \overline{\mathcal{F}\ell_i}$) of the canonical line bundle on $\overline{\mathcal{F}\ell_i} \simeq \mathbb{P}^1$ equipped with the natural $\mathbb{C}^\times \times \mathbf{I} \rtimes \mathbb{C}^\times$-equivariant structure. Finally, we set $\mathsf{T}_i := -1 - t\omega_{\mathbf{\Lambda}_i} \in K^{\mathbb{C}^\times \times \mathbf{I} \rtimes \mathbb{C}^\times}(\mathbf{\Lambda})$.

Definition 3.1 (Cf. [36, Definition 5.6]) The *double affine Hecke algebra (DAHA)* $\mathcal{H}(W_a, \widetilde{X})$ is the $\mathbb{C}[q^{\pm 1}, t^{\pm 1}]$-algebra generated by $\{\mathsf{X}_{\tilde{\lambda}}, T_w | \tilde{\lambda} \in \widetilde{X}, w \in W_a\}$ with the following defining relations:

(a) T_w's satisfy the braid relations of W_a;
(b) $\mathsf{X}_{\tilde{\lambda}}\mathsf{X}_{\tilde{\mu}} = \mathsf{X}_{\tilde{\lambda}+\tilde{\mu}}$, and $\mathsf{X}_\delta = q$;

(c) $(T_i - t)(T_i + 1) = 0$ for $i \in \widetilde{I}$, where we set $T_i = T_{s_i}$;
(d) $\mathsf{X}_{\tilde{\lambda}} T_i - T_i \mathsf{X}_{\tilde{\lambda}-r\check{\alpha}_i} = (t-1)\mathsf{X}_{\tilde{\lambda}}(1 + \mathsf{X}_{-\check{\alpha}_i} + \ldots + \mathsf{X}^{r-1}_{-\check{\alpha}_i})$ where $\langle \tilde{\lambda}, \alpha_i \rangle = r \geq 0$.

Theorem 3.2 *There is a unique isomorphism* $\Phi\colon \mathcal{H}(W_a, \widetilde{X}) \xrightarrow{\sim} K^{\mathbb{C}^\times \times \mathbf{I} \rtimes \mathbb{C}^\times}(\mathbf{\Lambda})$ *such that* $\Phi(\mathsf{X}_{\tilde{\lambda}}) = \mathcal{O}_{\mathbf{\Lambda}_e}\langle\tilde{\lambda}\rangle$, *and* $\Phi(T_i) = \mathsf{T}_i$, *for any* $i \in \widetilde{I}$.

Proof Same as the one of [64, Theorem 2.5.6]. □

3.4 Nil-DAHA, Symmetric Case

The complexified equivariant K-theory $K^{\mathbf{I} \rtimes \mathbb{C}^\times}(\mathcal{F}\ell)$ forms a $\mathbb{C}[q^{\pm 1}]$-algebra with respect to the convolution. We denote by $\mathcal{O}_{\mathcal{F}\ell_e}\langle\tilde{\lambda}\rangle$ the (class of the) structure sheaf of the point orbit $\mathcal{F}\ell_e \in \mathcal{F}\ell$ twisted by a character $\tilde{\lambda} \in \widetilde{X}$. We denote by $\omega_{\overline{\mathcal{F}\ell}_i}$ the (class of the) direct image (wrt the closed embedding $\overline{\mathcal{F}\ell}_i \hookrightarrow \mathcal{F}\ell$) of the canonical line bundle on $\overline{\mathcal{F}\ell}_i$ equipped with the natural $\mathbf{I} \rtimes \mathbb{C}^\times$-equivariant structure. We set $\mathbf{T}_i := -1 - \omega_{\overline{\mathcal{F}\ell}_i} \in K^{\mathbf{I} \rtimes \mathbb{C}^\times}(\mathcal{F}\ell)$.

Definition 3.3 (Cf. [13, § 1.1]) The *nil-DAHA* $\mathcal{HH}(W_a, \widetilde{X})$ is the $\mathbb{C}[q^{\pm 1}]$-algebra generated by $\{\mathsf{X}_{\tilde{\lambda}}, \mathcal{T}_w | \tilde{\lambda} \in \widetilde{X}, w \in W_a\}$ with the following defining relations:

(a) $\mathcal{T}_w$'s satisfy the braid relations of W_a;
(b) $\mathsf{X}_{\tilde{\lambda}}\mathsf{X}_{\tilde{\mu}} = \mathsf{X}_{\tilde{\lambda}+\tilde{\mu}}$, and $\mathsf{X}_\delta = q$;
(c) $\mathcal{T}_i(\mathcal{T}_i + 1) = 0$ for $i \in \widetilde{I}$, where we set $\mathcal{T}_i = \mathcal{T}_{s_i}$;
(d) $\mathsf{X}_{\tilde{\lambda}}\mathcal{T}_i - \mathcal{T}_i\mathsf{X}_{\tilde{\lambda}-r\check{\alpha}_i} = -\mathsf{X}_{\tilde{\lambda}}(1 + \mathsf{X}_{-\check{\alpha}_i} + \ldots + \mathsf{X}^{r-1}_{-\check{\alpha}_i})$ where $\langle\tilde{\lambda}, \alpha_i\rangle = r \geq 0$.

Theorem 3.4 *There is a unique isomorphism* $\Phi\colon \mathcal{HH}(W_a, \widetilde{X}) \xrightarrow{\sim} K^{\mathbf{I} \rtimes \mathbb{C}^\times}(\mathcal{F}\ell)$ *such that* $\Phi(\mathsf{X}_{\tilde{\lambda}}) = \mathcal{O}_{\mathcal{F}\ell_e}\langle\tilde{\lambda}\rangle$, *and* $\Phi(\mathcal{T}_i) = \mathbf{T}_i$, *for any* $i \in \widetilde{I}$.

Proof Same as the one of [64, Theorem 2.5.6]. □

3.5 Extended Nil-DAHA

We consider the $2m_{\mathrm{ad}}$-fold cover $\widetilde{\mathbb{C}}^\times \to \mathbb{C}^\times$ of the loop rotation group (see the end of Sect. 3.1), and set $\widehat{\mathbf{I}} := \mathbf{I} \rtimes \widetilde{\mathbb{C}}^\times$. The group of characters of $T \times \widetilde{\mathbb{C}}^\times$ is $\widehat{X} := X \oplus \mathbb{Z}\frac{1}{2m_{\mathrm{ad}}}\delta$. The extended affine Weyl group is $W_e = Y_{\mathrm{ad}} \rtimes W = W_a \rtimes \Pi$. The extended nil-DAHA $\mathcal{HH}(W_e, \widehat{X})$ is the (extended) semidirect product $(\mathcal{HH}(W_a, \widehat{X}) \rtimes \Pi) \otimes_{\mathbb{C}[q^{\pm 1}]} \mathbb{C}[q^{\frac{\pm 1}{2m_{\mathrm{ad}}}}]$. That is, it has generators $\mathsf{X}_{\hat{\lambda}}$, $\hat{\lambda} \in \widehat{X}$, and $\mathcal{T}_i$, $i \in \widetilde{I}$, and $\pi \in \Pi$; with additional relations $\pi\mathcal{T}_i\pi^{-1} = \mathcal{T}_{\pi(i)}$, and $\pi\mathsf{X}_{\hat{\lambda}}\pi^{-1} = \mathsf{X}_{\pi(\hat{\lambda})}$.

Remark 3.5 The definition of [13, § 1.1] is equivalent to our Sect. 3.5: the generators T_i of loc. cit. correspond to $-\mathcal{T}_i - 1$; geometrically, $T_i = [\omega_{\overline{\mathcal{F}\ell}_i}]$.

3.6 Residue Construction

Let $\mathcal{A} := \mathbb{C}[q^{\frac{\pm 1}{2m_{\mathrm{ad}}}}]$, and $\mathcal{Q} := \mathbb{C}(q^{\frac{1}{2m_{\mathrm{ad}}}})$. Let $\mathcal{O}_q(T \times T)$ be an $\mathcal{A}$-algebra with generators $[\lambda, \mu],\ \lambda, \mu \in X$, and relations $[\lambda, \mu] \cdot [\lambda', \mu'] = q^{\frac{(\mu,\lambda')-(\mu',\lambda)}{2}}[\lambda + \lambda', \mu + \mu']$. This is the subalgebra of endomorphisms of $\mathcal{A}[T]$ generated by multiplications by $\mathsf{X}_\lambda,\ \lambda \in X$, and q-shift operators $D_q^\mu f(t) := f(q^\mu t)$ where we view q^μ as a homomorphism $\widetilde{\mathbb{C}}^\times \to T$. In other words, $D_q^\mu \mathsf{X}_\lambda = q^{(\mu,\lambda)}\mathsf{X}_\lambda$. We may and will view $\mathcal{O}_q(T \times T)$ as a subalgebra of endomorphisms of the field of rational functions $\mathcal{Q}(T)$ as well. It embeds into the subalgebra $\mathbb{C}_q(T \times T) \subset \mathrm{End}(\mathcal{Q}(T))$ generated by $D_q^\mu,\ \mu \in X$, and multiplications by $f \in \mathcal{Q}(T)$. We consider the semidirect product $\mathbb{C}_q(T \times T) \rtimes \mathbb{C}[W]$ with respect to the diagonal action of W on $T \times T$. Inside we consider the linear subspace $\mathcal{H}\mathcal{H}_{\mathrm{res}}(W_e, \widehat{X})$ formed by the finite sums $\sum_{w \in W}^{\mu \in X} h_{w,\mu} D_q^\mu \cdot [w],\ h_{w,\mu} \in \mathcal{Q}(T)$, satisfying the following conditions:

(a) $h_{w,\mu}$ is regular except at the divisors $T_{\alpha,q^k} := \{t : \alpha(t) = q^k\},\ \alpha \in R,\ k \in \mathbb{Z}$, where they are allowed to have only first order poles.
(b) $\mathrm{Res}_{T_{\alpha,q^{-k}}}(h_{w,\mu}) + \mathrm{Res}_{T_{\alpha,q^{-k}}}(h_{s_\alpha w, k\alpha + s_\alpha \mu}) = 0$ for any $\alpha \in R$.

The algebra of regular functions $\mathbb{C}[T \times \widetilde{\mathbb{C}}^\times]$ is embedded into $\mathcal{H}\mathcal{H}_{\mathrm{res}}(W_e, \widehat{X})$ via the assignment $f \mapsto f \cdot [1]$. Furthermore, for $i \in I \subset \widetilde{I}$, we consider the *Demazure operator* [13, § 1.3] $\tau_i := \frac{1}{1-\mathsf{X}_{\alpha_i}} \cdot ([s_i] - [1]) \in \mathcal{H}\mathcal{H}_{\mathrm{res}}(W_e, \widehat{X})$, and for $i_0 \in \widetilde{I} \setminus I$ we consider the *Demazure operator* [13, § 1.3] $\tau_{i_0} := \frac{1}{1-q\mathsf{X}_\theta^{-1}} \cdot ([s_\theta] \cdot D_q^\theta - [1]) \in \mathcal{H}\mathcal{H}_{\mathrm{res}}(W_e, \widehat{X})$, where $\theta \in R$ is the dominant short root, $(\theta, \theta) = 2$.

Theorem 3.6

(a) *$\mathcal{H}\mathcal{H}_{\mathrm{res}}(W_e, \widehat{X})$ is a subalgebra of $\mathbb{C}_q(T \times T) \rtimes \mathbb{C}[W]$.*
(b) *The assignment $f \mapsto f \cdot [1]$; $\mathcal{T}_i \mapsto \tau_i,\ i \in \widetilde{I}$; $\Pi \ni \pi \mapsto$ the corresponding automorphism of $\mathcal{Q}(T) = \mathbb{Q}(\widehat{X} \otimes \mathbb{C}^\times)$ (arising from the automorphism of the extended Dynkin diagram), defines an isomorphism $\varphi\colon \mathcal{H}\mathcal{H}(W_e, \widehat{X}) \xrightarrow{\sim} \mathcal{H}\mathcal{H}_{\mathrm{res}}(W_e, \widehat{X})$.*

Proof Same as the one of [5, Theorem 7.2]. □

Remark 3.7 Nil-DAHA $\mathcal{H}\mathcal{H}(W_e, \widehat{X})$ is *not* isomorphic to the degeneration $\ddot{\mathsf{H}}|_{v=0}$ of [5, Section 6].

3.7 K-theory of Disconnected Flags

We define $\mathbf{I}_{\mathrm{ad}}$ as the image of $\mathbf{I}$ in $G'_{\mathrm{ad}}(\mathcal{K})^\varsigma$, and we consider the adjoint version of the affine flags $\mathcal{F}\ell_{\mathrm{ad}} := G'_{\mathrm{ad}}(\mathcal{K})^\varsigma / \mathbf{I}_{\mathrm{ad}}$. This is an ind-scheme having $|\Pi|$ connected components, each one isomorphic to $\mathcal{F}\ell$. The isomorphism of Theorem 3.4 extends to the same named isomorphism $\mathcal{H}\mathcal{H}(W_e, \widehat{X}) \xrightarrow{\sim} K^{\widehat{\mathbf{I}}}(\mathcal{F}\ell_{\mathrm{ad}})$. Let us explain why the

RHS forms an algebra. We consider an algebra $K(\widehat{\mathbf{I}}\backslash G'_{\mathrm{ad}}(\mathcal{K})^{\varsigma}/\widehat{\mathbf{I}}) = K^{\widehat{\mathbf{I}}}(\mathcal{F}\ell_{\mathrm{ad}}/\Pi)$. Here we view $\Pi = Z(G'^{\sigma})$ as the center of the simply-connected group G'^{σ} acting trivially on $\mathcal{F}\ell_{\mathrm{ad}}$. Now $K^{\widehat{\mathbf{I}}}(\mathcal{F}\ell_{\mathrm{ad}}/\Pi)$ contains a subalgebra $K^{\widehat{\mathbf{I}}}(\mathcal{F}\ell_{\mathrm{ad}}/\Pi)_{\mathrm{diag}}$ formed by the classes of bi-equivariant sheaves on $\mathcal{F}\ell_{\mathrm{ad}}$ such that the Π-equivariance coincides with the $Z(G'^{\sigma})$-equivariance obtained by the restriction of $\widehat{\mathbf{I}}$-equivariance. Finally, $K^{\widehat{\mathbf{I}}}(\mathcal{F}\ell_{\mathrm{ad}}/\Pi)_{\mathrm{diag}} \simeq K^{\widehat{\mathbf{I}}}(\mathcal{F}\ell_{\mathrm{ad}})$.

3.8 Spherical Nil-DAHA

We define the new generators $\hat{\mathcal{T}}_i := -\mathcal{T}_i - 1,\ i \in \widetilde{I}$ (they correspond to the generators T_i of [13, Definition 1.1]). Geometrically, $\hat{\mathcal{T}}_i = [\omega_{\overline{\mathcal{F}\ell_i}}]$. They still satisfy the braid relations of W_a. So for any $w \in W_a$ we have a well-defined element (product of the generators) $\hat{\mathcal{T}}_w$. We also define $\hat{\mathcal{T}}'_i := \hat{\mathcal{T}}_i + 1 = -\mathcal{T}_i,\ i \in \widetilde{I}$. Geometrically, for $i \in I \subset \widetilde{I}$, we have $\hat{\mathcal{T}}'_i = \mathsf{X}_{\rho^\vee}[\mathcal{O}_{\overline{\mathcal{F}\ell_i}}]\mathsf{X}_{\rho^\vee}^{-1}$. These generators also satisfy the braid relations of W_a, so for any $w \in W_a$ we have a well-defined element (product of the generators) $\hat{\mathcal{T}}'_w$.

Given a reduced decomposition $w = s_{i_1}\cdots s_{i_l}$ we have for the class of the structure sheaf of the Schubert variety $[\mathcal{O}_{\overline{\mathcal{F}\ell_w}}] = [\mathcal{O}_{\overline{\mathcal{F}\ell_{i_1}}}]\cdots[\mathcal{O}_{\overline{\mathcal{F}\ell_{i_l}}}]$ since $\overline{\mathcal{F}\ell_w}$ has rational singularities. Hence, for $w \in W \subset W_a$, we have $[\mathcal{O}_{\overline{\mathcal{F}\ell_w}}] = \mathsf{X}_{\rho^\vee}^{-1}\hat{\mathcal{T}}'_w\mathsf{X}_{\rho^\vee}$. In particular, for the longest element $w_0 \in W$ we set $\mathbf{e} := [\mathcal{O}_{\overline{\mathcal{F}\ell_{w_0}}}] = \mathsf{X}_{\rho^\vee}^{-1}\hat{\mathcal{T}}'_{w_0}\mathsf{X}_{\rho^\vee}$, an idempotent in $\mathcal{H}\mathcal{H}(W_e, \widehat{X})$. Indeed, calculating $[\mathcal{O}_{\overline{\mathcal{F}\ell_{w_0}}}][\mathcal{O}_{\overline{\mathcal{F}\ell_{w_0}}}]$ as the pushforward of the structure sheaf from the convolution diagram $\overline{\mathcal{F}\ell_{w_0}}\widetilde{\times}\overline{\mathcal{F}\ell_{w_0}} \to \overline{\mathcal{F}\ell_{w_0}}$ we get $\mathcal{O}_{\overline{\mathcal{F}\ell_{w_0}}}$ since $R\Gamma(\overline{\mathcal{F}\ell_{w_0}}, \mathcal{O}_{\overline{\mathcal{F}\ell_{w_0}}}) = \mathbb{C}$.

We define the spherical nil-DAHA $\mathcal{H}\mathcal{H}^{\mathrm{sph}}(W_a, \widetilde{X}) := \mathbf{e}\mathcal{H}\mathcal{H}(W_a, \widetilde{X})\mathbf{e}$, and the spherical extended nil-DAHA $\mathcal{H}\mathcal{H}^{\mathrm{sph}}(W_e, \widehat{X}) := \mathbf{e}\mathcal{H}\mathcal{H}(W_e, \widehat{X})\mathbf{e}$.

3.9 Equivariant K-theory of the Affine Grassmannian

We denote by $\mathrm{Gr}_{\mathrm{ad}}$ the twisted affine Grassmannian $G'_{\mathrm{ad}}(\mathcal{K})^{\varsigma}/G'_{\mathrm{ad}}(\mathcal{O})^{\varsigma}$: an ind-proper ind-scheme of ind-finite type, see [55]. The complexified equivariant coherent K-theory $K^{\mathbf{K}\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{Gr}_{\mathrm{ad}}) = K^{G'(\mathcal{O})^{\varsigma}\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{Gr}_{\mathrm{ad}})$ forms a $\mathbb{C}[q^{\pm\frac{1}{2m_{\mathrm{ad}}}}]$-algebra with respect to the convolution (see Sect. 3.7). We have the smooth projection $p\colon \mathcal{F}\ell_{\mathrm{ad}} \to \mathrm{Gr}_{\mathrm{ad}}$, and the natural embedding $K^{\mathbf{K}\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{Gr}_{\mathrm{ad}}) \hookrightarrow K^{\mathbf{I}\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{Gr}_{\mathrm{ad}}) \stackrel{p^*}{\hookrightarrow} K^{\mathbf{I}\rtimes\widetilde{\mathbb{C}}^\times}(\mathcal{F}\ell_{\mathrm{ad}})$.

Corollary 3.8 *The isomorphism* Φ *of Sect. 3.7 takes the spherical subalgebra* $\mathcal{H}\mathcal{H}^{\mathrm{sph}}(W_e, \widehat{X}) \subset \mathcal{H}\mathcal{H}(W_e, \widehat{X})$ *isomorphically onto* $K^{\mathbf{K}\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{Gr}_{\mathrm{ad}}) \subset K^{\mathbf{I}\rtimes\widetilde{\mathbb{C}}^\times}(\mathcal{F}\ell_{\mathrm{ad}})$. *The right ideal* $\mathbf{e}\mathcal{H}\mathcal{H}(W_e, \widehat{X})$ *corresponds to* $K^{\mathbf{K}\rtimes\mathbb{C}^\times}(\mathcal{F}\ell_{\mathrm{ad}}) = (K^{\mathbf{I}\rtimes\widetilde{\mathbb{C}}^\times}(\mathcal{F}\ell_{\mathrm{ad}}))^W \subset K^{\mathbf{I}\rtimes\widetilde{\mathbb{C}}^\times}(\mathcal{F}\ell_{\mathrm{ad}})$. □

3.10 Classical Limit

The following theorem is proved similarly to [8, Theorem 2.15]:

Theorem 3.9

(a) *The algebra* $K^{\mathbf{K}}(\mathrm{Gr}_{\mathrm{ad}})$ *is commutative.*
(b) *Its spectrum together with the projection onto* T/W *is naturally isomorphic to* $\mathfrak{Z}^G_G \xrightarrow{\mathrm{pr}} T/W$.
(c) *The Poisson structure on* $K^{\mathbf{K}}(\mathrm{Gr}_{\mathrm{ad}})$ *arising from the deformation* $K^{\mathbf{K}\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{\mathrm{ad}})$ *corresponds under the above identification to the Poisson (symplectic) structure of Remark 2.4 on* $\mathfrak{Z}^G_G$. □

Corollary 3.10

(a) *The algebra* $\mathcal{H}\mathcal{H}^{\mathrm{sph}}(W_e, \widehat{X})|_{q=1}$ *is commutative.*
(b) *This algebra with the subalgebra* $\mathbb{C}[X]^W$ *is naturally isomorphic to* $\mathbb{C}[\mathfrak{Z}^G_G] \supset \mathbb{C}[T/W]$.
(c) *The Poisson structure on* $\mathcal{H}\mathcal{H}^{\mathrm{sph}}(W_e, \widehat{X})|_{q=1}$ *arising from the deformation* $\mathcal{H}\mathcal{H}^{\mathrm{sph}}(W_e, \widehat{X})$ *corresponds under the above identification to the Poisson (symplectic) structure of Remark 2.4 on* $\mathfrak{Z}^G_G$. □

3.11 Nil-DAHA, General Linear Group

In case $G = GL(n) \simeq G^\vee$, the general definition of $\mathcal{H}\mathcal{H}(W_e, \widehat{X})$ takes a particularly explicit form.

Definition 3.11 The nil-DAHA $\mathcal{H}\mathcal{H}(GL(n))$ is the $\mathbb{C}[q^{\pm 1}]$-algebra with generators $\mathcal{T}_0, \ldots, \mathcal{T}_{n-1},\ \mathsf{X}_1^{\pm 1}, \ldots, \mathsf{X}_n^{\pm 1},\ \pi^{\pm 1}$, and the following relations:

(a) $\mathcal{T}_i$'s for $i \in \mathbb{Z}/n\mathbb{Z}$ satisfy the braid relations of the affine braid group of type $\tilde{A}_{n-1}$;
(b) $\mathsf{X}_i^{\pm 1},\ i = 1, \ldots, n$, all commute;
(c) $\mathcal{T}_i(\mathcal{T}_i + 1) = 0$ for $i \in \mathbb{Z}/n\mathbb{Z}$;
(d) $\pi \mathsf{X}_i \pi^{-1} = \mathsf{X}_{i+1}$ for $i = 1, \ldots, n-1$, and $\pi \mathsf{X}_n \pi^{-1} = q\mathsf{X}_1$;
(e) $\pi \mathcal{T}_i \pi^{-1} = \mathcal{T}_{i+1}$ for $i \in \mathbb{Z}/n\mathbb{Z}$;

(f) $\mathsf{X}_{i+1}\mathcal{T}_i - \mathcal{T}_i\mathsf{X}_i = \mathsf{X}_i$, and $\mathsf{X}_i^{-1}\mathcal{T}_i - \mathcal{T}_i\mathsf{X}_{i+1}^{-1} = \mathsf{X}_{i+1}^{-1}$ for $i = 1, \ldots, n-1$;
(h) $q\mathsf{X}_1\mathcal{T}_0 - \mathcal{T}_0\mathsf{X}_n = \mathsf{X}_n$, and $q\mathsf{X}_n^{-1}\mathcal{T}_0 - \mathcal{T}_0\mathsf{X}_1^{-1} = \mathsf{X}_1^{-1}$;
(fh) $\mathsf{X}_i^{\pm 1}$ and $\mathcal{T}_j$ commute for all the pairs i, j not listed in (f,h) above.

Note that $\mathsf{X} := \mathsf{X}_1 \cdots \mathsf{X}_n$ commutes with all the $\mathcal{T}_i$'s, while $\pi\mathsf{X}\pi^{-1} = q\mathsf{X}$. For a future use we give the following

Definition 3.12 The extended nil-DAHA $\mathcal{H}\mathcal{H}_e(GL(n))$ is the $\mathbb{C}[\boldsymbol{v}^{\pm 1}]$-algebra, $q = \boldsymbol{v}^2$, with generators $\mathcal{T}_0, \ldots, \mathcal{T}_{n-1}$, $\mathsf{X}_1^{\pm 1}, \ldots, \mathsf{X}_n^{\pm 1}$, $\pi^{\pm 1}$, $\sqrt{\mathsf{X}}^{\pm 1}$, and relations (a–fh) of Definition 3.11 plus

(i) $(\sqrt{\mathsf{X}}^{\pm 1})^2 = \mathsf{X}^{\pm 1} := \mathsf{X}_1^{\pm 1} \cdots \mathsf{X}_n^{\pm 1}$;
(j) $\sqrt{\mathsf{X}}^{\pm 1}$ commutes with all the $\mathsf{X}_i^{\pm 1}$ and all the $\mathcal{T}_i$;
(k) $\pi\sqrt{\mathsf{X}}\pi^{-1} = \boldsymbol{v}\sqrt{\mathsf{X}}$.

We interpret X_i, $i = 1, \ldots, n$, as the i-th diagonal matrix entry character of the diagonal torus $T \subset GL(n)$. It gives rise to the same named character of the Iwahori subgroup $\mathbf{I} \subset GL(n, \mathcal{K})$. We denote by $\mathcal{O}_{\mathcal{F}\ell_e}\langle \mathsf{X}_i \rangle$ the (class of the) structure sheaf of the point orbit $\mathcal{F}\ell_e \subset \mathcal{F}\ell = \mathcal{F}\ell_{GL(n)}$ (the affine flag variety of $GL(n)$) twisted by the character X_i. We denote by $\omega_{\overline{\mathcal{F}\ell}_i}$, $i = 0, \ldots, n-1$, the (class of the) direct image (wrt the closed embedding $\overline{\mathcal{F}\ell}_i \hookrightarrow \mathcal{F}\ell_{SL(n)} \hookrightarrow \mathcal{F}\ell_{GL(n)}$) of the canonical line bundle on $\overline{\mathcal{F}\ell}_i$ equipped with the natural $\mathbf{I} \rtimes \mathbb{C}^\times$-equivariant structure. We set $\mathbf{T}_i := -1 - \omega_{\overline{\mathcal{F}\ell}_i} \in K^{\mathbf{I} \rtimes \mathbb{C}^\times}(\mathcal{F}\ell)$ as in Sect. 3.4. Finally, note that the fixed point set $\mathcal{F}\ell^T$ is naturally identified with the extended affine Weyl group of $GL(n)$, that is the group of n-periodic permutations of $\mathbb{Z}$: $\sigma(k+n) = \sigma(k) + n$, and the fixed point ϖ corresponding to the shift permutation $\sigma(k) = k+1$ is a point $\mathbf{I} \rtimes \mathbb{C}^\times$-orbit $\mathcal{F}\ell_\varpi$. We denote by $\varpi \in K^{\mathbf{I} \rtimes \mathbb{C}^\times}(\mathcal{F}\ell)$ the class of the structure sheaf $\mathcal{O}_{\mathcal{F}\ell_\varpi}$.

Theorem 3.13 *There is a unique isomorphism* $\Phi \colon \mathcal{H}\mathcal{H}(GL(n)) \xrightarrow{\sim} K^{\mathbf{I} \rtimes \mathbb{C}^\times}(\mathcal{F}\ell)$ *such that* $\Phi(\mathsf{X}_i) = \mathcal{O}_{\mathcal{F}\ell_e}\langle \mathsf{X}_i \rangle$, $i = 1, \ldots, n$, *and* $\Phi(\mathcal{T}_i) = \mathbf{T}_i$, $i = 0, \ldots, n-1$, *and* $\Phi(\pi) = \varpi$.

Proof Same as the one of [64, Theorem 2.5.6]. □

As in Sect. 3.8, we have an idempotent $\mathbf{e} = [\mathcal{O}_{\overline{\mathcal{F}\ell}_{w_0}}] \in K^{\mathbf{I} \rtimes \mathbb{C}^\times}(\mathcal{F}\ell_{SL(n)}) \subset K^{\mathbf{I} \rtimes \mathbb{C}^\times}(\mathcal{F}\ell) \simeq \mathcal{H}\mathcal{H}(GL(n))$, and we define the spherical subalgebras $\mathcal{H}\mathcal{H}^{\mathrm{sph}}(GL(n)) := \mathbf{e}\mathcal{H}\mathcal{H}(GL(n))\mathbf{e}$, and $\mathcal{H}\mathcal{H}_e^{\mathrm{sph}}(GL(n)) := \mathbf{e}\mathcal{H}\mathcal{H}_e(GL(n))\mathbf{e}$. We also define a two-fold cover $\tilde{G} := \{(g \in GL(n),\ y \in \mathbb{C}^\times) : \det(g) = y^2\} \twoheadrightarrow G$, $\mathbf{K} := GL(n, \mathcal{O})$, $\tilde{\mathbf{K}} := \tilde{G}(\mathcal{O})$, and finally $\tilde{\mathbb{C}}^\times$ as the two-fold cover (with coordinate $\boldsymbol{v}$) of $\mathbb{C}^\times$ (with coordinate q).

Corollary 3.14 *The isomorphism* Φ *of Theorem 3.13 takes the spherical subalgebra* $\mathcal{H}\mathcal{H}^{\mathrm{sph}}(GL(n)) \subset \mathcal{H}\mathcal{H}(GL(n))$ *isomorphically onto* $K^{\mathbf{K} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL(n)}) \subset K^{\mathbf{I} \rtimes \mathbb{C}^\times}(\mathcal{F}\ell_{GL(n)})$. *This isomorphism extends uniquely to* $\mathcal{H}\mathcal{H}_e^{\mathrm{sph}}(GL(n)) \xrightarrow{\sim} K^{\tilde{\mathbf{K}} \rtimes \tilde{\mathbb{C}}^\times}(\mathrm{Gr}_{GL(n)})$ *where the right-hand side is equipped with the algebra structure as in Sect. 3.7.* □

The following theorem is proved similarly to [10, Theorem 3.1, Proposition 3.18]:

Theorem 3.15

(a) *The algebras* $K^{\mathbf{K}}(\mathrm{Gr}_{GL(n)}),\ K^{\tilde{\mathbf{K}}}(\mathrm{Gr}_{GL(n)})$ *are commutative.*
(b) *The spectrum of* $K^{\mathbf{K}}(\mathrm{Gr}_{GL(n)})$ *together with the projection onto* $(\mathbb{C}^\times)^{(n)} = \mathrm{Spec}(K_{GL(n)}(\mathrm{pt}))$ *is naturally isomorphic to* ${}^\dagger\mathring{Z}^n \xrightarrow{\mathrm{pr}} (\mathbb{C}^\times)^{(n)}$ *(see Sect. 2.5).*
(c) *The spectrum of* $K^{\tilde{\mathbf{K}}}(\mathrm{Gr}_{GL(n)})$ *together with the projection onto* $\mathrm{Spec}(K^{\mathbf{K}}(\mathrm{Gr}_{GL(n)}))$ *is naturally isomorphic to* ${}^\dagger\hat{Z}^n \to {}^\dagger\mathring{Z}^n$ *(see Sect. 2.5).*
(d) *The Poisson structure on* $K^{\mathbf{K}}(\mathrm{Gr}_{GL(n)})$ *arising from the deformation* $K^{\mathbf{K}\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL(n)})$ *corresponds under the above identification to the negative of the Poisson (symplectic) structure of [25, 34] on* ${}^\dagger\mathring{Z}^n$. *The Poisson (symplectic) structure on* $K^{\tilde{\mathbf{K}}}(\mathrm{Gr}_{GL(n)})$ *arising from the deformation* $K^{\tilde{\mathbf{K}}\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{Gr}_{GL(n)})$ *is the negative of the pull-back of the symplectic structure on* ${}^\dagger\mathring{Z}^n$. □

Corollary 3.16

(a) *The algebras* $\mathcal{H}\mathcal{C}^{\mathrm{sph}}(GL(n))|_{q=1},\ \mathcal{H}\mathcal{C}_e^{\mathrm{sph}}(GL(n))|_{\mathfrak{v}=1}$ *are commutative.*
(b) *The algebra* $\mathcal{H}\mathcal{C}^{\mathrm{sph}}(GL(n))|_{q=1}$ *with the subalgebra* $\mathbb{C}[\mathsf{X}_1^{\pm 1},\ldots,\mathsf{X}_n^{\pm 1}]^{\mathfrak{S}_n}$ *is naturally isomorphic to* $\mathbb{C}[{}^\dagger\mathring{Z}^n] \supset \mathbb{C}[(\mathbb{C}^\times)^{(n)}]$.
(c) *The Poisson structures on* $\mathcal{H}\mathcal{C}^{\mathrm{sph}}(GL(n))|_{q=1},\ \mathcal{H}\mathcal{C}_e^{\mathrm{sph}}(GL(n))|_{\mathfrak{v}=1}$ *arising from the deformations* $\mathcal{H}\mathcal{C}^{\mathrm{sph}}(GL(n)),\ \mathcal{H}\mathcal{C}_e^{\mathrm{sph}}(GL(n))$ *correspond under the above identification to the negative of the Poisson (symplectic) structures of [25, 34] on* ${}^\dagger\mathring{Z}^n,\ {}^\dagger\hat{Z}^n$. □

3.12 Quantum Poisson Reduction

Now again G is an almost simple simply-connected algebraic group. We consider Lusztig's integral form $U_q(\mathfrak{g})$ of the quantized universal enveloping algebra over $\mathbb{C}[q^{\pm 1}]$ with Cartan elements $K_\lambda,\ \lambda \in X$. It is denoted $\dot{\mathbb{U}}_{\mathcal{A}}$ in [65, § 2.2]. We extend the scalars to $\mathbb{C}[q^{\frac{\pm 1}{2m_{\mathrm{ad}}}}]$ and consider the integrable representations of $U_q(\mathfrak{g})$ with weights in X. We consider the reflection equation algebra $\mathcal{O}_q(G)$ spanned by the matrix coefficients of integrable $U_q(\mathfrak{g})$-modules (with weights in X); it is denoted $\mathbb{F}_{\mathcal{A}}$ in [65, § 2.2]. The corresponding integral form $\mathcal{D}_q(G)$ of the Heisenberg double [59, Section 3] (quantum differential operators) is denoted $\mathbb{D}_{\mathcal{A}}$ in [65, § 2.2]. The quasiclassical limit of $\mathcal{D}_q(G)$ is $D_+(G)$ with the Poisson structure $\{,\}_+$ considered in Sect. 2.2. The moment map $\mu\colon (D_+(G),\{,\}_+) \to (G,\pi)\times(G,\pi)$ is the quasiclassical limit of $\mu_q\colon U_q(\mathfrak{g})\otimes U_q(\mathfrak{g}) \to \mathcal{D}_q(G)$ (see, e.g., [48]). The Poisson action of $(G,\pi_G)\times(G,\pi_G)$ on $D_+(G)$ is the quasiclassical limit of the comodule structure of $\mathcal{D}_q(G)$ over $\mathcal{O}_q(G)\otimes\mathcal{O}_q(G)$.

Recall the subalgebra $U_q^c(\mathfrak{n}) \subset U_q(\mathfrak{g})$ [56, § 2.2] associated to a Coxeter element c (we shall omit its dependence on $\{n_{ij}\}_{i,j\in I}$ satisfying [56, § 2.2.2]). The $U_q(\mathfrak{g})$-module $U_q(\mathfrak{g})/(U_q(\mathfrak{g})\cdot[U_q^c(\mathfrak{n}), U_q^c(\mathfrak{n})])$ is the quantization of the coisotropic subvariety $C_c \subset (G,\pi)$ of Sect. 2.2. Given a pair of Coxeter elements c, c', we consider the left ideal $\mathcal{I}_{c',c}$ of $\mathcal{D}_q(G)$ generated by $\mu_q([U_q^{c'}(\mathfrak{n}), U_q^{c'}(\mathfrak{n})] \otimes S[U_q^c(\mathfrak{n}), U_q^c(\mathfrak{n})])$ where S stands for the antipode. The invariants of $\mathcal{D}_q(G)/\mathcal{I}_{c',c}$ with respect to the coaction of $\mathcal{O}_q(B_-) \otimes \mathcal{O}_q(B_-)$ form an algebra denoted $\mathcal{O}_q({}^\dagger\mathfrak{Z}^{c',c}(G))$.

Conjecture 3.17 There is an isomorphism $\mathcal{H}\mathcal{H}^{\mathrm{sph}}(W_e, \widehat{X}) \xrightarrow{\sim} \mathcal{O}_q({}^\dagger\mathfrak{Z}^{c,c}(G))$ equal to $\mathrm{id}_{{}^\dagger\mathfrak{Z}^{c,c}(G)}$ at $q = 1$.

4 Multiplicative Slices

4.1 Asymmetric Definition

We closely follow the exposition in [10, Section 2]. Let G be an adjoint simple complex algebraic group. We fix a Borel and a Cartan subgroup $G \supset B \supset T$. Let Λ be the coweight lattice, and let $\Lambda_+ \subset \Lambda$ be the submonoid spanned by the simple coroots $\alpha_i,\ i \in I$. The involution $\alpha \mapsto -w_0\alpha$ of Λ restricts to an involution of Λ_+ and induces an involution $\alpha_i \mapsto \alpha_{i^*}$ of the set of simple coroots. We will sometimes write $\alpha^* := -w_0\alpha$ for short. Let λ be a dominant coweight of G, and $\mu \le \lambda$ an arbitrary coweight of G, not necessarily dominant, such that $\alpha := \lambda - \mu = \sum_{i\in I} a_i\alpha_i,\ a_i \in \mathbb{N}$. We will define the *multiplicative (trigonometric)* analogues ${}^\dagger\overline{\mathcal{W}}{}^\lambda_\mu$ of the generalized slices $\overline{\mathcal{W}}{}^\lambda_\mu$ of [10, 2(ii)].

Namely, ${}^\dagger\overline{\mathcal{W}}{}^\lambda_\mu$ is the moduli space of the following data:

(a) a G-bundle $\mathcal{P}$ on $\mathbb{P}^1$;
(b) a trivialization $\sigma\colon \mathcal{P}_{\mathrm{triv}}|_{\mathbb{P}^1\setminus\{1\}} \xrightarrow{\sim} \mathcal{P}|_{\mathbb{P}^1\setminus\{1\}}$ having a pole of degree $\le \lambda$ at $1 \in \mathbb{P}^1$. This means that for an irreducible G-module $V^{\lambda^\vee}$ and the associated vector bundle $\mathcal{V}^{\lambda^\vee}_{\mathcal{P}}$ on $\mathbb{P}^1$ we have $V^{\lambda^\vee} \otimes \mathcal{O}_{\mathbb{P}^1}(-\langle\lambda,\lambda^\vee\rangle\cdot 1) \subset \mathcal{V}^{\lambda^\vee}_{\mathcal{P}} \subset V^{\lambda^\vee} \otimes \mathcal{O}_{\mathbb{P}^1}(-\langle w_0\lambda,\lambda^\vee\rangle\cdot 1)$;
(c) a reduction ϕ of $\mathcal{P}$ to a B-bundle (B-structure ϕ on $\mathcal{P}$) such that the induced T-bundle ϕ^T has degree $w_0\mu$, and the fiber of ϕ at $\infty \in \mathbb{P}^1$ is $B_- \subset G$ (with respect to the trivialization σ of $\mathcal{P}$ at $\infty \in \mathbb{P}^1$). This means in particular that for an irreducible G-module $V^{\lambda^\vee}$ and the associated vector bundle $\mathcal{V}^{\lambda^\vee}_{\mathcal{P}}$ on $\mathbb{P}^1$ we are given an invertible subsheaf $\mathcal{L}_{\lambda^\vee} \subset \mathcal{V}^{\lambda^\vee}_{\mathcal{P}}$ of degree $-\langle w_0\mu,\lambda^\vee\rangle$. We require ϕ to be transversal at $0 \in \mathbb{P}^1$ to the trivial B-structure B in $\mathcal{P}_{\mathrm{triv}}$.

4.2 Multiplicative BD Slices

Let $\underline{\lambda} = (\omega_{i_1}, \dots, \omega_{i_N})$ be a sequence of fundamental coweights of G such that $\sum_{s=1}^{N} \omega_{i_s} = \lambda$. We define ${}^\dagger\overline{\mathcal{W}}^{\underline{\lambda}}_\mu$ as the moduli space of the following data:

(a) a collection of points $(\mathsf{z}_1, \dots, \mathsf{z}_N) \in (\mathbb{C}^\times)^N$;
(b) a G-bundle $\mathcal{P}$ on $\mathbb{P}^1$;
(c) a trivialization (a section) σ of $\mathcal{P}$ on $\mathbb{P}^1 \setminus \{\mathsf{z}_1, \dots, \mathsf{z}_N\}$ with a pole of degree $\leq \sum_{s=1}^{N} \omega_{i_s} \cdot \mathsf{z}_s$ on the complement;
(d) a reduction ϕ of $\mathcal{P}$ to a B-bundle (B-structure ϕ on $\mathcal{P}$) such that the induced T-bundle ϕ^T has degree $w_0\mu$, and the fiber of ϕ at $\infty \in \mathbb{P}^1$ is $B_- \subset G$ and transversal to B at $0 \in \mathbb{P}^1$ (with respect to the trivialization σ).

Remark 4.1 The definition of multiplicative BD slices differs from the definition of BD slices in [10, § 2(x)] only by the open condition of transversality at $0 \in \mathbb{P}^1$. Thus ${}^\dagger\overline{\mathcal{W}}^{\underline{\lambda}}_\mu$ is an open subvariety in $\overline{\mathcal{W}}^{\underline{\lambda}}_\mu$ (and similarly, ${}^\dagger\overline{\mathcal{W}}^{\lambda}_\mu$ is an open subvariety in $\overline{\mathcal{W}}^{\lambda}_\mu$). Hence, the favorable properties of the slices of [10] (e.g., the Cohen–Macaulay property) are inherited by the multiplicative slices.

4.3 A Symmetric Definition

Given arbitrary coweights μ_-, μ_+ such that $\mu_- + \mu_+ = \mu$, we consider the moduli space ${}^\dagger\overline{\mathcal{W}}^{\underline{\lambda}}_{\mu_-,\mu_+}$ of the following data:

(a) a collection of points $(\mathsf{z}_1, \dots, \mathsf{z}_N) \in (\mathbb{C}^\times)^N$;
(b) G-bundles $\mathcal{P}_-, \mathcal{P}_+$ on $\mathbb{P}^1$;
(c) an isomorphism $\sigma\colon \mathcal{P}_-|_{\mathbb{P}^1 \setminus \{\mathsf{z}_1,\dots,\mathsf{z}_N\}} \xrightarrow{\sim} \mathcal{P}_+|_{\mathbb{P}^1 \setminus \{\mathsf{z}_1,\dots,\mathsf{z}_N\}}$ with a pole of degree $\leq \sum_{s=1}^{N} \omega_{i_s} \cdot \mathsf{z}_s$ on the complement;
(d) a trivialization of $\mathcal{P}_- = \mathcal{P}_+$ at $\infty \in \mathbb{P}^1$;
(e) a reduction ϕ_- of $\mathcal{P}_-$ to a B_--bundle (a B_--structure on $\mathcal{P}_-$) such that the induced T-bundle ϕ_-^T has degree $-w_0\mu_-$, and the fiber of ϕ_- at $\infty \in \mathbb{P}^1$ is $B \subset G$;
(f) a reduction ϕ_+ of $\mathcal{P}_+$ to a B-bundle (a B-structure on $\mathcal{P}_+$) such that the induced T-bundle ϕ_+^T has degree $w_0\mu_+$, and the fiber of ϕ_+ at $\infty \in \mathbb{P}^1$ is $B_- \subset G$. We require ϕ_- and ϕ_+ to be transversal at $0 \in \mathbb{P}^1$ (with respect to the isomorphism σ).

Note that the trivial G-bundle on $\mathbb{P}^1$ has a unique B_--reduction of degree 0 with fiber B at ∞. Conversely, a G-bundle $\mathcal{P}_-$ with a B_--structure of degree 0 is necessarily trivial, and its trivialization at ∞ uniquely extends to the whole of $\mathbb{P}^1$. Hence ${}^\dagger\overline{\mathcal{W}}^{\underline{\lambda}}_{0,\mu} = {}^\dagger\overline{\mathcal{W}}^{\underline{\lambda}}_\mu$.

For arbitrary ${}^\dagger\overline{\mathcal{W}}^{\underline{\lambda}}_{\mu_-,\mu_+}$, the G-bundles $\mathcal{P}_-, \mathcal{P}_+$ are identified via σ on $\mathbb{P}^1 \setminus \{\mathsf{z}_1, \dots, \mathsf{z}_N\}$, so they are both equipped with B and B_--structures transversal around $0, \infty \in \mathbb{P}^1$, that is they are both equipped with a reduction to a T-bundle around $0, \infty \in \mathbb{P}^1$. So $\mathcal{P}_\pm = \mathcal{P}^T_\pm \times^T G$ for certain T-bundles $\mathcal{P}^T_\pm$ around $0, \infty \in \mathbb{P}^1$, trivialized at $\infty \in \mathbb{P}^1$. The modified T-bundles $'\mathcal{P}^T_\pm := \mathcal{P}^T_\pm(w_0\mu_- \cdot \infty)$ are canonically isomorphic to $\mathcal{P}^T_\pm$ off $\infty \in \mathbb{P}^1$ and trivialized at $\infty \in \mathbb{P}^1$. We define $'\mathcal{P}_\pm$ as the result of gluing $\mathcal{P}_\pm$ and $'\mathcal{P}^T_\pm \times^T G$ in the punctured neighborhood of $\infty \in \mathbb{P}^1$. Then the isomorphism $\sigma\colon {'\mathcal{P}_-}|_{\mathbb{P}^1\setminus\{\infty,\mathsf{z}_1,\ldots,\mathsf{z}_N\}} \xrightarrow{\sim} {'\mathcal{P}_+}|_{\mathbb{P}^1\setminus\{\infty,\mathsf{z}_1,\ldots,\mathsf{z}_N\}}$ extends to $\mathbb{P}^1 \setminus \{\mathsf{z}_1, \ldots, \mathsf{z}_N\}$, and $\phi_\pm$ also extends from $\mathbb{P}^1 \setminus \{\infty\}$ to a B-structure $'\phi_+$ in $'\mathcal{P}_+$ of degree $w_0\mu$ (resp. a B_--structure $'\phi_-$ on $'\mathcal{P}_-$ of degree 0).

This defines an isomorphism ${}^\dagger\overline{\mathcal{W}}{}^\lambda_{\mu_-,\mu_+} \simeq {}^\dagger\overline{\mathcal{W}}{}^\lambda_\mu$. Similarly, for the nondeformed slices we have an isomorphism ${}^\dagger\mathcal{W}^\lambda_{\mu_-,\mu_+} \simeq {}^\dagger\mathcal{W}^\lambda_\mu$.

4.4 *Multiplication of Slices*

Given $\lambda_1 \geq \mu_1$ and $\lambda_2 \geq \mu_2$ with λ_1, λ_2 dominant, we think of ${}^\dagger\overline{\mathcal{W}}{}^{\lambda_1}_{\mu_1}$ (resp. ${}^\dagger\overline{\mathcal{W}}{}^{\lambda_2}_{\mu_2}$) in the incarnation ${}^\dagger\overline{\mathcal{W}}{}^{\lambda_1}_{\mu_1,0}$ (resp. ${}^\dagger\overline{\mathcal{W}}{}^{\lambda_2}_{0,\mu_2}$). Note that $\mathcal{P}^2_-$ is canonically trivialized as in Sect. 4.3, and $\mathcal{P}^1_+$ is canonically trivialized for the same reason. Given $(\mathcal{P}^1_\pm, \sigma_1, \phi^1_\pm) \in {}^\dagger\overline{\mathcal{W}}{}^{\lambda_1}_{\mu_1,0}$, we change the trivialization of $\mathcal{P}^1_+$ by a (uniquely determined) element of U_- (the unipotent radical of B_-) so that the value $\phi^1_-(0)$ becomes B (while $\phi^1_+(0)$ remains equal to B_-). Now the value $\phi^1_-(\infty)$ is not B anymore; it is only transversal to B_-. In order to distinguish the data obtained by the composition with the above trivialization change, we denote them by $('\mathcal{P}^1_\pm, '\sigma_1, '\phi^1_\pm)$. Given $(\mathcal{P}^2_\pm, \sigma_2, \phi^2_\pm) \in {}^\dagger\overline{\mathcal{W}}{}^{\lambda_2}_{0,\mu_2}$, we consider $('\mathcal{P}^1_-, \mathcal{P}^2_+, \sigma_2 \circ '\sigma_1, '\phi^1_-, \phi^2_+)$ (recall that $'\mathcal{P}^1_+ = \mathcal{P}_{\mathrm{triv}} = \mathcal{P}^2_-$). These data *do not* lie in ${}^\dagger\overline{\mathcal{W}}{}^{\lambda_1+\lambda_2}_{\mu_1,\mu_2}$ since the value $'\phi^1_-(\infty)$ is not necessarily equal to B, it is only transversal to B_-. However, we change the trivialization of $'\mathcal{P}^1_-(\infty) = \mathcal{P}^2_+(\infty)$ by a (uniquely determined) element of U_-, so that the value of $'\phi^1_-(\infty)$ becomes B, and we end up in ${}^\dagger\overline{\mathcal{W}}{}^{\lambda_1+\lambda_2}_{\mu_1,\mu_2} = {}^\dagger\overline{\mathcal{W}}{}^{\lambda_1+\lambda_2}_{\mu_1+\mu_2}$.

This defines a multiplication morphism ${}^\dagger\overline{\mathcal{W}}{}^{\lambda_1}_{\mu_1} \times {}^\dagger\overline{\mathcal{W}}{}^{\lambda_2}_{\mu_2} \to {}^\dagger\overline{\mathcal{W}}{}^{\lambda_1+\lambda_2}_{\mu_1+\mu_2}$.

In particular, taking $\mu_2 = \lambda_2$ so that ${}^\dagger\overline{\mathcal{W}}{}^{\lambda_2}_{\lambda_2}$ is a point and ${}^\dagger\overline{\mathcal{W}}{}^{\lambda_1}_{\mu_1} \times {}^\dagger\overline{\mathcal{W}}{}^{\lambda_2}_{\lambda_2} = {}^\dagger\overline{\mathcal{W}}{}^{\lambda_1}_{\mu_1}$, we get a stabilization morphism ${}^\dagger\overline{\mathcal{W}}{}^{\lambda_1}_{\mu_1} \to {}^\dagger\overline{\mathcal{W}}{}^{\lambda_1+\lambda_2}_{\mu_1+\lambda_2}$.

Remark 4.2 The multiplication of slices in [10, § 2(vi)] *does not* preserve the multiplicative slices viewed as open subvarieties according to Remark 4.1 (in particular, it *does not* induce the above multiplication on multiplicative slices).

4.5 *Scattering Matrix*

Given a collection $(\mathsf{z}_1, \ldots, \mathsf{z}_N) \in (\mathbb{C}^\times)^N$, we define $P_{\underline{\mathsf{z}}}(z) := \prod_{s=1}^N (z - \mathsf{z}_s) \in \mathbb{C}[z]$. We also define a closed subvariety ${}^\dagger\overline{\mathcal{W}}{}^{\lambda,\underline{\mathsf{z}}}_\mu \subset {}^\dagger\overline{\mathcal{W}}{}^\lambda_\mu$ as the fiber of the latter over $\underline{\mathsf{z}} = (\mathsf{z}_1, \dots, \mathsf{z}_N)$. We construct a locally closed embedding $\Psi\colon {}^\dagger\overline{\mathcal{W}}^{\lambda,\underline{\mathsf{z}}}_\mu \hookrightarrow G[z, P^{-1}]$ into an ind-affine scheme as follows. According to Sect. 4.3, we have an isomorphism $\zeta\colon {}^\dagger\overline{\mathcal{W}}^{\lambda,\underline{\mathsf{z}}}_\mu = {}^\dagger\overline{\mathcal{W}}^{\lambda,\underline{\mathsf{z}}}_{0,\mu} \xrightarrow{\sim} {}^\dagger\overline{\mathcal{W}}^{\lambda,\underline{\mathsf{z}}}_{\mu,0}$. We denote $\zeta(\mathcal{P}_\pm, \sigma, \phi_\pm)$ by $(\mathcal{P}'_\pm, \sigma', \phi'_\pm)$. Note that $\mathcal{P}_-$ and $\mathcal{P}'_+$ are trivialized, and $\mathcal{P}'_+$ is obtained from $\mathcal{P}_+$ by an application of a certain Hecke transformation at $\infty \in \mathbb{P}^1$. In particular, we obtain an isomorphism $\mathcal{P}_+|_{\mathbb{A}^1} \xrightarrow{\sim} \mathcal{P}'_+|_{\mathbb{A}^1} = \mathcal{P}_{\mathrm{triv}}|_{\mathbb{A}^1}$. As in Sect. 4.4, we change the trivialization of $\mathcal{P}'_+$ by a uniquely defined element of U_- so that the value of $\phi'_-(0)$ becomes B. Now we compose this change of trivialization with the above isomorphism $\mathcal{P}_+|_{\mathbb{A}^1} \xrightarrow{\sim} \mathcal{P}'_+|_{\mathbb{A}^1} = \mathcal{P}_{\mathrm{triv}}|_{\mathbb{A}^1}$ and with $\sigma\colon \mathcal{P}_{\mathrm{triv}}|_{\mathbb{A}^1\setminus\underline{\mathsf{z}}} = \mathcal{P}_-|_{\mathbb{A}^1\setminus\underline{\mathsf{z}}} \xrightarrow{\sim} \mathcal{P}_+|_{\mathbb{A}^1\setminus\underline{\mathsf{z}}}$ to obtain an isomorphism $\mathcal{P}_{\mathrm{triv}}|_{\mathbb{A}^1\setminus\underline{\mathsf{z}}} \xrightarrow{\sim} \mathcal{P}_{\mathrm{triv}}|_{\mathbb{A}^1\setminus\underline{\mathsf{z}}}$, i.e. an element of $G[z, P^{-1}]$.

Here is an equivalent construction of the above embedding. Given $(\mathcal{P}_\pm, \sigma, \phi_\pm) \in {}^\dagger\overline{\mathcal{W}}^{\lambda,\underline{\mathsf{z}}}_{\mu_-,\mu_+} = {}^\dagger\overline{\mathcal{W}}^{\lambda,\underline{\mathsf{z}}}_\mu$, we choose a trivialization of the B-bundle $\phi_+|_{\mathbb{A}^1}$ (resp. of the B_--bundle $\phi_-|_{\mathbb{A}^1}$). This trivialization gives rise to a trivialization of the G-bundle $\mathcal{P}_+|_{\mathbb{A}^1}$ (resp. of $\mathcal{P}_-|_{\mathbb{A}^1}$), so that σ becomes an element of $G(z)$ regular at $0 \in \mathbb{P}^1$; moreover, the value of $\sigma(0)$ lies in the big Bruhat cell $B \cdot B_- \subset G$. We require that $\sigma(0) \in B \subset G$. Then σ is well-defined up to the left multiplication by an element of $B[z]$ and the right multiplication by an element of $B_{-,1}[z]$ (the kernel of evaluation at $0 \in \mathbb{P}^1\colon B_-[z] \to B_-$), i.e. σ is a well-defined element of $B[z]\backslash G(z)/B_{-,1}[z]$. Clearly, this element of $G(z)$ lies in the closure of the double coset $\overline{G[z]z^{\underline{\lambda},\underline{\mathsf{z}}}G[z]}$ where $z^{\underline{\lambda},\underline{\mathsf{z}}} := \prod_{s=1}^N (z - \mathsf{z}_s)^{\omega_{i_s}}$. Moreover, it lies in $\overline{G[z]z^{\underline{\lambda},\underline{\mathsf{z}}}G[z]} \cap \mathrm{ev}_0^{-1}(B)$. Thus we have constructed an embedding

$$\Psi'\colon {}^\dagger\overline{\mathcal{W}}^{\lambda,\underline{\mathsf{z}}}_\mu \hookrightarrow B[z]\backslash(\overline{G[z]z^{\underline{\lambda},\underline{\mathsf{z}}}G[z]} \cap \mathrm{ev}_0^{-1}(B))/B_{-,1}[z]$$

If we compose with an embedding $G(z) \hookrightarrow G((z^{-1}))$, then the image of Ψ' lies in $B[z]\backslash U_1[[z^{-1}]]T_1[[z^{-1}]]z^\mu U_-[[z^{-1}]]/U_{-,1}[z]$ where $U_1[[z^{-1}]] \subset U[[z^{-1}]]$ (resp. $T_1[[z^{-1}]] \subset T[[z^{-1}]]$) stands for the kernel of evaluation at $\infty \in \mathbb{P}^1$. However, the projection

$$U_1[[z^{-1}]]T_1[[z^{-1}]]z^\mu U_-[[z^{-1}]] \to B[z]\backslash U_1[[z^{-1}]]T_1[[z^{-1}]]z^\mu U_-[[z^{-1}]]/U_{-,1}[z]$$

is clearly one-to-one. Summing up, we obtain an embedding

$$\Psi\colon {}^\dagger\overline{\mathcal{W}}^{\lambda,\underline{\mathsf{z}}}_\mu \to U_1[[z^{-1}]]T_1[[z^{-1}]]z^\mu U_-[[z^{-1}]] \cap \overline{G[z]z^{\underline{\lambda},\underline{\mathsf{z}}}G[z]} \cap \mathrm{ev}_0^{-1}(B).$$

We claim that Ψ is an isomorphism. To see it, we construct the inverse map to ${}^\dagger\overline{\mathcal{W}}^{\lambda,\underline{\mathsf{z}}}_{0,\mu}$: given $g(z) \in U_1[[z^{-1}]]T_1[[z^{-1}]]z^\mu U_-[[z^{-1}]] \cap \overline{G[z]z^{\underline{\lambda},\underline{\mathsf{z}}}G[z]} \cap \mathrm{ev}_0^{-1}(B)$ we use it to glue $\mathcal{P}_+$ together with a rational isomorphism $\sigma\colon \mathcal{P}_{\mathrm{triv}} = \mathcal{P}_- \to \mathcal{P}_+$, and define ϕ_+ as the image of the standard trivial B-structure in $\mathcal{P}_{\mathrm{triv}}$ under σ.

Remark 4.3 The embedding $\overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_\mu \hookrightarrow G(z)$ of [10, § 2(xi)] restricted to the open subvariety ${}^\dagger\overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_\mu \subset \overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_\mu$ *does not* give the above embedding ${}^\dagger\overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_\mu \hookrightarrow G(z)$.

4.6 A Cover of a Slice

We define a T-torsor ${}^\dagger\widetilde{\mathcal{W}}^{\underline{\lambda}}_\mu \to {}^\dagger\overline{\mathcal{W}}^{\underline{\lambda}}_\mu$ as the moduli space of data (a–d) as in Sect. 4.2 plus

(e) a collection of nowhere vanishing sections $u_{\lambda^\vee} \in \Gamma(\mathbb{P}^1 \setminus \{\infty\}, \mathcal{L}_{\lambda^\vee})$ satisfying Plücker relations (cf. Sect. 4.1(c)).

The construction of Sect. 4.5 defines an isomorphism

$$\widetilde{\Psi}\colon {}^\dagger\widetilde{\mathcal{W}}^{\underline{\lambda},\underline{z}}_\mu \xrightarrow{\sim} U_1[[z^{-1}]]T[[z^{-1}]]z^\mu U_-[[z^{-1}]] \cap \overline{G[z]z^{\underline{\lambda},\underline{z}}G[z]} \cap \mathrm{ev}_0^{-1}(B).$$

Let $T_{[2]} \subset T$ be the subgroup of 2-torsion. For a future use we define a $T_{[2]}$-torsor ${}^\dagger\widetilde{\mathcal{W}}^{\underline{\lambda},\underline{z}}_\mu \supset {}^\dagger\hat{\mathcal{W}}^{\underline{\lambda},\underline{z}}_\mu \to {}^\dagger\overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_\mu$ as follows. The evaluation at $0 \in \mathbb{P}^1$ gives rise to a projection $\mathrm{pr}_0\colon \overline{G[z]z^{\underline{\lambda},\underline{z}}G[z]} \cap \mathrm{ev}_0^{-1}(B) \to B \to T$. The leading coefficient (at z^μ) gives rise to a projection $\mathrm{pr}_\infty\colon U_1[[z^{-1}]]T[[z^{-1}]]z^\mu U_-[[z^{-1}]] \to T$, and ${}^\dagger\hat{\mathcal{W}}^{\underline{\lambda},\underline{z}}_\mu$ is cut out by the equation $\mathrm{pr}_0 \cdot \mathrm{pr}_\infty = (-1)^{\lambda-\mu} \in T_{[2]}$, where $\lambda = \sum_{s=1}^N \omega_{i_s}$, see Sect. 4.2. As $\underline{z}$ varies, we obtain a $T_{[2]}$-torsor ${}^\dagger\widetilde{\mathcal{W}}^{\underline{\lambda}}_\mu \supset {}^\dagger\hat{\mathcal{W}}^{\underline{\lambda}}_\mu \to {}^\dagger\overline{\mathcal{W}}^{\underline{\lambda}}_\mu$.

4.7 An Example

This section is parallel to [10, § 2(xii)], but our present conventions are slightly different. Let $G = GL(2) = GL(V)$ with $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. Let B be the stabilizer of $\mathbb{C}e_2$ (the lower triangular matrices), and let B_- be the stabilizer of $\mathbb{C}e_1$ (the upper triangular matrices). Let $N, m \in \mathbb{N}$; $\underline{\lambda}$ be an N-tuple of fundamental coweights $(0, 1)$, and $\mu = (m, N - m)$, so that $w_0\mu = (N - m, m)$. Let $\mathcal{O} := \mathcal{O}_{\mathbb{P}^1}$. We fix a collection $(\mathsf{z}_1, \ldots, \mathsf{z}_N) \in (\mathbb{C}^\times)^N$ and define $P_{\underline{\mathsf{z}}}(z) := \prod_{s=1}^N (z - \mathsf{z}_s) \in \mathbb{C}[z]$. Then ${}^\dagger\overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_\mu$ is the moduli space of flags $(\mathcal{O} \otimes V \supset \mathcal{V} \supset \mathcal{L})$, where

(a) $\mathcal{V}$ is a 2-dimensional locally free subsheaf in $\mathcal{O} \otimes V$ coinciding with $\mathcal{O} \otimes V$ around $0, \infty \in \mathbb{P}^1$ and such that on $\mathbb{A}^1 \subset \mathbb{P}^1$ the global sections of $\det \mathcal{V}$ coincide with $P_{\underline{\mathsf{z}}}\mathbb{C}[z]e_1 \wedge e_2$ as a $\mathbb{C}[z]$-submodule of $\Gamma(\mathbb{A}^1, \det(\mathcal{O}_{\mathbb{A}^1} \otimes V)) = \mathbb{C}[z]e_1 \wedge e_2$.
(b) $\mathcal{L}$ is a line subbundle in $\mathcal{V}$ of degree $-m$, assuming the value $\mathbb{C}e_1$ at $\infty \in \mathbb{P}^1$, and such that the value of $\mathcal{L}$ at $0 \in \mathbb{P}^1$ is transversal to $\mathbb{C}e_2$. In particular, $\deg \mathcal{V}/\mathcal{L} = m - N$.

On the other hand, let us introduce a closed subvariety ${}^\dagger\hat{\mathcal{W}}^{\lambda,\underline{z}}_\mu$ in $\mathrm{Mat}_2[z]$ formed by all the matrices $\mathsf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $A(z) = a_m z^m + \ldots + a_0$, and $a_m \cdot a_0 = (-1)^m$, while $\deg C(z) < m \geq \deg B(z)$, and $B(0) = 0$; furthermore, $\det \mathsf{M} = P_{\underline{z}}(z)$.

Then we have a two-fold cover $\mho\colon {}^\dagger\hat{\mathcal{W}}^{\lambda,\underline{z}}_\mu \to {}^\dagger\overline{\mathcal{W}}^{\lambda,\underline{z}}_\mu$: given $\mathsf{M} \in {}^\dagger\hat{\mathcal{W}}^{\lambda,\underline{z}}_\mu$ we view it as a transition matrix in a punctured neighborhood of $\infty \in \mathbb{P}^1$ to glue a vector bundle $\mathcal{V}$ which embeds, by construction, as a locally free subsheaf into $\mathcal{O} \otimes V$. The morphism $\mathsf{M}\mathcal{O}_{\mathbb{A}^1} e_1 \hookrightarrow \mathcal{O}_{\mathbb{A}^1} \otimes V$ naturally extends to $\infty \in \mathbb{P}^1$ with a pole of degree m, hence it extends to an embedding of $\mathcal{O}(-m \cdot \infty)$ into $\mathcal{V} \subset \mathcal{O} \otimes V$. The image of this embedding is the desired line subbundle $\mathcal{L} \subset \mathcal{V}$.

4.8 Thick Slices

We define *thick* multiplicative (trigonometric) slices ${}^\dagger\mathcal{W}_\mu$ as the moduli space of the following data:

(a) a G-bundle $\mathcal{P}$ on $\mathbb{P}^1$;

(b) a trivialization $\sigma\colon \mathcal{P}_{\mathrm{triv}}|_{\widehat{\mathbb{P}}^1_\infty} \xrightarrow{\sim} \mathcal{P}|_{\widehat{\mathbb{P}}^1_\infty}$ in the formal neighborhood of $\infty \in \mathbb{P}^1$;

(c) a reduction ϕ of $\mathcal{P}$ to a B-bundle (B-structure ϕ on $\mathcal{P}$) such that the induced T-bundle ϕ^T has degree $w_0\mu$, and the fiber of ϕ at $\infty \in \mathbb{P}^1$ is transversal to B (with respect to the trivialization σ of $\mathcal{P}$ at $\infty \in \mathbb{P}^1$);

(d) a collection of nowhere vanishing sections $u_{\lambda^\vee} \in \Gamma(\mathbb{P}^1 \setminus \{\infty\}, \mathcal{L}_{\lambda^\vee})$ satisfying Plücker relations (cf. Sect. 4.1(c)).

The construction of Sect. 4.6 identifies ${}^\dagger\mathcal{W}_\mu$ with the infinite type scheme (cf. [24, § 5.9])

$$ {}^\dagger\mathcal{W}_\mu \simeq U_1[[z^{-1}]]T[[z^{-1}]]z^\mu U_-[[z^{-1}]] \subset G((z^{-1})). \tag{4.1} $$

As the inclusion $U_1[[z^{-1}]] \hookrightarrow U((z^{-1}))$ gives rise to an isomorphism $U_1[[z^{-1}]] \simeq U[z]\backslash U((z^{-1}))$, we can identify ${}^\dagger\mathcal{W}_\mu$ with the quotient $U[z]\backslash U((z^{-1}))T[[z^{-1}]]z^\mu U_-((z^{-1}))/U_{-,1}[z]$, and we write π for this isomorphism. The construction of Sect. 4.5 (resp. of Sect. 4.6) defines a closed embedding ${}^\dagger\overline{\mathcal{W}}^\lambda_\mu \hookrightarrow {}^\dagger\mathcal{W}_\mu$ (resp. ${}^\dagger\hat{\mathcal{W}}^\lambda_\mu \hookrightarrow {}^\dagger\mathcal{W}_\mu$). We define the multiplication morphism $m_{\mu_1,\mu_2}\colon {}^\dagger\mathcal{W}_{\mu_1} \times {}^\dagger\mathcal{W}_{\mu_2} \to {}^\dagger\mathcal{W}_{\mu_1+\mu_2}$ by the formula $m_{\mu_1,\mu_2}(g_1, g_2) = \pi(g_1 g_2)$. Then the multiplication morphism $m^{\lambda_1,\lambda_2}_{\mu_1,\mu_2}\colon {}^\dagger\overline{\mathcal{W}}^{\lambda_1}_{\mu_1} \times {}^\dagger\overline{\mathcal{W}}^{\lambda_2}_{\mu_2} \to {}^\dagger\overline{\mathcal{W}}^{\lambda_1+\lambda_2}_{\mu_1+\mu_2}$ of Sect. 4.4 is the restriction of m_{μ_1,μ_2}. Similarly, m_{μ_1,μ_2} restricts to a multiplication ${}^\dagger\hat{\mathcal{W}}^{\lambda_1}_{\mu_1} \times {}^\dagger\hat{\mathcal{W}}^{\lambda_2}_{\mu_2} \to {}^\dagger\hat{\mathcal{W}}^{\lambda_1+\lambda_2}_{\mu_1+\mu_2}$.

For ν_1, ν_2 antidominant, we define the *shift maps* $\iota_{\mu,\nu_1,\nu_2}\colon {}^\dagger\mathcal{W}_{\mu+\nu_1+\nu_2} \to {}^\dagger\mathcal{W}_\mu$ by $g \mapsto \pi(z^{-\nu_1} g z^{-\nu_2})$.

5 Shifted Quantum Affine Algebras

Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$, and $(\cdot,\cdot)$ be a non-degenerate invariant bilinear symmetric form on $\mathfrak{g}$ (with a square length of the shortest root equal to 2). Let $\{\alpha_i^\vee\}_{i\in I} \subset \mathfrak{h}^*$ be the simple positive roots of $\mathfrak{g}$ relative to $\mathfrak{h}$, and $c_{ij} = 2\frac{(\alpha_i^\vee,\alpha_j^\vee)}{(\alpha_i^\vee,\alpha_i^\vee)}$–the entries of the corresponding Cartan matrix. Set $d_i := \frac{(\alpha_i^\vee,\alpha_i^\vee)}{2} \in \mathbb{Z}_{>0}$ so that $d_i c_{ij} = d_j c_{ji}$ for any $i, j \in I$. Let $\nu\colon \mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^*$ be the isomorphism determined by the symmetric form $(\cdot,\cdot)$ so that $\alpha_i = h_i = \nu^{-1}(\alpha_i^\vee)/d_i$ are the simple coroots of $\mathfrak{g}$.

5.1 Algebras $\mathcal{U}^{\mathrm{sc}}_{\mu_1,\mu_2}$ and $\mathcal{U}^{\mathrm{ad}}_{\mu_1,\mu_2}$

Given coweights $\mu^+,\mu^- \in \Lambda$, set $\underline{b}^\pm = \{b_i^\pm\}_{i\in I} \in \mathbb{Z}^I$ with $b_i^\pm := \alpha_i^\vee(\mu^\pm)$. Define the *simply-connected version of shifted quantum affine algebra*, denoted by $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$ or $\mathcal{U}^{\mathrm{sc}}_{\underline{b}^+,\underline{b}^-}$, to be the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by $\{e_{i,r}, f_{i,r}, \psi^\pm_{i,\pm s_i^\pm}, (\psi^\pm_{i,\mp b_i^\pm})^{-1}\}_{i\in I}^{r\in\mathbb{Z}, s_i^\pm\geq -b_i^\pm}$ with the following defining relations (for all $i, j \in I$ and $\epsilon,\epsilon' \in \{\pm\}$):

$$[\psi_i^\epsilon(z), \psi_j^{\epsilon'}(w)] = 0,\ \psi^\pm_{i,\mp b_i^\pm}\cdot(\psi^\pm_{i,\mp b_i^\pm})^{-1} = (\psi^\pm_{i,\mp b_i^\pm})^{-1}\cdot\psi^\pm_{i,\mp b_i^\pm} = 1, \tag{U1}$$

$$(z - \boldsymbol{v}_i^{c_{ij}} w) e_i(z) e_j(w) = (\boldsymbol{v}_i^{c_{ij}} z - w) e_j(w) e_i(z), \tag{U2}$$

$$(\boldsymbol{v}_i^{c_{ij}} z - w) f_i(z) f_j(w) = (z - \boldsymbol{v}_i^{c_{ij}} w) f_j(w) f_i(z), \tag{U3}$$

$$(z - \boldsymbol{v}_i^{c_{ij}} w) \psi_i^\epsilon(z) e_j(w) = (\boldsymbol{v}_i^{c_{ij}} z - w) e_j(w) \psi_i^\epsilon(z), \tag{U4}$$

$$(\boldsymbol{v}_i^{c_{ij}} z - w) \psi_i^\epsilon(z) f_j(w) = (z - \boldsymbol{v}_i^{c_{ij}} w) f_j(w) \psi_i^\epsilon(z), \tag{U5}$$

$$[e_i(z), f_j(w)] = \frac{\delta_{ij}}{\boldsymbol{v}_i - \boldsymbol{v}_i^{-1}} \delta\left(\frac{z}{w}\right)\left(\psi_i^+(z) - \psi_i^-(z)\right), \tag{U6}$$

$$\underset{z_1,\dots,z_{1-c_{ij}}}{\mathrm{Sym}} \sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix}_{\boldsymbol{v}_i} e_i(z_1)\cdots e_i(z_r) e_j(w) e_i(z_{r+1}) \cdots e_i(z_{1-c_{ij}}) = 0, \tag{U7}$$

$$\underset{z_1,\dots,z_{1-c_{ij}}}{\mathrm{Sym}} \sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix}_{\boldsymbol{v}_i} f_i(z_1)\cdots f_i(z_r) f_j(w) f_i(z_{r+1}) \cdots f_i(z_{1-c_{ij}}) = 0, \tag{U8}$$

where $\boldsymbol{v}_i := \boldsymbol{v}^{d_i}$, $[a,b]_x := ab - x\cdot ba$, $[m]_{\boldsymbol{v}} := \frac{\boldsymbol{v}^m-\boldsymbol{v}^{-m}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$, $\begin{bmatrix}a\\b\end{bmatrix}_{\boldsymbol{v}} := \frac{[a-b+1]_{\boldsymbol{v}}\cdots[a]_{\boldsymbol{v}}}{[1]_{\boldsymbol{v}}\cdots[b]_{\boldsymbol{v}}}$, $\mathrm{Sym}_{z_1,\ldots,z_s}$ stands for the symmetrization in $z_1,\ldots,z_s$, and the generating series are defined as follows:

$$e_i(z) := \sum_{r\in\mathbb{Z}} e_{i,r}z^{-r},\ f_i(z) := \sum_{r\in\mathbb{Z}} f_{i,r}z^{-r},\ \psi_i^{\pm}(z) := \sum_{r\geq -b_i^{\pm}} \psi_{i,\pm r}^{\pm}z^{\mp r},\ \delta(z) := \sum_{r\in\mathbb{Z}} z^r.$$

Let us introduce another set of Cartan generators $\{h_{i,\pm r}\}_{i\in I}^{r>0}$ instead of $\{\psi_{i,\pm s_i^{\pm}}^{\pm}\}_{i\in I}^{s_i^{\pm}>-b_i^{\pm}}$ via

$$(\psi_{i,\mp b_i^{\pm}}^{\pm}z^{\pm b_i^{\pm}})^{-1}\psi_i^{\pm}(z) = \exp\left(\pm(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})\sum_{r>0}h_{i,\pm r}z^{\mp r}\right).$$

Then, relations (U4, U5) are equivalent to the following:

$$\psi_{i,\mp b_i^{\pm}}^{\pm}e_{j,s} = \boldsymbol{v}_i^{\pm c_{ij}}e_{j,s}\psi_{i,\mp b_i^{\pm}}^{\pm},\ [h_{i,r},e_{j,s}] = \frac{[rc_{ij}]_{\boldsymbol{v}_i}}{r}\cdot e_{j,s+r} \text{ for } r\neq 0, \tag{U4$'$}$$

$$\psi_{i,\mp b_i^{\pm}}^{\pm}f_{j,s} = \boldsymbol{v}_i^{\mp c_{ij}}f_{j,s}\psi_{i,\mp b_i^{\pm}}^{\pm},\ [h_{i,r},f_{j,s}] = -\frac{[rc_{ij}]_{\boldsymbol{v}_i}}{r}\cdot f_{j,s+r} \text{ for } r\neq 0. \tag{U5$'$}$$

Let $\mathcal{U}^{\mathrm{sc},<}_{\mu^+,\mu^-}$, $\mathcal{U}^{\mathrm{sc},>}_{\mu^+,\mu^-}$, and $\mathcal{U}^{\mathrm{sc},0}_{\mu^+,\mu^-}$ be the $\mathbb{C}(\boldsymbol{v})$-subalgebras of $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$ generated by $\{f_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$, $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$, and $\{\psi_{i,\pm s_i^{\pm}}^{\pm},(\psi_{i,\mp b_i^{\pm}}^{\pm})^{-1}\}_{i\in I}^{s_i^{\pm}\geq -b_i^{\pm}}$, respectively. The following is proved completely analogously to [37, Theorem 2]:

Proposition 5.1

(a) (Triangular decomposition of $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$) The multiplication map

$$m\colon \mathcal{U}^{\mathrm{sc},<}_{\mu^+,\mu^-}\otimes\mathcal{U}^{\mathrm{sc},0}_{\mu^+,\mu^-}\otimes\mathcal{U}^{\mathrm{sc},>}_{\mu^+,\mu^-}\longrightarrow\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$$

is an isomorphism of $\mathbb{C}(\boldsymbol{v})$-vector spaces.

(b) The algebra $\mathcal{U}^{\mathrm{sc},0}_{\mu^+,\mu^-}$ (resp. $\mathcal{U}^{\mathrm{sc},<}_{\mu^+,\mu^-}$ and $\mathcal{U}^{\mathrm{sc},>}_{\mu^+,\mu^-}$) is isomorphic to the $\mathbb{C}(\boldsymbol{v})$-algebra generated by $\{\psi_{i,\pm s_i^{\pm}}^{\pm},(\psi_{i,\mp b_i^{\pm}}^{\pm})^{-1}\}_{i\in I}^{s_i^{\pm}\geq -b_i^{\pm}}$ (resp. $\{f_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ and $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$) with the defining relations (U1) (resp. (U3, U8) and (U2, U7)). In particular, $\mathcal{U}^{\mathrm{sc},<}_{\mu^+,\mu^-}$ and $\mathcal{U}^{\mathrm{sc},>}_{\mu^+,\mu^-}$ are independent of $\mu^{\pm}$.

Following the terminology of [50], we also define the *adjoint version of shifted quantum affine algebra*, denoted by $\mathcal{U}^{\mathrm{ad}}_{\mu^+,\mu^-}$ or $\mathcal{U}^{\mathrm{ad}}_{\underline{b}^+,\underline{b}^-}$, by adding extra generators $\{(\phi_i^\pm)^{\pm 1}\}_{i\in I}$ to $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$, which satisfy the following extra relations:

$$(\psi^\epsilon_{i,-\epsilon b_i^\epsilon})^{\pm 1} = (\phi_i^\epsilon)^{\pm 2} \cdot \prod_{j-i} (\phi_j^\epsilon)^{\pm c_{ji}},\ (\phi_i^\epsilon)^{\pm 1}\cdot(\phi_i^\epsilon)^{\mp 1} = 1,\ [\phi_i^\epsilon, \phi_j^{\epsilon'}] = 0, \tag{U9}$$

$$\phi_i^\epsilon \psi_j^{\epsilon'}(z) = \psi_j^{\epsilon'}(z)\phi_i^\epsilon,\ \phi_i^\epsilon e_j(z) = \boldsymbol{v}_i^{\epsilon\delta_{ij}} e_j(z)\phi_i^\epsilon,\ \phi_i^\epsilon f_j(z) = \boldsymbol{v}_i^{-\epsilon\delta_{ij}} f_j(z)\phi_i^\epsilon, \tag{U10}$$

for any $i, j \in I$ and $\epsilon, \epsilon' \in \{\pm\}$.

Both algebras $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$ and $\mathcal{U}^{\mathrm{ad}}_{\mu^+,\mu^-}$ depend only on $\mu := \mu^+ + \mu^-$ up to an isomorphism[4]. Let $\Lambda^\pm \subset \Lambda$ be the submonoids spanned by $\{\pm\omega_i\}_{i\in I}$, that is, Λ^+ (resp. Λ^-) consists of dominant (resp. antidominant) coweights of Λ. We will say that the algebras $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$, $\mathcal{U}^{\mathrm{ad}}_{\mu^+,\mu^-}$ are dominantly (resp. antidominantly) shifted if $\mu \in \Lambda^+$ (resp. $\mu \in \Lambda^-$). We note that $\mu \in \Lambda^+ \Leftrightarrow b_i^+ + b_i^- = \alpha_i^\vee(\mu) \geq 0$, $\mu \in \Lambda^- \Leftrightarrow b_i^+ + b_i^- = \alpha_i^\vee(\mu) \leq 0$ for all $i \in I$.

Remark 5.2 One of the key reasons to consider $\mathcal{U}^{\mathrm{ad}}_{\mu^+,\mu^-}$, not only $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$, is to construct quantizations of the thick slices ${}^\dagger\mathcal{W}_{\mu^*}$ of Sect. 4.8 and the multiplicative slice covers ${}^\dagger\hat{\mathcal{W}}^{\lambda^*}_{\mu^*}$ of Sect. 4.6, see our Conjecture 8.14. On the technical side, we also need an alternative set of Cartan generators, whose generating series $A_i^\pm(z)$ are defined via (6.1) of Sect. 6 and whose definition requires to work with $\mathcal{U}^{\mathrm{ad}}_{\mu^+,\mu^-}$ (see also Remark 6.7(b)).

Remark 5.3

(a) The elements $\{\psi^+_{i,-b_i^+}\psi^-_{i,b_i^-}\}_{i\in I}$ (resp. $\{\phi_i^+\phi_i^-\}_{i\in I}$) and their inverses are central elements of $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$ (resp. $\mathcal{U}^{\mathrm{ad}}_{\mu^+,\mu^-}$).
(b) We have $\mathcal{U}^{\mathrm{sc}}_{0,0}/(\psi^+_{i,0}\psi^-_{i,0} - 1) \simeq U_{\boldsymbol{v}}(L\mathfrak{g})$, the standard quantum loop algebra of $\mathfrak{g}$, while $\mathcal{U}^{\mathrm{ad}}_{0,0}/(\phi_i^+\phi_i^- - 1) \simeq U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{g})$, the adjoint version of $U_{\boldsymbol{v}}(L\mathfrak{g})$.
(c) We note that defining relations (U1–U8, U10) are independent of μ^+, μ^-.
(d) An equivalent definition of $\mathcal{U}^{\mathrm{sc}}_{\mu_1,\mu_2}$ was suggested to us by Boris Feigin in Spring 2010. In this definition, we take the same generators as for $U_{\boldsymbol{v}}(L\mathfrak{g})$ and just modify relation (U6) by requesting $p_i(z)[e_i(z), f_j(w)] = \frac{\delta_{ij}\delta(z/w)}{\boldsymbol{v}_i - \boldsymbol{v}_i^{-1}}\left(\psi_i^+(z) - \psi_i^-(z)\right)$ for any collection $\{p_i(z)\}_{i\in I}$ of rational functions.

[4]For example, there is an isomorphism $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-} \xrightarrow{\sim} \mathcal{U}^{\mathrm{sc}}_{0,\mu^++\mu^-}$ such that $f_i(z) \mapsto f_i(z)$, $e_i(z) \mapsto z^{b_i^+} e_i(z)$, $\psi_i^\pm(z) \mapsto z^{b_i^+}\psi_i^\pm(z)$.

5.2 Levendorskii Type Presentation of $\mathcal{U}^{\mathrm{sc}}_{0,\mu}$ for $\mu \in \Lambda^-$

In Sect. 10, we will crucially need a presentation of the shifted quantum affine algebras via a finite number of generators and defining relations. This is the purpose of this subsection.

Fix antidominant coweights $\mu_1, \mu_2 \in \Lambda^-$ and set $\mu := \mu_1 + \mu_2$. Define $b_{1,i} := \alpha_i^\vee(\mu_1)$, $b_{2,i} := \alpha_i^\vee(\mu_2)$, $b_i := b_{1,i} + b_{2,i}$. Denote by $\hat{\mathcal{U}}_{\mu_1,\mu_2}$ the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by

$$\{e_{i,r}, f_{i,s}, (\psi^+_{i,0})^{\pm 1}, (\psi^-_{i,b_i})^{\pm 1}, h_{i,\pm 1} | i \in I, b_{2,i} - 1 \le r \le 0, b_{1,i} \le s \le 1\}$$

and with the following defining relations:

$$\begin{gathered}\{(\psi^+_{i,0})^{\pm 1}, (\psi^-_{i,b_i})^{\pm 1}, h_{i,\pm 1}\}_{i\in I} \text{ pairwise commute,}\\ (\psi^+_{i,0})^{\pm 1} \cdot (\psi^+_{i,0})^{\mp 1} = (\psi^-_{i,b_i})^{\pm 1} \cdot (\psi^-_{i,b_i})^{\mp 1} = 1,\end{gathered} \tag{Û1}$$

$$e_{i,r+1}e_{j,s} - \boldsymbol{v}_i^{c_{ij}} e_{i,r}e_{j,s+1} = \boldsymbol{v}_i^{c_{ij}} e_{j,s}e_{i,r+1} - e_{j,s+1}e_{i,r}, \tag{Û2}$$

$$\boldsymbol{v}_i^{c_{ij}} f_{i,r+1}f_{j,s} - f_{i,r}f_{j,s+1} = f_{j,s}f_{i,r+1} - \boldsymbol{v}_i^{c_{ij}} f_{j,s+1}f_{i,r}, \tag{Û3}$$

$$\psi^+_{i,0}e_{j,r} = \boldsymbol{v}_i^{c_{ij}} e_{j,r}\psi^+_{i,0},\ \psi^-_{i,b_i}e_{j,r} = \boldsymbol{v}_i^{-c_{ij}} e_{j,r}\psi^-_{i,b_i},\ [h_{i,\pm 1}, e_{j,r}] = [c_{ij}]_{\boldsymbol{v}_i} \cdot e_{j,r\pm 1}, \tag{Û4}$$

$$\psi^+_{i,0}f_{j,s} = \boldsymbol{v}_i^{-c_{ij}} f_{j,s}\psi^+_{i,0},\ \psi^-_{i,b_i}f_{j,s} = \boldsymbol{v}_i^{c_{ij}} f_{j,s}\psi^-_{i,b_i},\ [h_{i,\pm 1}, f_{j,s}] = -[c_{ij}]_{\boldsymbol{v}_i} \cdot f_{j,s\pm 1}, \tag{Û5}$$

$$[e_{i,r}, f_{j,s}] = 0 \text{ if } i \ne j \text{ and } [e_{i,r}, f_{i,s}] = \begin{cases} \psi^+_{i,0}h_{i,1} & \text{if } r+s=1,\\ \psi^-_{i,b_i}h_{i,-1} & \text{if } r+s=b_i-1,\\ \frac{\psi^+_{i,0}-\delta_{b_i,0}\psi^-_{i,b_i}}{\boldsymbol{v}_i-\boldsymbol{v}_i^{-1}} & \text{if } r+s=0,\\ \frac{-\psi^-_{i,b_i}+\delta_{b_i,0}\psi^+_{i,0}}{\boldsymbol{v}_i-\boldsymbol{v}_i^{-1}} & \text{if } r+s=b_i,\\ 0 & \text{if } b_i < r+s < 0,\end{cases} \tag{Û6}$$

$$[e_{i,0}, [e_{i,0}, \cdots, [e_{i,0}, e_{j,0}]_{\boldsymbol{v}_i^{c_{ij}}} \cdots]_{\boldsymbol{v}_i^{-c_{ij}-2}}]_{\boldsymbol{v}_i^{-c_{ij}}} = 0 \text{ for } i \ne j, \tag{Û7}$$

$$[f_{i,0}, [f_{i,0}, \cdots, [f_{i,0}, f_{j,0}]_{\boldsymbol{v}_i^{c_{ij}}} \cdots]_{\boldsymbol{v}_i^{-c_{ij}-2}}]_{\boldsymbol{v}_i^{-c_{ij}}} = 0 \text{ for } i \ne j, \tag{Û8}$$

$$[h_{i,1}, [f_{i,1}, [h_{i,1}, e_{i,0}]]] = 0,\ [h_{i,-1}, [e_{i,b_{2,i}-1}, [h_{i,-1}, f_{i,b_{1,i}}]]] = 0, \tag{Û9}$$

for any $i, j \in I$ and r, s such that the above relations make sense.

Remark 5.4 One can rewrite relations (Û7, Û8) in the form similar to (U7, U8) as

$$\sum_{r=0}^{1-c_{ij}}(-1)^r\begin{bmatrix}1-c_{ij}\\ r\end{bmatrix}_{\boldsymbol{v}_i} e_{i,0}^r e_{j,0} e_{i,0}^{1-c_{ij}-r}=0,\quad \sum_{r=0}^{1-c_{ij}}(-1)^r\begin{bmatrix}1-c_{ij}\\ r\end{bmatrix}_{\boldsymbol{v}_i} f_{i,0}^r f_{j,0} f_{i,0}^{1-c_{ij}-r}=0.$$

Define inductively

$$e_{i,r}:=[2]_{\boldsymbol{v}_i}^{-1}\cdot\begin{cases}[h_{i,1},e_{i,r-1}] & \text{if } r>0,\\ [h_{i,-1},e_{i,r+1}] & \text{if } r<b_{2,i}-1,\end{cases}$$

$$f_{i,r}:=-[2]_{\boldsymbol{v}_i}^{-1}\cdot\begin{cases}[h_{i,1},f_{i,r-1}] & \text{if } r>1,\\ [h_{i,-1},f_{i,r+1}] & \text{if } r<b_{1,i},\end{cases}$$

$$\psi_{i,r}^+:=(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})\cdot[e_{i,r-1},f_{i,1}] \text{ for } r>0,$$

$$\psi_{i,r}^-:=(\boldsymbol{v}_i^{-1}-\boldsymbol{v}_i)\cdot[e_{i,r-b_{1,i}},f_{i,b_{1,i}}] \text{ for } r<b_i.$$

Theorem 5.5 *There is a unique* $\mathbb{C}(\boldsymbol{v})$*-algebra isomorphism* $\hat{\mathcal{U}}_{\mu_1,\mu_2}\xrightarrow{\sim}\mathcal{U}^{\mathrm{sc}}_{0,\mu}$, *such that*

$$e_{i,r}\mapsto e_{i,r},\ f_{i,r}\mapsto f_{i,r},\ \psi^{\pm}_{i,\pm s_i^{\pm}}\mapsto\psi^{\pm}_{i,\pm s_i^{\pm}}\ \text{ for } i\in I, r\in\mathbb{Z}, s_i^+\geq 0, s_i^-\geq -b_i.$$

This provides a new presentation of $\mathcal{U}^{\mathrm{sc}}_{0,\mu}$ via a finite number of generators and relations. The proof of this result is presented in Appendix A. Motivated by Guay et al. [33], we also provide a slight modification of this presentation of $\mathcal{U}^{\mathrm{sc}}_{0,\mu}$ in Theorem A.3.

Remark 5.6 Theorem 5.5 can be viewed as a $\boldsymbol{v}$-version of the corresponding result for the shifted Yangians of [24, Theorem 4.3]. In the particular case $\mu_1=\mu_2=0$, the latter is the standard Levendorskii presentation of the Yangian, see [47]. However, we are not aware of the reference for Theorem 5.5 even in the *unshifted case* $\mu_1=\mu_2=0$.

6 *ABCD* Generators of $\mathcal{U}^{\mathrm{ad}}_{\mu^+,\mu^-}$

In this section, we introduce an alternative set of generators of $\mathcal{U}^{\mathrm{ad}}_{\mu^+,\mu^-}$, which will be used later in the paper (they are also of independent interest), and deduce the defining relations among them. While the definition works for any two coweights $\mu^+,\mu^-\in\Lambda$, the relations hold only for antidominant $\mu^+,\mu^-\in\Lambda^-$, which we assume from now on.

First, we define the Cartan generators $\{A^{\pm}_{i,\pm r}\}^{r\geq 0}_{i\in I}$ via

$$z^{\mp b_i^{\pm}}\psi_i^{\pm}(z)=\frac{\prod_{j-i}\prod_{p=1}^{-c_{ji}}A_j^{\pm}(\boldsymbol{v}_j^{-c_{ji}-2p}z)}{A_i^{\pm}(z)A_i^{\pm}(\boldsymbol{v}_i^{-2}z)}\ \text{with}\ A^{\pm}_{i,0}:=(\phi_i^{\pm})^{-1}, \tag{6.1}$$

where we set $A_i^{\pm}(z)=\sum_{r\geq 0}A^{\pm}_{i,\pm r}z^{\mp r}$. Using non-degeneracy of the $\boldsymbol{v}$-version of the Cartan matrix (c_{ij}) and arguing by induction in $r>0$, one can easily see that relations (6.1) for all $i\in I$ determine uniquely all $A^{\pm}_{i,\pm r}$, see Remark B.2 (cf. [30, Lemma 2.1]). An explicit formula for $A_i^{\pm}(z)$ is given by (B.2) in Appendix B.

Next, we introduce the generating series $B_i^{\pm}(z)$, $C_i^{\pm}(z)$, $D_i^{\pm}(z)$ via

$$B_i^{\pm}(z):=(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})A_i^{\pm}(z)e_i^{\pm}(z), \tag{6.2}$$

$$C_i^{\pm}(z):=(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})f_i^{\pm}(z)A_i^{\pm}(z), \tag{6.3}$$

$$D_i^{\pm}(z):=A_i^{\pm}(z)\psi_i^{\pm}(z)+(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})^2f_i^{\pm}(z)A_i^{\pm}(z)e_i^{\pm}(z), \tag{6.4}$$

where the *Drinfeld half-currents* are defined as follows:

$$\begin{gathered}e_i^{+}(z):=\sum_{r\geq 0}e_{i,r}z^{-r},\ e_i^{-}(z):=-\sum_{r<0}e_{i,r}z^{-r},\\ f_i^{+}(z):=\sum_{r>0}f_{i,r}z^{-r},\ f_i^{-}(z):=-\sum_{r\leq 0}f_{i,r}z^{-r}.\end{gathered} \tag{6.5}$$

It is clear that coefficients of the generating series $\{A_i^{\pm}(z), B_i^{\pm}(z), C_i^{\pm}(z), D_i^{\pm}(z)\}_{i\in I}$ together with $\{\phi_i^{\pm}\}_{i\in I}$ generate (over $\mathbb{C}(\boldsymbol{v})$) the shifted quantum affine algebra $\mathcal{U}^{\mathrm{ad}}_{\mu^+,\mu^-}$. The following is the key result of this section.

Theorem 6.6 *Assume $\mu^+,\mu^-\in\Lambda^-$ and define $\{b_i^{\pm}\}_{i\in I}$ via $b_i^{\pm}:=\alpha_i^{\vee}(\mu^{\pm})$ as before.*

(a) The generating series $A_i^{\pm}(z)$, $B_i^{\pm}(z)$, $C_i^{\pm}(z)$, $D_i^{\pm}(z)$ satisfy the following relations:

$$\begin{gathered}\phi_i^{\epsilon}A_j^{\epsilon'}(w)=A_j^{\epsilon'}(w)\phi_i^{\epsilon},\ \phi_i^{\epsilon}D_j^{\epsilon'}(w)=D_j^{\epsilon'}(w)\phi_i^{\epsilon},\\ \phi_i^{\epsilon}B_j^{\epsilon'}(w)=\boldsymbol{v}_i^{\epsilon\delta_{ij}}B_j^{\epsilon'}(w)\phi_i^{\epsilon},\ \phi_i^{\epsilon}C_j^{\epsilon'}(w)=\boldsymbol{v}_i^{-\epsilon\delta_{ij}}C_j^{\epsilon'}(w)\phi_i^{\epsilon},\end{gathered} \tag{6.6}$$

$$[A_i^{\epsilon}(z),A_j^{\epsilon'}(w)]=0, \tag{6.7}$$

$$[A_i^{\epsilon}(z),B_j^{\epsilon'}(w)]=[A_i^{\epsilon}(z),C_j^{\epsilon'}(w)]=[B_i^{\epsilon}(z),C_j^{\epsilon'}(w)]=0\ \text{for}\ i\neq j, \tag{6.8}$$

$$[B_i^\epsilon(z), B_i^{\epsilon'}(w)] = [C_i^\epsilon(z), C_i^{\epsilon'}(w)] = [D_i^\epsilon(z), D_i^{\epsilon'}(w)] = 0, \tag{6.9}$$

$$(z-w)[B_i^{\epsilon'}(w), A_i^\epsilon(z)]_{\boldsymbol{v}_i^{-1}} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})\left(zA_i^\epsilon(z)B_i^{\epsilon'}(w) - wA_i^{\epsilon'}(w)B_i^\epsilon(z)\right), \tag{6.10}$$

$$(z-w)[A_i^\epsilon(z), C_i^{\epsilon'}(w)]_{\boldsymbol{v}_i} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})\left(wC_i^{\epsilon'}(w)A_i^\epsilon(z) - zC_i^\epsilon(z)A_i^{\epsilon'}(w)\right), \tag{6.11}$$

$$(z-w)[B_i^\epsilon(z), C_i^{\epsilon'}(w)] = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})z\left(D_i^{\epsilon'}(w)A_i^\epsilon(z) - D_i^\epsilon(z)A_i^{\epsilon'}(w)\right), \tag{6.12}$$

$$(z-w)[B_i^\epsilon(z), D_i^{\epsilon'}(w)]_{\boldsymbol{v}_i} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})\left(wD_i^{\epsilon'}(w)B_i^\epsilon(z) - zD_i^\epsilon(z)B_i^{\epsilon'}(w)\right), \tag{6.13}$$

$$(z-w)[D_i^{\epsilon'}(w), C_i^\epsilon(z)]_{\boldsymbol{v}_i^{-1}} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})\left(zC_i^\epsilon(z)D_i^{\epsilon'}(w) - wC_i^{\epsilon'}(w)D_i^\epsilon(z)\right), \tag{6.14}$$

$$(z-w)[A_i^\epsilon(z), D_i^{\epsilon'}(w)] = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})\left(wC_i^{\epsilon'}(w)B_i^\epsilon(z) - zC_i^\epsilon(z)B_i^{\epsilon'}(w)\right), \tag{6.15}$$

$$A_i^\epsilon(z)D_i^\epsilon(\boldsymbol{v}_i^{-2}z) - \boldsymbol{v}_i^{-1}B_i^\epsilon(z)C_i^\epsilon(\boldsymbol{v}_i^{-2}z) = z^{\epsilon b_i^\epsilon} \cdot \prod_{j-i}\prod_{p=1}^{-c_{ji}} A_j^\epsilon(\boldsymbol{v}_j^{-c_{ji}-2p}z), \tag{6.16}$$

$$\begin{aligned}&(z-\boldsymbol{v}_i^{c_{ij}}w)B_i^\epsilon(z)B_j^{\epsilon'}(w) - (\boldsymbol{v}_i^{c_{ij}}z - w)B_j^{\epsilon'}(w)B_i^\epsilon(z) = \\ &zA_i^\epsilon(z)[\phi_i^+B_{i,0}^+, B_j^{\epsilon'}(w)]_{\boldsymbol{v}_i^{c_{ij}}} + wA_j^{\epsilon'}(w)[\phi_j^+B_{j,0}^+, B_i^\epsilon(z)]_{\boldsymbol{v}_i^{c_{ij}}} \text{ for } i \neq j,\end{aligned} \tag{6.17}$$

$$\begin{aligned}&(\boldsymbol{v}_i^{c_{ij}}z - w)C_i^\epsilon(z)C_j^{\epsilon'}(w) - (z-\boldsymbol{v}_i^{c_{ij}}w)C_j^{\epsilon'}(w)C_i^\epsilon(z) = \\ &-[C_i^\epsilon(z), C_{j,1}^+\phi_j^+]_{\boldsymbol{v}_i^{c_{ij}}}A_j^{\epsilon'}(w) - [C_j^{\epsilon'}(w), C_{i,1}^+\phi_i^+]_{\boldsymbol{v}_i^{c_{ij}}}A_i^\epsilon(z) \text{ for } i \neq j,\end{aligned} \tag{6.18}$$

$$\begin{aligned}&\underset{z_1,\ldots,z_{1-c_{ij}}}{\mathrm{Sym}}\left\{\prod_{a<b}(\boldsymbol{v}_iz_a - \boldsymbol{v}_i^{-1}z_b)(z_a-z_b)\cdot\right. \\ &\left.\sum_{r=0}^{1-c_{ij}}(-1)^r\begin{bmatrix}1-c_{ij}\\ r\end{bmatrix}_{\boldsymbol{v}_i} B_i^{\epsilon_1}(z_1)\cdots B_i^{\epsilon_r}(z_r)B_j^{\epsilon'}(w)B_i^{\epsilon_{r+1}}(z_{r+1})\cdots B_i^{\epsilon_{1-c_{ij}}}(z_{1-c_{ij}})\right\} = 0,\end{aligned} \tag{6.19}$$

$$\begin{aligned}&\underset{z_1,\ldots,z_{1-c_{ij}}}{\mathrm{Sym}}\left\{\prod_{a<b}(\boldsymbol{v}_iz_b - \boldsymbol{v}_i^{-1}z_a)(z_b-z_a)\cdot\right. \\ &\left.\sum_{r=0}^{1-c_{ij}}(-1)^r\begin{bmatrix}1-c_{ij}\\ r\end{bmatrix}_{\boldsymbol{v}_i} C_i^{\epsilon_1}(z_1)\cdots C_i^{\epsilon_r}(z_r)C_j^{\epsilon'}(w)C_i^{\epsilon_{r+1}}(z_{r+1})\cdots C_i^{\epsilon_{1-c_{ij}}}(z_{1-c_{ij}})\right\} = 0,\end{aligned} \tag{6.20}$$

for any $i, j \in I$ *and* $\epsilon, \epsilon', \epsilon_1, \ldots, \epsilon_{1-c_{ij}} \in \{\pm\}$.

(b) *Relations (6.6–6.20) are the defining relations. In other words, the associative* $\mathbb{C}(v)$*-algebra generated by* $\{\phi_i^{\pm}, A_{i,\pm r}^{\pm}, B_{i,r}^{+}, B_{i,-r-1}^{-}, C_{i,r+1}^{+}, C_{i,-r}^{-}, D_{i,\pm r\pm b_i^{\pm}}^{\pm}\}_{i\in I}^{r\in\mathbb{N}}$ *with the defining relations (6.6–6.20) is isomorphic to* $\mathcal{U}^{\mathrm{ad}}_{\mu^+,\mu^-}$.

We sketch the proof in Appendix B. In the unshifted case, more precisely for $U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{g})$, the above construction should be viewed as a $\boldsymbol{v}$-version of that of [30]. In *loc.cit.*, the authors introduced analogous generating series $\{A_i(u), B_i(u), C_i(u), D_i(u)\}_{i\in I}$ with coefficients in the Yangian $Y(\mathfrak{g})$ and stated (without a proof) the relations between them, similar to (6.7–6.16).[5] Meanwhile, we note that adding rational analogues of (6.17–6.20) to their list of relations, we get a complete list of the defining relations among these generating series.

Remark 6.7

(a) For $\mathfrak{g} = \mathfrak{sl}_2$, relations (6.7, 6.9–6.15) are equivalent to the RTT-relations (with the trigonometric R-matrix of (11.3)), see our proof of Theorem 11.11 below.
(b) This construction can be adapted to the setting of $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$. First, we redefine the generating series $A_i^{\pm}(z) = 1 + \sum_{r>0} A_{i,\pm r}^{\pm} z^{\mp r}$ which have to satisfy

$$z^{\mp b_i^{\pm}}(\psi_{i,\mp b_i^{\pm}}^{\pm})^{-1}\psi_i^{\pm}(z) = \frac{\prod_{j-i}\prod_{p=1}^{-c_{ji}} A_j^{\pm}(\boldsymbol{v}_j^{-c_{ji}-2p}z)}{A_i^{\pm}(z)A_i^{\pm}(\boldsymbol{v}_i^{-2}z)}. \tag{6.21}$$

Next, we define $B_i^{\pm}(z), C_i^{\pm}(z)$ via formulas (6.2, 6.3). Finally, we define $D_i^{\pm}(z)$ via

$$D_i^{\pm}(z) := A_i^{\pm}(z)\psi_i^{\pm}(z) + \boldsymbol{v}_i^{\mp 1}(\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})^2 f_i^{\pm}(z)A_i^{\pm}(z)e_i^{\pm}(z). \tag{6.22}$$

The coefficients of these generating series together with $\{(\psi_{i,-\epsilon b_i^{\epsilon}}^{\epsilon})^{\pm 1}\}_{i\in I}^{\epsilon=\pm}$ generate $\mathcal{U}^{\mathrm{sc}}_{\mu^+,\mu^-}$. For $\mu^+, \mu^- \in \Lambda^-$ one can write a complete list of the defining relations among these generators, which look similar to (6.7–6.20).

7 Homomorphism to Difference Operators

In this section, we construct homomorphisms from the shifted quantum affine algebras to the algebras of difference operators.

[5] We note that the relation $[D_i(u), D_i(v)] = 0$ was missing in their list.

7.1 Homomorphism $\widetilde{\Phi}{}^{\lambda}_{\mu}$

Let $\mathrm{Dyn}(\mathfrak{g})$ be the graph obtained from the Dynkin diagram of $\mathfrak{g}$ by replacing all multiple edges by simple ones. We fix an orientation of $\mathrm{Dyn}(\mathfrak{g})$ and we fix a dominant coweight $\lambda \in \Lambda^+$ and a coweight $\mu \in \Lambda$, such that $\lambda - \mu = \sum_{i \in I} a_i \alpha_i$ with $a_i \in \mathbb{N}$. We also fix a sequence $\underline{\lambda} = (\omega_{i_1}, \ldots, \omega_{i_N})$ of fundamental coweights, such that $\sum_{s=1}^{N} \omega_{i_s} = \lambda$.

Consider the associative $\mathbb{C}[\boldsymbol{v}^{\pm 1}]$-algebra $\hat{\mathcal{A}}^{\boldsymbol{v}}$ generated by $\{D_{i,r}^{\pm 1}, \mathsf{w}_{i,r}^{\pm 1/2}\}_{i \in I}^{1 \le r \le a_i}$ with the defining relations (for all $i, j \in I,\ 1 \le r \le a_i,\ 1 \le s \le a_j$):

$$[D_{i,r}, D_{j,s}] = [\mathsf{w}_{i,r}^{1/2}, \mathsf{w}_{j,s}^{1/2}] = 0,\ D_{i,r}^{\pm 1} D_{i,r}^{\mp 1} = \mathsf{w}_{i,r}^{\pm 1/2} \mathsf{w}_{i,r}^{\mp 1/2} = 1,\ D_{i,r} \mathsf{w}_{j,s}^{1/2} = \boldsymbol{v}_i^{\delta_{ij}\delta_{rs}} \mathsf{w}_{j,s}^{1/2} D_{i,r}.$$

Let $\widetilde{\mathcal{A}}^{\boldsymbol{v}}$ be the localization of $\hat{\mathcal{A}}^{\boldsymbol{v}}$ by the multiplicative set generated by $\{\mathsf{w}_{i,r} - \boldsymbol{v}_i^m \mathsf{w}_{i,s}\}_{i \in I, m \in \mathbb{Z}}^{1 \le r \ne s \le a_i} \cup \{1 - \boldsymbol{v}^m\}_{m \in \mathbb{Z} \setminus \{0\}}$ (which obviously satisfies Ore conditions). We also define their $\mathbb{C}(\boldsymbol{v})$-counterparts $\hat{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}} := \hat{\mathcal{A}}^{\boldsymbol{v}} \otimes_{\mathbb{C}[\boldsymbol{v}^{\pm 1}]} \mathbb{C}(\boldsymbol{v})$ and $\widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}} := \widetilde{\mathcal{A}}^{\boldsymbol{v}} \otimes_{\mathbb{C}[\boldsymbol{v}^{\pm 1}]} \mathbb{C}(\boldsymbol{v})$.

In what follows, we will work with the larger algebra $\mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$, which is obtained from $\mathcal{U}^{\mathrm{sc}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] := \mathcal{U}^{\mathrm{sc}}_{0,\mu} \otimes_{\mathbb{C}(\boldsymbol{v})} \mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ by adding extra generators $\{(\phi_i^{\epsilon})^{\pm 1}\}_{i \in I}^{\epsilon = \pm}$ satisfying relations (U9, U10) with the only change:

$$\prod_{s: i_s = i} (-\boldsymbol{v}_i \mathsf{z}_s)^{\mp 1} \cdot (\psi^-_{i, \alpha_i^\vee(\mu)})^{\pm 1} = (\phi_i^-)^{\pm 2} \cdot \prod_{j - i} (\phi_j^-)^{\pm c_{ji}}.$$

We will also work with the larger algebras $\widetilde{\mathcal{A}}^{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] := \widetilde{\mathcal{A}}^{\boldsymbol{v}} \otimes_{\mathbb{C}[\boldsymbol{v}^{\pm 1}]} \mathbb{C}[\boldsymbol{v}^{\pm 1}][\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ and $\widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] := \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}} \otimes_{\mathbb{C}(\boldsymbol{v})} \mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$.

Define

$$Z_i(z) := \prod_{1 \le s \le N}^{i_s = i} \left(1 - \frac{\boldsymbol{v}_i \mathsf{z}_s}{z}\right),\ W_i(z) := \prod_{r=1}^{a_i} \left(1 - \frac{\mathsf{w}_{i,r}}{z}\right),\ W_{i,r}(z) := \prod_{1 \le s \le a_i}^{s \ne r} \left(1 - \frac{\mathsf{w}_{i,s}}{z}\right),$$

$$\hat{Z}_i(z) := \prod_{1 \le s \le N}^{i_s = i} \left(1 - \frac{z}{\boldsymbol{v}_i \mathsf{z}_s}\right),\ \hat{W}_i(z) := \prod_{r=1}^{a_i} \left(1 - \frac{z}{\mathsf{w}_{i,r}}\right),\ \hat{W}_{i,r}(z) := \prod_{1 \le s \le a_i}^{s \ne r} \left(1 - \frac{z}{\mathsf{w}_{i,s}}\right).$$

The following is the key result of this section.

Theorem 7.1 *There exists a unique $\mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$-algebra homomorphism*

$$\widetilde{\Phi}^{\lambda}_{\mu} \colon \mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] \longrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}],$$

such that

$$e_i(z) \mapsto \frac{-\boldsymbol{v}_i}{1-\boldsymbol{v}_i^2}\prod_{t=1}^{a_i}\mathsf{w}_{i,t}\prod_{j\to i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{c_{ji}/2}\cdot\sum_{r=1}^{a_i}\delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\frac{Z_i(\mathsf{w}_{i,r})}{W_{i,r}(\mathsf{w}_{i,r})}\prod_{j\to i}\prod_{p=1}^{-c_{ji}}W_j(\boldsymbol{v}_j^{-c_{ji}-2p}z)D_{i,r}^{-1},$$

$$f_i(z) \mapsto \frac{1}{1-\boldsymbol{v}_i^2}\prod_{j\leftarrow i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{c_{ji}/2}\cdot\sum_{r=1}^{a_i}\delta\left(\frac{\boldsymbol{v}_i^2\mathsf{w}_{i,r}}{z}\right)\frac{1}{W_{i,r}(\mathsf{w}_{i,r})}\prod_{j\leftarrow i}\prod_{p=1}^{-c_{ji}}W_j(\boldsymbol{v}_j^{-c_{ji}-2p}z)D_{i,r},$$

$$\psi_i^{\pm}(z) \mapsto \prod_{t=1}^{a_i}\mathsf{w}_{i,t}\prod_{j-i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{c_{ji}/2}\cdot\left(\frac{Z_i(z)}{W_i(z)W_i(\boldsymbol{v}_i^{-2}z)}\prod_{j-i}\prod_{p=1}^{-c_{ji}}W_j(\boldsymbol{v}_j^{-c_{ji}-2p}z)\right)^{\pm},$$

$$(\phi_i^+)^{\pm 1} \mapsto \prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{\pm 1/2},\ \ (\phi_i^-)^{\pm 1} \mapsto (-\boldsymbol{v}_i)^{\mp a_i}\prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{\mp 1/2}.$$

We write $\gamma(z)^{\pm}$ *for the expansion of a rational function* $\gamma(z)$ *in* $z^{\mp 1}$*, respectively.*

In the *unshifted case*, more precisely for $U_{\boldsymbol{v}}(L\mathfrak{g})$, this result was stated (without a proof) in [31]. The above formulas simplify for simply-laced $\mathfrak{g}$, in which case this result can be viewed as a $\boldsymbol{v}$-version of [10, Corollary B.17]. We present the proof in Appendix C.

7.2 Homomorphism $\tilde{\Phi}^{\lambda}_{\mu}$ in ABC Generators

Generalizing the construction of Sect. 6, we define new Cartan generators $\{A^{\pm}_{i,\pm r}\}_{i\in I}^{r\geq 0}$ of $\mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1},\ldots,\mathsf{z}_N^{\pm 1}]$ via

$$A^{\pm}_{i,0} := (\phi_i^{\pm})^{-1},$$

$$\frac{\psi_i^+(z)}{Z_i(z)} = \frac{\prod_{j-i}\prod_{p=1}^{-c_{ji}}A_j^+(\boldsymbol{v}_j^{-c_{ji}-2p}z)}{A_i^+(z)A_i^+(\boldsymbol{v}_i^{-2}z)},$$

$$\frac{z^{\alpha_i^\vee(\mu)}\psi_i^-(z)}{\prod_{s:i_s=i}(-\boldsymbol{v}_i\mathsf{z}_s)\cdot\hat{Z}_i(z)} = \frac{\prod_{j-i}\prod_{p=1}^{-c_{ji}}A_j^-(\boldsymbol{v}_j^{-c_{ji}-2p}z)}{A_i^-(z)A_i^-(\boldsymbol{v}_i^{-2}z)},$$

where we set $A_i^{\pm}(z) := \sum_{r\geq 0}A^{\pm}_{i,\pm r}z^{\mp r}$. We also define the generating series $B_i^{\pm}(z)$, $C_i^{\pm}(z)$, and $D_i^{\pm}(z)$ via formulas (6.2), (6.3), and (6.4), respectively.

Lemma 7.2 *For antidominant* $\mu \in \Lambda^-$, *the generating series* $A_i^\pm(z), B_i^\pm(z), C_i^\pm(z), D_i^\pm(z)$ *satisfy relations (6.7–6.15).*

Proof Let c be the determinant of the Cartan matrix of $\mathfrak{g}$. Choose unique $\lambda_i^+(z) \in 1+z^{-1}\mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm1},\ldots,\mathsf{z}_N^{\pm1}][[z^{-1}]]$, such that $Z_i(z) = \frac{\lambda_i^+(z)\lambda_i^+(\boldsymbol{v}_i^{-2}z)}{\prod_{j-i}\prod_{p=1}^{-c_{ji}}\lambda_j^+(\boldsymbol{v}_j^{-c_{ji}-2p}z)}$. Also choose $\lambda_i^-(z) \in \mathbb{C}(\boldsymbol{v}^{1/c})[\mathsf{z}_1^{\pm1/c},\ldots,\mathsf{z}_N^{\pm1/c}][[z]]$, such that $\hat{Z}_i(z)\cdot\prod_{s:i_s=i}(-\boldsymbol{v}_i\mathsf{z}_s) = \frac{\lambda_i^-(z)\lambda_i^-(\boldsymbol{v}_i^{-2}z)}{\prod_{j-i}\prod_{p=1}^{-c_{ji}}\lambda_j^-(\boldsymbol{v}_j^{-c_{ji}-2p}z)}$.

Then, the series $\lambda_i^\pm(z)^{-1}X_i^\pm(z)$ for $X = A, B, C, D$ are those of Sect. 6. The result follows from Theorem 6.6(a) (compare with the proof of [44, Proposition 5.5]). □

Corollary 7.3 *The following equalities hold in* $\mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm1},\ldots,\mathsf{z}_N^{\pm1}]$:

$$B_i^+(z) = [e_{i,0}, A_i^+(z)]_{\boldsymbol{v}_i^{-1}},\ \ C_i^+(z) = [z^{-1}A_i^+(z), f_{i,1}]_{\boldsymbol{v}_i^{-1}},$$

$$B_i^-(z) = [e_{i,-1}, zA_i^-(z)]_{\boldsymbol{v}_i},\ \ C_i^-(z) = [A_i^-(z), f_{i,0}]_{\boldsymbol{v}_i}.$$

Proof The above formula for $B_i^+(z)$ (resp. $C_i^+(z)$) follows by evaluating the terms of degree 1 (resp. 0) in w in the equality (6.10) (resp. (6.11)) with $\epsilon = \epsilon' = +$.

The formulas for $B_i^-(z), C_i^-(z)$ are proved analogously. □

The following result is straightforward.

Proposition 7.4 *The homomorphism* $\widetilde{\Phi}^\lambda_\mu$ *maps the* ABC *currents as follows:*

$$A_i^+(z) \mapsto \prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{-1/2}\cdot W_i(z),\ \ A_i^-(z) \mapsto (-\boldsymbol{v}_i)^{a_i}\prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{1/2}\cdot \hat{W}_i(z),$$

$$B_i^+(z) \mapsto \prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{1/2}\prod_{j\to i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{c_{ji}/2}\cdot\sum_{r=1}^{a_i}\frac{W_{i,r}(z)Z_i(\mathsf{w}_{i,r})}{W_{i,r}(\mathsf{w}_{i,r})}\prod_{j\to i}\prod_{p=1}^{-c_{ji}}W_j(\boldsymbol{v}_j^{-c_{ji}-2p}\mathsf{w}_{i,r})D_{i,r}^{-1},$$

$$B_i^-(z) \mapsto -(-\boldsymbol{v}_i)^{a_i}\prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{3/2}\prod_{j\to i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{c_{ji}/2}\cdot\sum_{r=1}^{a_i}\frac{z\hat{W}_{i,r}(z)Z_i(\mathsf{w}_{i,r})}{\mathsf{w}_{i,r}W_{i,r}(\mathsf{w}_{i,r})}\prod_{j\to i}\prod_{p=1}^{-c_{ji}}W_j(\boldsymbol{v}_j^{-c_{ji}-2p}\mathsf{w}_{i,r})D_{i,r}^{-1},$$

$$C_i^+(z) \mapsto -\prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{-1/2}\prod_{j\leftarrow i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{c_{ji}/2}\cdot\sum_{r=1}^{a_i}\frac{\mathsf{w}_{i,r}W_{i,r}(z)}{zW_{i,r}(\mathsf{w}_{i,r})}\prod_{j\leftarrow i}\prod_{p=1}^{-c_{ji}}W_j(\boldsymbol{v}_j^{-c_{ji}-2p}\boldsymbol{v}_i^2\mathsf{w}_{i,r})D_{i,r},$$

$$C_i^-(z) \mapsto (-\boldsymbol{v}_i)^{a_i}\prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{1/2}\prod_{j\leftarrow i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{c_{ji}/2}\cdot\sum_{r=1}^{a_i}\frac{\hat{W}_{i,r}(z)}{W_{i,r}(\mathsf{w}_{i,r})}\prod_{j\leftarrow i}\prod_{p=1}^{-c_{ji}}W_j(\boldsymbol{v}_j^{-c_{ji}-2p}\boldsymbol{v}_i^2\mathsf{w}_{i,r})D_{i,r}.$$

In particular, all these images belong to $\widetilde{\mathcal{A}}^{\boldsymbol{v}}[\mathsf{z}_1^{\pm1},\ldots,\mathsf{z}_N^{\pm1}] \subset \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm1},\ldots,\mathsf{z}_N^{\pm1}]$.

8 K-theoretic Coulomb Branch

8.1 Quiver Gauge Theories

We follow the notations and setup of [10, Appendix A], so that $(GL(V), \mathbf{N})$ is a quiver gauge theory. As in Sect. 7, we fix a sequence $(\omega_{i_1}, \ldots, \omega_{i_N})$ of fundamental coweights of G which is assumed to be simply-laced for the current discussion. We choose a basis $w_1, \ldots, w_N$ in $W = \bigoplus_{i \in I} W_i$ such that $w_s \in W_{i_s}$. This defines a maximal torus $T_W \subset \prod_i GL(W_i)$, and $K_{T_W}(\mathrm{pt}) = \mathbb{C}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$. We consider the (quantized) K-theoretic Coulomb branch with flavor deformation $\mathcal{A}^q = K^{(GL(V)\times T_W)_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathcal{R}_{GL(V),\mathbf{N}})$ equipped with the convolution algebra structure as in [9, Remark 3.9(3)]. It is a $K_{\mathbb{C}^\times \times T_W}(\mathrm{pt})$-algebra; we denote $K_{\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{C}[q^{\pm 1}]$. We will also need $\boldsymbol{v} = q^{1/2}$, the generator of the equivariant K-theory of a point with respect to the two-fold cover $\widetilde{\mathbb{C}}^\times \to \mathbb{C}^\times$. Recall that $GL(V) = \prod_{i \in I} GL(V_i)$. We will need its 2^I-cover $\widetilde{GL}(V) = \prod_{i \in I} \widetilde{GL}(V_i)$ where $\widetilde{GL}(V_i) := \{(g \in GL(V_i),\ y \in \mathbb{C}^\times) : \det(g) = y^2\}$. We consider the extended Coulomb branch $\mathcal{A}^{\boldsymbol{v}} := K^{(\widetilde{GL}(V)\times T_W)_{\mathcal{O}} \rtimes \widetilde{\mathbb{C}}^\times}(\mathcal{R}_{GL(V),\mathbf{N}}) = \mathcal{A}^q \otimes_{K_{GL(V)\times\mathbb{C}^\times}(\mathrm{pt})} K_{\widetilde{GL}(V)\times\widetilde{\mathbb{C}}^\times}(\mathrm{pt})$. It is equipped with an algebra structure as in Sect. 3.7.

Recall from [10] that $w_{i,r}^*$ is the cocharacter of the Lie algebra of $GL(V) = \prod GL(V_i)$, which is equal to 0 except at the vertex i, and is $(0, \ldots, 0, 1, 0, \ldots, 0)$ at i. Here 1 is at the r-th entry ($r = 1, \ldots, a_i = \dim V_i$). We denote the corresponding coordinates of T_V and $T_V^\vee$ by $\mathsf{w}_{i,r}$ and $D_{i,r}$ ($i \in I$, $1 \le r \le a_i$). The roots are $\mathsf{w}_{i,r}\mathsf{w}_{i,s}^{-1}$ ($r \neq s$). Furthermore, $K^{(T_V\times T_W)_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathcal{R}_{T_V,0})$ with scalars extended by $\boldsymbol{v}, \mathsf{w}_{i,r}^{\pm 1/2}$ is nothing but the algebra $\hat{\mathcal{A}}^{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] := \hat{\mathcal{A}}^{\boldsymbol{v}} \otimes_{\mathbb{C}[\boldsymbol{v}^{\pm 1}]} \mathbb{C}[\boldsymbol{v}^{\pm 1}][\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$, where $\hat{\mathcal{A}}^{\boldsymbol{v}}$ was defined in Sect. 7. We thus have an algebra embedding

$$\mathbf{z}^*(\iota_*)^{-1} \colon \mathcal{A}^{\boldsymbol{v}} \hookrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}].$$

Let $\varpi_{i,n}$ be the n-th fundamental coweight of the factor $GL(V_i)$, i.e., $w_{i,1}^* + \ldots + w_{i,n}^* = (1, \ldots, 1, 0, \ldots, 0)$ where 1 appears n times ($1 \le n \le a_i$). Then $\mathrm{Gr}_{GL(V)}^{\varpi_{i,n}}$ is closed and isomorphic to the Grassmannian $\mathrm{Gr}(V_i, n)$ of n-dimensional quotients of V_i. Let $\mathcal{Q}_i$ be the tautological rank n quotient bundle on $\mathrm{Gr}_{GL(V)}^{\varpi_{i,n}}$. Its pull-back to $\mathcal{R}_{\varpi_{i,n}}$ is also denoted by $\mathcal{Q}_i$ for brevity. Let $\Lambda^p(\mathcal{Q}_i)$ denote the class of its p-th external power in $\mathcal{A}^{\boldsymbol{v}}$. More generally, we can consider a class $f(\mathcal{Q}_i)$ for a symmetric function f in n variables so that $\Lambda^p(\mathcal{Q}_i)$ corresponds to the p-th elementary symmetric polynomial e_p.

Similarly, we consider $\varpi_{i,n}^* = -w_0\varpi_{i,n}$, where the corresponding orbit $\mathrm{Gr}_{GL(V)}^{\varpi_{i,n}^*}$ is closed and isomorphic to the Grassmannian $\mathrm{Gr}(n, V_i)$ of n-dimensional subspaces in V_i. Let $\mathcal{S}_i$ be the tautological rank n subbundle on $\mathrm{Gr}_{GL(V)}^{\varpi_{i,n}^*}$. Its pull-back to $\mathcal{R}_{\varpi_{i,n}^*}$

is also denoted by $\mathcal{S}_i$. Now similarly to [10, (A.3), (A.5)], cf. [10, Remark A.8], we obtain

$$\mathbf{z}^*(\iota_*)^{-1}\left(f(\mathcal{Q}_i)\otimes\mathcal{O}_{\mathcal{R}_{\varpi_{i,n}}}\right)=\sum_{\substack{J\subset\{1,\ldots,a_i\}\\ \#J=n}} f(\mathsf{w}_{i,J})\frac{\prod\limits_{\substack{j\leftarrow i\\ r\in J}}\prod\limits_{\substack{s=1\\ (j,s)\neq(i,r)}}^{a_j}(1-\boldsymbol{v}\mathsf{w}_{i,r}\mathsf{w}_{j,s}^{-1})}{\prod\limits_{r\in J,s\notin J}(1-\mathsf{w}_{i,s}\mathsf{w}_{i,r}^{-1})}\prod_{r\in J}D_{i,r} \tag{8.1}$$

(the appearance of $\boldsymbol{v}$ is due to the convention before [9, Remark 2.1]);

$$\mathbf{z}^*(\iota_*)^{-1}\left(f(\mathcal{S}_i)\otimes\mathcal{O}_{\mathcal{R}_{\varpi^*_{i,n}}}\right)=$$

$$\sum_{\substack{J\subset\{1,\ldots,a_i\}\\ \#J=n}} f(\boldsymbol{v}^{-2}\mathsf{w}_{i,J})\prod_{\substack{r\in J\\ t:i_t=i}}(1-\boldsymbol{v}\mathsf{z}_t\mathsf{w}_{i,r}^{-1})\frac{\prod\limits_{\substack{j\rightarrow i\\ r\in J}}\prod\limits_{\substack{s=1\\ (j,s)\neq(i,r)}}^{a_j}(1-\boldsymbol{v}\mathsf{w}_{j,s}\mathsf{w}_{i,r}^{-1})}{\prod\limits_{r\in J,s\notin J}(1-\mathsf{w}_{i,r}\mathsf{w}_{i,s}^{-1})}\prod_{r\in J}D_{i,r}^{-1}, \tag{8.2}$$

where $f(\boldsymbol{v}^{-2}\mathsf{w}_{i,J})$ means that we substitute $\{\boldsymbol{v}^{-2}\mathsf{w}_{i,r}\}_{r\in J}$ to f.

Also, for the vector bundles $\Omega^p_{\varpi_{i,1}}, \Omega^p_{\varpi^*_{i,1}}$ of p-forms on $\mathrm{Gr}^{\varpi_{i,1}}_{GL(V)}, \mathrm{Gr}^{\varpi^*_{i,1}}_{GL(V)}$ we obtain

$$\mathbf{z}^*(\iota_*)^{-1}\left(\Omega^p_{\varpi_{i,1}}\otimes\mathcal{Q}_i^{\otimes p'}\otimes\mathcal{O}_{\mathcal{R}_{\varpi_{i,1}}}\right)=$$

$$\sum_{1\leq r\leq a_i}\mathsf{w}_{i,r}^{p'-p}\left(\sum_{\substack{J\subset\{1,\ldots,a_i\}\backslash\{r\}\\ \#J=p}}\prod_{s\in J}\mathsf{w}_{i,s}\right)\frac{\prod\limits_{j\leftarrow i}\prod\limits_{\substack{s=1\\ (j,s)\neq(i,r)}}^{a_j}(1-\boldsymbol{v}\mathsf{w}_{i,r}\mathsf{w}_{j,s}^{-1})}{\prod\limits_{s\neq r}(1-\mathsf{w}_{i,s}\mathsf{w}_{i,r}^{-1})}D_{i,r}, \tag{8.3}$$

$$\mathbf{z}^*(\iota_*)^{-1}\left(\Omega^p_{\varpi^*_{i,1}}\otimes\mathcal{S}_i^{\otimes p'}\otimes\mathcal{O}_{\mathcal{R}_{\varpi^*_{i,1}}}\right)=$$

$$\sum_{1\leq r\leq a_i}\boldsymbol{v}^{-2p'}\mathsf{w}_{i,r}^{p'+p}\prod_{t:i_t=i}(1-\boldsymbol{v}\mathsf{z}_t\mathsf{w}_{i,r}^{-1})\left(\sum_{\substack{J\subset\{1,\ldots,a_i\}\backslash\{r\}\\ \#J=p}}\prod_{s\in J}\mathsf{w}_{i,s}^{-1}\right)\frac{\prod\limits_{j\rightarrow i}\prod\limits_{\substack{s=1\\ (j,s)\neq(i,r)}}^{a_j}(1-\boldsymbol{v}\mathsf{w}_{j,s}\mathsf{w}_{i,r}^{-1})}{\prod\limits_{s\neq r}(1-\mathsf{w}_{i,r}\mathsf{w}_{i,s}^{-1})}D_{i,r}^{-1}. \tag{8.4}$$

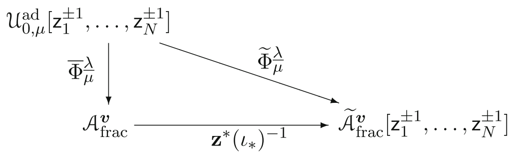
8.2 Homomorphism $\overline{\Phi}^{\lambda}_{\mu}$

We set $\mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}} := \mathcal{A}^{\boldsymbol{v}} \otimes_{\mathbb{C}[\boldsymbol{v}^{\pm 1}]} \mathbb{C}(\boldsymbol{v})$. The key result of this section asserts that the homomorphism $\widetilde{\Phi}^{\lambda}_{\mu}$ of Theorem 7.1 factors through the above embedding $\mathbf{z}^*(\iota_*)^{-1} \colon \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}} \hookrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$, similarly to [10, Theorem B.18].

Theorem 8.1 *There exists a unique* $\mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$*-algebra homomorphism*

$$\overline{\Phi}^{\lambda}_{\mu} \colon \mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] \longrightarrow \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}},$$

such that the following diagram commutes:



Explicitly, $\overline{\Phi}^{\lambda}_{\mu}$ *maps the generators as follows:*

$$e_{i,r} \mapsto \frac{(-1)^{a_i}\boldsymbol{v}}{1-\boldsymbol{v}^2} \prod_{j \to i} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{-1/2} \cdot (\boldsymbol{v}^2 \mathcal{S}_i)^{\otimes(r+a_i)} \otimes \mathcal{O}_{\mathcal{R}_{\varpi^*_{i,1}}},$$

$$f_{i,r} \mapsto \frac{(-\boldsymbol{v})^{-\sum_{j \leftarrow i} a_j}}{1-\boldsymbol{v}^2} \prod_{j \leftarrow i} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{1/2} \cdot \mathcal{Q}_i^{\otimes(-\sum_{j \leftarrow i} a_j)} \otimes (\boldsymbol{v}^2 \mathcal{Q}_i)^{\otimes r} \otimes \mathcal{O}_{\mathcal{R}_{\varpi_{i,1}}},$$

$$A^+_{i,r} \mapsto (-1)^r \prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{-1/2} \cdot e_r(\{\mathsf{w}_{i,t}\}_{t=1}^{a_i}),$$

$$A^-_{i,-r} \mapsto (-1)^r (-\boldsymbol{v})^{a_i} \prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{1/2} \cdot e_r(\{\mathsf{w}_{i,t}^{-1}\}_{t=1}^{a_i}),$$

$$\phi_i^+ \mapsto \prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{1/2},\ \ \phi_i^- \mapsto (-\boldsymbol{v})^{-a_i} \prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{-1/2}.$$

Proof For $X \in \{e_{i,r}, f_{i,r}, A^{\pm}_{i,\pm s}, \phi_i^{\pm} | i \in I, r \in \mathbb{Z}, s \in \mathbb{N}\}$ consider the assignment $X \mapsto \overline{\Phi}^{\lambda}_{\mu}(X)$ with the right-hand side defined as above. Since $\mathbf{z}^*(\iota_*)^{-1} \colon \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}} \hookrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ is injective and $\widetilde{\Phi}^{\lambda}_{\mu} \colon \mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ is an algebra homomorphism, it suffices to check that $\mathbf{z}^*(\iota_*)^{-1}\left(\overline{\Phi}^\lambda_\mu(X)\right) = \widetilde{\Phi}^\lambda_\mu(X)$ for X as above. This is a straightforward verification based on formulas (8.1) and (8.2). □

Combining Proposition 7.4 with formulas (8.3) and (8.4), we immediately find the images of the generators $\{B^+_{i,r}, C^+_{i,r+1}\}^{r\geq 0}_{i\in I}$ under $\overline{\Phi}^\lambda_\mu$.

Corollary 8.2 *For $r \in \mathbb{N}$, we have*

$$\overline{\Phi}^\lambda_\mu(B^+_{i,r}) = (-1)^{r+a_i+1}\boldsymbol{v}^{2r}\prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{1/2}\prod_{j\to i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{-1/2}\cdot\left(\Omega^{a_i-1-r}_{\varpi^*_{i,1}}\otimes\mathcal{S}_i^{\otimes r}\otimes\mathcal{O}_{\mathcal{R}_{\varpi^*_{i,1}}}\right),$$

$$\overline{\Phi}^\lambda_\mu(C^+_{i,r+1}) = (-1)^{r+1}(-\boldsymbol{v})^{-\sum_{j\leftarrow i}a_j}\prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{-1/2}\prod_{j\leftarrow i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{1/2}\cdot\left(\Omega^r_{\varpi_{i,1}}\otimes\mathcal{Q}_i^{\otimes(r+1-\sum_{j\leftarrow i}a_j)}\otimes\mathcal{O}_{\mathcal{R}_{\varpi_{i,1}}}\right).$$

In particular, the images of $\{A^+_{i,r}, B^+_{i,r}, C^+_{i,r+1}, \phi^+_i\}^{r\in\mathbb{N}}_{i\in I}$ under $\overline{\Phi}^\lambda_\mu$ belong to $\mathcal{A}^{\boldsymbol{v}} \subset \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$. In fact, the images of $\{A^-_{i,-r}, B^-_{i,-r-1}, C^-_{i,-r}, \phi^-_i\}^{r\in\mathbb{N}}_{i\in I}$ under $\overline{\Phi}^\lambda_\mu$ also belong to $\mathcal{A}^{\boldsymbol{v}}$.

Remark 8.3 (A. Weekes) In the case of shifted Yangians, the images of the generating series $B_i(z), C_i(z)$ [44, Section 5.3] in the quantized (cohomological) Coulomb branch $\mathcal{A}_\hbar$ under the homomorphism $\overline{\Phi}^\lambda_\mu$ of [10, Theorem B.18] are equal to

$$\overline{\Phi}^\lambda_\mu(B_i(z)) = (-1)^{a_i}z^{-1}\cdot c(\widetilde{\mathcal{Q}}_i, -z^{-1})\cap[\mathcal{R}_{\varpi^*_{i,1}}],$$

$$\overline{\Phi}^\lambda_\mu(C_i(z)) = (-1)^{\sum_{j\leftarrow i}a_j}z^{-1}\cdot c(\mathcal{S}_i, -z^{-1})\cap[\mathcal{R}_{\varpi_{i,1}}],$$

where $c(\mathcal{F}, z)$ denotes the Chern polynomial of a vector bundle $\mathcal{F}$. Here we view $\mathcal{Q}_i, \mathcal{S}_i$ as rank $n-1$ vector bundles on $\mathcal{R}_{\varpi^*_{i,1}}, \mathcal{R}_{\varpi_{i,1}}$, respectively, while $\widetilde{\mathcal{Q}}_i$ denotes the vector bundle $\mathcal{Q}_i$ with the equivariance structure twisted by $\hbar$.

Remark 8.4 Note that $\mathrm{Gr}^{\varpi_{i,1}}_{GL(V)} \simeq \mathbb{P}^{a_i-1} \simeq \mathrm{Gr}^{\varpi^*_{i,1}}_{GL(V)}$, and if we forget the equivariance, then up to sign, $\overline{\Phi}^\lambda_\mu(f_{i,r}),\ 1\leq r\leq a_i$, is the collection of classes of pull-backs of the line bundles $\mathcal{O}_{\mathbb{P}^{a_i-1}}(1-\sum_{j\leftarrow i}a_j),\ldots,\mathcal{O}_{\mathbb{P}^{a_i-1}}(a_i-\sum_{j\leftarrow i}a_j)$, while $\overline{\Phi}^\lambda_\mu(C^+_{i,r}),\ 1\leq r\leq a_i$, is the collection of classes of pull-backs of the vector bundles $\Omega^{r-1}_{\mathbb{P}^{a_i-1}}(r-\sum_{j\leftarrow i}a_j)$. These two collections are the dual exceptional collections of vector bundles on $\mathbb{P}^{a_i-1}$ (more precisely, the former collection is left dual to the latter one). In fact, this is the historically first example of dual exceptional collections, [3]. Similarly, up to sign and forgetting equivariance, $\overline{\Phi}^{\lambda}_{\mu}(e_{i,r}),\ 0 \leq r < a_i$, are the classes of the exceptional collection of line bundles right dual to the exceptional collection of vector bundles whose classes are $\overline{\Phi}^{\lambda}_{\mu}(B^{+}_{i,r}),\ 0 \leq r < a_i$.

Remark 8.5 An action of the quantized K-theoretic Coulomb branch $\mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$ of the type A quiver gauge theory on the localized equivariant K-theory of parabolic Laumon spaces was constructed in [4]. Combining this construction with Theorem 8.1, we see that there should be a natural action of $\mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ (with $\mathfrak{g} = \mathfrak{sl}_n$) on the aforementioned K-theory. We construct explicitly such an action of $\mathcal{U}^{\mathrm{sc}}_{0,\mu}$ in Theorem 12.2 by adapting the arguments of [61] to the current setting (the adjoint version is achieved by considering equivariant K-theory with respect to a larger torus).

8.3 Truncated Shifted Quantum Affine Algebras

We consider a 2-sided ideal $\mathcal{I}^{\lambda}_{\mu}$ of $\mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ generated over $\mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ by the following elements:

$$A^{\pm}_{i,\pm s}\ (s > a_i),\ \ A^{+}_{i,0}A^{+}_{i,a_i} - (-1)^{a_i},\ \ A^{-}_{i,0}A^{-}_{i,-a_i} - (-1)^{a_i}\boldsymbol{v}_i^{2a_i}, \tag{8.5}$$

$$A^{-}_{i,-r} - \boldsymbol{v}_i^{a_i} A^{+}_{i,a_i-r}\ (0 \leq r \leq a_i). \tag{8.6}$$

Definition 8.6 $\mathcal{U}^{\lambda}_{\mu} := \mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]/\mathcal{I}^{\lambda}_{\mu}$ is called the *truncated shifted quantum affine algebra*.

Note that the homomorphism $\widetilde{\Phi}^{\lambda}_{\mu} \colon \mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ factors through the same named homomorphism $\widetilde{\Phi}^{\lambda}_{\mu} \colon \mathcal{U}^{\lambda}_{\mu} \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$, due to Proposition 7.4. Similarly to [10, Remark B.21], we expect this homomorphism to be injective:

Conjecture 8.7 $\widetilde{\Phi}^{\lambda}_{\mu} \colon \mathcal{U}^{\lambda}_{\mu} \hookrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$.

Remark 8.8 As a first indication of the validity of this conjecture, we note that the elements $\{B^{+}_{i,r}, C^{+}_{i,r+1}, B^{-}_{i,-r-1}, C^{-}_{i,-r}\}^{r \geq a_i}_{i \in I}$ which belong to $\mathrm{Ker}(\widetilde{\Phi}^{\lambda}_{\mu})$ (due to Proposition 7.4) also belong to $\mathcal{I}^{\lambda}_{\mu}$, due to Corollary 7.3 and relation (U10).

Moreover, we expect the following result:

Conjecture 8.9 $\overline{\Phi}^{\lambda}_{\mu} \colon \mathcal{U}^{\lambda}_{\mu} \xrightarrow{\sim} \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$.

8.4 Truncated Shifted v-Yangians

Recall that $\mathfrak{g}$ is assumed to be simply-laced. Recall an explicit identification of the Drinfeld-Jimbo and the new Drinfeld realizations of the standard quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{g})$. To this end, choose a decomposition of the highest root θ of $\mathfrak{g}$ into a sum of simple roots $\theta = \alpha^\vee_{i_1} + \alpha^\vee_{i_2} + \ldots + \alpha^\vee_{i_{h-1}}$ such that $\epsilon_k := \langle \alpha_{i_{k+1}}, \alpha^\vee_{i_1} + \ldots + \alpha^\vee_{i_k} \rangle \in \mathbb{Z}_{<0}$ for any $1 \leq k \leq h-2$ (here h is the Coxeter number of $\mathfrak{g}$). We encode a choice of such a decomposition by a sequence $\mathbf{i} = (i_1, \ldots, i_{h-1})$. Let $U^{\mathrm{DJ}}_{\boldsymbol{v}}(L\mathfrak{g})$ denote the Drinfeld-Jimbo quantum group of $\widehat{\mathfrak{g}}$ (affinization of $\mathfrak{g}$) with a trivial central charge, generated by $\{E_i, F_i, K_i^{\pm 1}\}_{i \in \widetilde{I}}$ (here $\widetilde{I} = I \cup \{i_0\}$ is the vertex set of the extended Dynkin diagram), see [50]. The following result is due to [16] (proved in [41]).

Theorem 8.10 *There is a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $U^{\mathrm{DJ}}_{\boldsymbol{v}}(L\mathfrak{g}) \xrightarrow{\sim} U_{\boldsymbol{v}}(L\mathfrak{g})$, such that*

$$E_i \mapsto e_{i,0},\ F_i \mapsto f_{i,0},\ K_i^{\pm 1} \mapsto \psi^{\pm}_{i,0} \text{ for } i \in I,$$

$$E_{i_0} \mapsto [f_{i_{h-1},0}, [f_{i_{h-2},0}, \cdots, [f_{i_2,0}, f_{i_1,1}]_{\boldsymbol{v}^{\epsilon_1}} \cdots]_{\boldsymbol{v}^{\epsilon_{h-3}}}]_{\boldsymbol{v}^{\epsilon_{h-2}}} \cdot \psi^-_\theta,$$

$$F_{i_0} \mapsto (-\boldsymbol{v})^{-\epsilon} \psi^+_\theta \cdot [e_{i_{h-1},0}, [e_{i_{h-2},0}, \cdots, [e_{i_2,0}, e_{i_1,-1}]_{\boldsymbol{v}^{\epsilon_1}} \cdots]_{\boldsymbol{v}^{\epsilon_{h-3}}}]_{\boldsymbol{v}^{\epsilon_{h-2}}},$$

$$K^{\pm}_{i_0} \mapsto \psi^{\mp}_\theta,$$

where $\psi^\pm_\theta := \psi^\pm_{i_1,0} \cdots \psi^\pm_{i_{h-1},0}$, $\epsilon := \epsilon_1 + \ldots + \epsilon_{h-2}$.

In particular, the image of the negative Drinfeld-Jimbo Borel subalgebra of $U^{\mathrm{DJ}}_{\boldsymbol{v}}(L\mathfrak{g})$ generated by $\{F_i, K_i^{\pm 1}\}_{i \in \widetilde{I}}$ under the above isomorphism is the subalgebra $U^-_{\boldsymbol{v}}$ of $U_{\boldsymbol{v}}(L\mathfrak{g})$, generated by $\{f_{i,0}, (\psi^-_{i,0})^{\pm 1}, F\}_{i \in I}$ with $F := [e_{i_{h-1},0}, [e_{i_{h-2},0}, \cdots, [e_{i_2,0}, e_{i_1,-1}]_{\boldsymbol{v}^{\epsilon_1}} \cdots]_{\boldsymbol{v}^{\epsilon_{h-3}}}]_{\boldsymbol{v}^{\epsilon_{h-2}}}$. Motivated by this observation, we introduce the following definition.

Definition 8.11

(a) Fix $\mathbf{i} = (i_1, \ldots, i_{h-1})$ as above. The shifted $\boldsymbol{v}$-Yangian ${}_{\mathbf{i}}\mathcal{Y}^{\boldsymbol{v}}_\mu[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ is the $\mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$-subalgebra of $\mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ generated by $\{f_{i,0}, (\psi^-_{i,b_i})^{\pm 1}, \hat{F}\}_{i \in I}$, where

$$\hat{F} := [e_{i_{h-1},b_{i_{h-1}}}, [e_{i_{h-2},b_{i_{h-2}}}, \cdots, [e_{i_2,b_{i_2}}, e_{i_1,b_{i_1}-1}]_{\boldsymbol{v}^{\epsilon_1}} \cdots]_{\boldsymbol{v}^{\epsilon_{h-3}}}]_{\boldsymbol{v}^{\epsilon_{h-2}}}$$

and $b_i := \alpha_i^\vee(\mu)$.

(b) The truncated shifted $\boldsymbol{v}$-Yangian ${}_{\mathbf{i}}\mathcal{Y}^\lambda_\mu$ is the quotient of ${}_{\mathbf{i}}\mathcal{Y}^{\boldsymbol{v}}_\mu[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ by the 2-sided ideal ${}_{\mathbf{i}}\mathcal{I}^{\lambda,+}_\mu := \mathcal{I}^\lambda_\mu \cap {}_{\mathbf{i}}\mathcal{Y}^{\boldsymbol{v}}_\mu[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$.

Remark 8.12 For $\mathfrak{g} = \mathfrak{gl}_n$ and $\mu = 0$, our definition of the $\boldsymbol{v}$-Yangian is consistent with that of the *quantum Yangian* $Y_q(\mathfrak{gl}_n)$ of [54] (in particular, independent of the choice of $\mathbf{i}$). The latter is defined via the RTT presentation, see our discussion in Appendix G, and corresponds to the subalgebra generated by the coefficients of the matrix $T^-(z)$.

Conjecture 8.13 $\overline{\Phi}^{\lambda}_{\mu} \colon {}_{\mathbf{i}}\mathcal{Y}^{\lambda}_{\mu} \xrightarrow{\sim} \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$.

8.5 Integral Forms

If we believe Conjectures 8.9 and 8.13, we can transfer the integral forms $\mathcal{A}^{\boldsymbol{v}} \subset \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$ to the truncated shifted quantum affine algebras and the truncated shifted $\boldsymbol{v}$-Yangians to obtain the $\mathbb{C}[\boldsymbol{v}^{\pm 1}]$-subalgebras $'\mathfrak{U}^{\underline{\lambda}}_{\mu} \subset \mathcal{U}^{\underline{\lambda}}_{\mu}$ and ${}_{\mathbf{i}}\mathfrak{Y}^{\underline{\lambda}}_{\mu} \subset {}_{\mathbf{i}}\mathcal{Y}^{\underline{\lambda}}_{\mu}$. Finally, we define the integral form $'\mathfrak{U}^{\mathrm{ad}}_{0,\mu} \subset \mathcal{U}^{\mathrm{ad}}_{0,\mu}$ as an intersection of all the preimages of $'\mathfrak{U}^{\underline{\lambda}}_{\mu}|_{\mathsf{z}_1=\ldots=\mathsf{z}_N=1}$ under projections $\mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] \twoheadrightarrow \mathcal{U}^{\underline{\lambda}}_{\mu}$ as $\underline{\lambda}$ varies, and ${}_{\mathbf{i}}\mathfrak{Y}^{\boldsymbol{v}}_{\mu} := {}'\mathfrak{U}^{\mathrm{ad}}_{0,\mu} \cap {}_{\mathbf{i}}\mathcal{Y}^{\boldsymbol{v}}_{\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]|_{\mathsf{z}_1=\ldots=\mathsf{z}_N=1}$. Unfortunately, we cannot define these integral forms by generators and relations in general. In the case of $\mathfrak{sl}_2$ see Sect. 9.1.

Recall that $*$ stands for the involution $\mu \mapsto -w_0\mu$ of the coweight lattice Λ. Similarly to [10, Remark 3.17], one can construct an isomorphism from the non-quantized extended K-theoretic Coulomb branch $\mathrm{Spec}\, K^{(\widetilde{GL(V)\times T_W})_{\mathcal{O}}}(\mathcal{R}_{GL(V),\mathbf{N}})$ of Sect. 8.1 to the multiplicative slice cover ${}^{\dagger}\hat{\mathcal{W}}^{\underline{\lambda}^*}_{\mu^*}$ of Sect. 4.6. Its quantization is the subject of the following

Conjecture 8.14

(a) The shifted $\boldsymbol{v}$-Yangian ${}_{\mathbf{i}}\mathfrak{Y}^{\boldsymbol{v}}_{\mu}$ is a quantization of the thick multiplicative slice ${}^{\dagger}\mathcal{W}_{\mu^*}$ of Sect. 4.8, that is ${}_{\mathbf{i}}\mathfrak{Y}^{\boldsymbol{v}}_{\mu}|_{\boldsymbol{v}=1} \simeq \mathbb{C}[{}^{\dagger}\mathcal{W}_{\mu^*}]$.
(b) The truncated shifted $\boldsymbol{v}$-Yangian ${}_{\mathbf{i}}\mathfrak{Y}^{\underline{\lambda}}_{\mu}$ and the truncated shifted quantum affine algebra $'\mathfrak{U}^{\underline{\lambda}}_{\mu}$ are quantizations of the multiplicative slice cover ${}^{\dagger}\hat{\mathcal{W}}^{\underline{\lambda}^*}_{\mu^*}$ of Sect. 4.6, that is ${}_{\mathbf{i}}\mathfrak{Y}^{\underline{\lambda}}_{\mu}|_{\boldsymbol{v}=1} \simeq {}'\mathfrak{U}^{\underline{\lambda}}_{\mu}|_{\boldsymbol{v}=1} \simeq \mathbb{C}[{}^{\dagger}\hat{\mathcal{W}}^{\underline{\lambda}^*}_{\mu^*}]$.

8.6 An Example

Let $\mathfrak{g} = \mathfrak{sl}_n$, $\mu = 0$, $\underline{\lambda} = (\omega_1, \ldots, \omega_1)$ (the first fundamental coweight taken n times). Note that the symmetric group $\mathfrak{S}_n$ acts naturally on $'\mathfrak{U}^{\underline{\lambda}}_{\mu}$, permuting the parameters $\mathsf{z}_1, \ldots, \mathsf{z}_n$. This action induces the one on the quotient algebra $'\overline{\mathfrak{U}}^{\underline{\lambda}}_{\mu}$ by the relation $\mathsf{z}_1 \cdots \mathsf{z}_n = 1$. Then we expect that the evaluation homomorphism

$U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \twoheadrightarrow U_{\boldsymbol{v}}(\mathfrak{sl}_n)$ [40] gives rise to an isomorphism $({}'\overline{\mathfrak{U}}^{\underline{\lambda}}_{\mu})^{\mathfrak{S}_n} \xrightarrow{\sim} {}_{\mathcal{A}}\mathbf{O}_{\mathrm{loc}}$, where ${}_{\mathcal{A}}\mathbf{O}$ is the integral form of the quantum coordinate algebra of $SL(N)$ introduced in [50, 29.5.2], and ${}_{\mathcal{A}}\mathbf{O}_{\mathrm{loc}}$ stands for its localization by inverting the quantum minors $\{c_\nu\}_{\nu\in\Lambda^+}$, see [42, 9.1.10].

9 Shifted Quantum Affine $\mathfrak{sl}_2$ and Nil-DAHA for $GL(n)$

9.1 Integral Form

In this section $\mathfrak{g} = \mathfrak{sl}_2$, whence we denote $A^{\pm}_{i,r}, B^{\pm}_{i,r}, C^{\pm}_{i,r}, \phi^{\pm}_i$ simply by $A^{\pm}_r, B^{\pm}_r, C^{\pm}_r, \phi^{\pm}$. The shift $\mu \in \Lambda = \mathbb{Z}$ is an integer. Furthermore, $\underline{\lambda} = (\omega_1, \ldots, \omega_1)$ (a collection of N copies of the fundamental coweight). The corresponding shifted quantum affine algebra is $\mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$. We define a $\mathbb{C}[\boldsymbol{v}^{\pm 1}]$-subalgebra $\mathfrak{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] \subset \mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ generated by $\{A^{\pm}_{\pm r}, B^+_r, B^-_{-r-1}, C^+_{r+1}, C^-_{-r}, \phi^{\pm}\}_{r\in\mathbb{N}}$ and its quotient algebra (an integral version of the truncated shifted quantum affine algebra)

$$\mathfrak{U}^{\underline{\lambda}}_{\mu} := \mathfrak{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]/(\mathcal{I}^{\underline{\lambda}}_{\mu} \cap \mathfrak{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]).$$

Let $V = \mathbb{C}^n,\ W = \mathbb{C}^N$. According to Corollary 8.2, the homomorphism

$$\overline{\Phi}^N_{N-2n} \colon \mathcal{U}^{\mathrm{ad}}_{0,N-2n}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] \longrightarrow \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}} = K^{(\widetilde{GL}(V)\times T_W)_{\mathcal{O}} \rtimes \widetilde{\mathbb{C}}^\times}(\mathcal{R}_{GL(V),\mathrm{Hom}(W,V)}) \otimes_{\mathbb{C}[\boldsymbol{v}^{\pm 1}]} \mathbb{C}(\boldsymbol{v})$$

takes $\mathfrak{U}^{\mathrm{ad}}_{0,N-2n}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] \subset \mathcal{U}^{\mathrm{ad}}_{0,N-2n}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ to $\mathcal{A}^{\boldsymbol{v}} \subset \mathcal{A}^{\boldsymbol{v}}_{\mathrm{frac}}$. In particular, we have $\mathfrak{U}^{\mathrm{ad}}_{0,N-2n}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] \subset {}'\mathfrak{U}^{\mathrm{ad}}_{0,N-2n}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ (cf. Sect. 8.5). We also define a $\mathbb{C}[\boldsymbol{v}^{\pm 1}]$-subalgebra $\mathfrak{Y}^{\boldsymbol{v}}_{N-2n}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] \subset \mathcal{Y}^{\boldsymbol{v}}_{N-2n}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ generated by $\{A^-_{-r}, B^-_{-r-1}, C^-_{-r}, \phi^-\}_{r\in\mathbb{N}}$. Furthermore, we define the shifted Borel $\boldsymbol{v}$-Yangian $\mathfrak{Y}^{\boldsymbol{v}}_{N-2n,-}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ as the $\mathbb{C}[\boldsymbol{v}^{\pm 1}]$-subalgebra of $\mathfrak{Y}^{\boldsymbol{v}}_{N-2n}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}]$ generated by $\{A^-_{-r}, C^-_{-r}, \phi^-\}_{r\in\mathbb{N}}$. Finally, we have their truncated quotients $\mathfrak{Y}^{\underline{\lambda}}_{N-2n},\ \mathfrak{Y}^{\underline{\lambda}}_{N-2n,-}$. We expect that

$$\begin{aligned}\mathfrak{U}^{\mathrm{ad}}_{0,N-2n}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] &= {}'\mathfrak{U}^{\mathrm{ad}}_{0,N-2n}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}],\\ \mathfrak{Y}^{\boldsymbol{v}}_{N-2n}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}] &= {}'\mathfrak{Y}^{\boldsymbol{v}}_{N-2n}[\mathsf{z}_1^{\pm 1}, \ldots, \mathsf{z}_N^{\pm 1}],\\ \mathfrak{U}^{\underline{\lambda}}_{N-2n} = {}'\mathfrak{U}^{\underline{\lambda}}_{N-2n},\ \mathfrak{Y}^{\underline{\lambda}}_{N-2n} &= {}'\mathfrak{Y}^{\underline{\lambda}}_{N-2n}.\end{aligned}$$

Conjecture 9.1 The natural homomorphisms induce isomorphisms

$$\mathfrak{Y}^{\underline{\lambda}}_{N-2n} \xrightarrow{\sim} \mathfrak{U}^{\underline{\lambda}}_{N-2n} \xrightarrow{\sim} \mathcal{A}^{\boldsymbol{v}}.$$

From now on, we specialize to the case $N = 0, \mu = -2n$. According to Corollary 3.14, the corresponding Coulomb branch $\mathcal{A}^{\boldsymbol{v}} = K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{Gr}_{GL(n)})$ is nothing but the spherical extended nil-DAHA $\mathcal{H}\mathcal{H}_e^{\mathrm{sph}}(GL(n))$. We define $\mathbb{C}[\boldsymbol{v}^{\pm1}]_{\mathrm{loc}}$ inverting $(1-\boldsymbol{v}^{2m}),\ m = 1, 2, \dots, n$. We extend the scalars to $\mathbb{C}[\boldsymbol{v}^{\pm1}]_{\mathrm{loc}}$ to obtain

$$\overline{\Phi}^0_{-2n,\mathrm{loc}}\colon \mathfrak{U}^{\mathrm{ad}}_{0,-2n,\mathrm{loc}} \longrightarrow K_{\mathrm{loc}}^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{Gr}_{GL(n)}).$$

The following theorem and Proposition 9.8 is a supportive evidence in favor of Conjecture 9.1.

Theorem 9.2 $\overline{\Phi}^0_{-2n,\mathrm{loc}}\colon \mathfrak{U}^{\mathrm{ad}}_{0,-2n,\mathrm{loc}} \to K_{\mathrm{loc}}^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{Gr}_{GL(n)})$ *is surjective.*[6]

Proof We must prove that $K_{\mathrm{loc}}^{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}_{GL(n)})$ is generated by $K_{GL(n)}(\mathrm{pt}) = R(GL(n))$, and $\mathcal{O}(a)_{\varpi_1}, \mathcal{O}(a)_{\varpi_1^*},\ a \in \mathbb{Z}$. Here $\varpi_1 = (1, 0, \dots, 0)$ denotes the first fundamental coweight of $GL(n)$, and $\mathrm{Gr}^{\varpi_1} \simeq \mathbb{P}^{n-1}$ is the corresponding minuscule orbit, so that $\mathrm{Gr}^{\varpi_1^*} \simeq \check{\mathbb{P}}^{n-1}$. Finally, $\mathcal{Q}$ is the tautological quotient bundle on Gr^{ϖ_1}, isomorphic to the ample line bundle $\mathcal{O}(1)$ on $\mathbb{P}^{n-1}$, and $\mathcal{O}(a)_{\varpi_1}$ stands for $\mathcal{Q}^{\otimes a}$. Similarly, $\mathcal{S}$ is the tautological line subbundle on $\mathrm{Gr}^{\varpi_1^*}$ isomorphic to $\mathcal{O}(-1)$ on $\check{\mathbb{P}}^{n-1}$, and $\mathcal{O}(a)_{\varpi_1^*}$ stands for $\mathcal{S}^{\otimes -a}$. Note that $\mathcal{O}(1)_{\varpi_1}, \mathcal{O}(1)_{\varpi_1^*}$ are isomorphic to the restrictions of the determinant line bundle on $\mathrm{Gr}_{GL(n)}$.

Given an arbitrary sequence $\nu_1, \dots, \nu_N$ with $\nu_i \in \{\varpi_1, \dots, \varpi_n, \varpi_1^*, \dots, \varpi_n^*\}$, the equivariant K-theory of the iterated convolution diagram

$$K^{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}^{\nu_1}\widetilde{\times}\dots\widetilde{\times}\mathrm{Gr}^{\nu_N})$$

is isomorphic to

$$K^{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}^{\nu_1}) \otimes_{K_{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{pt})} \cdots \otimes_{K_{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{pt})} K^{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}^{\nu_N}).$$

By the projection formula and rationality of singularities of $\overline{\mathrm{Gr}}^{\nu_1+\dots+\nu_N}$, the convolution pushforward morphism

$$m_*\colon K^{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}^{\nu_1}\widetilde{\times}\dots\widetilde{\times}\mathrm{Gr}^{\nu_N}) \longrightarrow K^{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}(\overline{\mathrm{Gr}}^{\nu_1+\dots+\nu_N})$$

is surjective. Hence in order to prove the surjectivity statement of the theorem, it suffices to express $K_{\mathrm{loc}}^{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}^{\nu}),\ \nu \in \{\varpi_1, \dots, \varpi_n, \varpi_1^*, \dots, \varpi_n^*\}$, in terms of $\mathcal{O}(a)_{\varpi_1}, \mathcal{O}(a)_{\varpi_1^*},\ a \in \mathbb{Z}$, and $K_{GL(n)}(\mathrm{pt})$. We will consider $\nu = \varpi_m,\ 1 \le m \le n$, the case of ϖ_m^* being similar. Note that $\mathcal{O}_{\varpi_n}$ is the structure sheaf of a point $GL(n,\mathcal{O})$-orbit corresponding to the coweight $(1, \dots, 1)$. We argue by induction in m.

[6] A stronger version of the theorem (over $\mathbb{Z}[\boldsymbol{v}^{\pm1}]$ as opposed to over $\mathbb{C}[\boldsymbol{v}^{\pm1}]_{\mathrm{loc}}$) is proved independently in [15, Corollary 2.21, Remark 2.22].

For ν as above, the Picard group of Gr^ν is $\mathbb{Z}$, and we denote the ample generator by $\mathcal{O}(1)_\nu$. It is isomorphic to the restriction of the determinant line bundle on $\mathrm{Gr}_{GL(n)}$. We start with an explicit expression for $\mathcal{O}_{\varpi_m} := \mathcal{O}_{\mathrm{Gr}^{\varpi_m}}$, $1 \le m \le n$, in terms of $\mathcal{O}(a)_{\varpi_1}$, $a \in \mathbb{Z}$. Recall that $\overline{\Phi}^0_{-2n}(f_r) = \frac{\boldsymbol{v}^{2r}}{1-\boldsymbol{v}^2}\mathcal{O}(r)_{\varpi_1}$ and $\overline{\Phi}^0_{-2n}(e_r) = \frac{(-1)^n\boldsymbol{v}^{2r+2n+1}}{1-\boldsymbol{v}^2}\mathcal{O}(-r-n)_{\varpi_1^*}$. We denote $\mathrm{ad}_x^{\boldsymbol{v}^r}\, y := [x,y]_{\boldsymbol{v}^r} = xy - \boldsymbol{v}^r yx$.

Proposition 9.3 *For any* $1 \le m \le n$, *we have*

$$\mathcal{O}_{\varpi_m} = (-1)^{\frac{m(m-1)}{2}}(1-\boldsymbol{v}^2)\overline{\Phi}^0_{-2n}(\mathrm{ad}^{\boldsymbol{v}^{2m}}_{f_{1-m}}\mathrm{ad}^{\boldsymbol{v}^{2(m-1)}}_{f_{3-m}}\cdots\mathrm{ad}^{\boldsymbol{v}^4}_{f_{m-3}} f_{m-1}), \qquad (9.1)$$

$$\mathcal{O}_{\varpi_m^*} = (-1)^{nm+\frac{m(m+1)}{2}+1}\boldsymbol{v}^{m^2-2}(1-\boldsymbol{v}^2)\times$$
$$\overline{\Phi}^0_{-2n}(\mathrm{ad}^{\boldsymbol{v}^{-2m}}_{e_{-n+1-m}}\mathrm{ad}^{\boldsymbol{v}^{-2(m-1)}}_{e_{-n+3-m}}\cdots\mathrm{ad}^{\boldsymbol{v}^{-4}}_{e_{-n+m-3}} e_{-n+m-1}). \qquad (9.2)$$

Proof We prove (9.1); the proof of (9.2) is similar. We will compare the images of the LHS and the RHS in $\widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}$. According to (8.1), the image of the LHS equals

$$\sum_{\#J=m}\prod_{r\in J}^{s\notin J}(1-\mathsf{w}_s\mathsf{w}_r^{-1})^{-1}\prod_{r\in J}D_r. \qquad (9.3)$$

Here $J \subset \{1,\ldots,n\}$ is a subset of cardinality m. Let us denote the iterated $\boldsymbol{v}$-commutator $\mathrm{ad}^{\boldsymbol{v}^{2m}}_{f_{1-m}}\mathrm{ad}^{\boldsymbol{v}^{2(m-1)}}_{f_{3-m}}\cdots\mathrm{ad}^{\boldsymbol{v}^4}_{f_{m-3}} f_{m-1}$ by F_m. We want to prove

$$\widetilde{\Phi}^0_{-2n}(F_m) = (-1)^{\frac{m(m-1)}{2}}(1-\boldsymbol{v}^2)^{-1}\cdot\sum_{\#J=m}\prod_{r\in J}^{s\notin J}(1-\mathsf{w}_s\mathsf{w}_r^{-1})^{-1}\prod_{r\in J}D_r. \qquad (9.4)$$

The proof proceeds by induction in m. So we assume (9.4) known for an integer $k < n$, and want to deduce (9.4) for $m = k+1$. We introduce a "shifted" $\boldsymbol{v}$-commutator $F'_k := \mathrm{ad}^{\boldsymbol{v}^{2k}}_{f_{2-k}}\mathrm{ad}^{\boldsymbol{v}^{2(k-1)}}_{f_{4-k}}\cdots\mathrm{ad}^{\boldsymbol{v}^4}_{f_{k-2}} f_k$. Then

$$\widetilde{\Phi}^0_{-2n}(F'_k) = (-1)^{\frac{k(k-1)}{2}}(1-\boldsymbol{v}^2)^{-1}\boldsymbol{v}^{2k}\cdot\sum_{\#J=k}\prod_{r\in J}\mathsf{w}_r\prod_{r\in J}^{s\notin J}\left(1-\frac{\mathsf{w}_s}{\mathsf{w}_r}\right)^{-1}\prod_{r\in J}D_r.$$

Now

$$\widetilde{\Phi}^0_{-2n}(F_{k+1}) = \widetilde{\Phi}^0_{-2n}([f_{-k},F'_k]_{\boldsymbol{v}^{2(k+1)}}) = [\widetilde{\Phi}^0_{-2n}(f_{-k}),\widetilde{\Phi}^0_{-2n}(F'_k)]_{\boldsymbol{v}^{2(k+1)}} =$$

$$(-1)^{\frac{k(k-1)}{2}}(1-\boldsymbol{v}^2)^{-2}\boldsymbol{v}^{2k}\cdot\left[\sum_{p=1}^n\frac{(\boldsymbol{v}^2\mathsf{w}_p)^{-k}}{\prod\limits_{t\neq p}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_p}\right)}D_p,\ \sum_{\#J=k}\prod_{r\in J}\mathsf{w}_r\prod_{r\in J}^{s\notin J}\left(1-\frac{\mathsf{w}_s}{\mathsf{w}_r}\right)^{-1}\prod_{r\in J}D_r\right]_{\boldsymbol{v}^{2(k+1)}}.$$

First we check that the summands corresponding to $p \in J$ vanish. Due to the symmetry reasons, we may assume $p = 1,\ J = \{1, 2, \ldots, k\}$. Then

$$\left[\frac{(\boldsymbol{v}^2\mathsf{w}_1)^{-k}}{\prod\limits_{t>1}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_1}\right)}D_1, \prod_{r=1}^{k}\mathsf{w}_r \prod_{r\le k}^{s>k}\left(1-\frac{\mathsf{w}_s}{\mathsf{w}_r}\right)^{-1} D_1\cdots D_k\right]_{\boldsymbol{v}^{2(k+1)}} =$$

$$\left[\frac{(\boldsymbol{v}^2\mathsf{w}_1)^{-k}}{\prod\limits_{t>k}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_1}\right)\prod\limits_{1<r\le k}\left(1-\frac{\mathsf{w}_r}{\mathsf{w}_1}\right)}D_1, \frac{\mathsf{w}_1\cdots\mathsf{w}_k}{\prod\limits_{s>k}\left(1-\frac{\mathsf{w}_s}{\mathsf{w}_1}\right)\prod\limits_{1<r\le k}^{s>k}\left(1-\frac{\mathsf{w}_s}{\mathsf{w}_r}\right)}D_1\cdots D_k\right]_{\boldsymbol{v}^{2(k+1)}} =$$

$$\left(\frac{(\boldsymbol{v}^2\mathsf{w}_1)^{-k}\boldsymbol{v}^2\mathsf{w}_1\cdots\mathsf{w}_k}{\prod\limits_{t>k}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_1}\right)\prod\limits_{1<r\le k}\left(1-\frac{\mathsf{w}_r}{\mathsf{w}_1}\right)\prod\limits_{s>k}\left(1-\boldsymbol{v}^{-2}\frac{\mathsf{w}_s}{\mathsf{w}_1}\right)\prod\limits_{1<r\le k}^{s>k}\left(1-\frac{\mathsf{w}_s}{\mathsf{w}_r}\right)}-\right.$$

$$\left.-\frac{\boldsymbol{v}^{2(k+1)}\mathsf{w}_1\cdots\mathsf{w}_k(\boldsymbol{v}^2\mathsf{w}_1)^{-k}\boldsymbol{v}^{-2k}}{\prod\limits_{s>k}\left(1-\frac{\mathsf{w}_s}{\mathsf{w}_1}\right)\prod\limits_{1<r\le k}^{s>k}\left(1-\frac{\mathsf{w}_s}{\mathsf{w}_r}\right)\prod\limits_{t>k}\left(1-\boldsymbol{v}^{-2}\frac{\mathsf{w}_t}{\mathsf{w}_1}\right)\prod\limits_{1<r\le k}\left(1-\frac{\mathsf{w}_r}{\mathsf{w}_1}\right)}\right)D_1^2D_2\cdots D_k = 0.$$

Therefore,

$$(-1)^{\frac{k(k-1)}{2}}(1-\boldsymbol{v}^2)^2\widetilde{\Phi}^0_{-2n}(F_{k+1}) =$$
$$\sum_{\#J=k}^{p\notin J}\left[\frac{(\boldsymbol{v}^2\mathsf{w}_p)^{-k}}{\prod_{t\neq p}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_p}\right)}D_p, \boldsymbol{v}^{2k}\prod_{r\in J}\mathsf{w}_r\prod_{r\in J}^{s\notin J}\left(1-\frac{\mathsf{w}_s}{\mathsf{w}_r}\right)^{-1}\prod_{r\in J}D_r\right]_{\boldsymbol{v}^{2(k+1)}}.$$

We expand this combination of $\boldsymbol{v}^{2(k+1)}$-commutators as a sum

$$\sum_{\#J=k+1}\phi_J(\mathsf{w}_1,\ldots,\mathsf{w}_n)\prod_{r\in J}D_r.$$

For the symmetry reasons, it suffices to calculate the rational function ϕ_J for a single $J = \{1, \ldots, k+1\}$. We have

$$\phi_J(\mathsf{w}_1,\ldots,\mathsf{w}_n)D_1\cdots D_{k+1} =$$

$$\sum_{r=1}^{k+1}\left[\frac{(\boldsymbol{v}^2\mathsf{w}_r)^{-k}}{\prod\limits_{t\neq r}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_r}\right)}D_r, \frac{\boldsymbol{v}^{2k}\mathsf{w}_1\cdots\widehat{\mathsf{w}_r}\cdots\mathsf{w}_{k+1}}{\prod\limits_{r\neq p\le k+1}^{t>k+1}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_p}\right)\prod\limits_{r\neq p\le k+1}\left(1-\frac{\mathsf{w}_r}{\mathsf{w}_p}\right)}D_1\cdots\widehat{D_r}\cdots D_{k+1}\right]_{\boldsymbol{v}^{2(k+1)}} =$$

$$\sum_{r=1}^{k+1}\left(\frac{\mathsf{w}_r^{-k}\mathsf{w}_1\cdots\widehat{\mathsf{w}}_r\cdots\mathsf{w}_{k+1}}{\prod\limits_{t>k+1}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_r}\right)\prod\limits_{r\neq p\leq k+1}\left(1-\frac{\mathsf{w}_p}{\mathsf{w}_r}\right)\prod\limits_{r\neq p\leq k+1}^{t>k+1}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_p}\right)\prod\limits_{r\neq p\leq k+1}\left(1-\frac{\boldsymbol{v}^2\mathsf{w}_r}{\mathsf{w}_p}\right)}-\right.$$

$$\left.-\frac{\boldsymbol{v}^{2(k+1)}\mathsf{w}_r^{-k}\mathsf{w}_1\cdots\widehat{\mathsf{w}}_r\cdots\mathsf{w}_{k+1}}{\prod\limits_{r\neq p\leq k+1}^{t>k+1}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_p}\right)\prod\limits_{r\neq p\leq k+1}\left(1-\frac{\mathsf{w}_r}{\mathsf{w}_p}\right)\prod\limits_{t>k+1}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_r}\right)\prod\limits_{r\neq p\leq k+1}\left(1-\frac{\boldsymbol{v}^2\mathsf{w}_p}{\mathsf{w}_r}\right)}\right)D_1\cdots D_{k+1}=$$

$$-\boldsymbol{v}^{2(k+1)}\mathsf{w}_1\cdots\mathsf{w}_{k+1}\prod_{r\leq k+1}^{t>k+1}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_r}\right)^{-1}\times$$

$$\sum_{r=1}^{k+1}\left(\frac{\mathsf{w}_r^{-k-1}}{\prod\limits_{r\neq p\leq k+1}\left(1-\frac{\mathsf{w}_r}{\mathsf{w}_p}\right)\left(1-\frac{\boldsymbol{v}^2\mathsf{w}_p}{\mathsf{w}_r}\right)}-\frac{\boldsymbol{v}^{-2(k+1)}\mathsf{w}_r^{-k-1}}{\prod\limits_{r\neq p\leq k+1}\left(1-\frac{\mathsf{w}_p}{\mathsf{w}_r}\right)\left(1-\frac{\boldsymbol{v}^2\mathsf{w}_r}{\mathsf{w}_p}\right)}\right)D_1\cdots D_{k+1}.$$

This is equal to the following expression, by Lemma 9.4 below:

$$-\boldsymbol{v}^{2(k+1)}\mathsf{w}_1\cdots\mathsf{w}_{k+1}\prod_{r\leq k+1}^{t>k+1}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_r}\right)^{-1}\frac{(-1)^k(\boldsymbol{v}^2-1)}{\boldsymbol{v}^{2(k+1)}\prod\limits_{r\leq k+1}\mathsf{w}_r}D_1\cdots D_{k+1}=$$

$$(-1)^k(1-\boldsymbol{v}^2)\prod_{r\leq k+1}^{t>k+1}\left(1-\frac{\mathsf{w}_t}{\mathsf{w}_r}\right)^{-1}D_1\cdots D_{k+1}.$$

We conclude that

$$\widetilde{\Phi}^0_{-2n}(F_{k+1})=(-1)^{\frac{k(k+1)}{2}}(1-\boldsymbol{v}^2)^{-1}\cdot\sum_{\#J=k+1}\prod_{r\in J}^{s\notin J}(1-\mathsf{w}_s\mathsf{w}_r^{-1})^{-1}\prod_{r\in J}D_r,$$

and (9.4) is proved. It remains to check

Lemma 9.4 *We have*

$$\sum_{r=1}^{k+1}\left(\frac{\mathsf{w}_r^{-k-1}}{\prod\limits_{s\neq r}(1-\mathsf{w}_r/\mathsf{w}_s)(1-\boldsymbol{v}^2\mathsf{w}_s/\mathsf{w}_r)}-\frac{\boldsymbol{v}^{-2(k+1)}\mathsf{w}_r^{-k-1}}{\prod\limits_{s\neq r}(1-\mathsf{w}_s/\mathsf{w}_r)(1-\boldsymbol{v}^2\mathsf{w}_r/\mathsf{w}_s)}\right)=\frac{(-1)^k(\boldsymbol{v}^2-1)}{\boldsymbol{v}^{2(k+1)}\prod\limits_{r=1}^{k+1}\mathsf{w}_r}.$$

Proof The LHS is a degree $-k-1$ rational function of $\mathsf{w}_1,\ldots,\mathsf{w}_{k+1}$ with poles at the hyperplanes given by equations $\mathsf{w}_r-\mathsf{w}_s,\ \mathsf{w}_r-\boldsymbol{v}^2\mathsf{w}_s,\ \mathsf{w}_r\ (1\leq r\neq s\leq$

$k+1)$. One can check $\mathrm{Res}_{\mathsf{w}_r-\mathsf{w}_s}\,\mathrm{LHS} = \mathrm{Res}_{\mathsf{w}_r-\boldsymbol{v}^2\mathsf{w}_s}\,\mathrm{LHS} = 0$, so that LHS $= f\cdot\prod_{1\le r\le k+1}\mathsf{w}_r^{-1}$ for a rational function $f\in\mathbb{C}(\boldsymbol{v})$. To compute f, we specialize $\mathsf{w}_1\mapsto 0$ in the equality

$$f=\prod_{t=1}^{k+1}\mathsf{w}_t\cdot\sum_{r=1}^{k+1}\left(\frac{\prod\limits_{s\neq r}\mathsf{w}_s}{\prod\limits_{s\neq r}(\mathsf{w}_s-\mathsf{w}_r)(\mathsf{w}_r-\boldsymbol{v}^2\mathsf{w}_s)}\cdot\frac{1}{\mathsf{w}_r}-\frac{\boldsymbol{v}^{-2(k+1)}\prod\limits_{s\neq r}\mathsf{w}_s}{\prod\limits_{s\neq r}(\mathsf{w}_r-\mathsf{w}_s)(\mathsf{w}_s-\boldsymbol{v}^2\mathsf{w}_r)}\cdot\frac{1}{\mathsf{w}_r}\right).$$

The only summands surviving under this specialization correspond to $r=1$, and so we get

$$f=\prod_{t=2}^{k+1}\mathsf{w}_t\cdot\left(\frac{\prod\limits_{s=2}^{k+1}\mathsf{w}_s}{(-\boldsymbol{v}^2)^k\cdot\prod\limits_{s=2}^{k+1}\mathsf{w}_s^2}-\frac{\boldsymbol{v}^{-2(k+1)}\cdot\prod\limits_{s=2}^{k+1}\mathsf{w}_s}{(-1)^k\cdot\prod\limits_{s=2}^{k+1}\mathsf{w}_s^2}\right)=(-1)^k(\boldsymbol{v}^{-2k}-\boldsymbol{v}^{-2(k+1)}).$$

The lemma is proved. □

The proposition is proved. □

Returning to the proof of Theorem 9.2, we need to prove that $K^{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}_{\mathrm{loc}}(\mathrm{Gr}^{\varpi_m})$ lies in the image $\overline{\Phi}^0_{-2n,\mathrm{loc}}(\mathfrak{U}^{\mathrm{ad}}_{0,-2n,\mathrm{loc}})$ for $1\le m\le n$. We know that the class of the structure sheaf $\mathcal{O}_{\varpi_m}\in K^{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}_{\mathrm{loc}}(\mathrm{Gr}^{\varpi_m})$ lies in $\overline{\Phi}^0_{-2n,\mathrm{loc}}(\mathfrak{U}^{\mathrm{ad}}_{0,-2n,\mathrm{loc}})$. It is also known that $K^{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{Gr}^{\varpi_m})$ as a left $K_{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{pt})$-module is generated by the classes $\Sigma^\lambda(\mathcal{Q})$ where $\mathcal{Q}$ is the tautological quotient bundle on $\mathrm{Gr}^{\varpi_m}\simeq\mathrm{Gr}(m,n)$, and Σ^λ is the polynomial Schur functor corresponding to a Young diagram λ with $\le m$ rows (in fact, it is enough to consider λ's with $\le n-m$ columns). Given such λ, it suffices to check that $\mathrm{Sym}\left(\mathsf{w}_1^{\lambda_1}\cdots\mathsf{w}_m^{\lambda_m}\prod\limits_{r\le m}^{s>m}\left(1-\frac{\mathsf{w}_s}{\mathsf{w}_r}\right)^{-1}D_1\cdots D_m\right)$ lies in $\widetilde{\Phi}^0_{-2n,\mathrm{loc}}(\mathfrak{U}^{\mathrm{ad}}_{0,-2n,\mathrm{loc}})$ (here Sym stands for the symmetrization with respect to the symmetric group $\mathfrak{S}_n$). More generally, for a Young diagram μ with $\le n$ rows we will show that $\mathrm{Sym}\left(\mathsf{w}_1^{\mu_1}\cdots\mathsf{w}_n^{\mu_n}\cdot\prod\limits_{r\le m}^{s>m}\left(1-\frac{\mathsf{w}_s}{\mathsf{w}_r}\right)^{-1}D_1\cdots D_m\right)$ lies in $\widetilde{\Phi}^0_{-2n,\mathrm{loc}}(\mathfrak{U}^{\mathrm{ad}}_{0,-2n,\mathrm{loc}})$. To this end, we use the *right* multiplication by $K_{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{pt})$. It suffices to check that the $K_{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}(\mathrm{pt})_{\mathrm{loc}}$-bimodule generated by $X_{1,m}:=\mathrm{Sym}\left(\prod\limits_{r\le m}^{s>m}\left(1-\frac{\mathsf{w}_s}{\mathsf{w}_r}\right)^{-1}D_1\cdots D_m\right)$ contains elements $X_{F,m}:=\mathrm{Sym}\left(F\prod\limits_{r\le m}^{s>m}\left(1-\frac{\mathsf{w}_s}{\mathsf{w}_r}\right)^{-1}D_1\cdots D_m\right)$ for any polynomial $F\in\mathbb{C}[\mathsf{w}_1,\ldots,\mathsf{w}_n]$. We can assume that $F\in\mathbb{C}[\mathsf{w}_1,\ldots,\mathsf{w}_n]^{\mathfrak{S}_m\times\mathfrak{S}_{n-m}}$, where the symmetric groups act by permuting $\{\mathsf{w}_r,\ 1\le r\le m\}$ and $\{\mathsf{w}_s,\ m+1\le s\le n\}$. Note that $\mathbb{C}[\mathsf{w}_1,\ldots,\mathsf{w}_n]^{\mathfrak{S}_m\times\mathfrak{S}_{n-m}}$ is generated by $\mathbb{C}[\mathsf{w}_1,\ldots,\mathsf{w}_m]^{\mathfrak{S}_m}$

as a left $\mathbb{C}[\mathsf{w}_1, \ldots, \mathsf{w}_n]^{\mathfrak{S}_n}$-module. Hence, it suffices to treat the case $F \in \mathbb{C}[\mathsf{w}_1, \ldots, \mathsf{w}_m]^{\mathfrak{S}_m} = \mathbb{C}[p_1, \ldots, p_m]$, where $p_k := \sum_{r=1}^m \mathsf{w}_r^k$. The latter case follows from the equality

$$\left[\sum_{r=1}^n \mathsf{w}_r^k, X_{F,m}\right] = (1 - \boldsymbol{v}^{2k}) X_{F p_k, m}$$

for $F \in \mathbb{C}[\mathsf{w}_1, \ldots, \mathsf{w}_m]^{\mathfrak{S}_m}$.

The theorem is proved. □

Remark 9.5 The end of our proof of Theorem 9.2 is a variation of the following argument we learned from P. Etingof. We define $\mathbb{C}[\boldsymbol{v}^{\pm 1}]_{\mathrm{Loc}}$ inverting $(1 - \boldsymbol{v}^m)$, $m \in \mathbb{Z}$. We consider a $\mathbb{C}[\boldsymbol{v}^{\pm 1}]_{\mathrm{Loc}}$-algebra $\mathfrak{A}$ of finite difference operators with generators $\{\mathsf{w}_i^{\pm 1}, D_i^{\pm 1}\}_{i=1}^n$ and defining relations $D_i \mathsf{w}_j = \boldsymbol{v}^{2\delta_{ij}} \mathsf{w}_j D_i$, $[D_i, D_j] = [\mathsf{w}_i, \mathsf{w}_j] = 0$. Then the algebra of $\mathfrak{S}_n$-invariants $\mathfrak{A}^{\mathfrak{S}_n}$ is generated by its subalgebras $\mathbb{C}[\boldsymbol{v}^{\pm 1}]_{\mathrm{Loc}}[D_1^{\pm 1}, \ldots, D_n^{\pm 1}]^{\mathfrak{S}_n}$ and $\mathbb{C}[\boldsymbol{v}^{\pm 1}]_{\mathrm{Loc}}[\mathsf{w}_1^{\pm 1}, \ldots, \mathsf{w}_n^{\pm 1}]^{\mathfrak{S}_n}$.

Indeed, let $\mathfrak{B}$ be the $\mathbb{C}[\boldsymbol{v}^{\pm 1}]_{\mathrm{Loc}}$-algebra generated by $\mathsf{w}^{\pm 1}$, $D^{\pm 1}$ subject to $D\mathsf{w} = \boldsymbol{v}^2 \mathsf{w} D$. Then $\mathfrak{A} = \mathfrak{B}^{\otimes n}$ (tensor product over $\mathbb{C}[\boldsymbol{v}^{\pm 1}]_{\mathrm{Loc}}$), and $\mathfrak{A}^{\mathfrak{S}_n} = \mathrm{Sym}^n \mathfrak{B}$ (symmetric power over $\mathbb{C}[\boldsymbol{v}^{\pm 1}]_{\mathrm{Loc}}$). Now $\mathrm{Sym}^n \mathfrak{B}$ is spanned by the elements $\{b^{\otimes n}\}_{b \in \mathfrak{B}}$, and hence $\mathrm{Sym}^n \mathfrak{B}$ is generated by the elements $\{b_{(1)} + \ldots + b_{(n)}\}_{b \in \mathfrak{B}}$, where $b_{(r)} = 1 \otimes \cdots \otimes 1 \otimes b \otimes 1 \otimes \cdots \otimes 1$ (b at the r-th entry). Indeed, it suffices to verify the generation claim for an algebra $\mathbb{C}[\boldsymbol{v}^{\pm 1}]_{\mathrm{Loc}}[b]$ where it is nothing but the fundamental theorem on symmetric functions.

We conclude that $\mathrm{Sym}^n \mathfrak{B}$ is generated by the elements $\{p_{m,k} = \sum_{r=1}^n \mathsf{w}_r^m D_r^k\}_{m,k \in \mathbb{Z}}$. However, $p_{m,k} = (\boldsymbol{v}^{2mk} - 1)^{-1} [\sum_{r=1}^n D_r^k, \sum_{s=1}^n \mathsf{w}_s^m]$ for $m \neq 0 \neq k$.

Remark 9.6 Motivated by [10, Remark 3.5] we call $\mathcal{O}_{\varpi_n} \in K^{GL(n,\mathcal{O}) \rtimes \mathbb{C}^\times}(\mathrm{Gr}_{GL(n)})$ *the quantum resultant*. In fact, it is a quantization of the boundary equation for the trigonometric zastava ${}^{\dagger}\mathring{Z}^n_{SL(2)}$ which is nothing but the resultant of two polynomials. Note that, up to multiplication by an element of $\mathbb{C}[\boldsymbol{v}^{\pm 1}]$, the quantum resultant is uniquely characterized by the property

$$\mathcal{O}_{\varpi_n} \overline{\Phi}^0_{-2n}(A^{\pm}_{\pm r}) = \boldsymbol{v}^{\pm(2r-n)} \overline{\Phi}^0_{-2n}(A^{\pm}_{\pm r}) \mathcal{O}_{\varpi_n}, \ \mathcal{O}_{\varpi_n} \overline{\Phi}^0_{-2n}(f_p) = \boldsymbol{v}^{2p} \overline{\Phi}^0_{-2n}(f_p) \mathcal{O}_{\varpi_n}. \tag{9.5}$$

Remark 9.7 Here is a geometric explanation of the equality

$$\mathcal{O}(-k-1)_{\varpi_1} * \mathcal{O}_{\varpi_k} - \boldsymbol{v}^{2(k+1)} \mathcal{O}_{\varpi_k} * \mathcal{O}(-k-1)_{\varpi_1} = (-1)^k (1 - \boldsymbol{v}^2) \boldsymbol{v}^{-2(k+1)} \mathcal{O}(-1)_{\varpi_{k+1}}, \tag{9.6}$$

established as an induction step during our proof of Proposition 9.3. We have the convolution morphisms

$$\mathrm{Gr}^{\varpi_1} \widetilde{\times} \mathrm{Gr}^{\varpi_k} \xrightarrow{m} \overline{\mathrm{Gr}}^{\varpi_1 + \varpi_k} \xleftarrow{m'} \mathrm{Gr}^{\varpi_k} \widetilde{\times} \mathrm{Gr}^{\varpi_1},$$

and $\overline{\mathrm{Gr}}^{\varpi_1+\varpi_k} = \mathrm{Gr}^{\varpi_1+\varpi_k} \sqcup \mathrm{Gr}^{\varpi_{k+1}}$. Let us consider the transversal slice $\overline{\mathcal{W}}^{\varpi_1+\varpi_k}_{\varpi_{k+1}} \subset \overline{\mathrm{Gr}}^{\varpi_1+\varpi_k}$ through the point $\varpi_{k+1} = (1,\ldots,1,0,\ldots,0)$ ($k+1$ 1's). It suffices to check that

$$m_*\left(\mathcal{O}(-k-1)_{\varpi_1}\widetilde{\boxtimes}\mathcal{O}_{\varpi_k}|_{m^{-1}\overline{\mathcal{W}}^{\varpi_1+\varpi_k}_{\varpi_{k+1}}}\right) - \boldsymbol{v}^{2(k+1)}m'_*\left(\mathcal{O}_{\varpi_k}\widetilde{\boxtimes}\mathcal{O}(-k-1)_{\varpi_1}|_{m'^{-1}\overline{\mathcal{W}}^{\varpi_1+\varpi_k}_{\varpi_{k+1}}}\right) = (-1)^k(1-\boldsymbol{v}^2)\boldsymbol{v}^{-2(k+1)}\mathsf{w}_1^{-1}\cdots\mathsf{w}_{k+1}^{-1},$$

where we view $\boldsymbol{v}^{-2(k+1)}\mathsf{w}_1^{-1}\cdots\mathsf{w}_{k+1}^{-1}$ as a character of $T\times\mathbb{C}^\times$ ($T\subset GL(n)$ is the diagonal Cartan torus). According to [52, Corollary 3.4], $\overline{\mathcal{W}}^{\varpi_1+\varpi_k}_{\varpi_{k+1}}$ is naturally isomorphic to the slice $\overline{\mathcal{W}}^{\theta}_0 \subset \mathrm{Gr}_{GL(k+1)\times(\mathbb{C}^\times)^{n-k-1}}$ where $\theta = (1,0,\ldots,0,-1)$ is the highest coroot of $GL(k+1)$. Moreover, the preimages of $\overline{\mathcal{W}}^{\varpi_1+\varpi_k}_{\varpi_{k+1}}$ in the two convolution diagrams are isomorphic to the cotangent bundles $T^*\mathbb{P}^k$ and $T^*\check{\mathbb{P}}^k$, respectively. We will keep the following notation for the convolution morphisms restricted to the slice:

$$T^*\mathbb{P}^k \xrightarrow{m} \overline{\mathcal{W}}^\theta_0 \xleftarrow{m'} T^*\check{\mathbb{P}}^k.$$

Note also that $\overline{\mathcal{W}}^\theta_0$ is isomorphic to the minimal nilpotent orbit closure $\overline{\mathbb{O}}_{\min} \subset \mathfrak{sl}_{k+1}$. Finally, $\mathcal{O}(-k-1)_{\varpi_1}\widetilde{\boxtimes}\mathcal{O}_{\varpi_k}|_{m^{-1}\overline{\mathcal{W}}^{\varpi_1+\varpi_k}_{\varpi_{k+1}}}$ and $\mathcal{O}_{\varpi_k}\widetilde{\boxtimes}\mathcal{O}(-k-1)_{\varpi_1}|_{m'^{-1}\overline{\mathcal{W}}^{\varpi_1+\varpi_k}_{\varpi_{k+1}}}$ are isomorphic to the pull-backs of $\mathcal{O}_{\mathbb{P}^k}(-k-1)$ and $\mathcal{O}_{\check{\mathbb{P}}^k}(-k-1)$, respectively, but with *nontrivial* $\mathbb{C}^\times$-equivariant structures.

Let us explain our choice of the line bundles. According to [8, Proposition 8.2], the convolutions in question are $GL(k+1)\times\mathbb{C}^\times$-equivariant perverse coherent sheaves on $\overline{\mathbb{O}}_{\min} \subset \mathfrak{sl}_{k+1}$. Since $\dim H^k(T^*\mathbb{P}^k, \mathcal{O}_{T^*\mathbb{P}^k}(-k-1)) = 1$, while $H^k(T^*\mathbb{P}^k, \mathcal{O}_{T^*\mathbb{P}^k}(k+1)) = 0$, we have an exact sequence of perverse coherent sheaves[7] on $\overline{\mathbb{O}}_{\min} \subset \mathfrak{sl}_{k+1}$:

$$0 \to j_{!*}\mathcal{O}_{\mathbb{O}_{\min}}(-k-1)[k] \to m_*\mathcal{O}_{T^*\mathbb{P}^k}(-k-1)[k] \to \delta_0 \to 0,$$

where $j\colon \mathbb{O}_{\min} \hookrightarrow \overline{\mathbb{O}}_{\min}$ is the open embedding, and δ_0 is an irreducible skyscraper sheaf at $0\in\overline{\mathbb{O}}_{\min}$ with certain $\mathbb{C}^\times$-equivariant structure. The same exact sequence holds for $m'_*\mathcal{O}_{T^*\check{\mathbb{P}}^k}(-k-1)[k]$, but the quotient δ_0 has a *different* $\mathbb{C}^\times$-equivariant structure.

Proposition 9.8 *The restriction of* $\widetilde{\Phi}^0_{-2n}$ *to* $\mathfrak{Y}^0_{-2n,-}$ *is injective.*

Proof Consider an ordering $A^-_0 \prec A^-_{-1} \prec \ldots \prec A^-_{-n+1} \prec C^-_0 \prec \ldots \prec C^-_{-n+1}$. We set $(A^-_0)^{-k} := ((-\boldsymbol{v}^2)^{-n}A^-_{-n})^k$ for $k>0$. For

[7] We are grateful to R. Bezrukavnikov for his explanations about perverse coherent sheaves.

$\vec{r} = (r_1, \ldots, r_{2n}) \in \mathbb{Z} \times \mathbb{N}^{2n-1}$, we define the ordered monomial $m_{\vec{r}} := (A_0^-)^{r_1}(A_{-1}^-)^{r_2} \cdots (A_{-n+1}^-)^{r_n}(C_0^-)^{r_{n+1}} \cdots (C_{-n+1}^-)^{r_{2n}}$.

Lemma 9.9 *The ordered monomials* $\{m_{\vec{r}}\}$ *span* $\mathfrak{Y}^0_{-2n,-}$.

Proof According to relations (6.7, 6.9), we have $[A_t^-, A_s^-] = [C_t^-, C_s^-] = 0$ for $s, t \leq 0$. Due to Remark 8.8, we also have $C_s^- = 0$ for $s \leq -n$. It remains to prove that all A_t^- can be taken to the left of all C_s^-. This is implied by the fact that $C_s^- A_t^-$ can be written as a linear combination of normally ordered monomials $A_{t'}^- C_{s'}^-$. The latter claim follows from relation (6.11) by induction in $\min\{-t, -s\}$. The lemma is proved. □

The following result will be proved in Sect. 9.2:

Lemma 9.10

(a) The ordered monomials $\{m_{\vec{r}}\}$ *form a* $K_{\mathbb{C}^\times}(\mathrm{pt})$*-basis of* $\mathfrak{Y}^0_{-2n,-}$.
(b) $\{\overline{\Phi}{}^0_{-2n}(m_{\vec{r}})\}$ *form a* $K_{\mathbb{C}^\times}(\mathrm{pt})$*-basis of* $\overline{\Phi}{}^0_{-2n}(\mathfrak{Y}^0_{-2n,-})$.

The proposition is proved. □

9.2 Positive Grassmannian

Recall the positive part of the affine Grassmannian $\mathrm{Gr}^+_{GL(n)} \subset \mathrm{Gr}_{GL(n)}$ [10, § 3(ii)] parametrizing the *sublattices* in the standard one. Recall also that $K^{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}_{\mathrm{loc}}(\mathrm{Gr}^{\varpi_1}) = K^{GL(n,\mathcal{O})\rtimes\mathbb{C}^\times}_{\mathrm{loc}}(\mathbb{P}^{n-1})$ is generated over $K_{GL(n)}(\mathrm{pt})$ by the classes of $\mathcal{O}(a)_{\varpi_1}, -n+1 \leq a \leq 0$. The proof of Theorem 9.2 shows that $\overline{\Phi}{}^0_{-2n,\mathrm{loc}}\colon \mathfrak{U}^{\mathrm{ad}}_{0,-2n,\mathrm{loc}} \twoheadrightarrow K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}_{\mathrm{loc}}(\mathrm{Gr}_{GL(n)})$ restricts to a surjective homomorphism $\overline{\Phi}{}^0_{-2n,\mathrm{loc}}\colon \mathfrak{Y}^0_{-2n,-,\mathrm{loc}} \twoheadrightarrow K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}_{\mathrm{loc}}(\mathrm{Gr}^+_{GL(n)})$.

Proposition 9.11 $\overline{\Phi}{}^0_{-2n,\mathrm{loc}}\colon \mathfrak{Y}^0_{-2n,-,\mathrm{loc}} \xrightarrow{\sim} K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}_{\mathrm{loc}}(\mathrm{Gr}^+_{GL(n)})$.

Proof We have to check that $\overline{\Phi}{}^0_{-2n,\mathrm{loc}}\colon \mathfrak{Y}^0_{-2n,-,\mathrm{loc}} \to K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}_{\mathrm{loc}}(\mathrm{Gr}^+_{GL(n)})$ is injective. To this end, note that $\mathrm{Gr}^+_{GL(n)}$ is a union of connected components numbered by nonnegative integers: $\mathrm{Gr}^+_{GL(n)} = \bigsqcup_{r\in\mathbb{N}} \mathrm{Gr}^{+,r}_{GL(n)}$, where $\mathrm{Gr}^{+,r}_{GL(n)}$ parametrizes the sublattices of codimension r in the standard one. The direct sum decomposition $K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}_{\mathrm{loc}}(\mathrm{Gr}^+_{GL(n)}) = \bigoplus_{r\in\mathbb{N}} K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}_{\mathrm{loc}}(\mathrm{Gr}^{+,r}_{GL(n)})$ is a grading of the convolution algebra. For any connected component, $K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}_{\mathrm{loc}}(\mathrm{Gr}^{+,r}_{GL(n)})$ is a free $K_{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{pt})_{\mathrm{loc}}$-module of rank d_r, where d_r is the number of T-fixed points in $\mathrm{Gr}^{+,r}_{GL(n)}$, that is the number of weights of the irreducible $GL(n)$-module with the highest weight $(r, 0, \ldots, 0)$, isomorphic to $\mathrm{Sym}^r(\mathbb{C}^n)$. Note that all the weights of $\mathrm{Sym}^r(\mathbb{C}^n)$ have multiplicity one; in other words, $d_r = \dim \mathrm{Sym}^r(\mathbb{C}^n)$.

According to Lemma 9.9, we can introduce a grading $\mathfrak{Y}^0_{-2n,-,\mathrm{loc}} = \bigoplus_{r\in\mathbb{N}} \mathfrak{Y}^{0,r}_{-2n,-,\mathrm{loc}}$: a monomial $m_{\vec{r}}$ has degree r if $r_{n+1}+\ldots+r_{2n} = r$. It is immediate from the relations between $A^-_\bullet$, $C^-_\bullet$-generators that this grading is well-defined. Also, it is clear that $\overline{\Phi}^0_{-2n,\mathrm{loc}}(\mathfrak{Y}^{0,r}_{-2n,-,\mathrm{loc}}) \subset K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}_{\mathrm{loc}}(\mathrm{Gr}^{+,r}_{GL(n)})$. Meanwhile, we know from Theorem 9.2 that $\overline{\Phi}^0_{-2n,\mathrm{loc}}(\mathfrak{Y}^{0,r}_{-2n,-,\mathrm{loc}}) = K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}_{\mathrm{loc}}(\mathrm{Gr}^{+,r}_{GL(n)})$. On the other hand, we know from Lemma 9.9 that $\mathfrak{Y}^{0,r}_{-2n,-,\mathrm{loc}}$ as a left $K_{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{pt})_{\mathrm{loc}}$-module has no more than d'_r generators, where d'_r is the number of compositions of r into n (ordered) summands. Since $d_r = d'_r$, we conclude that $\overline{\Phi}^0_{-2n,\mathrm{loc}}\colon \mathfrak{Y}^{0,r}_{-2n,-,\mathrm{loc}} \to K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}_{\mathrm{loc}}(\mathrm{Gr}^{+,r}_{GL(n)})$ must be an isomorphism, and $\mathfrak{Y}^{0,r}_{-2n,-,\mathrm{loc}}$ is a free left $K_{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{pt})_{\mathrm{loc}}$-module of rank $d_r = d'_r$. This completes the proof of Proposition 9.11, Lemma 9.10 (and Proposition 9.8). □

Remark 9.12 One can check that the natural morphism

$$K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{Gr}^+_{GL(n)})[\mathcal{O}^{-1}_{\varpi_n}] \to K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{Gr}_{GL(n)})$$

is an isomorphism. Now it follows from the proof of Proposition 9.11 and Theorem 9.2 that in order to check Conjectures 8.7, 8.9 and 8.13 in our case: $\mathrm{Ker}(\widetilde{\Phi}^0_{-2n,\mathrm{loc}}) = \mathcal{I}^0_{-2n,\mathrm{loc}}$, it suffices to check the following equality in $\mathfrak{U}^{\mathrm{ad}}_{0,-2n}/\mathcal{I}^0_{-2n}$:

$$-\boldsymbol{v}^{n^2-2}(1-\boldsymbol{v}^2)^2\cdot(\mathrm{ad}^{\boldsymbol{v}^{2n}}_{f_{1-n}}\,\mathrm{ad}^{\boldsymbol{v}^{2(n-1)}}_{f_{3-n}}\cdots\mathrm{ad}^{\boldsymbol{v}^4}_{f_{n-3}}\,f_{n-1})(\mathrm{ad}^{\boldsymbol{v}^{-2n}}_{e_{1-2n}}\,\mathrm{ad}^{\boldsymbol{v}^{-2(n-1)}}_{e_{3-2n}}\cdots\mathrm{ad}^{\boldsymbol{v}^{-4}}_{e_{-3}}\,e_{-1}) = 1.$$

Remark 9.13 Consider a subalgebra $\mathfrak{U}^<_{0,-2n} \subset \mathfrak{U}^{\mathrm{ad}}_{0,-2n}$ generated by $\{(\boldsymbol{v}-\boldsymbol{v}^{-1})f_s\}_{s\in\mathbb{Z}}$. Note that it is independent of n, cf. Proposition 5.1. The image $\overline{\Phi}^0_{-2n}(\mathfrak{U}^<_{0,-2n})$ in $K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{Gr}_{GL(n)})$ is isomorphic to the *M-system algebra* $\mathcal{U}'_{n-1}$ of [18]. In particular, the generators $M_{m,s} \in \mathcal{U}'_{n-1}$ of [18, § 2.1] correspond to scalar multiples of the classes $\mathcal{O}(-s)_{\varpi_m} \in K^{\widetilde{GL}(n,\mathcal{O})\rtimes\widetilde{\mathbb{C}}^\times}(\mathrm{Gr}_{GL(n)})$, cf. (9.1) and [18, (2.23)].

10 Coproducts on Shifted Quantum Affine Algebras

Throughout this section, we work mainly with simply-connected shifted quantum affine algebras. However, all the results can be obviously generalized to the adjoint versions.

10.1 Drinfeld Formal Coproduct

The standard quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{g})$ admits the Drinfeld formal coproduct

$$\widetilde{\Delta}\colon U_{\boldsymbol{v}}(L\mathfrak{g}) \longrightarrow U_{\boldsymbol{v}}(L\mathfrak{g})\widehat{\otimes}U_{\boldsymbol{v}}(L\mathfrak{g}),$$

defined in the new Drinfeld realization of $U_{\boldsymbol{v}}(L\mathfrak{g})$ via

$$\begin{aligned}\widetilde{\Delta}(e_i(z)) &:= e_i(z)\otimes 1+\psi_i^-(z)\otimes e_i(z),\\ \widetilde{\Delta}(f_i(z)) &:= f_i(z)\otimes \psi_i^+(z)+1\otimes f_i(z),\\ \widetilde{\Delta}(\psi_i^\pm(z)) &:= \psi_i^\pm(z)\otimes\psi_i^\pm(z).\end{aligned} \tag{10.1}$$

Remark 10.1 Composing $\widetilde{\Delta}$ with the $\mathbb{C}^\times$-action on the first factor, D. Hernandez obtained a deformed coproduct $\Delta_\zeta\colon U_{\boldsymbol{v}}(L\mathfrak{g})\to U_{\boldsymbol{v}}(L\mathfrak{g})\otimes U_{\boldsymbol{v}}(L\mathfrak{g})((\zeta))$, where ζ is a formal variable, see [37, Section 6].

This can be obviously generalized to the shifted setting.

Lemma 10.2 *For any coweights $\mu_1^\pm,\mu_2^\pm\in\Lambda$, there is a $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism*

$$\widetilde{\Delta}\colon \mathcal{U}^{\mathrm{sc}}_{\mu_1^++\mu_2^+,\mu_1^-+\mu_2^-}\longrightarrow \mathcal{U}^{\mathrm{sc}}_{\mu_1^+,\mu_1^-}\widehat{\otimes}\,\mathcal{U}^{\mathrm{sc}}_{\mu_2^+,\mu_2^-},$$

defined via (10.1).

We call this homomorphism a formal coproduct for shifted quantum affine algebras. Given two representations V_1, V_2 of $\mathcal{U}^{\mathrm{sc}}_{\mu_1^+,\mu_1^-}, \mathcal{U}^{\mathrm{sc}}_{\mu_2^+,\mu_2^-}$, respectively, we will use $V_1\widetilde{\otimes}V_2$ to denote the representation of $\mathcal{U}^{\mathrm{sc}}_{\mu_1^++\mu_2^+,\mu_1^-+\mu_2^-}$ on the vector space $V_1\otimes V_2$ induced by $\widetilde{\Delta}$, whenever the action of the infinite sums representing $\widetilde{\Delta}(e_{i,r}), \widetilde{\Delta}(f_{i,r})$ are well-defined. We will discuss a particular example of this construction in Sect. 12.6.

10.2 Drinfeld-Jimbo Coproduct

The standard quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{g})$ also admits the Drinfeld-Jimbo coproduct

$$\Delta\colon U_{\boldsymbol{v}}(L\mathfrak{g})\longrightarrow U_{\boldsymbol{v}}(L\mathfrak{g})\otimes U_{\boldsymbol{v}}(L\mathfrak{g}),$$

defined in the Drinfeld-Jimbo realization of $U_{\boldsymbol{v}}(L\mathfrak{g})$ via

$$\Delta\colon E_i\mapsto E_i\otimes K_i+1\otimes E_i,\ F_i\mapsto F_i\otimes 1+K_i^{-1}\otimes F_i,\ K_i^{\pm1}\mapsto K_i^{\pm1}\otimes K_i^{\pm1},\ i\in\widetilde{I}.$$

Recall that $\widetilde{I}=I\cup\{i_0\}$ is the vertex set of the extended Dynkin diagram and $\{E_i,F_i,K_i^{\pm1}\}_{i\in\widetilde{I}}$ are the standard Drinfeld-Jimbo generators of $U_{\boldsymbol{v}}^{\mathrm{DJ}}(L\mathfrak{g})\simeq U_{\boldsymbol{v}}(L\mathfrak{g})$.

We also denote the Drinfeld-Jimbo coproduct on $U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{g})$ by Δ^{ad}: the natural inclusion $U_{\boldsymbol{v}}(L\mathfrak{g}) \hookrightarrow U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{g})$ intertwines Δ and Δ^{ad}, while $\Delta^{\mathrm{ad}}(\phi_i^\pm) = \phi_i^\pm \otimes \phi_i^\pm$.

The goal of this section is to generalize these coproducts to the shifted setting. In other words, given $\mathfrak{g}$ and coweights $\mu_1, \mu_2 \in \Lambda$, we would like to construct homomorphisms

$$\Delta_{\mu_1,\mu_2} \colon \mathcal{U}^{\mathrm{sc}}_{0,\mu_1+\mu_2} \longrightarrow \mathcal{U}^{\mathrm{sc}}_{0,\mu_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,\mu_2},$$

which coincide with Δ in the particular case $\mu_1 = \mu_2 = 0$. We provide such a construction for the simplest case $\mathfrak{g} = \mathfrak{sl}_2$ in Sects. 10.3 ($\mu_1, \mu_2 \in \Lambda^-$) and 10.4 (general μ_1, μ_2). Using the RTT presentation of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, we generalize this to obtain Δ_{μ_1,μ_2} for $\mathfrak{g} = \mathfrak{sl}_n$ in Sects. 10.6 ($\mu_1, \mu_2 \in \Lambda^-$) and 10.7 (general μ_1, μ_2).

Remark 10.3

(a) This result is nontrivial due to an absence of the Drinfeld-Jimbo type presentation of shifted quantum affine algebras.
(b) A similar coproduct for the shifted Yangians has been constructed in [24] for arbitrary simply-laced $\mathfrak{g}$.
(c) Once Δ_{μ_1,μ_2} is constructed, one should be able to immediately extend it to the homomorphism $\Delta^{\mathrm{ad}}_{\mu_1,\mu_2} \colon \mathcal{U}^{\mathrm{ad}}_{0,\mu_1+\mu_2} \to \mathcal{U}^{\mathrm{ad}}_{0,\mu_1} \otimes \mathcal{U}^{\mathrm{ad}}_{0,\mu_2}$ by setting $\Delta^{\mathrm{ad}}_{\mu_1,\mu_2}(\phi_i^\pm) = \phi_i^\pm \otimes \phi_i^\pm$.

10.3 Homomorphisms Δ_{b_1,b_2} for $b_1, b_2 \in \mathbb{Z}_{\leq 0}$, $\mathfrak{g} = \mathfrak{sl}_2$

We start this subsection by explicitly computing the Drinfeld-Jimbo coproduct of the Drinfeld generators $e_0, e_{-1}, f_0, f_1, \psi_0^\pm$ of $U_{\boldsymbol{v}}(L\mathfrak{sl}_2)$ and $h_{\pm 1} = \pm \frac{\psi_0^\mp \psi_{\pm 1}^\pm}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$, which generate the quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{sl}_2)$.

Lemma 10.4 *We have*

$$\Delta(e_0) = e_0 \otimes \psi_0^+ + 1 \otimes e_0,\ \ \Delta(e_{-1}) = e_{-1} \otimes \psi_0^- + 1 \otimes e_{-1},$$

$$\Delta(f_0) = f_0 \otimes 1 + \psi_0^- \otimes f_0,\ \ \Delta(f_1) = f_1 \otimes 1 + \psi_0^+ \otimes f_1,\ \ \Delta(\psi_0^\pm) = \psi_0^\pm \otimes \psi_0^\pm,$$

$$\Delta(h_1) = h_1 \otimes 1 + 1 \otimes h_1 - (\boldsymbol{v}^2 - \boldsymbol{v}^{-2}) e_0 \otimes f_1,\ \ \Delta(h_{-1}) = h_{-1} \otimes 1 + 1 \otimes h_{-1} + (\boldsymbol{v}^2 - \boldsymbol{v}^{-2}) e_{-1} \otimes f_0.$$

Proof This is a straightforward computation based on the explicit identification between the Drinfeld-Jimbo and the new Drinfeld realizations of the quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{sl}_2)$ of Theorem 8.10: $e_0 = E_{i_1}, f_0 = F_{i_1}, \psi_0^\pm = K_{i_1}^{\pm 1}, e_{-1} = K_{i_1}^{-1} F_{i_0}, f_1 = E_{i_0} K_{i_1}$. □

The key result of this subsection provides analogues of Δ for antidominantly shifted quantum affine algebras of $\mathfrak{sl}_2$. For $\mu_1, \mu_2 \in \Lambda^-$, we construct homomorphisms $\Delta_{b_1,b_2}\colon \mathcal{U}^{\mathrm{sc}}_{0,b_1+b_2} \to \mathcal{U}^{\mathrm{sc}}_{0,b_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2}$, where $b_1 := \alpha^\vee(\mu_1)$, $b_2 := \alpha^\vee(\mu_2)$ (so that $b_1, b_2 \in \mathbb{Z}_{\leq 0}$).

Theorem 10.5 *For any $b_1, b_2 \in \mathbb{Z}_{\leq 0}$, there is a unique $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism*

$$\Delta_{b_1,b_2}\colon \mathcal{U}^{\mathrm{sc}}_{0,b_1+b_2} \longrightarrow \mathcal{U}^{\mathrm{sc}}_{0,b_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2}$$

(we will denote $\Delta = \Delta_{b_1,b_2}$ when the algebras involved are clear), such that

$$\Delta(e_r) = 1 \otimes e_r,\ \Delta(f_s) = f_s \otimes 1 \text{ for } b_2 \leq r < 0, b_1 < s \leq 0,$$

$$\Delta(e_0) = e_0 \otimes \psi_0^+ + 1 \otimes e_0,\ \Delta(e_{b_2-1}) = e_{-1} \otimes \psi_{b_2}^- + 1 \otimes e_{b_2-1},$$

$$\Delta(f_1) = f_1 \otimes 1 + \psi_0^+ \otimes f_1,\ \Delta(f_{b_1}) = f_{b_1} \otimes 1 + \psi_{b_1}^- \otimes f_0,$$

$$\Delta((\psi_0^+)^{\pm 1}) = (\psi_0^+)^{\pm 1} \otimes (\psi_0^+)^{\pm 1},\ \Delta((\psi_{b_1+b_2}^-)^{\pm 1}) = (\psi_{b_1}^-)^{\pm 1} \otimes (\psi_{b_2}^-)^{\pm 1},$$

$$\Delta(h_1) = h_1 \otimes 1 + 1 \otimes h_1 - (\boldsymbol{v}^2 - \boldsymbol{v}^{-2}) e_0 \otimes f_1,\ \Delta(h_{-1}) = h_{-1} \otimes 1 + 1 \otimes h_{-1} + (\boldsymbol{v}^2 - \boldsymbol{v}^{-2}) e_{-1} \otimes f_0.$$

These homomorphisms generalize the Drinfeld-Jimbo coproduct, since we recover the formulas of Lemma 10.4 for $b_1 = b_2 = 0$. The proof of Theorem 10.5 is presented in Appendix D and is crucially based on Theorem 5.5 which provides a presentation of the shifted quantum affine algebras via a finite number of generators and relations.

Remark 10.6 The similarity between the formulas for Δ_{b_1,b_2} of Theorem 10.5 and Drinfeld-Jimbo coproduct Δ of Lemma 10.4 can be explained as follows. Let $U_{\boldsymbol{v}}^-$ (resp. $\mathcal{U}^{\mathrm{sc},-}_{0,b_1,b_2}$) be the subalgebra of $U_{\boldsymbol{v}}(L\mathfrak{sl}_2)$ (resp. $\mathcal{U}^{\mathrm{sc}}_{0,b_1+b_2}$) generated by $\{e_{-1}, f_0, (\psi_0^-)^{\pm 1}\}$, or equivalently, by $\{e_{-r-1}, f_{-r}, (\psi_0^-)^{\pm 1}, \psi_{-r-1}^-\}_{r \in \mathbb{N}}$ (resp. by $\{e_{b_2-1}, f_{b_1}, (\psi_{b_1+b_2}^-)^{\pm 1}\}$, or equivalently, by $\{e_{b_2-r-1}, f_{b_1-r}, (\psi_{b_1+b_2}^-)^{\pm 1}, \psi_{b_1+b_2-r-1}^-\}_{r \in \mathbb{N}}$). Analogously, let $U_{\boldsymbol{v}}^+$ (resp. $\mathcal{U}^{\mathrm{sc},+}_{0,b_1,b_2}$) be the subalgebra of $U_{\boldsymbol{v}}(L\mathfrak{sl}_2)$ (resp. $\mathcal{U}^{\mathrm{sc}}_{0,b_1+b_2}$) generated by $\{e_0, f_1, (\psi_0^+)^{\pm 1}\}$ in both cases, or equivalently, by $\{e_r, f_{r+1}, (\psi_0^+)^{\pm 1}, \psi_{r+1}^+\}_{r \in \mathbb{N}}$. Then, there are unique $\mathbb{C}(\boldsymbol{v})$-algebra homomorphisms $\jmath^{\pm}_{b_1,b_2}\colon U_{\boldsymbol{v}}^{\pm} \to \mathcal{U}^{\mathrm{sc},\pm}_{0,b_1,b_2}$, such that

$$\jmath^+_{b_1,b_2}\colon e_0 \mapsto e_0,\ f_1 \mapsto f_1,\ (\psi_0^+)^{\pm 1} \mapsto (\psi_0^+)^{\pm 1},$$

$$\jmath^-_{b_1,b_2}\colon e_{-1} \mapsto e_{b_2-1},\ f_0 \mapsto f_{b_1},\ (\psi_0^-)^{\pm 1} \mapsto (\psi_{b_1+b_2}^-)^{\pm 1}.$$

Moreover, the following diagram is commutative:

$$\begin{array}{ccc} U_{\boldsymbol{v}}^{\pm} & \xrightarrow{\quad\Delta\quad} & U_{\boldsymbol{v}}^{\pm} \otimes U_{\boldsymbol{v}}^{\pm} \\ \Big\downarrow {\scriptstyle \jmath^{\pm}_{b_1,b_2}} & & \Big\downarrow {\scriptstyle \jmath^{\pm}_{b_1,0} \otimes \jmath^{\pm}_{0,b_2}} \\ \mathcal{U}^{\mathrm{sc},\pm}_{0,b_1,b_2} & \xrightarrow[\quad\Delta_{b_1,b_2}\quad]{} & \mathcal{U}^{\mathrm{sc},\pm}_{0,b_1,0} \otimes \mathcal{U}^{\mathrm{sc},\pm}_{0,0,b_2} \end{array}$$

Remark 10.7 The aforementioned homomorphism Δ_{b_1,b_2} can be naturally extended to the homomorphism $\Delta^{\mathrm{ad}}_{b_1,b_2} \colon \mathcal{U}^{\mathrm{ad}}_{0,b_1+b_2} \to \mathcal{U}^{\mathrm{ad}}_{0,b_1} \otimes \mathcal{U}^{\mathrm{ad}}_{0,b_2}$ by setting $\Delta^{\mathrm{ad}}_{b_1,b_2}(\phi^{\pm}) = \phi^{\pm} \otimes \phi^{\pm}$.

10.4 Homomorphisms Δ_{b_1,b_2} for Arbitrary $b_1, b_2 \in \mathbb{Z}$, $\mathfrak{g} = \mathfrak{sl}_2$

In this subsection, we generalize the construction of Δ_{b_1,b_2} of Theorem 10.5 ($b_1, b_2 \in \mathbb{Z}_{\leq 0}$) to the general case $b_1, b_2 \in \mathbb{Z}$. We follow the corresponding construction for the shifted Yangians of [24, Theorem 4.12].

The key ingredient of our approach are the *shift homomorphisms* ι_{n,m_1,m_2} (the trigonometric analogues of the shift homomorphisms of [24]).

Proposition 10.8 *For any $n \in \mathbb{Z}$ and $m_1, m_2 \in \mathbb{Z}_{\leq 0}$, there is a unique $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism $\iota_{n,m_1,m_2} \colon \mathcal{U}^{\mathrm{sc}}_{0,n} \to \mathcal{U}^{\mathrm{sc}}_{0,n+m_1+m_2}$, which maps the currents as follows:*

$$e(z) \mapsto (1-z^{-1})^{-m_1} e(z),\ \ f(z) \mapsto (1-z^{-1})^{-m_2} f(z),\ \ \psi^{\pm}(z) \mapsto (1-z^{-1})^{-m_1-m_2}\psi^{\pm}(z).$$

Proof The above assignment is obviously compatible with defining relations (U1–U8). Moreover, we have $\iota_{n,m_1,m_2} \colon \psi_0^+ \mapsto \psi_0^+,\ \psi_n^- \mapsto (-1)^{m_1+m_2}\psi^-_{n+m_1+m_2}$. □

These homomorphisms satisfy two important properties:

Lemma 10.9

(a) We have $\iota_{n+m_1+m_2,m_1',m_2'} \circ \iota_{n,m_1,m_2} = \iota_{n,m_1+m_1',m_2+m_2'}$ for any $n \in \mathbb{Z}$ and $m_1, m_2, m_1', m_2' \in \mathbb{Z}_{\leq 0}$.
(b) The homomorphism ι_{n,m_1,m_2} is injective for any $n \in \mathbb{Z}$ and $m_1, m_2 \in \mathbb{Z}_{\leq 0}$.

Part (a) is obvious, while part (b) is proved in Appendix E and follows from the PBW property for $\mathcal{U}^{\mathrm{sc}}_{0,n}$ (cf. Theorem 10.19). The following is the key result of this subsection.

Theorem 10.10 *For any $b_1, b_2 \in \mathbb{Z}$ and $b := b_1 + b_2$, there is a unique $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism*

$$\Delta_{b_1,b_2} \colon \mathcal{U}^{\mathrm{sc}}_{0,b} \longrightarrow \mathcal{U}^{\mathrm{sc}}_{0,b_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2},$$

such that for any $m_1, m_2 \in \mathbb{Z}_{\leq 0}$ *the following diagram is commutative:*

$$\begin{CD}
\mathcal{U}^{\mathrm{sc}}_{0,b} @>\Delta_{b_1,b_2}>> \mathcal{U}^{\mathrm{sc}}_{0,b_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2} \\
@V{\iota_{b,m_2,m_1}}VV @VV{\iota_{b_1,0,m_1} \otimes \iota_{b_2,m_2,0}}V \\
\mathcal{U}^{\mathrm{sc}}_{0,b+m_1+m_2} @>>\Delta_{b_1+m_1,b_2+m_2}> \mathcal{U}^{\mathrm{sc}}_{0,b_1+m_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2+m_2}
\end{CD}$$

The proof of this theorem is presented in Appendix F and is similar to the proof of [24, Theorem 4.12].

Corollary 10.11 *For any* $b_1, b_2 \in \mathbb{Z}$, *we have*

$$\Delta_{b_1,b_2}(h_1) = h_1 \otimes 1 + 1 \otimes h_1 - (\boldsymbol{v}^2 - \boldsymbol{v}^{-2})e_0 \otimes f_1,$$

$$\Delta_{b_1,b_2}(h_{-1}) = h_{-1} \otimes 1 + 1 \otimes h_{-1} + (\boldsymbol{v}^2 - \boldsymbol{v}^{-2})e_{-1} \otimes f_0.$$

Proof In the antidominant case $b_1, b_2 \in \mathbb{Z}_{\leq 0}$, both equalities are due to our definition of Δ_{b_1,b_2} of Theorem 10.5. For general b_1, b_2, choose $m_1, m_2 \in \mathbb{Z}_{\leq 0}$ such that $b_1 + m_1, b_2 + m_2 \in \mathbb{Z}_{\leq 0}$. By the definition of ι_{b,m_2,m_1}, we have $\iota_{b,m_2,m_1}(h_{\pm 1}) = h_{\pm 1} \pm \frac{m_1+m_2}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$. Meanwhile, we also have

$$\iota_{b_1,0,m_1} \otimes \iota_{b_2,m_2,0}(h_{\pm 1} \otimes 1 + 1 \otimes h_{\pm 1}) = h_{\pm 1} \otimes 1 + 1 \otimes h_{\pm 1} \pm \frac{m_1 + m_2}{\boldsymbol{v} - \boldsymbol{v}^{-1}},$$

while $\iota_{b_1,0,m_1}(e_r) = e_r, \ \iota_{b_2,m_2,0}(f_s) = f_s$ for any $r, s \in \mathbb{Z}$. The result follows by combining the formula for $\Delta_{b_1+m_1,b_2+m_2}(h_{\pm 1})$ with the commutativity of the diagram of Theorem 10.10 (we also use injectivity of the vertical arrows, due to Lemma 10.9(b)). □

The following result is analogous to [24, Proposition 4.14] and we leave its proof to the interested reader.

Lemma 10.12 *For* $b = b_1 + b_2 + b_3$ *with* $b_1, b_3 \in \mathbb{Z}, b_2 \in \mathbb{Z}_{\leq 0}$, *the following diagram is commutative:*

$$\begin{CD}
\mathcal{U}^{\mathrm{sc}}_{0,b} @>\Delta_{b_1,b_2+b_3}>> \mathcal{U}^{\mathrm{sc}}_{0,b_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2+b_3} \\
@V{\Delta_{b_1+b_2,b_3}}VV @VV{\mathrm{Id} \otimes \Delta_{b_2,b_3}}V \\
\mathcal{U}^{\mathrm{sc}}_{0,b_1+b_2} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_3} @>>\Delta_{b_1,b_2} \otimes \mathrm{Id}> \mathcal{U}^{\mathrm{sc}}_{0,b_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_3}
\end{CD}$$

10.5 Drinfeld-Jimbo Coproduct on $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ via Drinfeld Generators

According to Theorem 5.5, the quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is generated by the elements $\{e_{i,0}, f_{i,0}, e_{i,-1}, f_{i,1}, \psi^{\pm}_{i,0}, h_{i,\pm 1}\}_{i=1}^{n-1}$. The key result of this subsection provides explicit formulas for the action of the Drinfeld-Jimbo coproduct Δ on these generators of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. Since $e_{i,0} = E_i$, $f_{i,0} = F_i$, $\psi^{\pm}_{i,0} = K_i^{\pm 1}$ (for $i \in I = \{1, 2, \cdots, n-1\}$), we obviously have

$$\Delta(e_{i,0}) = 1 \otimes e_{i,0} + e_{i,0} \otimes \psi^{+}_{i,0},\ \Delta(f_{i,0}) = f_{i,0} \otimes 1 + \psi^{-}_{i,0} \otimes f_{i,0},\ \Delta(\psi^{\pm}_{i,0}) = \psi^{\pm}_{i,0} \otimes \psi^{\pm}_{i,0}.$$

It remains to compute the coproduct of the remaining generators above.

Theorem 10.13 *Let Δ be the Drinfeld-Jimbo coproduct on $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. Then, we have*

$$\begin{aligned}
&\Delta(h_{i,1}) = \\
&h_{i,1} \otimes 1 + 1 \otimes h_{i,1} - (\boldsymbol{v}^2 - \boldsymbol{v}^{-2}) E^{(0)}_{i,i+1} \otimes F^{(1)}_{i+1,i} + (\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{l>i+1} E^{(0)}_{i+1,l} \otimes F^{(1)}_{l,i+1} + \\
&(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{k+1-i} \widetilde{E}^{(0)}_{ki} \otimes F^{(1)}_{ik} + \boldsymbol{v}^{-2} (\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{l>i+1} [E^{(0)}_{i,i+1}, E^{(0)}_{i+1,l}]_{\boldsymbol{v}^3} \otimes F^{(1)}_{li} - \\
&(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{k-i-1} [E^{(0)}_{i,i+1}, \widetilde{E}^{(0)}_{ki}]_{\boldsymbol{v}^3} \otimes F^{(1)}_{i+1,k} + \\
&(\boldsymbol{v} - \boldsymbol{v}^{-1})^2 \sum_{l>i+1}^{k<i} \boldsymbol{v}^{k-i} \left(E^{(0)}_{il} \widetilde{E}^{(0)}_{ki} - E^{(0)}_{i+1,l} \widetilde{E}^{(0)}_{k,i+1} \right) \otimes F^{(1)}_{lk},
\end{aligned} \tag{10.2}$$

$$\begin{aligned}
&\Delta(h_{i,-1}) = \\
&h_{i,-1} \otimes 1 + 1 \otimes h_{i,-1} + (\boldsymbol{v}^2 - \boldsymbol{v}^{-2}) E^{(-1)}_{i,i+1} \otimes F^{(0)}_{i+1,i} - (\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{l>i+1} E^{(-1)}_{i+1,l} \otimes F^{(0)}_{l,i+1} - \\
&(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{i-k-1} E^{(-1)}_{ki} \otimes \widetilde{F}^{(0)}_{ik} - \boldsymbol{v}^2 (\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{l>i+1} E^{(-1)}_{il} \otimes [F^{(0)}_{l,i+1}, F^{(0)}_{i+1,i}]_{\boldsymbol{v}^{-3}} + \\
&(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{i+1-k} E^{(-1)}_{k,i+1} \otimes [\widetilde{F}^{(0)}_{ik}, F^{(0)}_{i+1,i}]_{\boldsymbol{v}^{-3}} - \\
&(\boldsymbol{v} - \boldsymbol{v}^{-1})^2 \sum_{l>i+1}^{k<i} \boldsymbol{v}^{i-k} E^{(-1)}_{kl} \otimes \left(\widetilde{F}^{(0)}_{i+1,k} F^{(0)}_{l,i+1} - \widetilde{F}^{(0)}_{ik} F^{(0)}_{li} \right),
\end{aligned} \tag{10.3}$$

$$\Delta(e_{i,-1}) = 1 \otimes e_{i,-1} + e_{i,-1} \otimes \psi^-_{i,0} - (\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{l>i+1} E^{(-1)}_{il} \otimes F^{(0)}_{l,i+1}\psi^-_{i,0} +$$

$$(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{i-k-1} E^{(-1)}_{k,i+1} \otimes \widetilde{F}^{(0)}_{ik}\psi^-_{i,0} - (\boldsymbol{v} - \boldsymbol{v}^{-1})^2 \sum_{l>i+1}^{k<i} \boldsymbol{v}^{i-k-1} E^{(-1)}_{kl} \otimes \widetilde{F}^{(0)}_{ik} F^{(0)}_{l,i+1}\psi^-_{i,0}, \tag{10.4}$$

$$\Delta(f_{i,1}) = f_{i,1} \otimes 1 + \psi^+_{i,0} \otimes f_{i,1} + \boldsymbol{v}^{-1}(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{l>i+1} E^{(0)}_{i+1,l}\psi^+_{i,0} \otimes F^{(1)}_{li} -$$

$$(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{k-i} \widetilde{E}^{(0)}_{ki}\psi^+_{i,0} \otimes F^{(1)}_{i+1,k} - (\boldsymbol{v} - \boldsymbol{v}^{-1})^2 \sum_{l>i+1}^{k<i} \boldsymbol{v}^{k-i-1} E^{(0)}_{i+1,l} \widetilde{E}^{(0)}_{ki}\psi^+_{i,0} \otimes F^{(1)}_{lk}, \tag{10.5}$$

where for $1 \leq j < i \leq n$ *we set*

$$\begin{aligned}
E^{(0)}_{ji} &:= [e_{i-1,0}, \cdots, [e_{j+1,0}, e_{j,0}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}} = [\cdots[e_{i-1,0}, e_{i-2,0}]_{\boldsymbol{v}^{-1}}, \cdots, e_{j,0}]_{\boldsymbol{v}^{-1}},\\
F^{(0)}_{ij} &:= [f_{j,0}, \cdots, [f_{i-2,0}, f_{i-1,0}]_{\boldsymbol{v}} \cdots]_{\boldsymbol{v}} = [\cdots[f_{j,0}, f_{j+1,0}]_{\boldsymbol{v}}, \cdots, f_{i-1,0}]_{\boldsymbol{v}},\\
E^{(-1)}_{ji} &:= [e_{i-1,0}, \cdots, [e_{j+1,0}, e_{j,-1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}}\\
&= [[\cdots[e_{i-1,0}, e_{i-2,0}]_{\boldsymbol{v}^{-1}}, \cdots, e_{j+1,0}]_{\boldsymbol{v}^{-1}}, e_{j,-1}]_{\boldsymbol{v}^{-1}},\\
F^{(1)}_{ij} &:= [f_{j,1}, [f_{j+1,0}, \cdots, [f_{i-2,0}, f_{i-1,0}]_{\boldsymbol{v}} \cdots]_{\boldsymbol{v}}]_{\boldsymbol{v}} = [\cdots[f_{j,1}, f_{j+1,0}]_{\boldsymbol{v}}, \cdots, f_{i-1,0}]_{\boldsymbol{v}},\\
\widetilde{E}^{(0)}_{ji} &:= [e_{i-1,0}, \cdots, [e_{j+1,0}, e_{j,0}]_{\boldsymbol{v}} \cdots]_{\boldsymbol{v}} = [\cdots[e_{i-1,0}, e_{i-2,0}]_{\boldsymbol{v}}, \cdots, e_{j,0}]_{\boldsymbol{v}},\\
\widetilde{F}^{(0)}_{ij} &:= [f_{j,0}, \cdots, [f_{i-2,0}, f_{i-1,0}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}} = [\cdots[f_{j,0}, f_{j+1,0}]_{\boldsymbol{v}^{-1}}, \cdots, f_{i-1,0}]_{\boldsymbol{v}^{-1}}.
\end{aligned} \tag{10.6}$$

The proof of this result is based on the RTT realization of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and is presented in Appendix G.

Remark 10.14 The right equalities in each of the lines of (10.6) are not obvious and are established during our proof of Theorem 10.13. They play an important role in the proof of Theorem 10.16 below.

Let $U^>_{\boldsymbol{v}}(L\mathfrak{g})$ and $U^{\geq}_{\boldsymbol{v}}(L\mathfrak{g})$ (resp. $U^<_{\boldsymbol{v}}(L\mathfrak{g})$ and $U^{\leq}_{\boldsymbol{v}}(L\mathfrak{g})$) be the $\mathbb{C}(\boldsymbol{v})$-subalgebras of $U_{\boldsymbol{v}}(L\mathfrak{g})$ generated by $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ and $\{e_{i,r}, \psi^{\pm}_{i,\pm s}\}_{i\in I}^{r\in\mathbb{Z},s\in\mathbb{N}}$ (resp. $\{f_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ and $\{f_{i,r}, \psi^{\pm}_{i,\pm s}\}_{i\in I}^{r\in\mathbb{Z},s\in\mathbb{N}}$).

Corollary 10.15 *For any* $1 \leq i < n$ *and* $r \in \mathbb{Z}$*, we have*

$$\Delta(h_{i,\pm1}) - h_{i,\pm1} \otimes 1 - 1 \otimes h_{i,\pm1} \in U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \otimes U^<_{\boldsymbol{v}}(L\mathfrak{sl}_n),$$

$$\Delta(e_{i,r}) - 1 \otimes e_{i,r} \in U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \otimes U^{\leq}_{\boldsymbol{v}}(L\mathfrak{sl}_n),$$

$$\Delta(f_{i,r}) - f_{i,r} \otimes 1 \in U^{\geq}_{\boldsymbol{v}}(L\mathfrak{sl}_n) \otimes U^<_{\boldsymbol{v}}(L\mathfrak{sl}_n).$$

Proof The claim is clear for $\Delta(h_{i,\pm 1}), \Delta(e_{i,-1}), \Delta(f_{i,1})$, due to (10.2–10.5). Applying iteratively $[\Delta(h_{i,\pm 1}), \Delta(e_{i,r})] = [2]_{\boldsymbol{v}} \cdot \Delta(e_{i,r\pm 1}), [\Delta(h_{i,\pm 1}), \Delta(f_{i,r})] = -[2]_{\boldsymbol{v}} \cdot \Delta(f_{i,r\pm 1})$, we deduce the claim for $\Delta(e_{i,r})$ and $\Delta(f_{i,r})$. □

10.6 Homomorphisms Δ_{μ_1,μ_2} for $\mu_1, \mu_2 \in \Lambda^-$, $\mathfrak{g} = \mathfrak{sl}_n$

In this subsection, we construct homomorphisms $\Delta_{\mu_1,\mu_2} \colon \mathcal{U}^{\mathrm{sc}}_{0,\mu_1+\mu_2} \to \mathcal{U}^{\mathrm{sc}}_{0,\mu_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$ for $\mu_1, \mu_2 \in \Lambda^-$, which coincide with the Drinfeld-Jimbo coproduct on $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ for $\mu_1 = \mu_2 = 0$. Set $b_{1,i} := \alpha_i^\vee(\mu_1)$ and $b_{2,i} := \alpha_i^\vee(\mu_2)$ (so that $b_{1,i}, b_{2,i} \in \mathbb{Z}_{\leq 0}$).

Theorem 10.16 *For any $\mu_1, \mu_2 \in \Lambda^-$, there is a unique $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism*

$$\Delta_{\mu_1,\mu_2} \colon \mathcal{U}^{\mathrm{sc}}_{0,\mu_1+\mu_2} \longrightarrow \mathcal{U}^{\mathrm{sc}}_{0,\mu_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$$

(we will denote $\Delta = \Delta_{\mu_1,\mu_2}$ when the algebras involved are clear), such that

$$\Delta(e_{i,r}) = 1 \otimes e_{i,r},\ \Delta(f_{i,s}) = f_{i,s} \otimes 1 \text{ for } b_{2,i} \leq r < 0, b_{1,i} < s \leq 0,$$

$$\Delta(e_{i,0}) = 1 \otimes e_{i,0} + e_{i,0} \otimes \psi^+_{i,0},\ \Delta(f_{i,b_{1,i}}) = f_{i,b_{1,i}} \otimes 1 + \psi^-_{i,b_{1,i}} \otimes f_{i,0},$$

$$\Delta(e_{i,b_{2,i}-1}) = 1 \otimes e_{i,b_{2,i}-1} + e_{i,-1} \otimes \psi^-_{i,b_{2,i}} - (\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{l>i+1} E^{(-1)}_{il} \otimes F^{(0)}_{l,i+1} \psi^-_{i,b_{2,i}} +$$

$$(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{i-k-1} E^{(-1)}_{k,i+1} \otimes \widetilde{F}^{(0)}_{ik} \psi^-_{i,b_{2,i}} - (\boldsymbol{v} - \boldsymbol{v}^{-1})^2 \sum_{l>i+1}^{k<i} \boldsymbol{v}^{i-k-1} E^{(-1)}_{kl} \otimes \widetilde{F}^{(0)}_{ik} F^{(0)}_{l,i+1} \psi^-_{i,b_{2,i}},$$

$$\Delta(f_{i,1}) = f_{i,1} \otimes 1 + \psi^+_{i,0} \otimes f_{i,1} + \boldsymbol{v}^{-1}(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{l>i+1} E^{(0)}_{i+1,l} \psi^+_{i,0} \otimes F^{(1)}_{li} -$$

$$(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{k-i} \widetilde{E}^{(0)}_{ki} \psi^+_{i,0} \otimes F^{(1)}_{i+1,k} - (\boldsymbol{v} - \boldsymbol{v}^{-1})^2 \sum_{l>i+1}^{k<i} \boldsymbol{v}^{k-i-1} E^{(0)}_{i+1,l} \widetilde{E}^{(0)}_{ki} \psi^+_{i,0} \otimes F^{(1)}_{lk},$$

$$\Delta((\psi^+_{i,0})^{\pm 1}) = (\psi^+_{i,0})^{\pm 1} \otimes (\psi^+_{i,0})^{\pm 1},\ \Delta((\psi^-_{i,b_{1,i}+b_{2,i}})^{\pm 1}) = (\psi^-_{i,b_{1,i}})^{\pm 1} \otimes (\psi^-_{i,b_{2,i}})^{\pm 1},$$

$$\Delta(h_{i,1}) = h_{i,1}\otimes 1 + 1\otimes h_{i,1} - (\boldsymbol{v}^2-\boldsymbol{v}^{-2})E^{(0)}_{i,i+1}\otimes F^{(1)}_{i+1,i} + (\boldsymbol{v}-\boldsymbol{v}^{-1})\sum_{l>i+1} E^{(0)}_{i+1,l}\otimes F^{(1)}_{l,i+1} +$$

$$(\boldsymbol{v}-\boldsymbol{v}^{-1})\sum_{k<i}\boldsymbol{v}^{k+1-i}\widetilde{E}^{(0)}_{ki}\otimes F^{(1)}_{ik} + \boldsymbol{v}^{-2}(\boldsymbol{v}-\boldsymbol{v}^{-1})\sum_{l>i+1}[E^{(0)}_{i,i+1}, E^{(0)}_{i+1,l}]_{\boldsymbol{v}^3}\otimes F^{(1)}_{li} -$$

$$(\boldsymbol{v}-\boldsymbol{v}^{-1})\sum_{k<i}\boldsymbol{v}^{k-i-1}[E^{(0)}_{i,i+1}, \widetilde{E}^{(0)}_{ki}]_{\boldsymbol{v}^3}\otimes F^{(1)}_{i+1,k} +$$

$$(\boldsymbol{v}-\boldsymbol{v}^{-1})^2\sum_{l>i+1}^{k<i}\boldsymbol{v}^{k-i}\left(E^{(0)}_{il}\widetilde{E}^{(0)}_{ki} - E^{(0)}_{i+1,l}\widetilde{E}^{(0)}_{k,i+1}\right)\otimes F^{(1)}_{lk},$$

$$\Delta(h_{i,-1}) = h_{i,-1}\otimes 1 + 1\otimes h_{i,-1} + (\boldsymbol{v}^2-\boldsymbol{v}^{-2})E^{(-1)}_{i,i+1}\otimes F^{(0)}_{i+1,i} -$$

$$(\boldsymbol{v}-\boldsymbol{v}^{-1})\sum_{l>i+1}E^{(-1)}_{i+1,l}\otimes F^{(0)}_{l,i+1} -$$

$$(\boldsymbol{v}-\boldsymbol{v}^{-1})\sum_{k<i}\boldsymbol{v}^{i-k-1}E^{(-1)}_{ki}\otimes\widetilde{F}^{(0)}_{ik} - \boldsymbol{v}^2(\boldsymbol{v}-\boldsymbol{v}^{-1})\sum_{l>i+1}E^{(-1)}_{il}\otimes[F^{(0)}_{l,i+1}, F^{(0)}_{i+1,i}]_{\boldsymbol{v}^{-3}} +$$

$$(\boldsymbol{v}-\boldsymbol{v}^{-1})\sum_{k<i}\boldsymbol{v}^{i+1-k}E^{(-1)}_{k,i+1}\otimes[\widetilde{F}^{(0)}_{ik}, F^{(0)}_{i+1,i}]_{\boldsymbol{v}^{-3}} -$$

$$(\boldsymbol{v}-\boldsymbol{v}^{-1})^2\sum_{l>i+1}^{k<i}\boldsymbol{v}^{i-k}E^{(-1)}_{kl}\otimes\left(\widetilde{F}^{(0)}_{i+1,k}F^{(0)}_{l,i+1} - \widetilde{F}^{(0)}_{ik}F^{(0)}_{li}\right),$$

where $E^{(0)}_{ji}, \widetilde{E}^{(0)}_{ji}, E^{(-1)}_{ji}, F^{(0)}_{ij}, \widetilde{F}^{(0)}_{ij}, F^{(1)}_{ij}$ *are defined as in (10.6).*

The proof of this result is similar to our proof of Theorem 10.5, but is much more tedious; we sketch it in Appendix H.

Remark 10.17 The similarity between the formulas for Δ_{μ_1,μ_2} of Theorem 10.16 and Δ of Theorem 10.13 can be explained via an analogue of Remark 10.6. To be more precise, let $U^{\pm}_{\boldsymbol{v}}$ be the positive/negative Borel subalgebras in the Drinfeld-Jimbo presentation of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, while their analogues $\mathcal{U}^{\mathrm{sc},\pm}_{0,\mu_1,\mu_2}$ (subalgebras of $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1+\mu_2}$) will be introduced in Appendix H. There are natural $\mathbb{C}(\boldsymbol{v})$-algebra homomorphisms $\jmath^{\pm}_{\mu_1,\mu_2}\colon U^{\pm}_{\boldsymbol{v}} \to \mathcal{U}^{\mathrm{sc},\pm}_{0,\mu_1,\mu_2}$, see Proposition H.1. According to Proposition H.16, the following diagram is commutative:

$$\begin{array}{ccc}
U^{\pm}_{\boldsymbol{v}} & \xrightarrow{\quad\Delta\quad} & U^{\pm}_{\boldsymbol{v}}\otimes U^{\pm}_{\boldsymbol{v}} \\
\downarrow{\scriptstyle \jmath^{\pm}_{\mu_1,\mu_2}} & & \downarrow{\scriptstyle \jmath^{\pm}_{\mu_1,0}\otimes\jmath^{\pm}_{0,\mu_2}} \\
\mathcal{U}^{\mathrm{sc},\pm}_{0,\mu_1,\mu_2} & \xrightarrow[\Delta_{\mu_1,\mu_2}]{} & \mathcal{U}^{\mathrm{sc},\pm}_{0,\mu_1,0}\otimes\mathcal{U}^{\mathrm{sc},\pm}_{0,0,\mu_2}
\end{array}$$

10.7 Homomorphisms Δ_{μ_1,μ_2} for Arbitrary $\mu_1, \mu_2 \in \Lambda$, $\mathfrak{g} = \mathfrak{sl}_n$

Let us first generalize the *shift homomorphisms* of Proposition 10.8.

Lemma 10.18 *For any $\mu \in \Lambda$ and $\nu_1, \nu_2 \in \Lambda^-$, there is a unique $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism $\iota_{\mu,\nu_1,\nu_2} \colon \mathcal{U}^{\mathrm{sc}}_{0,\mu} \to \mathcal{U}^{\mathrm{sc}}_{0,\mu+\nu_1+\nu_2}$, which maps the currents as follows:*

$$\iota_{\mu,\nu_1,\nu_2} \colon e_i(z) \mapsto (1-z^{-1})^{-\alpha_i^\vee(\nu_1)} e_i(z),\ \ f_i(z) \mapsto (1-z^{-1})^{-\alpha_i^\vee(\nu_2)} f_i(z),$$
$$\psi_i^\pm(z) \mapsto (1-z^{-1})^{-\alpha_i^\vee(\nu_1+\nu_2)} \psi_i^\pm(z).$$

Proof The proof is analogous to that of Proposition 10.8. □

The proof of the following technical result is presented in Appendix I and is based on the shuffle realization of the quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, see [53] (cf. [63]).

Theorem 10.19 *The homomorphism ι_{μ,ν_1,ν_2} is injective for any $\mu \in \Lambda$ and $\nu_1, \nu_2 \in \Lambda^-$.*

Combining this theorem with Corollary 10.15 and our arguments from the proof of Theorem 10.10, we get the key result of this section.

Theorem 10.20 *For any $\mu_1, \mu_2 \in \Lambda$ and $\mu := \mu_1 + \mu_2$, there is a unique $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism*

$$\Delta_{\mu_1,\mu_2} \colon \mathcal{U}^{\mathrm{sc}}_{0,\mu} \longrightarrow \mathcal{U}^{\mathrm{sc}}_{0,\mu_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,\mu_2},$$

such that for any $\nu_1, \nu_2 \in \Lambda^-$ the following diagram is commutative:

$$\begin{array}{ccc}
\mathcal{U}^{\mathrm{sc}}_{0,\mu} & \xrightarrow{\ \Delta_{\mu_1,\mu_2}\ } & \mathcal{U}^{\mathrm{sc}}_{0,\mu_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,\mu_2} \\
\Big\downarrow{\scriptstyle \iota_{\mu,\nu_2,\nu_1}} & & \Big\downarrow{\scriptstyle \iota_{\mu_1,0,\nu_1} \otimes \iota_{\mu_2,\nu_2,0}} \\
\mathcal{U}^{\mathrm{sc}}_{0,\mu+\nu_1+\nu_2} & \xrightarrow[\ \Delta_{\mu_1+\nu_1,\mu_2+\nu_2}\ ]{} & \mathcal{U}^{\mathrm{sc}}_{0,\mu_1+\nu_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,\mu_2+\nu_2}
\end{array}$$

The following is proved analogously to Corollary 10.11:

Proposition 10.21 *For arbitrary $\mu_1, \mu_2 \in \Lambda$, the images $\Delta_{\mu_1,\mu_2}(h_{i,\pm 1})$ are given by formulas (10.2) and (10.3).*

10.8 Open Problems

Following [24], we expect that homomorphisms $\Delta_{\mu_1,\mu_2}\colon \mathcal{U}^{\mathrm{sc}}_{0,\mu_1+\mu_2} \to \mathcal{U}^{\mathrm{sc}}_{0,\mu_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$ (specializing to the Drinfeld-Jimbo coproduct for $\mu_1 = \mu_2 = 0$) exist for any simply-laced Lie algebra $\mathfrak{g}$ and its two coweights $\mu_1, \mu_2 \in \Lambda$. Moreover, their construction should proceed in the same way as for the aforementioned case $\mathfrak{g} = \mathfrak{sl}_n$. To be more precise, for antidominant $\mu_1, \mu_2 \in \Lambda^-$, we expect that the homomorphism Δ_{μ_1,μ_2} is characterized by the following two properties:

(a) $\Delta_{\mu_1,\mu_2}(e_{i,r}) = 1 \otimes e_{i,r},\ \Delta_{\mu_1,\mu_2}(f_{i,s}) = f_{i,s} \otimes 1$ for $\alpha_i^\vee(\mu_2) \le r < 0, \alpha_i^\vee(\mu_1) < s \le 0$;
(b) an analogue of the commutative diagram of Remark 10.17 holds.

For general μ_1, μ_2, we expect that the construction of Δ_{μ_1,μ_2} should be easily deduced from the antidominant case with the help of *shift homomorphisms* ι_{μ,ν_1,ν_2} ($\mu \in \Lambda, \nu_1, \nu_2 \in \Lambda^-$) as in Theorems 10.10 and 10.20.

The outlined construction of Δ_{μ_1,μ_2} for a general $\mathfrak{g}$ lacks explicit formulas for the Drinfeld-Jimbo coproduct of $\{e_{i,0}, e_{i,-1}, f_{i,0}, f_{i,1}, \psi^{\pm}_{i,0}, h_{i,\pm 1}\}_{i\in I}$–the generators of $U_{\boldsymbol{v}}(L\mathfrak{g})$, similar to those of Lemma 10.4 and Theorem 10.13.

11 Ubiquity of RTT Relations

11.1 Rational Lax Matrix

Before we proceed to the *trigonometric* setting, let us recall the classical relation between rational Lax matrices and type A quantum open Toda systems, which goes back to [28].

Let $R_{\mathrm{rat}}(z) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ be the standard rational R-matrix:

$$R_{\mathrm{rat}}(z) = \mathrm{Id} + \frac{\hbar}{z} P, \text{ where } P \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \text{ is the permutation map.}$$

Let $\hat{\mathcal{A}}^\hbar_n$ be the associative $\mathbb{C}[\hbar]$-algebra generated by $\{\mathsf{u}_i^{\pm 1}, w_i\}_{i=1}^n$ with the defining relations $[\mathsf{u}_i, \mathsf{u}_j] = [w_i, w_j] = 0, \mathsf{u}_i^{\pm 1}\mathsf{u}_i^{\mp 1} = 1, [\mathsf{u}_i, w_j] = \delta_{ij}\hbar \mathsf{u}_i$. Define the *(local) rational Lax matrix*

$$L_i^\hbar(z) = \begin{pmatrix} z - w_i & \mathsf{u}_i^{-1} \\ -\mathsf{u}_i & 0 \end{pmatrix} \in \mathrm{Mat}(2, \hat{\mathcal{A}}^\hbar_n[z]) \tag{11.1}$$

and introduce the *complete monodromy matrix* $T_n^\hbar(z) := L_n^\hbar(z) \cdots L_1^\hbar(z)$. Then, the monodromy matrix $T_n^\hbar(z)$ satisfies the rational RTT-relation:

$$R_{\mathrm{rat}}(z-w)(T_n^\hbar(z) \otimes 1)(1 \otimes T_n^\hbar(w)) = (1 \otimes T_n^\hbar(w))(T_n^\hbar(z) \otimes 1) R_{\mathrm{rat}}(z-w).$$

Due to this relation, the coefficients (in z) of the matrix element $T_n^{\hbar}(z)_{11}$ generate a commutative subalgebra of $\hat{\mathcal{A}}_n^{\hbar}$, known as the quantum open Toda system of $\mathfrak{gl}_n$. The coefficient of z^{n-2} equals

$$\mathsf{H}_2^{\mathrm{rat}} = \frac{1}{2}\left(\sum_{i=1}^{n} w_i\right)^2 - \frac{1}{2}\sum_{i=1}^{n} w_i^2 - \sum_{i=1}^{n-1} \mathsf{u}_i \mathsf{u}_{i+1}^{-1}. \tag{11.2}$$

We recover the standard quantum open Toda hamiltonian of $\mathfrak{sl}_n$ once we set $w_1 + \ldots + w_n = 0$.

11.2 Trigonometric/Relativistic Lax Matrices

Let $R_{\mathrm{trig}}(z) \in \mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2)$ be the standard trigonometric R-matrix (see [17, (3.7)]):

$$R_{\mathrm{trig}}(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{z-1}{\boldsymbol{v}z-\boldsymbol{v}^{-1}} & \frac{z(\boldsymbol{v}-\boldsymbol{v}^{-1})}{\boldsymbol{v}z-\boldsymbol{v}^{-1}} & 0 \\ 0 & \frac{\boldsymbol{v}-\boldsymbol{v}^{-1}}{\boldsymbol{v}z-\boldsymbol{v}^{-1}} & \frac{z-1}{\boldsymbol{v}z-\boldsymbol{v}^{-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11.3}$$

Let $\hat{\mathcal{A}}_n^{\boldsymbol{v}}$ be the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by $\{\widetilde{\mathsf{w}}_i^{\pm 1}, D_i^{\pm 1}\}_{i=1}^n$ with the defining relations $[\widetilde{\mathsf{w}}_i, \widetilde{\mathsf{w}}_j] = [D_i, D_j] = 0$, $\widetilde{\mathsf{w}}_i^{\pm 1}\widetilde{\mathsf{w}}_i^{\mp 1} = D_i^{\pm 1} D_i^{\mp 1} = 1$, $D_i \widetilde{\mathsf{w}}_j = \boldsymbol{v}^{\delta_{ij}} \widetilde{\mathsf{w}}_j D_i$. If we set $\mathsf{w}_i^{\pm 1} = \widetilde{\mathsf{w}}_i^{\pm 2}$, we see that $\hat{\mathcal{A}}_n^{\boldsymbol{v}}$ is a particular example of the algebras $\hat{\mathcal{A}}_{\mathrm{frac}}^{\boldsymbol{v}}$ of Sect. 7. Define the *(local) relativistic Lax matrix*

$$L_i^{\boldsymbol{v},0}(z) = \begin{pmatrix} \widetilde{\mathsf{w}}_i^{-1} z^{1/2} - \widetilde{\mathsf{w}}_i z^{-1/2} & D_i^{-1} z^{1/2} \\ -D_i z^{-1/2} & 0 \end{pmatrix} \in \mathrm{Mat}(2, z^{-1/2}\hat{\mathcal{A}}_n^{\boldsymbol{v}}[z]) \tag{11.4}$$

and introduce the *complete monodromy matrix* $T_n^{\boldsymbol{v},0}(z) := L_n^{\boldsymbol{v},0}(z) \cdots L_1^{\boldsymbol{v},0}(z)$.

Lemma 11.1 *The monodromy matrix $T_n^{\boldsymbol{v},0}(z)$ satisfies the trigonometric RTT-relation:*

$$R_{\mathrm{trig}}(z/w)(T_n^{\boldsymbol{v},0}(z) \otimes 1)(1 \otimes T_n^{\boldsymbol{v},0}(w)) = (1 \otimes T_n^{\boldsymbol{v},0}(w))(T_n^{\boldsymbol{v},0}(z) \otimes 1) R_{\mathrm{trig}}(z/w).$$

Proof It suffices to check the above relation for $n = 1$. The proof in the latter case is straightforward. □

Corollary 11.2 *The coefficients (in z) of the matrix element $z^{n/2}T_n^{\boldsymbol{v},0}(z)_{11}$ generate a commutative subalgebra of $\hat{\mathcal{A}}_n^{\boldsymbol{v}}$. The coefficient of z equals*

$$\mathsf{H}_2^0 = (-1)^{n-1}\widetilde{\mathsf{w}}_1\cdots\widetilde{\mathsf{w}}_n \cdot \left(\sum_{i=1}^{n}\widetilde{\mathsf{w}}_i^{-2} + \sum_{i=1}^{n-1}\widetilde{\mathsf{w}}_i^{-1}\widetilde{\mathsf{w}}_{i+1}^{-1}D_iD_{i+1}^{-1}\right). \tag{11.5}$$

This hamiltonian is equivalent to the quadratic hamiltonian of the q-difference quantum Toda lattice of [19, (5.7)] (see also [56]) once we set $\widetilde{\mathsf{w}}_1\cdots\widetilde{\mathsf{w}}_n = 1$.

Remark 11.3 The notion of a *relativistic Lax matrix* goes back to [43]. In particular, our choice of $L_i^{\boldsymbol{v},0}(z)$ is a slight variation of their construction, which is adapted to a different choice of the trigonometric R-matrix.

Now let us consider two *(local) trigonometric Lax matrices*

$$L_i^{\boldsymbol{v},-1}(z) = \begin{pmatrix} \widetilde{\mathsf{w}}_i^{-1} - \widetilde{\mathsf{w}}_i z^{-1} & \widetilde{\mathsf{w}}_i D_i^{-1} \\ -\widetilde{\mathsf{w}}_i D_i z^{-1} & \widetilde{\mathsf{w}}_i \end{pmatrix} \in \mathrm{Mat}(2, z^{-1}\hat{\mathcal{A}}_n^{\boldsymbol{v}}[z]), \tag{11.6}$$

$$L_i^{\boldsymbol{v},1}(z) = \begin{pmatrix} \widetilde{\mathsf{w}}_i^{-1} z - \widetilde{\mathsf{w}}_i & \widetilde{\mathsf{w}}_i^{-1} D_i^{-1} z \\ -\widetilde{\mathsf{w}}_i^{-1} D_i & -\widetilde{\mathsf{w}}_i^{-1} \end{pmatrix} \in \mathrm{Mat}(2, \hat{\mathcal{A}}_n^{\boldsymbol{v}}[z]). \tag{11.7}$$

Lemma 11.4 *The Lax matrices $L_i^{\boldsymbol{v},\pm 1}(z)$ satisfy the trigonometric RTT-relation:*

$$R_{\mathrm{trig}}(z/w)(L_i^{\boldsymbol{v},\pm1}(z)\otimes 1)(1\otimes L_i^{\boldsymbol{v},\pm1}(w)) = (1\otimes L_i^{\boldsymbol{v},\pm1}(w))(L_i^{\boldsymbol{v},\pm1}(z)\otimes 1)R_{\mathrm{trig}}(z/w).$$

Proof The proof is straightforward. □

11.3 Mixed Toda Hamiltonians

Now we construct 3^n Hamiltonians generalizing H_2^0 in spirit of [21, (90)], cf. also [11, (1.1) and Section 2]. For any $\vec{k} = (k_n, \ldots, k_1) \in \{-1, 0, 1\}^n$, define the *mixed complete monodromy matrix*

$$T_{\vec{k}}^{\boldsymbol{v}}(z) := L_n^{\boldsymbol{v},k_n}(z)\cdots L_1^{\boldsymbol{v},k_1}(z).$$

In particular, $T_{\vec{0}}^{\boldsymbol{v}}(z) = T_n^{\boldsymbol{v},0}(z)$. Since all three matrices $L_i^{\boldsymbol{v},-1}(z), L_i^{\boldsymbol{v},0}(z), L_i^{\boldsymbol{v},1}(z)$ satisfy the RTT-relation with the same R-matrix $R_{\mathrm{trig}}(z)$, the same is true for $T_{\vec{k}}^{\boldsymbol{v}}(z)$. Hence, the coefficients (in z) of the matrix element $T_{\vec{k}}^{\boldsymbol{v}}(z)_{11}$ generate a commutative subalgebra of $\hat{\mathcal{A}}_n^{\boldsymbol{v}}$. We have

$$T_{\vec{k}}^{\boldsymbol{v}}(z)_{11} = \mathsf{H}_1^{\vec{k}} z^s + \mathsf{H}_2^{\vec{k}} z^{s+1} + \text{higher powers of } z,$$

where $s = \sum_{i=1}^{n} \frac{k_i-1}{2}$. Here $\mathsf{H}_1^{\vec{k}} = (-1)^n \widetilde{\mathsf{w}}_1 \cdots \widetilde{\mathsf{w}}_n$, while the hamiltonian $\mathsf{H}_2^{\vec{k}}$ equals

$$\mathsf{H}_2^{\vec{k}} = (-1)^{n-1}\widetilde{\mathsf{w}}_1 \cdots \widetilde{\mathsf{w}}_n \cdot \left(\sum_{i=1}^{n} \widetilde{\mathsf{w}}_i^{-2} + \sum_{i=1}^{n-1} \sigma_{i,i+1} D_i D_{i+1}^{-1} + \sum_{1\le i<j-1<n}^{k_{i+1}=\ldots=k_{j-1}=1} \sigma_{i,j} D_i D_j^{-1} \right), \tag{11.8}$$

where $\sigma_{i,j} := \widetilde{\mathsf{w}}_i^{-k_i-1} \widetilde{\mathsf{w}}_{i+1}^{-k_{i+1}-1} \cdots \widetilde{\mathsf{w}}_j^{-k_j-1}$.

Remark 11.5 At the classical level, the birational Bäcklund-Darboux transformations interchanging various hamiltonians $\mathsf{H}_2^{\vec{k}}$ are given in [34, Theorem 6.1].

Lemma 11.6 *For any $\vec{k}$, set $\vec{k}' = (0, k_{n-1}, \ldots, k_2, 0)$. Then, $\mathsf{H}_2^{\vec{k}}$ is equivalent to $\mathsf{H}_2^{\vec{k}'}$.*

Proof It is straightforward to see that $\mathsf{H}_2^{\vec{k}'} = \mathrm{Ad}(F(\widetilde{\mathsf{w}}_1, \ldots, \widetilde{\mathsf{w}}_n))\mathsf{H}_2^{\vec{k}}$, where $F(\widetilde{\mathsf{w}}_1, \ldots, \widetilde{\mathsf{w}}_n) = \exp(k_1 f_-(\log(\widetilde{\mathsf{w}}_1)) + k_n f_+(\log(\widetilde{\mathsf{w}}_n)))$ with $f_\pm(t) = \pm\frac{t^2}{2\log(\boldsymbol{v})} + \frac{t}{2}$. □

Remark 11.7 It follows that among the aforementioned 3^n *mixed Toda hamiltonians* $\mathsf{H}_2^{\vec{k}}$, parameterized by $\vec{k} \in \{-1, 0, 1\}^n$, there are no more than 3^{n-2} different up to equivalence. In [35] these hamiltonians are identified with the modified versions of the q-Toda hamiltonian in [19, 56], which now depend on a choice of two orientations of the Dynkin diagram of type A_{n-1} (equivalently, a choice of a pair of Coxeter elements). There are 4^{n-2} such choices, but some of them are equivalent leading to exactly 3^{n-2} inequivalent hamiltonians, which turn out to be equivalent to the aforementioned $\mathsf{H}_2^{\vec{k}}$. All the q-Toda hamiltonians of [19, 56] correspond to the pairs of coinciding orientations, i.e. to $\vec{k} = (0, \ldots, 0)$, and they share the same eigenfunction J [22, Section 3], while our mixed Toda hamiltonians do not admit the common eigenfunctions. We are grateful to P. Etingof for his suggestion to study the construction of [56] for pairs of different orientations.

11.4 Shifted RTT Algebras of $\mathfrak{sl}_2$

Fix $n \in \mathbb{N}$. Following [17] (cf. also Remark G.1), we introduce the *(trigonometric) shifted RTT algebras of* $\mathfrak{sl}_2$, denoted by $\mathcal{U}^{\mathrm{rtt}}_{0,-2n}$. These are associative $\mathbb{C}(\boldsymbol{v})$-algebras generated by

$$\{t^+_{11}[r], t^+_{12}[r], t^+_{21}[r+1], t^+_{22}[r], t^-_{11}[-m], t^-_{12}[-m-1], t^-_{21}[-m], t^-_{22}[-m-1+\delta_{n,0}]\}_{r\geq 0}^{m\geq -n} \cup \{(t^+_{11}[0])^{-1}, (t^-_{11}[n])^{-1}\}$$

subject to the following defining relations:

$$(t_{11}^{+}[0])^{\pm 1}(t_{11}^{+}[0])^{\mp 1} = 1,\ (t_{11}^{-}[n])^{\pm 1}(t_{11}^{-}[n])^{\mp 1} = 1, \tag{R1}$$

$$R_{\mathrm{trig}}(z/w)(T^{\epsilon}(z)\otimes 1)(1\otimes T^{\epsilon'}(w)) = (1\otimes T^{\epsilon'}(w))(T^{\epsilon}(z)\otimes 1)R_{\mathrm{trig}}(z/w), \tag{R2}$$

$$\mathrm{qdet}\, T^{\pm}(z) = 1 \tag{R3}$$

for all $\epsilon, \epsilon' \in \{\pm\}$, where the two-by-two matrices $T^{\pm}(z)$ are given by

$$T^{\pm}(z) = \begin{pmatrix} T_{11}^{\pm}(z) & T_{12}^{\pm}(z) \\ T_{21}^{\pm}(z) & T_{22}^{\pm}(z) \end{pmatrix} \text{ with } T_{ij}^{\pm}(z) := \sum_{r} t_{ij}^{\pm}[r]z^{-r},$$

and the quantum determinant qdet is defined in a standard way as[8]

$$\mathrm{qdet}\, T^{\pm}(z) := T_{11}^{\pm}(z)T_{22}^{\pm}(\boldsymbol{v}^{-2}z) - \boldsymbol{v}^{-1}T_{12}^{\pm}(z)T_{21}^{\pm}(\boldsymbol{v}^{-2}z).$$

Note that $T^{\pm}(z)$ admits the following unique *Gauss* decomposition:

$$T^{\pm}(z) = \begin{pmatrix} 1 & 0 \\ \tilde{f}^{\pm}(z) & 1 \end{pmatrix} \begin{pmatrix} \tilde{g}_1^{\pm}(z) & 0 \\ 0 & \tilde{g}_2^{\pm}(z) \end{pmatrix} \begin{pmatrix} 1 & \tilde{e}^{\pm}(z) \\ 0 & 1 \end{pmatrix},$$

where coefficients of the *half-currents* $\tilde{e}^{\pm}(z)$, $\tilde{f}^{\pm}(z)$, $\tilde{g}_1^{\pm}(z)$, $\tilde{g}_2^{\pm}(z)$ are elements of $\mathcal{U}^{\mathrm{rtt}}_{0,-2n}$.

To establish the relation between $\mathcal{U}^{\mathrm{rtt}}_{0,-2n}$ and $\mathcal{U}^{\mathrm{ad}}_{0,-2n}$ (adjoint version of the shifted quantum affine algebra of $\mathfrak{sl}_2$), recall *Drinfeld half-currents* $e^{\pm}(z)$, $f^{\pm}(z)$ of (6.5).

Theorem 11.8

(a) The currents $\tilde{g}_1^{\pm}(z)$, $\tilde{g}_2^{\pm}(z)$ *pairwise commute and satisfy*

$$\tilde{g}_2^{\pm}(z)\tilde{g}_1^{\pm}(\boldsymbol{v}^{-2}z) = 1.$$

(b) There exists a unique $\mathbb{C}(\boldsymbol{v})$*-algebra homomorphism* $\Upsilon_{0,-2n} : \mathcal{U}^{\mathrm{ad}}_{0,-2n} \to \mathcal{U}^{\mathrm{rtt}}_{0,-2n}$*, defined by*

$$e^{\pm}(z) \mapsto \tilde{e}^{\pm}(z)/(\boldsymbol{v}-\boldsymbol{v}^{-1}),\ f^{\pm}(z) \mapsto \tilde{f}^{\pm}(z)/(\boldsymbol{v}-\boldsymbol{v}^{-1}),$$
$$\psi^{\pm}(z) \mapsto \tilde{g}_2^{\pm}(z)\tilde{g}_1^{\pm}(z)^{-1},\ (\phi^{+})^{\pm 1} \mapsto (t_{11}^{+}[0])^{\mp 1},\ (\phi^{-})^{\pm 1} \mapsto \boldsymbol{v}^{\mp n}(t_{11}^{-}[n])^{\mp 1}.$$

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[8] It is instructive to point out the difference with [51], where the author uses a different trigonometric R-matrix given by $R^{\mathrm{M}}_{\mathrm{trig}}(z/w) = (R_{\mathrm{trig}}(z/w)^{t})^{-1}$ as well as $T^{\mathrm{M},\pm}(z) = T^{\pm}(z)^{t}$. For this reason, the quantum determinant $\mathrm{qdet}^{\mathrm{M}}$ of [51, Exercise 1.6] is consistent with our definition of qdet, that is, $\mathrm{qdet}^{\mathrm{M}}\, T^{\mathrm{M},\pm}(z) := T_{11}^{\mathrm{M},\pm}(z)T_{22}^{\mathrm{M},\pm}(\boldsymbol{v}^{-2}z) - \boldsymbol{v}^{-1}T_{21}^{\mathrm{M},\pm}(z)T_{12}^{\mathrm{M},\pm}(\boldsymbol{v}^{-2}z) = \mathrm{qdet}\, T^{\pm}(z)$.

(c) *For any* $b_1, b_2 \in \mathbb{Z}_{\leq 0}$, *there exists a unique* $\mathbb{C}(\boldsymbol{v})$*-algebra homomorphism*

$$\Delta^{\mathrm{rtt}}_{2b_1,2b_2} : \mathcal{U}^{\mathrm{rtt}}_{0,2b_1+2b_2} \longrightarrow \mathcal{U}^{\mathrm{rtt}}_{0,2b_1} \otimes \mathcal{U}^{\mathrm{rtt}}_{0,2b_2},$$

defined by $T^{\pm}(z) \mapsto T^{\pm}(z) \otimes T^{\pm}(z)$.

Remark 11.9 The $n = 0$ case of this theorem was proved in [17], cf. Remark G.1.

Proof The verification of part (b) is analogous to the one for $n = 0$, dealt with in [17]. Once (b) is established, it is easy to see that $\mathrm{qdet}\, T^{\pm}(z) = \tilde{g}_2^{\pm}(z)\tilde{g}_1^{\pm}(\boldsymbol{v}^{-2}z)$, hence (a). It is clear that $\Delta^{\mathrm{rtt}}_{2b_1,2b_2}$ is well-defined on the generators. The compatibility of $\Delta^{\mathrm{rtt}}_{2b_1,2b_2}$ with the defining relations (R1–R3) is checked analogously to the case $n = 0$. □

Recall the generating series $A^{\pm}(z), B^{\pm}(z), C^{\pm}(z), D^{\pm}(z)$ with coefficients in $\mathcal{U}^{\mathrm{ad}}_{0,-2n}$, introduced in Sect. 6.

Corollary 11.10 *The homomorphism* $\Upsilon_{0,-2n}$ *maps these generating series as follows:*

$$A^{+}(z) \mapsto T^{+}_{11}(z),\ B^{+}(z) \mapsto T^{+}_{12}(z),\ C^{+}(z) \mapsto T^{+}_{21}(z),\ D^{+}(z) \mapsto T^{+}_{22}(z),$$

$$A^{-}(z) \mapsto (\boldsymbol{v}z)^{n}T^{-}_{11}(z),\ B^{-}(z) \mapsto (\boldsymbol{v}z)^{n}T^{-}_{12}(z),\ C^{-}(z) \mapsto (\boldsymbol{v}z)^{n}T^{-}_{21}(z),\ D^{-}(z) \mapsto (\boldsymbol{v}z)^{n}T^{-}_{22}(z).$$

Proof Due to Theorem 11.8(a, b), we have

$$\Upsilon_{0,-2n}(\psi^{\pm}(z)) = 1/\tilde{g}_1^{\pm}(z)\tilde{g}_1^{\pm}(\boldsymbol{v}^{-2}z),\ \Upsilon_{0,-2n}((\phi^{+})^{-1}) = t^{+}_{11}[0],\ \Upsilon_{0,-2n}((\phi^{-})^{-1}) = \boldsymbol{v}^{n}t^{-}_{11}[n].$$

Combining this with $\psi^{+}(z) = \frac{1}{A^{+}(z)A^{+}(\boldsymbol{v}^{-2}z)}$, $\psi^{-}(z) = \frac{z^{2n}}{A^{-}(z)A^{-}(\boldsymbol{v}^{-2}z)}$, and $A_0^{\pm} = (\phi^{\pm})^{-1}$, we get $\Upsilon_{0,-2n}(A^{+}(z)) = \tilde{g}_1^{+}(z) = T^{+}_{11}(z)$, $\Upsilon_{0,-2n}(A^{-}(z)) = (\boldsymbol{v}z)^{n}\tilde{g}_1^{-}(z) = (\boldsymbol{v}z)^{n}T^{-}_{11}(z)$. The computation of the images of the remaining generating series is straightforward, e.g. $\Upsilon_{0,-2n}(B^{-}(z)) = (\boldsymbol{v} - \boldsymbol{v}^{-1})\Upsilon_{0,-2n}(A^{-}(z))\Upsilon_{0,-2n}(e^{-}(z)) = (\boldsymbol{v}z)^{n}\tilde{g}_1^{-}(z)\tilde{e}^{-}(z) = (\boldsymbol{v}z)^{n}T^{-}_{12}(z)$. □

The following is the key result of this subsection.

Theorem 11.11 *For* $n \in \mathbb{N}$, $\Upsilon_{0,-2n} : \mathcal{U}^{\mathrm{ad}}_{0,-2n} \to \mathcal{U}^{\mathrm{rtt}}_{0,-2n}$ *is an isomorphism of* $\mathbb{C}(\boldsymbol{v})$*-algebras.*

Proof Due to Theorem 11.8 and Corollary 11.10, it suffices to prove that there exists a $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism $\mathcal{U}^{\mathrm{rtt}}_{0,-2n} \to \mathcal{U}^{\mathrm{ad}}_{0,-2n}$, such that

$$\begin{gathered}
(t^{+}_{11}[0])^{-1} \mapsto \phi^{+},\ (t^{-}_{11}[n])^{-1} \mapsto \boldsymbol{v}^{n}\phi^{-},\\
T^{+}_{11}(z) \mapsto A^{+}(z),\ T^{+}_{12}(z) \mapsto B^{+}(z),\ T^{+}_{21}(z) \mapsto C^{+}(z),\ T^{+}_{22}(z) \mapsto D^{+}(z),\\
T^{-}_{11}(z) \mapsto (\boldsymbol{v}z)^{-n}A^{-}(z),\ T^{-}_{12}(z) \mapsto (\boldsymbol{v}z)^{-n}B^{-}(z),\\
T^{-}_{21}(z) \mapsto (\boldsymbol{v}z)^{-n}C^{-}(z),\ T^{-}_{22}(z) \mapsto (\boldsymbol{v}z)^{-n}D^{-}(z).
\end{gathered} \tag{11.9}$$

This amounts to verifying that the assignment (11.9) preserves defining relations (R1–R3). Relation (R1) is preserved, due to $A_0^{\pm}\phi^{\pm} = \phi^{\pm}A_0^{\pm} = 1$, while (R3) is preserved, due to relation (6.16). Finally, (R2) is an equality in $\mathrm{End}(\mathbb{C}^2 \otimes \mathbb{C}^2) \otimes \mathcal{U}^{\mathrm{ad}}_{0,-2n}$ and thus can be viewed as a collection of 16 relations in $\mathcal{U}^{\mathrm{ad}}_{0,-2n}$ for each choice of $\epsilon, \epsilon' \in \{\pm\}$. It is straightforward to see that 6 of these relations follow from the rest, while the remaining 10 relations exactly match the 10 relations of (6.7, 6.9–6.15) under the assignment (11.9). □

Remark 11.12 The results of this subsection admit natural generalizations to the case of arbitrary $b_1, b_2 \in \mathbb{Z}_{\leq 0}$ such that $b_1 + b_2$ is even. In other words, one can define an analogous shifted RTT algebra of $\mathfrak{sl}_2$, denoted $\mathcal{U}^{\mathrm{rtt}}_{b_1,b_2}$, and construct a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $\Upsilon_{b_1,b_2}\colon \mathcal{U}^{\mathrm{ad}}_{b_1,b_2} \xrightarrow{\sim} \mathcal{U}^{\mathrm{rtt}}_{b_1,b_2}$. This observation is used in Remark 11.14 below, where we provide an alternative interpretation of the Lax matrices $L_1^{\boldsymbol{v},-1}(z), L_1^{\boldsymbol{v},0}(z), L_1^{\boldsymbol{v},1}(z)$ from Sect. 11.2.

11.5 Relation Between Two Different Appearances of RTT

Recall the local trigonometric Lax matrix $L_1^{\boldsymbol{v},-1}(z)$ of (11.6). Combining the equality $\mathrm{qdet}\, L_1^{\boldsymbol{v},-1}(z) = 1$ with Lemma 11.4, we see that $L_1^{\boldsymbol{v},-1}(z)$ gives rise to an algebra homomorphism $\Phi^{\mathrm{rtt}}_{0,-2}\colon \mathcal{U}^{\mathrm{rtt}}_{0,-2} \to \hat{\mathcal{A}}_1^{\boldsymbol{v}}$ defined by $T^{\pm}(z) \mapsto L_1^{\boldsymbol{v},-1}(z)$. Recall the homomorphism $\widetilde{\Phi}^0_{-2}\colon \mathcal{U}^{\mathrm{ad}}_{0,-2} \to \hat{\mathcal{A}}_1^{\boldsymbol{v}}$ of Theorem 7.1 (where $\mathsf{w}^{1/2} = \widetilde{\mathsf{w}}$). The following is straightforward.

Lemma 11.13 *The composition $\Phi^{\mathrm{rtt}}_{0,-2} \circ \Upsilon_{0,-2}$ coincides with $\widetilde{\Phi}^0_{-2}$.*

Remark 11.14 Let us provide a similar interpretation of the other two Lax matrices $L_1^{\boldsymbol{v},0}(z)$ and $L_1^{\boldsymbol{v},1}(z)$. Recall that the algebras $\mathcal{U}^{\mathrm{ad}}_{0,-2}$ and $\mathcal{U}^{\mathrm{ad}}_{b,-2-b}$ are isomorphic for any $b \in \mathbb{Z}$. In particular, one can pull back the homomorphism $\widetilde{\Phi}^0_{-2}$ to obtain a homomorphism $\widetilde{\Phi}_{b,-2-b}\colon \mathcal{U}^{\mathrm{ad}}_{b,-2-b} \to \hat{\mathcal{A}}_1^{\boldsymbol{v}}$, explicitly given by

$$e(z) \mapsto \frac{\widetilde{\mathsf{w}}^{2+b}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\delta\left(\frac{\widetilde{\mathsf{w}}^2}{z}\right)D^{-1},\ f(z) \mapsto \frac{\widetilde{\mathsf{w}}^b}{1-\boldsymbol{v}^2}\delta\left(\frac{\boldsymbol{v}^2\widetilde{\mathsf{w}}^2}{z}\right)D,$$

$$\psi^{\pm}(z) \mapsto \left(\frac{\boldsymbol{v}^{-b}\widetilde{\mathsf{w}}^2 z^b}{(1-\widetilde{\mathsf{w}}^2/z)(1-\boldsymbol{v}^2\widetilde{\mathsf{w}}^2/z)}\right)^{\pm},\ (\phi^+)^{\pm 1} \mapsto \boldsymbol{v}^{\mp b/2}\widetilde{\mathsf{w}}^{\pm 1},\ (\phi^-)^{\pm 1} \mapsto -\boldsymbol{v}^{\mp(b/2+1)}\widetilde{\mathsf{w}}^{\mp 1}.$$

Due to Remark 11.12, the algebra $\mathcal{U}^{\mathrm{ad}}_{b,-2-b}$ admits an RTT realization, that is there is an isomorphism $\Upsilon_{b,-2-b}\colon \mathcal{U}^{\mathrm{ad}}_{b,-2-b} \xrightarrow{\sim} \mathcal{U}^{\mathrm{rtt}}_{b,-2-b}$, only for $b = 0, -1, -2$. Analogously to Lemma 11.13, recasting the homomorphisms $\widetilde{\Phi}_{b,-2-b}$ as the homomorphisms $\mathcal{U}^{\mathrm{rtt}}_{b,-2-b} \to \hat{\mathcal{A}}_1^{\boldsymbol{v}}$, we recover the Lax matrix $L_1^{\boldsymbol{v},0}(z)$ (for $b = -1$) and $L_1^{\boldsymbol{v},1}(z)$ (for $b = -2$). Moreover, this also explains why we had exactly three Lax matrices in Sect. 11.2.

Fix $n \geq 1$ and consider the *complete monodromy matrix* $T_n^{\boldsymbol{v},-1}(z) = L_n^{\boldsymbol{v},-1}(z)\cdots L_1^{\boldsymbol{v},-1}(z)$. Applying iteratively $\Delta^{\mathrm{rtt}}_{\bullet,\bullet}$ of Theorem 11.8(c), we get $\Delta^{\mathrm{rtt}}_n\colon \mathcal{U}^{\mathrm{rtt}}_{0,-2n} \to (\mathcal{U}^{\mathrm{rtt}}_{0,-2})^{\otimes n}$. Composing it with the homomorphism $(\Phi^{\mathrm{rtt}}_{0,-2})^{\otimes n}\colon (\mathcal{U}^{\mathrm{rtt}}_{0,-2})^{\otimes n} \to (\hat{\mathcal{A}}^{\boldsymbol{v}}_1)^{\otimes n} \simeq \hat{\mathcal{A}}^{\boldsymbol{v}}_n$, we obtain the homomorphism $\Phi^{\mathrm{rtt}}_{0,-2n}\colon \mathcal{U}^{\mathrm{rtt}}_{0,-2n} \to \hat{\mathcal{A}}^{\boldsymbol{v}}_n$. The following is straightforward.

Lemma 11.15 *We have* $\Phi^{\mathrm{rtt}}_{0,-2n}(T^{\pm}(z)) = T_n^{\boldsymbol{v},-1}(z)$.

Remark 11.16 For $n > 1$, the composition $\Phi^{\mathrm{rtt}}_{0,-2n} \circ \Upsilon_{0,-2n}$ *does not* coincide with the homomorphism $\widetilde{\Phi}^0_{-2n}$ of Theorem 7.1.

Remark 11.17 The result of Lemma 11.13 admits a natural *rational* counterpart. Let $\mathbf{Y}_{-2}$ be the shifted Yangian of $\mathfrak{sl}_2$ with the shift $-\alpha$. Recall the homomorphism $\Phi^0_{-2}\colon \mathbf{Y}_{-2} \to \hat{\mathcal{A}}^{\hbar}_1$ of [10, Corollary B.17]. Consider a slight modification of it

$$\hat{\Phi}_{-2}\colon E(z) \mapsto (z-w)^{-1}\mathsf{u}^{-1},\ F(z) \mapsto -(z-w-\hbar)^{-1}\mathsf{u},\ H(z) \mapsto (z-w)^{-1}(z-w-\hbar)^{-1}.$$

One can also define a *(rational) shifted RTT algebra of* $\mathfrak{sl}_2$, denoted by $\mathcal{Y}^{\mathrm{rtt}}_{-2}$. This is an associative $\mathbb{C}[\hbar]$-algebra generated by $\{t_{11}[r-1], t_{12}[r], t_{21}[r], t_{22}[r+1], (t_{11}[-1])^{-1}\}_{r\geq 0}$ and with the defining relations $(t_{11}[-1])^{\pm 1}(t_{11}[-1])^{\mp 1} = 1$, $T_{11}(z)T_{22}(z-\hbar) - T_{12}(z)T_{21}(z-\hbar) = 1$, $R_{\mathrm{rat}}(z-w)(T(z)\otimes 1)(1\otimes T(w)) = (1\otimes T(w))(T(z)\otimes 1)R_{\mathrm{rat}}(z-w)$, where $T(z) = (T_{ij}(z))_{i,j=1}^2$ with $T_{ij}(z) := \sum_r t_{ij}[r]z^{-r}$. Consider the Gauss decomposition of $T(z)$:

$$T(z) = \begin{pmatrix} 1 & 0 \\ \tilde{f}(z) & 1 \end{pmatrix} \begin{pmatrix} \tilde{g}_1(z) & 0 \\ 0 & \tilde{g}_2(z) \end{pmatrix} \begin{pmatrix} 1 & \tilde{e}(z) \\ 0 & 1 \end{pmatrix}.$$

Analogously to Theorem 11.8(b), there is a $\mathbb{C}[\hbar]$-algebra homomorphism $\Upsilon^{\mathrm{rat}}_{-2}\colon \mathbf{Y}_{-2} \to \mathcal{Y}^{\mathrm{rtt}}_{-2}$, defined by $E(z) \mapsto \tilde{e}(z), F(z) \mapsto \tilde{f}(z), H(z) \mapsto \tilde{g}_2(z)\tilde{g}_1(z)^{-1}$. Composing $\Upsilon^{\mathrm{rat}}_{-2}$ with the homomorphism $\mathcal{Y}^{\mathrm{rtt}}_{-2} \to \hat{\mathcal{A}}^{\hbar}_1$ given by $T(z) \mapsto L^{\hbar}_1(z)$, we recover $\hat{\Phi}_{-2}$ from above.

11.6 Homomorphism Δ_{b_1,b_2} ($b_1, b_2 \in \mathbb{Z}_{\leq 0}$) via Drinfeld Half-Currents, $\mathfrak{g} = \mathfrak{sl}_2$

Recall the currents $e^{\pm}(z), f^{\pm}(z), \psi^{\pm}(z)$ of (6.5).

Proposition 11.18 *Let Δ be the Drinfeld-Jimbo coproduct on $U_{\boldsymbol{v}}(L\mathfrak{sl}_2)$. Then, we have*

$$\Delta(e^{\pm}(z)) = 1 \otimes e^{\pm}(z) + \sum_{r=0}^{\infty} (-\boldsymbol{v})^r (\boldsymbol{v} - \boldsymbol{v}^{-1})^{2r} \cdot e^{\pm}(z)^{r+1} \otimes f^{\pm}(\boldsymbol{v}^2 z)^r \psi^{\pm}(z), \tag{11.10}$$

$$\Delta(f^{\pm}(z)) = f^{\pm}(z) \otimes 1 + \sum_{r=0}^{\infty} (-\boldsymbol{v})^{-r} (\boldsymbol{v} - \boldsymbol{v}^{-1})^{2r} \cdot \psi^{\pm}(z) e^{\pm}(\boldsymbol{v}^2 z)^r \otimes f^{\pm}(z)^{r+1}, \tag{11.11}$$

$$\Delta(\psi^{\pm}(z)) = \sum_{r=0}^{\infty} (-1)^r [r+1]_{\boldsymbol{v}} (\boldsymbol{v} - \boldsymbol{v}^{-1})^{2r} \cdot \psi^{\pm}(z) e^{\pm}(\boldsymbol{v}^2 z)^r \otimes f^{\pm}(\boldsymbol{v}^2 z)^r \psi^{\pm}(z). \tag{11.12}$$

These formulas are analogous to those for the Yangian $Y_\hbar(\mathfrak{sl}_2)$ of [51, Exercise 3.2]. The proof of this result is based on the RTT realization of $U_{\boldsymbol{v}}(L\mathfrak{sl}_2)$ and is presented in Appendix J.

Proposition 11.19 *Let $b_1, b_2 \in \mathbb{Z}_{\leq 0}$ and $b = b_1 + b_2$. Then, the homomorphism $\Delta_{b_1,b_2} \colon \mathcal{U}^{\mathrm{sc}}_{0,b} \to \mathcal{U}^{\mathrm{sc}}_{0,b_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2}$ from Theorem 10.5 also satisfies the formulas (11.10–11.12), where by abuse of notation $e^{\pm}(z), f^{\pm}(z), \psi^{\pm}(z)$ denote the generating series for each respective algebra.*

Proof Our proof is based on the commutative diagram of Remark 10.6:

$$\begin{array}{ccc} U^{\pm}_{\boldsymbol{v}} & \xrightarrow{\ \Delta\ } & U^{\pm}_{\boldsymbol{v}} \otimes U^{\pm}_{\boldsymbol{v}} \\ \Big\downarrow {\scriptstyle \jmath^{\pm}_{b_1,b_2}} & & \Big\downarrow {\scriptstyle \jmath^{\pm}_{b_1,0} \otimes \jmath^{\pm}_{0,b_2}} \\ \mathcal{U}^{\mathrm{sc},\pm}_{0,b_1,b_2} & \xrightarrow[\ \Delta_{b_1,b_2}\ ]{} & \mathcal{U}^{\mathrm{sc},\pm}_{0,b_1,0} \otimes \mathcal{U}^{\mathrm{sc},\pm}_{0,0,b_2} \end{array}$$

Since $\jmath^{+}_{\bullet,\bullet} \colon e^+(z) \mapsto e^+(z), f^+(z) \mapsto f^+(z), \psi^+(z) \mapsto \psi^+(z)$, we immediately get the validity of (11.10–11.12) for the currents $e^+(z), f^+(z), \psi^+(z)$ and the homomorphism Δ_{b_1,b_2}.

Let us now treat the case of $e^-(z), f^-(z), \psi^-(z)$. Combining the commutativity of the above diagram (in the "−" case) with equality (11.10) yields

$$\Delta_{b_1,b_2,}(\underline{e}^-(z)) = 1 \otimes \underline{e}^-(z) + \sum_{r=0}^{\infty} (-\boldsymbol{v})^r (\boldsymbol{v} - \boldsymbol{v}^{-1})^{2r} \cdot e^-(z)^{r+1} \otimes f^-(\boldsymbol{v}^2 z)^r \psi^-(z),$$

where $\underline{e}^-(z) := e^-(z) + \sum_{r=b_2}^{-1} e_r z^{-r}$. Meanwhile, $\Delta_{b_1,b_2}(e_r) = 1 \otimes e_r$ for $b_2 \leq r \leq -1$. Hence, $\Delta_{b_1,b_2}(e^-(z))$ is given by the right-hand side of (11.10). Likewise, we get the validity of (11.11), (11.12) for the currents $f^-(z), \psi^-(z)$ and the homomorphism Δ_{b_1,b_2}. □

Since our proof of (11.10–11.12) in Appendix J is based on the RTT-type coproduct $\Delta^{\mathrm{rtt}}_{0,0}$, we immediately get

Corollary 11.20 *Let $b_1, b_2 \in \mathbb{Z}_{\leq 0}$ and $b = b_1 + b_2$. The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{U}^{\mathrm{ad}}_{0,2b} & \xrightarrow{\Delta^{\mathrm{ad}}_{2b_1,2b_2}} & \mathcal{U}^{\mathrm{ad}}_{0,2b_1} \otimes \mathcal{U}^{\mathrm{ad}}_{0,2b_2} \\ {\scriptstyle \Upsilon_{0,2b}}\downarrow \wr & & \wr \downarrow {\scriptstyle \Upsilon_{0,2b_1}\otimes\Upsilon_{0,2b_2}} \\ \mathcal{U}^{\mathrm{rtt}}_{0,2b} & \xrightarrow[\Delta^{\mathrm{rtt}}_{2b_1,2b_2}]{} & \mathcal{U}^{\mathrm{rtt}}_{0,2b_1} \otimes \mathcal{U}^{\mathrm{rtt}}_{0,2b_2} \end{array}$$

11.7 Coproduct for Truncated Shifted Algebras, $\mathfrak{g} = \mathfrak{sl}_2$

For $b_1, b_2 \in \mathbb{Z}_{\leq 0}$ and $b = b_1 + b_2$, recall the homomorphism $\Delta^{\mathrm{ad}}_{2b_1,2b_2} \colon \mathcal{U}^{\mathrm{ad}}_{0,2b} \to \mathcal{U}^{\mathrm{ad}}_{0,2b_1} \otimes \mathcal{U}^{\mathrm{ad}}_{0,2b_2}$ of Remark 10.7. Consider the truncated versions of the algebras involved $\mathcal{U}^0_{2b}, \mathcal{U}^0_{2b_1}, \mathcal{U}^0_{2b_2}$, see Definition 8.6. The goal of this subsection is to prove the following result.

Proposition 11.21 *For $b_1, b_2 \leq 0$, the homomorphism $\Delta^{\mathrm{ad}}_{2b_1,2b_2}$ descends to the same named homomorphism $\mathcal{U}^0_{2b} \to \mathcal{U}^0_{2b_1} \otimes \mathcal{U}^0_{2b_2}$.*

Proof Define a 2-sided ideal $\mathcal{J} \subset \mathcal{U}^{\mathrm{ad}}_{0,2b_1} \otimes \mathcal{U}^{\mathrm{ad}}_{0,2b_2}$ via $\mathcal{J} := \mathcal{J}^0_{2b_1} \otimes \mathcal{U}^{\mathrm{ad}}_{0,2b_2} + \mathcal{U}^{\mathrm{ad}}_{0,2b_1} \otimes \mathcal{J}^0_{2b_2}$. It suffices to show that $\Delta^{\mathrm{ad}}_{2b_1,2b_2}(X) \in \mathcal{J}$ for every generator X of the ideal $\mathcal{J}^0_{2b}$ of (8.5–8.6). To achieve this, recall the commutative diagram of Corollary 11.20.

Case $X = A^+_s (s > -b)$ Applying the aforementioned commutative diagram to the equality $\Delta^{\mathrm{rtt}}_{2b_1,2b_2}(t^+_{11}[s]) = \sum_{s_1,s_2\geq 0}^{s_1+s_2=s} t^+_{11}[s_1] \otimes t^+_{11}[s_2] + \sum_{s_1,s_2\geq 0}^{s_1+s_2=s} t^+_{12}[s_1] \otimes t^+_{21}[s_2]$, we get $\Delta^{\mathrm{ad}}_{2b_1,2b_2}(A^+_s) = \sum_{s_1,s_2\geq 0}^{s_1+s_2=s} A^+_{s_1} \otimes A^+_{s_2} + \sum_{s_1,s_2\geq 0}^{s_1+s_2=s} B^+_{s_1} \otimes C^+_{s_2}$. For $s_1 + s_2 = s > -b$, either $s_1 > -b_1$ or $s_2 > -b_2$. Hence, each summand in the right-hand side belongs to $\mathcal{J}$, due to Remark 8.8.

Case $X = A^+_0 A^+_{-b} - (-1)^b$ As above $\Delta^{\mathrm{ad}}_{2b_1,2b_2}(A^+_{-b}) \equiv A^+_{-b_1} \otimes A^+_{-b_2}$, where the notation $x \equiv y$ is used to denote $x - y \in \mathcal{J}$. We also have $\Delta^{\mathrm{ad}}_{2b_1,2b_2}(A^+_0) = A^+_0 \otimes A^+_0$. Thus $\Delta^{\mathrm{ad}}_{2b_1,2b_2}(A^+_0 A^+_{-b} - (-1)^b) \equiv A^+_0 A^+_{-b_1} \otimes A^+_0 A^+_{-b_2} - (-1)^b = (A^+_0 A^+_{-b_1} - (-1)^{b_1}) \otimes A^+_0 A^+_{-b_2} + (-1)^{b_1} \otimes (A^+_0 A^+_{-b_2} - (-1)^{b_2}) \equiv 0$. Hence, $\Delta^{\mathrm{ad}}_{2b_1,2b_2}(A^+_0 A^+_{-b} - (-1)^b) \in \mathcal{J}$.

Case $X = A^-_{-r} - \boldsymbol{v}^{-b} A^+_{-b-r}$ $(0 \leq r \leq -b)$ Analogously to the first case considered above, we have $\Delta^{\mathrm{ad}}_{2b_1,2b_2}(A^+_{-b-r}) \equiv \sum^{r_1+r_2=r}_{\substack{0\leq r_1\leq -b_1\\ 0\leq r_2\leq -b_2}} A^+_{-b_1-r_1} \otimes A^+_{-b_2-r_2} + \sum^{r_1+r_2=r}_{\substack{1\leq r_1\leq -b_1\\ 0\leq r_2\leq -b_2-1}} B^+_{-b_1-r_1} \otimes C^+_{-b_2-r_2}$, where the lower bounds on r_1, r_2 are due to Remark 8.8. Completely analogously, we obtain $\Delta^{\mathrm{ad}}_{2b_1,2b_2}(A^-_{-r}) \equiv$

$\sum_{\substack{0\le r_1\le -b_1\\ 0\le r_2\le -b_2}}^{r_1+r_2=r} A^-_{-r_1}\otimes A^-_{-r_2}+\sum_{\substack{1\le r_1\le -b_1\\ 0\le r_2\le -b_2-1}}^{r_1+r_2=r} B^-_{-r_1}\otimes C^-_{-r_2}$. Hence,

$$\Delta^{\mathrm{ad}}_{2b_1,2b_2}(A^-_{-r}-\boldsymbol{v}^{-b}A^+_{-b-r})\equiv\sum_{\substack{0\le r_1\le -b_1\\ 0\le r_2\le -b_2}}^{r_1+r_2=r}(A^-_{-r_1}\otimes A^-_{-r_2}-\boldsymbol{v}^{-b}A^+_{-b_1-r_1}\otimes A^+_{-b_2-r_2})+$$
$$\sum_{\substack{1\le r_1\le -b_1\\ 0\le r_2\le -b_2-1}}^{r_1+r_2=r}(B^-_{-r_1}\otimes C^-_{-r_2}-\boldsymbol{v}^{-b}B^+_{-b_1-r_1}\otimes C^+_{-b_2-r_2}). \tag{11.13}$$

The first sum of (11.13) belongs to $\mathcal{J}$ as $A^-_{-r_1}\otimes A^-_{-r_2}-\boldsymbol{v}^{-b}A^+_{-b_1-r_1}\otimes A^+_{-b_2-r_2}=(A^-_{-r_1}-\boldsymbol{v}^{-b_1}A^+_{-b_1-r_1})\otimes A^-_{-r_2}+\boldsymbol{v}^{-b_1}A^+_{-b_1-r_1}\otimes(A^-_{-r_2}-\boldsymbol{v}^{-b_2}A^+_{-b_2-r_2})\in\mathcal{J}$. Completely analogously, $B^-_{-r_1}\otimes C^-_{-r_2}-\boldsymbol{v}^{-b}B^+_{-b_1-r_1}\otimes C^+_{-b_2-r_2}=(B^-_{-r_1}-\boldsymbol{v}^{-b_1}B^+_{-b_1-r_1})\otimes C^-_{-r_2}+\boldsymbol{v}^{-b_1}B^+_{-b_1-r_1}\otimes(C^-_{-r_2}-\boldsymbol{v}^{-b_2}C^+_{-b_2-r_2})$. To complete the proof, it suffices to show

$$\begin{aligned}&B^-_{-r_1}-\boldsymbol{v}^{-b_1}B^+_{-b_1-r_1}\in\mathcal{J}^0_{2b_1}\ \text{ for } 1\le r_1\le -b_1,\\ &C^-_{-r_2}-\boldsymbol{v}^{-b_2}C^+_{-b_2-r_2}\in\mathcal{J}^0_{2b_2}\ \text{ for } 0\le r_2\le -b_2-1.\end{aligned} \tag{11.14}$$

To prove the first inclusion of (11.14), recall that $B^+(z)=[e_0,A^+(z)]_{\boldsymbol{v}^{-1}}$, due to Corollary 7.3. Likewise (comparing the terms of degree 1 in w in the equality (6.10) with $\epsilon=-,\epsilon'=+$), we obtain $B^-(z)=[e_0,A^-(z)]_{\boldsymbol{v}^{-1}}$. Therefore,

$$B^-_{-r_1}-\boldsymbol{v}^{-b_1}B^+_{-b_1-r_1}=[e_0,A^-_{-r_1}-\boldsymbol{v}^{-b_1}A^+_{-b_1-r_1}]_{\boldsymbol{v}^{-1}}\in\mathcal{J}^0_{2b_1}.$$

Similarly, applying the equalities $zC^\pm(z)=[A^\pm(z),f_1]_{\boldsymbol{v}^{-1}}$, we obtain

$$C^-_{-r_2}-\boldsymbol{v}^{-b_2}C^+_{-b_2-r_2}=[A^-_{-r_2-1}-\boldsymbol{v}^{-b_2}A^+_{-b_2-r_2-1},f_1]_{\boldsymbol{v}^{-1}}\in\mathcal{J}^0_{2b_2},$$

which implies the second inclusion of (11.14). Thus, $\Delta^{\mathrm{ad}}_{2b_1,2b_2}(A^-_{-r}-\boldsymbol{v}^{-b}A^+_{-b-r})\in\mathcal{J}$.

The cases when X is one of $A^-_{-s}(s>-b)$, $A^-_0A^-_b-(-\boldsymbol{v}^2)^{-b}$ are treated analogously to the above first two cases. This completes our proof. □

11.8 Coproduct for Truncated Shifted Algebras, General $\mathfrak{g}$

Recall the homomorphism $\Delta_{\mu_1,\mu_2}\colon\mathcal{U}^{\mathrm{sc}}_{0,\mu}\to\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}\otimes\mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$ of Theorem 10.20 ($\mu=\mu_1+\mu_2,\ \mathfrak{g}=\mathfrak{sl}_n$). Given $N=N_1+N_2$, this coproduct extends to

$$\Delta^{\mathrm{ad}}_{\mu_1,\mu_2}\colon\mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm1},\ldots,\mathsf{z}_N^{\pm1}]\longrightarrow\mathcal{U}^{\mathrm{ad}}_{0,\mu_1}[\mathsf{z}_1^{\pm1},\ldots,\mathsf{z}_{N_1}^{\pm1}]\otimes\mathcal{U}^{\mathrm{ad}}_{0,\mu_2}[\mathsf{z}_{N_1+1}^{\pm1},\ldots,\mathsf{z}_N^{\pm1}]$$

as in Remark 10.3(c). Given two sequences $\underline{\lambda}^{(1)} = (\omega_{i_1}, \ldots, \omega_{i_{N_1}}), \underline{\lambda}^{(2)} = (\omega_{i_{N_1+1}}, \ldots, \omega_{i_N})$, we concatenate them to $\underline{\lambda} = (\omega_{i_1}, \ldots, \omega_{i_N})$ and consider the corresponding truncated shifted algebras $\mathcal{U}_{\mu}^{\underline{\lambda}}, \mathcal{U}_{\mu_1}^{\underline{\lambda}^{(1)}}, \mathcal{U}_{\mu_2}^{\underline{\lambda}^{(2)}}$ as in Definition 8.6.

Conjecture 11.22 The aforementioned homomorphism $\Delta^{\mathrm{ad}}_{\mu_1,\mu_2}$ descends to the same named homomorphism $\Delta^{\mathrm{ad}}_{\mu_1,\mu_2} \colon \mathcal{U}_{\mu}^{\underline{\lambda}} \to \mathcal{U}_{\mu_1}^{\underline{\lambda}^{(1)}} \otimes \mathcal{U}_{\mu_2}^{\underline{\lambda}^{(2)}}$.

We hope that the comultiplication $\Delta^{\mathrm{ad}}_{\mu_1,\mu_2}$ can be defined for arbitrary simply-laced $\mathfrak{g}$ (see Sect. 10.8) and descends to the truncated shifted algebras.

12 *K*-theory of Parabolic Laumon Spaces

12.1 Parabolic Laumon Spaces

We recall the setup of [7]. Let $\mathbf{C}$ be a smooth projective curve of genus zero. We fix a coordinate z on $\mathbf{C}$, and consider the action of $\mathbb{C}^\times$ on $\mathbf{C}$ such that $\boldsymbol{v}(z) = \boldsymbol{v}^{-2}z$. We have $\mathbf{C}^{\mathbb{C}^\times} = \{0, \infty\}$.

We consider an N-dimensional vector space W with a basis $w_1, \ldots, w_N$. This defines a Cartan torus $T \subset G = GL(N) = GL(W)$. We also consider its 2^N-fold cover, the bigger torus $\widetilde{T}$, acting on W as follows: for $\widetilde{T} \ni \underline{t} = (t_1, \ldots, t_N)$ we have $\underline{t}(w_i) = t_i^2 w_i$.

We fix an n-tuple of positive integers $\pi = (p_1, \ldots, p_n) \in \mathbb{Z}^n_{>0}$ such that $p_1 + \ldots + p_n = N$. Let $P \subset G$ be a parabolic subgroup preserving the flag $0 \subset W_1 := \langle w_1, \ldots, w_{p_1}\rangle \subset W_2 := \langle w_1, \ldots, w_{p_1+p_2}\rangle \subset \cdots \subset W_{n-1} := \langle w_1, \ldots, w_{p_1+\ldots+p_{n-1}}\rangle \subset W_n := W$. Let $\mathcal{B} := G/P$ be the corresponding partial flag variety.

Given an $(n-1)$-tuple of nonnegative integers $\underline{d} = (d_1, \ldots, d_{n-1}) \in \mathbb{N}^{n-1}$, we consider the Laumon parabolic quasiflags' space $\mathcal{Q}_{\underline{d}}$, see [46, § 4.2]. It is the moduli space of flags of locally free subsheaves

$$0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W} = W \otimes \mathcal{O}_{\mathbf{C}}$$

such that $\mathrm{rank}(\mathcal{W}_i) = p_1 + \ldots + p_i$ and $\deg(\mathcal{W}_i) = -d_i$. It is known to be a smooth connected projective variety of dimension $\dim \mathcal{B} + \sum_{i=1}^{n-1} d_i(p_i + p_{i+1})$, see [46, § 2.10].

We consider the following locally closed subvariety $\mathfrak{Q}_{\underline{d}} \subset \mathcal{Q}_{\underline{d}}$ (parabolic quasiflags based at $\infty \in \mathbf{C}$) formed by the flags

$$0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W} = W \otimes \mathcal{O}_{\mathbf{C}}$$

such that $\mathcal{W}_i \subset \mathcal{W}$ is a vector subbundle in a neighborhood of $\infty \in \mathbf{C}$, and the fiber of $\mathcal{W}_i$ at ∞ equals the span $\langle w_1, \ldots, w_{p_1+\ldots+p_i} \rangle \subset W$. It is known to be a smooth connected quasiprojective variety of dimension $\sum_{i=1}^{n-1} d_i(p_i + p_{i+1})$.

12.2 Fixed Points

The group $G \times \mathbb{C}^\times$ acts naturally on $\mathcal{Q}_{\underline{d}}$, and the group $\widetilde{T} \times \mathbb{C}^\times$ acts naturally on $\mathfrak{Q}_{\underline{d}}$. The set of fixed points of $\widetilde{T} \times \mathbb{C}^\times$ on $\mathfrak{Q}_{\underline{d}}$ is finite; its description is given in [7, § 4.4].

Let $\vec{\underline{d}}$ be a collection of nonnegative integral vectors $\vec{d}_{ij} = (d_{ij}^{(1)}, \ldots, d_{ij}^{(p_j)})$, $n-1 \geq i \geq j \geq 1$, such that $d_i = \sum_{j=1}^{i} |d_{ij}| = \sum_{j=1}^{i} \sum_{a=1}^{p_j} d_{ij}^{(a)}$, and for $i \geq k \geq j$ we have $\vec{d}_{kj} \geq \vec{d}_{ij}$, i.e., $d_{kj}^{(a)} \geq d_{ij}^{(a)}$ for any $1 \leq a \leq p_j$. Abusing notation, we denote by $\vec{\underline{d}}$ the corresponding $\widetilde{T} \times \mathbb{C}^\times$-fixed point in $\mathfrak{Q}_{\underline{d}}$:

$\mathcal{W}_1 = \mathcal{O}_{\mathbf{C}}(-d_{11}^{(1)} \cdot 0)w_1 \oplus \cdots \oplus \mathcal{O}_{\mathbf{C}}(-d_{11}^{(p_1)} \cdot 0)w_{p_1}$,

$\mathcal{W}_2 = \mathcal{O}_{\mathbf{C}}(-d_{21}^{(1)} \cdot 0)w_1 \oplus \cdots \oplus \mathcal{O}_{\mathbf{C}}(-d_{21}^{(p_1)} \cdot 0)w_{p_1} \oplus \mathcal{O}_{\mathbf{C}}(-d_{22}^{(1)} \cdot 0)w_{p_1+1} \oplus \cdots \oplus \mathcal{O}_{\mathbf{C}}(-d_{22}^{(p_2)} \cdot 0)w_{p_1+p_2}$,

$\vdots$

$\mathcal{W}_{n-1} = \mathcal{O}_{\mathbf{C}}(-d_{n-1,1}^{(1)} \cdot 0)w_1 \oplus \cdots \oplus \mathcal{O}_{\mathbf{C}}(-d_{n-1,1}^{(p_1)} \cdot 0)w_{p_1} \oplus \cdots$
$\cdots \oplus \mathcal{O}_{\mathbf{C}}(-d_{n-1,n-1}^{(1)} \cdot 0)w_{p_1+\ldots+p_{n-2}+1} \oplus \cdots \oplus \mathcal{O}_{\mathbf{C}}(-d_{n-1,n-1}^{(p_{n-1})} \cdot 0)w_{p_1+\ldots+p_{n-1}}$.

Notation Given a collection $\vec{\underline{d}}$ as above, we will denote by $\vec{\underline{d}} \pm \delta_{ij}^{(p)}$ the collection $\vec{\underline{d}}'$, such that $d_{ij}'^{(p)} = d_{ij}^{(p)} \pm 1$, while $d_{kl}'^{(a)} = d_{kl}^{(a)}$ for $(a, k, l) \neq (p, i, j)$.

12.3 Correspondences

For $i \in \{1, \ldots, n-1\}$ and $\underline{d} = (d_1, \ldots, d_{n-1})$, we set $\underline{d} + i := (d_1, \ldots, d_i + 1, \ldots, d_{n-1})$. We have a correspondence $\mathcal{E}_{\underline{d},i} \subset \mathcal{Q}_{\underline{d}} \times \mathcal{Q}_{\underline{d}+i}$ formed by the pairs $(\mathcal{W}_\bullet, \mathcal{W}'_\bullet)$ such that $\mathcal{W}'_i \subset \mathcal{W}_i$ and we have $\mathcal{W}_j = \mathcal{W}'_j$ for $j \neq i$, see [7, § 4.5]. In other words, $\mathcal{E}_{\underline{d},i}$ is the moduli space of flags of locally free sheaves

$$0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{i-1} \subset \mathcal{W}'_i \subset \mathcal{W}_i \subset \mathcal{W}_{i+1} \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W}$$

such that $\operatorname{rank}(\mathcal{W}_j) = p_1 + \ldots + p_j$ and $\deg(\mathcal{W}_j) = -d_j$, while $\operatorname{rank}(\mathcal{W}'_i) = p_1 + \ldots + p_i$ and $\deg(\mathcal{W}'_i) = -d_i - 1$. According to [46, § 2.10], $\mathcal{E}_{\underline{d},i}$ is a smooth projective algebraic variety of dimension $\dim \mathcal{B} + \sum_{i=1}^{n-1} d_i(p_i + p_{i+1}) + p_i$.

We denote by $\mathbf{p}$ (resp. $\mathbf{q}$) the natural projection $\mathcal{E}_{\underline{d},i} \to \mathcal{Q}_{\underline{d}}$ (resp. $\mathcal{E}_{\underline{d},i} \to \mathcal{Q}_{\underline{d}+i}$). We also have a map $\mathbf{s}\colon \mathcal{E}_{\underline{d},i} \to \mathbf{C}$,

$$(0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{i-1} \subset \mathcal{W}'_i \subset \mathcal{W}_i \subset \mathcal{W}_{i+1} \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W}) \mapsto \operatorname{supp}(\mathcal{W}_i/\mathcal{W}'_i).$$

The correspondence $\mathcal{E}_{\underline{d},i}$ comes equipped with a natural line bundle $\mathcal{L}_i$ whose fiber at a point

$$(0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_{i-1} \subset \mathcal{W}'_i \subset \mathcal{W}_i \subset \mathcal{W}_{i+1} \subset \cdots \subset \mathcal{W}_{n-1} \subset \mathcal{W})$$

equals $\Gamma(\mathbf{C}, \mathcal{W}_i/\mathcal{W}'_i)$. Finally, we have a transposed correspondence ${}^{\mathsf{T}}\mathcal{E}_{\underline{d},i} \subset \mathcal{Q}_{\underline{d}+i} \times \mathcal{Q}_{\underline{d}}$.

Restricting to $\mathfrak{Q}_{\underline{d}} \subset \mathcal{Q}_{\underline{d}}$, we obtain the correspondence $\mathsf{E}_{\underline{d},i} \subset \mathfrak{Q}_{\underline{d}} \times \mathfrak{Q}_{\underline{d}+i}$ together with the line bundle L_i and the natural maps $\mathbf{p}\colon \mathsf{E}_{\underline{d},i} \to \mathfrak{Q}_{\underline{d}}$, $\mathbf{q}\colon \mathsf{E}_{\underline{d},i} \to \mathfrak{Q}_{\underline{d}+i}$, $\mathbf{s}\colon \mathsf{E}_{\underline{d},i} \to \mathbf{C}\backslash\{\infty\}$. We also have a transposed correspondence ${}^{\mathsf{T}}\mathsf{E}_{\underline{d},i} \subset \mathfrak{Q}_{\underline{d}+i} \times \mathfrak{Q}_{\underline{d}}$. It is a smooth quasiprojective variety of dimension $\sum_{i=1}^{n-1} d_i(p_i + p_{i+1}) + p_i$.

12.4 Equivariant K-groups

We denote by ${}'M(\pi)$ the direct sum of equivariant (complexified) K-groups:

$${}'M(\pi) = \bigoplus_{\underline{d}} K^{\widetilde{T}\times\mathbb{C}^\times}(\mathfrak{Q}_{\underline{d}}).$$

It is a module over $K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{C}[\widetilde{T} \times \mathbb{C}^\times] = \mathbb{C}[t_1^{\pm 1}, \ldots, t_N^{\pm 1}, \boldsymbol{v}^{\pm 1}]$. We define

$$M(\pi) := {}'M(\pi) \otimes_{K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt})} \mathrm{Frac}(K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt})).$$

It is naturally graded

$$M(\pi) = \oplus_{\underline{d}} M(\pi)_{\underline{d}}, \text{ where } M(\pi)_{\underline{d}} = K^{\widetilde{T}\times\mathbb{C}^\times}(\mathfrak{Q}_{\underline{d}}) \otimes_{K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt})} \mathrm{Frac}(K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt})).$$

According to the Thomason localization theorem, restriction to the $\widetilde{T} \times \mathbb{C}^\times$-fixed point set induces an isomorphism

$$K^{\widetilde{T}\times\mathbb{C}^\times}(\mathfrak{Q}_{\underline{d}}) \otimes_{K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt})} \mathrm{Frac}(K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt})) \xrightarrow{\sim} K^{\widetilde{T}\times\mathbb{C}^\times}(\mathfrak{Q}_{\underline{d}}^{\widetilde{T}\times\mathbb{C}^\times}) \otimes_{K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt})} \mathrm{Frac}(K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt})).$$

The classes of the structure sheaves $[\vec{\underline{d}}]$ of the $\widetilde{T} \times \mathbb{C}^\times$-fixed points $\vec{\underline{d}}$ (see Sect. 12.2) form a basis in $\bigoplus_{\underline{d}} K^{\widetilde{T}\times\mathbb{C}^\times}(\mathfrak{Q}_{\underline{d}}^{\widetilde{T}\times\mathbb{C}^\times}) \otimes_{K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt})} \mathrm{Frac}(K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt}))$. The embedding of a point $\vec{\underline{d}}$ into $\mathfrak{Q}_{\underline{d}}$ is a proper morphism, so the direct image in the equivariant K-theory is well-defined, and we will denote by $[\vec{\underline{d}}] \in M(\pi)_{\underline{d}}$ the direct image of the structure sheaf of the point $\vec{\underline{d}}$. The set $\{[\vec{\underline{d}}]\}$ forms a basis of $M(\pi)$.

12.5 Action of $\mathfrak{U}^{\boldsymbol{v}}_{\boldsymbol{\pi}}$ on $\boldsymbol{M(\pi)}$

From now on, we will denote by $\mathfrak{U}^{\boldsymbol{v}}_{\pi}$ the shifted quantum affine algebra $\mathfrak{U}^{\mathrm{sc}}_{0,\mu}$ for $\mathfrak{g} = \mathfrak{sl}_n$ and $\mu = \sum_{j=1}^{n-1}(p_{j+1} - p_j)\omega_j$. We will also need the characters T_i of $\widetilde{T} \times \mathbb{C}^\times$ defined via $T_i := \prod_{j=p_1+\ldots+p_{i-1}+1}^{p_1+\ldots+p_i} t_j$. Let $\boldsymbol{v}$ stand for the character of $\widetilde{T} \times \mathbb{C}^\times : (\underline{t}, \boldsymbol{v}) \mapsto \boldsymbol{v}$.

For any $0 \le i \le n$, we will denote by $\underline{\mathcal{W}}_i$ the tautological $(p_1 + \ldots + p_i)$-dimensional vector bundle on $\mathfrak{Q}_{\underline{d}} \times \mathbf{C}$. Let $\varpi : \mathfrak{Q}_{\underline{d}} \times (\mathbf{C}\backslash\{\infty\}) \to \mathfrak{Q}_{\underline{d}}$ denote the standard projection. We define the generating series $\mathbf{b}_i(z)$ with coefficients in the equivariant K-theory of $\mathfrak{Q}_{\underline{d}}$ as follows:

$$\mathbf{b}_i(z) := \Lambda^\bullet_{-1/z}(\varpi_*(\underline{\mathcal{W}}_i|_{\mathbf{C}\backslash\{\infty\}})) = 1 + \sum_{r\ge 1} \Lambda^r(\varpi_*(\underline{\mathcal{W}}_i|_{\mathbf{C}\backslash\{\infty\}}))(-z^{-1})^r.$$

We also define the operators

$$e_{i,r} := T_{i+1}^{-1}\boldsymbol{v}^{d_{i+1}-d_i+2-i}\mathbf{p}_*((\boldsymbol{v}^i\mathsf{L}_i)^{\otimes r} \otimes \mathbf{q}^*) \colon M(\pi)_{\underline{d}} \to M(\pi)_{\underline{d}-i}, \tag{12.1}$$

$$f_{i,r} := T_i^{-1}\boldsymbol{v}^{d_i-d_{i-1}+i}\mathbf{q}_*((-\mathsf{L}_i)^{\otimes p_i} \otimes (\boldsymbol{v}^i\mathsf{L}_i)^{\otimes r} \otimes \mathbf{p}^*) \colon M(\pi)_{\underline{d}} \to M(\pi)_{\underline{d}+i}, \tag{12.2}$$

and consider the following generating series of operators on $M(\pi)$:

$$e_i(z) = \sum_{r=-\infty}^{\infty} e_{i,r}z^{-r} \colon M(\pi)_{\underline{d}} \to M(\pi)_{\underline{d}-i}[[z, z^{-1}]], \tag{12.3}$$

$$f_i(z) = \sum_{r=-\infty}^{\infty} f_{i,r}z^{-r} \colon M(\pi)_{\underline{d}} \to M(\pi)_{\underline{d}+i}[[z, z^{-1}]]. \tag{12.4}$$

We define $\psi_i^+(z) \colon M(\pi)_{\underline{d}} \to M(\pi)_{\underline{d}}[[z^{-1}]]$ and $\psi_i^-(z) \colon M(\pi)_{\underline{d}} \to z^{p_i-p_{i+1}}M(\pi)_{\underline{d}}[[z]]$ via

$$\psi_i^\pm(z) := T_{i+1}^{-1}T_i\boldsymbol{v}^{d_{i+1}-2d_i+d_{i-1}} \cdot \left(\frac{\mathbf{b}_{i+1}(z\boldsymbol{v}^{-i-2})\mathbf{b}_{i-1}(z\boldsymbol{v}^{-i})}{\mathbf{b}_i(z\boldsymbol{v}^{-i-2})\mathbf{b}_i(z\boldsymbol{v}^{-i})}\right)^\pm, \tag{12.5}$$

where as before $\gamma(z)^\pm$ denotes the expansion of a rational function $\gamma(z)$ in $z^{\mp 1}$, respectively.

Notation To each $\vec{\underline{d}}$, we assign a collection of $\widetilde{T} \times \mathbb{C}^\times$-weights $s_{ij}^{(a)} := t^2_{p_1+\ldots+p_{j-1}+a}\boldsymbol{v}^{-2d_{ij}^{(a)}}$.

Proposition 12.1

(a) The matrix coefficients of the operators $f_{i,r}, e_{i,r}$ in the fixed point basis $\{[\underline{\vec{d}}]\}$ of $M(\pi)$ are as follows:

$$f_{i,r[\underline{\vec{d}},\underline{\vec{d}}']} = T_i^{-1}\boldsymbol{v}^{d_i-d_{i-1}+i}(1-\boldsymbol{v}^2)^{-1}(-s_{ij}^{(a)})^{p_i}(s_{ij}^{(a)}\boldsymbol{v}^i)^r\frac{\prod_{j'\le i-1}^{a'\le p_{j'}}(1-s_{ij}^{(a)}/s_{i-1,j'}^{(a')})}{\prod_{j'\le i,a'\le p_{j'}}^{(j',a')\ne(j,a)}(1-s_{ij}^{(a)}/s_{ij'}^{(a')})}$$

if $\underline{\vec{d}}' = \underline{\vec{d}} + \delta_{ij}^{(a)}$ for certain $j \le i$, $1 \le a \le p_j$;

$$e_{i,r[\underline{\vec{d}},\underline{\vec{d}}']} = T_{i+1}^{-1}\boldsymbol{v}^{d_{i+1}-d_i+2-i}(1-\boldsymbol{v}^2)^{-1}(s_{ij}^{(a)}\boldsymbol{v}^{i+2})^r\frac{\prod_{j'\le i+1}^{a'\le p_{j'}}(1-s_{i+1,j'}^{(a')}/s_{ij}^{(a)})}{\prod_{j'\le i,a'\le p_{j'}}^{(j',a')\ne(j,a)}(1-s_{ij'}^{(a')}/s_{ij}^{(a)})}$$

if $\underline{\vec{d}}' = \underline{\vec{d}} - \delta_{ij}^{(a)}$ for certain $j \le i$, $1 \le a \le p_j$.
All the other matrix coefficients of $e_{i,r}$, $f_{i,r}$ vanish.

(b) The eigenvalue $\psi_i^\pm(z)|_{\underline{\vec{d}}}$ of $\psi_i^\pm(z)$ on $[\underline{\vec{d}}]$ equals

$$T_{i+1}^{-1}T_i\boldsymbol{v}^{d_{i+1}-2d_i+d_{i-1}}\left(\frac{\prod_{j\le i+1}^{a\le p_j}(1-z^{-1}\boldsymbol{v}^{i+2}s_{i+1,j}^{(a)})\prod_{j\le i-1}^{a\le p_j}(1-z^{-1}\boldsymbol{v}^i s_{i-1,j}^{(a)})}{\prod_{j\le i}^{a\le p_j}(1-z^{-1}\boldsymbol{v}^{i+2}s_{ij}^{(a)})\prod_{j\le i}^{a\le p_j}(1-z^{-1}\boldsymbol{v}^i s_{ij}^{(a)})}\right)^{\pm}.$$

The proof is straightforward and is analogous to that of [61, Proposition 2.15]. The following is the key result of this section.

Theorem 12.2 *The generating series of operators $\{\psi_i^\pm(z), e_i(z), f_i(z)\}_{i=1}^{n-1}$ of (12.3–12.5) acting on $M(\pi)$ satisfy the relations in $\mathcal{U}_\pi^{\boldsymbol{v}}$, i.e., they give rise to the action of $\mathcal{U}_\pi^{\boldsymbol{v}}$ on $M(\pi)$.*

In the particular case $\pi = 1^n$, we recover [61, Theorem 2.12].

Proof First, note that $\psi_i^+(z)$ contains only nonpositive powers of z, while $\psi_i^-(z)$ contains only powers of z bigger or equal to $p_i - p_{i+1}$ (this follows from Proposition 12.1(b)). Moreover, the coefficients of z^0 in $\psi_i^+(z)$ and of $z^{p_i-p_{i+1}}$ in $\psi_i^-(z)$ are invertible operators.

Applying Proposition 12.1, the verification of all the defining relations of $\mathcal{U}_\pi^{\boldsymbol{v}}$, except for (U6), boils down to routine straightforward computations in the fixed point basis (compare to the proof of [61, Theorem 2.12]). The same arguments can be used to show that $[e_i(z), f_j(w)] = 0$ for $i \ne j$. It remains to prove $(\boldsymbol{v}-\boldsymbol{v}^{-1})[e_i(z), f_i(w)] = \delta\left(\frac{z}{w}\right)\left(\psi_i^+(z)-\psi_i^-(z)\right)$. Applying Proposition 12.1(a), we see that the left-hand side is diagonal in the fixed point basis and its eigenvalue on $[\underline{\vec{d}}]$ equals

$$T_{i+1}^{-1}T_i^{-1}\boldsymbol{v}^{d_{i+1}-d_{i-1}}(1-\boldsymbol{v}^2)^{-1}\cdot\delta\left(\frac{z}{w}\right)\times$$

$$\sum_{j\le i}^{a\le p_j}(-s_{ij}^{(a)})^{p_i}\left\{\boldsymbol{v}^{2p_i}\frac{\prod_{j'\le i+1}^{a'\le p_{j'}}(1-s_{i+1,j'}^{(a')}/s_{ij}^{(a)})\prod_{j'\le i-1}^{a'\le p_{j'}}(1-\boldsymbol{v}^2 s_{ij}^{(a)}/s_{i-1,j'}^{(a')})}{\prod_{j'\le i,a'\le p_{j'}}^{(j',a')\ne(j,a)}(1-s_{ij'}^{(a')}/s_{ij}^{(a)})(1-\boldsymbol{v}^2 s_{ij}^{(a)}/s_{ij'}^{(a')})}\delta\left(\frac{z}{\boldsymbol{v}^{i+2}s_{ij}^{(a)}}\right)-\right.$$

$$\left.\frac{\prod_{j'\le i+1}^{a'\le p_{j'}}(1-\boldsymbol{v}^2 s_{i+1,j'}^{(a')}/s_{ij}^{(a)})\prod_{j'\le i-1}^{a'\le p_{j'}}(1-s_{ij}^{(a)}/s_{i-1,j'}^{(a')})}{\prod_{j'\le i,a'\le p_{j'}}^{(j',a')\ne(j,a)}(1-\boldsymbol{v}^2 s_{ij'}^{(a')}/s_{ij}^{(a)})(1-s_{ij}^{(a)}/s_{ij'}^{(a')})}\delta\left(\frac{z}{\boldsymbol{v}^i s_{ij}^{(a)}}\right)\right\}.$$

To compare this expression with the eigenvalue of $\psi_i^+(z)-\psi_i^-(z)$ on $[\underline{\vec{d}}]$, it suffices to apply Lemma C.1 below to the particular case of $\gamma(z)$ chosen to be the rational function of Proposition 12.1(b).

The theorem is proved. □

Remark 12.3

(a) The above verification of (U6) by applying Lemma C.1 significantly simplifies our original indirect proof of this relation in [61].
(b) For $\pi = p^n$, this produces the action of the quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ on $M(\pi)$.
(c) According to [4], there is an action of $\mathcal{A}_{\mathrm{frac}}^{\boldsymbol{v}}$ on $M(\pi)$. Its pull-back along the homomorphism $\overline{\Phi}_\mu^{\underline{\lambda}}$ ($\underline{\lambda} = (\omega_{n-1},\ldots,\omega_{n-1})$ taken N times) yields essentially the action of $\mathcal{U}_\pi^{\boldsymbol{v}}$ on $M(\pi)$ established above. In particular, the kernel $\mathrm{Ker}(\overline{\Phi}_\mu^{\underline{\lambda}}) = \mathrm{Ker}(\widetilde{\Phi}_\mu^{\underline{\lambda}})$ acts trivially on $M(\pi)$. The first instance of that is the fact that the generators $\{A_{i,\pm r}^{\pm} : r > p_1+\ldots+p_i\}$ of $\mathcal{U}_\pi^{\boldsymbol{v}}$ (see Remark 6.7(b)) act trivially on $M(\pi)$, due to the observation that the eigenvalue of $A_i^{\pm}(z)$ on $[\underline{\vec{d}}]$ equals $\prod_{j\le i}^{a\le p_j}(1-(z^{-1}\boldsymbol{v}^i s_{ij}^{(a)})^{\pm1})$.

12.6 Tensor Products

Fix two n-tuples $\pi' = (p_1',\ldots,p_n'), \pi'' = (p_1'',\ldots,p_n'') \in \mathbb{Z}_{>0}^n$ and define $\pi = (p_1,\ldots,p_n)$ via $p_i := p_i' + p_i'' \in \mathbb{Z}_{>0}$. Let $\mathcal{U}_{\pi'}^{\boldsymbol{v}}, \mathcal{U}_{\pi''}^{\boldsymbol{v}}, \mathcal{U}_\pi^{\boldsymbol{v}}$ be the corresponding shifted quantum affine algebras of $\mathfrak{sl}_n$ as defined in Sect. 12.5. According to Theorem 12.2, we have natural actions of $\mathcal{U}_\pi^{\boldsymbol{v}}$ on $M(\pi)$, of $\mathcal{U}_{\pi'}^{\boldsymbol{v}}$ on $M(\pi')$, and of $\mathcal{U}_{\pi''}^{\boldsymbol{v}}$ on $M(\pi'')$. The vector spaces $M(\pi)$ and $M(\pi')\otimes M(\pi'')$ have natural fixed point bases $\{[\underline{\vec{d}}]\}$ and $\{[\underline{\vec{d}}']\otimes[\underline{\vec{d}}'']\}$, parameterized by $\underline{\vec{d}}$ and pairs $(\underline{\vec{d}}',\underline{\vec{d}}'')$ with $\underline{\vec{d}},\underline{\vec{d}}',\underline{\vec{d}}''$ satisfying the conditions of Sect. 12.2. The assignment $(\underline{\vec{d}}',\underline{\vec{d}}'')\mapsto\underline{\vec{d}}'\cup\underline{\vec{d}}''$ defined via $(d'\cup d'')_{ij}^{(a)} = d_{ij}'^{(a)}$, $(d'\cup d'')_{ij}^{(p_j'+b)} = d_{ij}''^{(b)}$ for $1\le a\le p_j'$, $1\le b\le p_j''$ provides a bijection between such pairs $(\underline{\vec{d}}',\underline{\vec{d}}'')$ and $\underline{\vec{d}}$. We also identify

$\widetilde{T}' \times \widetilde{T}'' \xrightarrow{\sim} \widetilde{T}$ via $t_{p_1+\ldots+p_{j-1}+a} = t'_{p'_1+\ldots+p'_{j-1}+a}, t_{p_1+\ldots+p_{j-1}+p'_j+b} = t''_{p''_1+\ldots+p''_{j-1}+b}$ for a, b as above. Finally, we use $\underline{\vec{0}}$ to denote the collection of zero vectors.

Recall the Drinfeld formal coproduct $\widetilde{\Delta}\colon \mathcal{U}^{\boldsymbol{v}}_{\pi} \to \mathcal{U}^{\boldsymbol{v}}_{\pi'} \widehat{\otimes} \mathcal{U}^{\boldsymbol{v}}_{\pi''}$ of Lemma 10.2.

Theorem 12.4 *There is a unique collection of* $c_{\underline{\vec{d}}',\underline{\vec{d}}''} \in \mathrm{Frac}(K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt}))$ *with* $c_{\underline{\vec{0}},\underline{\vec{0}}} = 1$, *such that the map* $[\underline{\vec{d}}'] \otimes [\underline{\vec{d}}''] \mapsto c_{\underline{\vec{d}}',\underline{\vec{d}}''} \cdot [\underline{\vec{d}}' \cup \underline{\vec{d}}'']$ *induces an isomorphism* $M(\pi')\widetilde{\otimes} M(\pi'') \xrightarrow{\sim} M(\pi)$ *of* $\mathcal{U}^{\boldsymbol{v}}_{\pi}$*-representations.*

First let us make sense of the $\mathcal{U}^{\boldsymbol{v}}_{\pi}$-module $M(\pi')\widetilde{\otimes} M(\pi'')$. The action of $e_i(z)$ in the fixed point basis $\{[\underline{\vec{d}}'']\}$ of $M(\pi'')$ can be written as $e_i(z)[\underline{\vec{d}}''] = \sum_{j\le i}^{a\le p_j} a_{\underline{\vec{d}}'',\delta^{(a)}_{ij}} \delta(s^{(a)}_{ij}\boldsymbol{v}^{i+2}/z)[\underline{\vec{d}}'' - \delta^{(a)}_{ij}]$ for certain $a_{\underline{\vec{d}}'',\delta^{(a)}_{ij}} \in \mathrm{Frac}(K_{\widetilde{T}''\times\mathbb{C}^\times}(\mathrm{pt}))$. According to the comultiplication formula (10.1), we have $\widetilde{\Delta}(e_i(z))([\underline{\vec{d}}'] \otimes [\underline{\vec{d}}'']) = e_i(z)([\underline{\vec{d}}'])\otimes[\underline{\vec{d}}''] + \psi^-_i(z)([\underline{\vec{d}}'])\otimes e_i(z)([\underline{\vec{d}}''])$. The first summand is well-defined. To make sense of the second summand, we just need to apply the formula $\gamma(z)\delta(a/z) = \gamma(a)\delta(a/z)$ to the rational function $\gamma(z)$ chosen to be the eigenvalue of $\psi^-_i(z)$ on $[\underline{\vec{d}}']$. The action of $f_i(z)$ on $M(\pi')\widetilde{\otimes} M(\pi'')$ is defined analogously. Finally, the formula $\widetilde{\Delta}(\psi^\pm_i(z)) = \psi^\pm_i(z) \otimes \psi^\pm_i(z)$ provides a well-defined action of $\psi^\pm_i(z)$. These formulas endow $M(\pi') \otimes M(\pi'')$ with a well-defined action of $\mathcal{U}^{\boldsymbol{v}}_{\pi}$.

Proof According to Proposition 12.1(b), the eigenvalue of $\widetilde{\Delta}(\psi^\pm_i(z)) = \psi^\pm_i(z) \otimes \psi^\pm_i(z)$ on $[\underline{\vec{d}}'] \otimes [\underline{\vec{d}}''] \in M(\pi') \otimes M(\pi)''$ equals the eigenvalue of $\psi^\pm_i(z)$ on $[\underline{\vec{d}}' \cup \underline{\vec{d}}''] \in M(\pi)$. Hence, the map $[\underline{\vec{d}}'] \otimes [\underline{\vec{d}}''] \mapsto c_{\underline{\vec{d}}',\underline{\vec{d}}''} \cdot [\underline{\vec{d}}' \cup \underline{\vec{d}}'']$ intertwines actions of $\psi^\pm_i(z)$ for any $c_{\underline{\vec{d}}',\underline{\vec{d}}''} \in \mathrm{Frac}(K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt}))$.

Consider $c_{\underline{\vec{d}}',\underline{\vec{d}}''} \in \mathrm{Frac}(K_{\widetilde{T}\times\mathbb{C}^\times}(\mathrm{pt}))$ such that $c_{\underline{\vec{0}},\underline{\vec{0}}} = 1$ and

$$\begin{aligned}
\frac{c_{\underline{\vec{d}}'-\delta^{(a)}_{ij},\underline{\vec{d}}''}}{c_{\underline{\vec{d}}',\underline{\vec{d}}''}} &= (T''_{i+1})^{-1}\boldsymbol{v}^{d''_{i+1}-d''_i} \cdot \frac{\prod_{j'\le i+1}^{a'\le p''_{j'}}(1 - s''^{(a')}_{i+1,j'}/s'^{(a)}_{ij})}{\prod_{j'\le i}^{a'\le p''_{j'}}(1 - s''^{(a')}_{ij'}/s'^{(a)}_{ij})},\\
\frac{c_{\underline{\vec{d}}',\underline{\vec{d}}''-\delta^{(a)}_{ij}}}{c_{\underline{\vec{d}}',\underline{\vec{d}}''}} &= (T'_i)^{-1}\boldsymbol{v}^{d'_i-d'_{i-1}} \cdot \frac{\prod_{j'\le i}^{a'\le p'_{j'}}(1 - \boldsymbol{v}^{-2}s'^{(a')}_{ij'}/s''^{(a)}_{ij})}{\prod_{j'\le i-1}^{a'\le p'_{j'}}(1 - \boldsymbol{v}^{-2}s'^{(a')}_{i-1,j'}/s''^{(a)}_{ij})}.
\end{aligned} \tag{12.6}$$

The existence of $c_{\underline{\vec{d}}',\underline{\vec{d}}''}$ satisfying these relations as well as a verification that $[\underline{\vec{d}}'] \otimes [\underline{\vec{d}}''] \mapsto c_{\underline{\vec{d}}',\underline{\vec{d}}''} \cdot [\underline{\vec{d}}' \cup \underline{\vec{d}}'']$ intertwines actions of $e_{i,r}$ and $f_{i,r}$ are left to the interested reader. □

Remark 12.5 In the particular case $p_1 = \ldots = p_n = p$, this implies the isomorphism $M(p^n) \simeq M(1^n)^{\widetilde{\otimes} p}$ of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$-representations. This isomorphism is reminiscent of the isomorphism between the action of the quantum toroidal algebra of $\mathfrak{gl}_1$ on the equivariant K-theory of the Gieseker moduli spaces $M(r, n)$ and the r-fold tensor product of such representation for $r = 1$, see [62, Theorem 4.6].

12.7 Shifted Quantum Affine Algebras of $\mathfrak{gl}_n$

Let $U_{\boldsymbol{v}}(\widehat{\mathfrak{gl}_n})$ be the quantum affine algebra of $\mathfrak{gl}_n$ as defined in [17, Definition 3.1], and let $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ be the quantum loop algebra of $\mathfrak{gl}_n$, that is, $U_{\boldsymbol{v}}(L\mathfrak{gl}_n) := U_{\boldsymbol{v}}(\widehat{\mathfrak{gl}_n})/(\boldsymbol{v}^{\pm c/2}-1)$. This is an associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by

$$\{X^{\pm}_{i,r}, k^{\pm}_{j,\mp s^{\pm}_j} | i=1,\ldots,n-1,\ j=1,\ldots,n,\ r\in\mathbb{Z},\ s^{\pm}_j\in\mathbb{N}\}$$

and with the defining relations as in [17, (3.3, 3.4)]. There is a natural injective $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)\hookrightarrow U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$, defined by

$$e_i(z)\mapsto \frac{X^-_i(\boldsymbol{v}^i z)}{\boldsymbol{v}-\boldsymbol{v}^{-1}},\ f_i(z)\mapsto \frac{X^+_i(\boldsymbol{v}^i z)}{\boldsymbol{v}-\boldsymbol{v}^{-1}},\ \psi^{\pm}_i(z)\mapsto (k^{\mp}_i(\boldsymbol{v}^i z))^{-1}k^{\mp}_{i+1}(\boldsymbol{v}^i z). \tag{12.7}$$

For $\pi=(p_1,\ldots,p_n)\in\mathbb{Z}^n_{>0}$, define the shifted quantum affine algebra $\mathcal{U}^{\boldsymbol{v}}_\pi(\mathfrak{gl}_n)$ in the same way as $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ except that now $s^+_j\geq -p_j$ and we formally add inverse elements $\{(k^-_{j,0})^{-1},(k^+_{j,p_j})^{-1}\}_{j=1}^n$ (as we no longer require $k^-_{j,0}k^+_{j,p_j}=1$). Note that the assignment (12.7) still gives rise to an injective[9] homomorphism $\varrho\colon \mathcal{U}^{\boldsymbol{v}}_\pi\hookrightarrow \mathcal{U}^{\boldsymbol{v}}_\pi(\mathfrak{gl}_n)$.

Consider the following generating series of operators on $M(\pi)$:

$$X^+_i(z):=(\boldsymbol{v}-\boldsymbol{v}^{-1})f_i(\boldsymbol{v}^{-i}z)\colon M(\pi)_{\underline{d}}\to M(\pi)_{\underline{d}+i}[[z,z^{-1}]],$$

$$X^-_i(z):=(\boldsymbol{v}-\boldsymbol{v}^{-1})e_i(\boldsymbol{v}^{-i}z)\colon M(\pi)_{\underline{d}}\to M(\pi)_{\underline{d}-i}[[z,z^{-1}]],$$

$$k^-_j(z):=T^{-1}_j\boldsymbol{v}^{d_j-d_{j-1}}\cdot(\mathbf{b}_j(z\boldsymbol{v}^{-2j})/\mathbf{b}_{j-1}(z\boldsymbol{v}^{-2j}))^+\colon M(\pi)_{\underline{d}}\to M(\pi)_{\underline{d}}[[z^{-1}]],$$

$$k^+_j(z):=T^{-1}_j\boldsymbol{v}^{d_j-d_{j-1}}\cdot(\mathbf{b}_j(z\boldsymbol{v}^{-2j})/\mathbf{b}_{j-1}(z\boldsymbol{v}^{-2j}))^-\colon M(\pi)_{\underline{d}}\to z^{-p_j}M(\pi)_{\underline{d}}[[z]]$$

with $e_i(z), f_i(z), \mathbf{b}_j(z)$ defined in Sect. 12.5.

The following is a simple generalization of Theorem 12.2.

Theorem 12.6 *The generating series of operators $X^{\pm}_i(z), k^{\pm}_j(z)$ acting on $M(\pi)$ satisfy the relations of $\mathcal{U}^{\boldsymbol{v}}_\pi(\mathfrak{gl}_n)$, i.e., they give rise to the action of $\mathcal{U}^{\boldsymbol{v}}_\pi(\mathfrak{gl}_n)$ on $M(\pi)$.*

The restriction of this action to the subalgebra $\mathcal{U}^{\boldsymbol{v}}_\pi$ (embedded into $\mathcal{U}^{\boldsymbol{v}}_\pi(\mathfrak{gl}_n)$ via ϱ) recovers the action of $\mathcal{U}^{\boldsymbol{v}}_\pi$ on $M(\pi)$ of Theorem 12.2.

[9] One can prove the injectivity of ϱ by using Proposition 5.1 for both algebras. Indeed, the homomorphism ϱ is "glued" from three homomorphisms: $\varrho^>\colon \mathcal{U}^{\boldsymbol{v},>}_\pi\to\mathcal{U}^{\boldsymbol{v},>}_\pi(\mathfrak{gl}_n)$, $\varrho^<\colon \mathcal{U}^{\boldsymbol{v},<}_\pi\to\mathcal{U}^{\boldsymbol{v},<}_\pi(\mathfrak{gl}_n)$, $\varrho^0\colon \mathcal{U}^{\boldsymbol{v},0}_\pi\to\mathcal{U}^{\boldsymbol{v},0}_\pi(\mathfrak{gl}_n)$. The homomorphisms $\varrho^>,\varrho^<$ are isomorphisms due to Proposition 5.1(b), while the injectivity of ϱ^0 is clear.

12.8 The Cohomology Case Revisited

The above results can be immediately generalized to the cohomological setting. Let $V(\pi)$ be the direct sum of localized $T \times \mathbb{C}^\times$-equivariant cohomology of type π Laumon parabolic based quasiflags' spaces:

$$V(\pi) := \bigoplus_{\underline{d}} H^\bullet_{T\times\mathbb{C}^\times}(\mathfrak{Q}_{\underline{d}}) \otimes_{H^\bullet_{T\times\mathbb{C}^\times}(\mathrm{pt})} \mathrm{Frac}(H^\bullet_{T\times\mathbb{C}^\times}(\mathrm{pt})).$$

It is a module over $\mathrm{Frac}(H^\bullet_{T\times\mathbb{C}^\times}(\mathrm{pt}))$, where $H^\bullet_{T\times\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{C}[\mathrm{Lie}(T \times \mathbb{C}^\times)] = \mathbb{C}[x_1, \ldots, x_N, \hbar]$.

Let $\mathcal{Y}^\hbar_\pi = \mathbf{Y}_\pi \otimes_{\mathbb{C}[\hbar]} \mathbb{C}(\hbar)$, where $\mathbf{Y}_\pi$ is the shifted Yangian of $\mathfrak{sl}_n$ in the sense of [10, Appendix B(i)]. It is the associative $\mathbb{C}(\hbar)$-algebra generated by $\{E_i^{(r+1)}, F_i^{(r+1)}, H_i^{(r+1+p_i-p_{i+1})}\}_{1\le i<n}^{r\in\mathbb{N}}$ with the same defining relations as in the standard Yangian $Y_\hbar(\mathfrak{sl}_n)$.

We define the generating series $\mathbf{a}_i(z)$with coefficients in the equivariant cohomology of $\mathfrak{Q}_{\underline{d}}$ as follows:

$$\mathbf{a}_i(z) := z^{p_1+\ldots+p_i} \cdot c(\varpi_*(\underline{\mathcal{W}}_i|_{\mathbf{C}\setminus\{\infty\}}), (-z\hbar)^{-1}),$$

where $c(\mathcal{V}, x)$ denotes the Chern polynomial (in x) of $\mathcal{V}$. We also define the operators

$$E_i^{(r+1)} := \mathbf{p}_*((c_1(\mathsf{L}_i) + i\hbar/2)^r \cdot \mathbf{q}^*) \colon V(\pi)_{\underline{d}} \to V(\pi)_{\underline{d}-i}, \tag{12.8}$$

$$F_i^{(r+1)} := (-1)^{p_i} \mathbf{q}_*((c_1(\mathsf{L}_i) + i\hbar/2)^r \cdot \mathbf{p}^*) \colon V(\pi)_{\underline{d}} \to V(\pi)_{\underline{d}+i}. \tag{12.9}$$

We define $H_i(z) = z^{p_{i+1}-p_i} + \sum_{r>p_i-p_{i+1}} H_i^{(r)} \hbar^{-r+p_i-p_{i+1}+1} z^{-r}$ via

$$H_i(z) := \left(\frac{\mathbf{a}_{i+1}(z - \frac{i+2}{2})\mathbf{a}_{i-1}(z - \frac{i}{2})}{\mathbf{a}_i(z - \frac{i+2}{2})\mathbf{a}_i(z - \frac{i}{2})}\right)^+ \colon V(\pi)_{\underline{d}} \to z^{p_{i+1}-p_i} V(\pi)_{\underline{d}}[[z^{-1}]]. \tag{12.10}$$

The following result is completely analogous to Theorem 12.2.

Theorem 12.7 *The operators* $\{E_i^{(r+1)}, F_i^{(r+1)}, H_i^{(r+1+p_i-p_{i+1})}\}_{1\le i<n}^{r\in\mathbb{N}}$ *of (12.8–12.10) acting on* $V(\pi)$ *satisfy the defining relations of* $\mathcal{Y}^\hbar_\pi$*, i.e., they give rise to the action of* $\mathcal{Y}^\hbar_\pi$ *on* $V(\pi)$*.*

A slight refinement of this theorem in the *dominant* case $p_1 \le \ldots \le p_n$ constituted the key result of [7]. In loc. cit., the authors constructed the action of the shifted Yangian of $\mathfrak{gl}_n$, denoted by $\mathcal{Y}^\hbar_\pi(\mathfrak{gl}_n)$, on $V(\pi)$. There is a natural (injective) homomorphism $\mathcal{Y}^\hbar_\pi \to \mathcal{Y}^\hbar_\pi(\mathfrak{gl}_n)$, such that $F_i^{(r+1)} \mapsto$

$\sum_{s=0}^{r}\binom{r}{s}\left(\frac{2-i}{2}\hbar\right)^{r-s}\mathsf{f}_i^{(s+1)}$, $E_i^{(r+1)} \mapsto \sum_{s=0}^{r}\binom{r}{s}\left(\frac{2-i}{2}\hbar\right)^{r-s}\mathsf{e}_i^{(s+1+p_{i+1}-p_i)}$. The pull-back of the action of [7] along this homomorphism recovers the action $\mathcal{Y}_\pi^\hbar$ on $V(\pi)$ of Theorem 12.7.

The proof of [7] was based on an explicit identification of the geometric action in the fixed point basis with the formulas of [27] for the action of $\mathcal{Y}_\pi^\hbar(\mathfrak{gl}_n)$ in the Gelfand-Tsetlin basis. The benefits of our straightforward proof of Theorem 12.7 are two-fold:

(1) we eliminate the crucial assumption $p_1 \le \ldots \le p_n$ of [7],
(2) we obtain an alternative proof of the formulas of [27] (cf. Proposition 12.8 below).

Moreover, we can derive $\boldsymbol{v}$-analogues of the Gelfand-Tsetlin formulas of [27] via a certain specialization of the parameters in Proposition 12.1 as explained below. We set $t_l = \boldsymbol{v}^{\beta_l}$ for $1 \le l \le N$. To a collection $\underline{\vec{d}} = (d_{ij}^{(a)})_{1\le j\le i\le n-1}^{1\le a\le p_j}$, we associate a Gelfand-Tsetlin pattern $\Lambda = \Lambda(\underline{\vec{d}}) = (\lambda_{ij}^{(a)})_{1\le j\le i\le n}^{1\le a\le p_j}$ as follows: $\lambda_{nj}^{(a)} = \beta_{p_1+\ldots+p_{j-1}+a}+j-1$, $\lambda_{ij}^{(a)} = \beta_{p_1+\ldots+p_{j-1}+a}+j-1-d_{ij}^{(a)}$. Set $\lambda_j^{(a)} := \lambda_{nj}^{(a)}$, which is independent of $\underline{\vec{d}}$. Note that the vector space $M(\pi)$ has a basis $\{[\Lambda]\}$ parametrized by $\Lambda = (\lambda_{ij}^{(a)})_{1\le j\le i\le n}^{1\le a\le p_j}$ with $\lambda_{nj}^{(a)} = \lambda_j^{(a)}$ and $\lambda_{i+1,j}^{(a)} - \lambda_{ij}^{(a)} \in \mathbb{N}$. Consider a specialization of $\{\beta_l\}_{1\le l\le N}$ such that $\lambda_j^{(a)} - \lambda_{j+1}^{(a)} \in \mathbb{N}$, while $\lambda_i^{(a)} - \lambda_j^{(b)} \notin \mathbb{Z}$ if $a \ne b$. Let S be the subset of those Λ from above such that $\lambda_{ij}^{(a)} - \lambda_{i+1,j+1}^{(a)} \in \mathbb{N}$ (note that S is finite), while $\bar{S}$ will denote the set of the remaining Gelfand-Tsetlin patterns Λ.

As before, we define

$$A_i^\pm(z) := k_1^\mp(\boldsymbol{v}^{2-i}z)k_2^\mp(\boldsymbol{v}^{4-i}z)\cdots k_i^\mp(\boldsymbol{v}^i z),$$

$$B_i^\pm(z) := (\boldsymbol{v}-\boldsymbol{v}^{-1})A_i^\pm(z)e_i^\pm(z),$$

$$C_i^\pm(z) := (\boldsymbol{v}-\boldsymbol{v}^{-1})f_i^\pm(z)A_i^\pm(z).$$

We set $\lambda_{ij}(z) := \prod_{a=1}^{p_j}(\boldsymbol{v}^{-\lambda_{ij}^{(a)}} - \boldsymbol{v}^{\lambda_{ij}^{(a)}}z^{-1})$. The next result follows from Proposition 12.1.

Proposition 12.8

(a) The vector subspace of $M(\pi)$ spanned by $\{[\Lambda]\}_{\Lambda\in\bar{S}}$ is $\mathcal{U}_\pi^{\boldsymbol{v}}(\mathfrak{gl}_n)$-invariant. We denote by $L(\pi)$ the corresponding quotient of $M(\pi)$.

(b) Let $\{\xi_\Lambda\}_{\Lambda\in S}$ be the basis of $L(\pi)$ inherited from $\{[\Lambda]\}_{\Lambda\in S}$. Then, we have:

$$A_i^\pm(\boldsymbol{v}^i z)\xi_\Lambda = \boldsymbol{v}^{m_i}\lambda_{i1}(z)\lambda_{i2}(\boldsymbol{v}^2 z)\cdots\lambda_{ii}(\boldsymbol{v}^{2(i-1)}z)\xi_\Lambda,$$

$$B_i^{\pm}(\boldsymbol{v}^i\cdot\boldsymbol{v}^{2l_{ij}^{(a)}})\xi_\Lambda=-\boldsymbol{v}^{m_{i+1}-i}\cdot\lambda_{i+1,1}(\boldsymbol{v}^{2l_{ij}^{(a)}})\lambda_{i+1,2}(\boldsymbol{v}^{2(l_{ij}^{(a)}+1)})\cdots\lambda_{i+1,i+1}(\boldsymbol{v}^{2(l_{ij}^{(a)}+i)})\xi_{\Lambda+\delta_{ij}^{(a)}},$$

$$C_i^{\pm}(\boldsymbol{v}^i\cdot\boldsymbol{v}^{2l_{ij}^{(a)}})\xi_\Lambda=\boldsymbol{v}^{m_{i-1}+i-1}\cdot\lambda_{i-1,1}(\boldsymbol{v}^{2l_{ij}^{(a)}})\lambda_{i-1,2}(\boldsymbol{v}^{2(l_{ij}^{(a)}+1)})\cdots\lambda_{i-1,i-1}(\boldsymbol{v}^{2(l_{ij}^{(a)}+i-2)})\xi_{\Lambda-\delta_{ij}^{(a)}},$$

where $m_j := \sum_{j'=1}^{j}(j'-1)p_{j'}$ *and* $l_{ij}^{(a)} := \lambda_{ij}^{(a)} - j + 1$.

Remark 12.9

(a) In the simplest case $\pi = 1^n$, the above homomorphism $\mathcal{Y}_\pi^\hbar \to \mathcal{Y}_\pi^\hbar(\mathfrak{gl}_n)$ is the classical embedding of the Yangian of $\mathfrak{sl}_n$ into the Yangian of $\mathfrak{gl}_n$.
(b) The injectivity of the above homomorphism $\mathcal{Y}_\pi^\hbar \to \mathcal{Y}_\pi^\hbar(\mathfrak{gl}_n)$ follows from the PBW property for $\mathcal{Y}_\pi^\hbar$ (see [24, Corollary 3.15]) and its analogue for $\mathcal{Y}_\pi^\hbar(\mathfrak{gl}_n)$.
(c) We take this opportunity to correct the sign in [7, (4.2)], where the '$-$' sign should be replaced by $(-1)^{p_k}$, that is, $\mathsf{f}_k^{(r+1)} := (-1)^{p_k}\mathbf{q}_*(c_1(\mathcal{L}_k')^r\cdot\mathbf{p}^*)$.
(d) We take this opportunity to correct the typos in [23]. First, the formulas for the eigenvalues of $\mathbf{h}_i(u)$ and $\mathbf{a}_{mi}(u)$ of Theorem 3.20 and its proof should be corrected by replacing $p_{i'j'} \rightsquigarrow \hbar^{-1}p_{i'j'}$. Second, the formulas defining $\mathbf{a}_m(u)$ (Section 2.11), $\mathbf{a}_{mi}(u)$ (Section 2.13), $\mathbf{a}_{mi}(u)$ (Section 3.17) should be modified by ignoring $\mathbf{p}_*$, $\mathbf{q}^*$.

Remark 12.10 Let $e_\pi \in \mathfrak{gl}_N$ be a nilpotent element of Jordan type π. For $p_1 \leq \ldots \leq p_n$, Brundan-Kleshchev proved that the finite W-algebra $W(\mathfrak{gl}_N, e_\pi)$ is the quotient of $\mathcal{Y}_\pi^\hbar(\mathfrak{gl}_n)$ by the 2-sided ideal generated by $\{d_1^{(r)}\}_{r>p_1}$, see [12]. Together with Theorem 12.7 this yields a natural action of $W(\mathfrak{gl}_N, e_\pi)$ on $V(\pi)$, referred to as a finite analogue of the AGT relation in [7]. We expect that the truncated version of $\mathcal{U}_\pi^{\boldsymbol{v}}(\mathfrak{gl}_n)$ with $\lambda = N\omega_{n-1}$ should be isomorphic to the $\boldsymbol{v}$-version of the W-algebra $W(\mathfrak{gl}_N, e_\pi)$ as defined by Sevostyanov in [57].

12.9 Shifted Quantum Toroidal $\mathfrak{sl}_n$ and Parabolic Affine Laumon Spaces

The second main result of [61] provides the action of the quantum toroidal algebra $U_{\boldsymbol{v},u}(\widehat{\mathfrak{sl}}_n)$ (denoted $\ddot{\mathrm{U}}_{\boldsymbol{v}}(\widehat{\mathfrak{sl}}_n)$ in loc. cit.) on the direct sum of localized equivariant K-groups of the affine Laumon spaces $\mathcal{P}_{\underline{d}}$. The cohomological counterpart of this was established in [23], where the action of the affine Yangian $Y_{\hbar,\hbar'}(\widehat{\mathfrak{sl}}_n)$ (denoted $\widehat{Y}$ in loc. cit.) on the direct sum of localized equivariant cohomology of $\mathcal{P}_{\underline{d}}$ was constructed.

Likewise, the results of Theorems 12.2 and 12.7 can be naturally generalized to provide the actions of the shifted quantum toroidal algebra $\mathcal{U}_\pi^{\boldsymbol{v},u}$ (resp. shifted affine Yangian $\mathcal{Y}_\pi^{\hbar,\hbar'}$) on the direct sum of localized equivariant K-groups (resp. cohomology) of parabolic affine Laumon spaces. Here $\mathcal{U}_\pi^{\boldsymbol{v},u}$ is the associative

$\mathbb{C}(\boldsymbol{v}, u)$-algebra generated by $\{e_{i,r}, f_{i,r}, \psi^{\pm}_{i,\pm s_i^{\pm}} | 1 \leq i \leq n, r \in \mathbb{Z}, s_i^+ \geq 0, s_i^- \geq p_i - p_{i+1}\}$ and with the same defining relations as for $U_{\boldsymbol{v},u}(\widehat{\mathfrak{sl}}_n)$, while $\mathcal{Y}_\pi^{\hbar,\hbar'}$ is the associative $\mathbb{C}(\hbar, \hbar')$-algebra generated by $\{E_i^{(r+1)}, F_i^{(r+1)}, H_i^{(r+1+p_i-p_{i+1})} | 1 \leq i \leq n, r \in \mathbb{N}\}$ and with the same defining relations as for $Y_{\hbar,\hbar'}(\widehat{\mathfrak{sl}}_n)$ (here we set $p_{n+1} := p_1$). On the geometric side, the parabolic affine Laumon spaces of type π are defined similarly to the case $\pi = 1^n$. We leave details to the interested reader.

12.10 Whittaker Vector

Consider the *Whittaker vector*

$$\mathfrak{m} := \sum_{\underline{d}} [\mathcal{O}_{\mathfrak{Q}_{\underline{d}}}] \in M(\pi)^\wedge,$$

where $M(\pi)^\wedge := \prod_{\underline{d}} M(\pi)_{\underline{d}}$. We also define the operators

$$e'_{i,r} := \mathbf{p}_*((\boldsymbol{v}^i \mathsf{L}_i)^{\otimes r} \otimes \mathbf{q}^*) = \boldsymbol{v}^{i-1}(k^-_{i+1,0})^{-1} e_{i,r} : M(\pi)_{\underline{d}} \to M(\pi)_{\underline{d}-i}.$$

Proposition 12.11 *For $1 \leq i \leq n-1$, we have*

$$e'_{i,0}(\mathfrak{m}) = (1-\boldsymbol{v}^2)^{-1}\mathfrak{m} \text{ and } e'_{i,1}(\mathfrak{m}) = \ldots = e'_{i,p_i-1}(\mathfrak{m}) = 0.$$

Proof According to the Bott-Lefschetz formula, we have:

(1) $\mathfrak{m} = \sum_{\vec{\underline{d}}} a_{\vec{\underline{d}}} [\vec{\underline{d}}]$, where $a_{\vec{\underline{d}}} = \prod_{w \in T_{\vec{\underline{d}}} \mathfrak{Q}_{\underline{d}}} (1-w)^{-1}$;

(2) $\frac{a_{\vec{\underline{d}}'}}{a_{\vec{\underline{d}}}} \mathbf{p}_*((\boldsymbol{v}^i \mathsf{L}_i)^{\otimes r} \otimes \mathbf{q}^*)_{[\vec{\underline{d}}',\vec{\underline{d}}]} = \mathbf{q}_*((\boldsymbol{v}^i \mathsf{L}_i)^{\otimes r} \otimes \mathbf{p}^*)_{[\vec{\underline{d}},\vec{\underline{d}}']}$.

Set $C_{i,0} := (1-\boldsymbol{v}^2)^{-1}$ and $C_{i,r} := 0$ for $0 < r < p_i$. It suffices to prove the equality $C_{i,r} = \sum_{j \leq i}^{a \leq p_j} \mathbf{q}_*((\boldsymbol{v}^i \mathsf{L}_i)^{\otimes r} \otimes \mathbf{p}^*)_{[\vec{\underline{d}},\vec{\underline{d}}+\delta_{ij}^{(a)}]}$ for any $\vec{\underline{d}}$ and any $1 \leq i \leq n-1, 0 \leq r \leq p_i - 1$. According to Proposition 12.1(a), we have

$$\mathbf{q}_*((\boldsymbol{v}^i \mathsf{L}_i)^{\otimes r} \otimes \mathbf{p}^*)_{[\vec{\underline{d}},\vec{\underline{d}}+\delta_{ij}^{(a)}]} = (1-\boldsymbol{v}^2)^{-1}(s_{ij}^{(a)}\boldsymbol{v}^i)^r \frac{\prod_{j' \leq i-1}^{a' \leq p_{j'}} (1 - s_{ij}^{(a)}/s_{i-1,j'}^{(a')})}{\prod_{j' \leq i, a' \leq p_{j'}}^{(j',a') \neq (j,a)} (1 - s_{ij}^{(a)}/s_{ij'}^{(a')})} =$$

$$\frac{\boldsymbol{v}^i}{1-\boldsymbol{v}^2} \frac{\prod_{j' \leq i}^{a' \leq p_{j'}} s_{ij'}^{(a')}}{\prod_{j' \leq i-1}^{a' \leq p_{j'}} s_{i-1,j'}^{(a')}} \cdot (s_{ij}^{(a)}\boldsymbol{v}^i)^{r-1} \frac{\prod_{j' \leq i-1}^{a' \leq p_{j'}} (s_{i-1,j'}^{(a')} - s_{ij}^{(a)})}{\prod_{j' \leq i, a' \leq p_{j'}}^{(j',a') \neq (j,a)} (s_{ij'}^{(a')} - s_{ij}^{(a)})}.$$

For $1 \le r \le p_i - 1$, the sum

$$\sum_{j\le i}^{a\le p_j} (s_{ij}^{(a)}\boldsymbol{v}^i)^{r-1} \frac{\prod_{j'\le i-1}^{a'\le p_{j'}} (s_{i-1,j'}^{(a')} - s_{ij}^{(a)})}{\prod_{j'\le i, a'\le p_{j'}}^{(j',a')\ne(j,a)} (s_{ij'}^{(a')} - s_{ij}^{(a)})}$$

is a rational function in $\{s_{ij'}^{(a')}\}_{j'\le i}^{a'\le p_{j'}}$ of degree $r - p_i < 0$ and without poles. Hence, it is zero. For $r = 0$, the same arguments imply

$$\sum_{j\le i}^{a\le p_j} (s_{ij}^{(a)}\boldsymbol{v}^i)^{-1} \frac{\prod_{j'\le i-1}^{a'\le p_{j'}} (s_{i-1,j'}^{(a')} - s_{ij}^{(a)})}{\prod_{j'\le i, a'\le p_{j'}}^{(j',a')\ne(j,a)} (s_{ij'}^{(a')} - s_{ij}^{(a)})} = \sum_{j\le i}^{a\le p_j} (s_{ij}^{(a)}\boldsymbol{v}^i)^{-1} \frac{\prod_{j'\le i-1}^{a'\le p_{j'}} s_{i-1,j'}^{(a')}}{\prod_{j'\le i, a'\le p_{j'}}^{(j',a')\ne(j,a)} (s_{ij'}^{(a')} - s_{ij}^{(a)})}.$$

It remains to compute $\sum_{j\le i}^{a\le p_j} \prod_{j'\le i, a'\le p_{j'}}^{(j',a')\ne(j,a)} \frac{s_{ij'}^{(a')}}{s_{ij'}^{(a')} - s_{ij}^{(a)}}$, which is a rational function in $\{s_{ij'}^{(a')}\}_{j'\le i}^{a'\le p_{j'}}$ of degree 0 and without poles, hence, a constant. Specializing $s_{i1}^{(1)} \mapsto 0$, we see that this constant is equal to 1 (note that only one summand is nonzero under this specialization).

The proposition is proved. □

Remark 12.12

(a) For $\pi = 1^n$, this result was proved in [6, Proposition 2.31].

(b) By the same arguments, we also find $e''_{i,p_i}(\mathfrak{m}) = \frac{(-1)^{p_i-1}\boldsymbol{v}^{ip_i}}{1-\boldsymbol{v}^2}\mathfrak{m}$, where $e''_{i,r} := (k_{i,0}^-)^2 e'_{i,r}$.

(c) Likewise, one can prove that $E_i^{(1)}(\mathfrak{v}) = \ldots = E_i^{(p_i-1)}(\mathfrak{v}) = 0$, $E_i^{(p_i)}(\mathfrak{v}) = \hbar^{-1}\mathfrak{v}$, where $\mathfrak{v} := \sum_{\underline{d}}[\mathfrak{Q}_{\underline{d}}] \in V(\pi)^\wedge$. This result was established in [7, Proposition 5.1].

Appendix A Proof of Theorem 5.5 and Its Modification

To prove Theorem 5.5, let us first note that relations ($\hat{\mathrm{U}}1$–$\hat{\mathrm{U}}9$) hold in $\mathcal{U}^{\mathrm{sc}}_{0,\mu}$. Hence, there exists an algebra homomorphism $\varepsilon\colon \hat{\mathcal{U}}_{\mu_1,\mu_2} \to \mathcal{U}^{\mathrm{sc}}_{0,\mu}$ such that $e_{i,r} \mapsto e_{i,r}, f_{i,s} \mapsto f_{i,s}, (\psi_{i,0}^+)^{\pm1} \mapsto (\psi_{i,0}^+)^{\pm1}, (\psi_{i,b_i}^-)^{\pm1} \mapsto (\psi_{i,b_i}^-)^{\pm1}, h_{i,\pm1} \mapsto h_{i,\pm1}$ for $i \in I, b_{2,i} - 1 \le r \le 0, b_{1,i} \le s \le 1$. Moreover, the way we defined $e_{i,r}, f_{i,r}, \psi_{i,r}^\pm \in \hat{\mathcal{U}}_{\mu_1,\mu_2}$ right before Theorem 5.5, it is clear that $\varepsilon\colon e_{i,r} \mapsto e_{i,r}, f_{i,r} \mapsto f_{i,r}, \psi_{i,\pm s_i^\pm}^\pm \mapsto \psi_{i,\pm s_i^\pm}^\pm$ for $i \in I, r \in \mathbb{Z}, s_i^+ \ge 0, s_i^- \ge -b_i$. In particular, ε is surjective. Injectivity of ε is equivalent to showing that relations (U1–U8) hold in $\hat{\mathcal{U}}_{\mu_1,\mu_2}$. This occupies the rest of this Appendix until A(iv), where we consider a slight modification of this presentation, see Theorem A.3 and its proof.

A(i) Derivation of Some Useful Relations in $\hat{\mathcal{U}}_{\mu_1,\mu_2}$

First, we note that (Û1, Û4, Û5) together with our definition of $e_{i,r}, f_{i,r}, \psi^+_{i,r}$ imply:

$$\psi^+_{i,0}e_{j,r} = \boldsymbol{v}_i^{c_{ij}} e_{j,r}\psi^+_{i,0},\ \psi^-_{i,b_i}e_{j,r} = \boldsymbol{v}_i^{-c_{ij}} e_{j,r}\psi^-_{i,b_i},\ [h_{i,\pm 1}, e_{j,r}] = [c_{ij}]_{\boldsymbol{v}_i} \cdot e_{j,r\pm 1}, \tag{v1}$$

$$\psi^+_{i,0}f_{j,r} = \boldsymbol{v}_i^{-c_{ij}} f_{j,r}\psi^+_{i,0},\ \psi^-_{i,b_i}f_{j,r} = \boldsymbol{v}_i^{c_{ij}} f_{j,r}\psi^-_{i,b_i},\ [h_{i,\pm 1}, f_{j,r}] = -[c_{ij}]_{\boldsymbol{v}_i} \cdot f_{j,r\pm 1}, \tag{v2}$$

$$[\psi^+_{i,0}, \psi^\pm_{j,\pm s_j^\pm}] = 0,\ [\psi^-_{i,b_i}, \psi^\pm_{j,\pm s_j^\pm}] = 0 \tag{v3}$$

for any $i, j \in I, r \in \mathbb{Z}, s_j^+ \geq 0, s_j^- \geq -b_j$.

Second, combining relations (Û1, Û4, Û5, Û6), we get

$$\begin{aligned} &[e_{i,1}, f_{i,0}] = [e_{i,0}, f_{i,1}] = \psi^+_{i,1}/(\boldsymbol{v}_i - \boldsymbol{v}_i^{-1}),\\ &[e_{i,b_{2,i}}, f_{i,b_{1,i}-1}] = [e_{i,b_{2,i}-1}, f_{i,b_{1,i}}] = \psi^-_{i,b_i-1}/(\boldsymbol{v}_i^{-1} - \boldsymbol{v}_i). \end{aligned} \tag{v4}$$

Note that $\psi^+_{i,1} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})[e_{i,0}, f_{i,1}] = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})\psi^+_{i,0}h_{i,1}$. Hence, $[h_{i,1}, \psi^+_{i,1}] = 0$. Combining this further with (v1, v2, v4) and our definition of $\psi^+_{i,2}$, we obtain

$$[e_{i,2}, f_{i,0}] = [e_{i,1}, f_{i,1}] = [e_{i,0}, f_{i,2}] = \psi^+_{i,2}/(\boldsymbol{v}_i - \boldsymbol{v}_i^{-1}). \tag{v5}$$

Likewise, we also get

$$[e_{i,b_{2,i}}, f_{i,b_{1,i}-2}] = [e_{i,b_{2,i}-1}, f_{i,b_{1,i}-1}] = [e_{i,b_{2,i}-2}, f_{i,b_{1,i}}] = \psi^-_{i,b_i-2}/(\boldsymbol{v}_i^{-1} - \boldsymbol{v}_i). \tag{v6}$$

Third, let us point out that relation (Û9) is equivalent to

$$[h_{i,1}, \psi^+_{i,2}] = 0,\ [h_{i,-1}, \psi^-_{i,b_i-2}] = 0. \tag{v7}$$

According to the above relations, for any $i, j \in I$ we also have

$$[h_{j,-1}, \psi^+_{i,2}] = 0,\ [h_{j,1}, \psi^-_{i,b_i-2}] = 0. \tag{v8}$$

Finally, we define elements $h_{i,\pm 2} \in \hat{\mathcal{U}}_{\mu_1,\mu_2}$ as follows:

$$\begin{aligned} h_{i,2} &:= (\psi^+_{i,0})^{-1}\psi^+_{i,2}/(\boldsymbol{v}_i - \boldsymbol{v}_i^{-1}) - (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})h^2_{i,1}/2, \\ h_{i,-2} &:= (\psi^-_{i,b_i})^{-1}\psi^-_{i,b_i-2}/(\boldsymbol{v}_i^{-1} - \boldsymbol{v}_i) - (\boldsymbol{v}_i^{-1} - \boldsymbol{v}_i)h^2_{i,-1}/2. \end{aligned} \tag{A.1}$$

Due to relations (Û1, v7, v8), for every $i, j \in I$ we have

$$[h_{i,\pm 1}, h_{i,\pm 2}] = 0,\ \ [h_{j,\mp 1}, h_{i,\pm 2}] = 0. \tag{v9}$$

Lemma A.1 *For any $i \in I, r \in \mathbb{Z}$, we have*

$$[h_{i,\pm 2}, e_{i,r}] = \frac{[4]_{\boldsymbol{v}_i}}{2} \cdot e_{i,r\pm 2},\ \ [h_{i,\pm 2}, f_{i,r}] = -\frac{[4]_{\boldsymbol{v}_i}}{2} \cdot f_{i,r\pm 2}.$$

Proof Due to (Û2), we have $[e_{i,0}, e_{i,-1}]_{\boldsymbol{v}_i^2} = 0$. Commuting this with $h_{i,1}$ and applying relation (Û4), we obtain $e_{i,1}e_{i,-1} - \boldsymbol{v}_i^2 e^2_{i,0} = \boldsymbol{v}_i^2 e_{i,-1}e_{i,1} - e^2_{i,0}$. Commuting this further with $f_{i,1}$ and applying relation (Û6), we obtain

$$\begin{aligned} &\psi^+_{i,2}e_{i,-1} - \boldsymbol{v}_i^2\psi^+_{i,1}e_{i,0} + e_{i,1}\psi^+_{i,0} - \boldsymbol{v}_i^2 e_{i,0}\psi^+_{i,1} - \delta_{b_i,0}e_{i,1}\psi^-_{i,b_i} = \\ &\boldsymbol{v}_i^2 e_{i,-1}\psi^+_{i,2} - e_{i,0}\psi^+_{i,1} + \boldsymbol{v}_i^2\psi^+_{i,0}e_{i,1} - \psi^+_{i,1}e_{i,0} - \boldsymbol{v}_i^2\delta_{b_i,0}\psi^-_{i,b_i}e_{i,1}. \end{aligned}$$

First, note that $e_{i,1}\psi^-_{i,b_i} = \boldsymbol{v}_i^2\psi^-_{i,b_i}e_{i,1}$, due to (Û4). Second, we have

$$e_{i,1}\psi^+_{i,0} - \boldsymbol{v}_i^2 e_{i,0}\psi^+_{i,1} = \boldsymbol{v}_i^2\psi^+_{i,0}e_{i,1} - \psi^+_{i,1}e_{i,0}. \tag{v10}$$

Indeed, due to the equality $\psi^+_{i,1} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})\psi^+_{i,0}h_{i,1}$ and relations (Û1, v1), we have

$$\psi^+_{i,1}e_{i,0} - \boldsymbol{v}_i^2 e_{i,0}\psi^+_{i,1} = \boldsymbol{v}_i^2(\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})[2]_{\boldsymbol{v}_i} \cdot e_{i,1}\psi^+_{i,0} = (\boldsymbol{v}_i^4 - 1)e_{i,1}\psi^+_{i,0} = \boldsymbol{v}_i^2\psi^+_{i,0}e_{i,1} - e_{i,1}\psi^+_{i,0}.$$

Therefore, we get

$$\psi^+_{i,2}e_{i,-1} - \boldsymbol{v}_i^2\psi^+_{i,1}e_{i,0} = \boldsymbol{v}_i^2 e_{i,-1}\psi^+_{i,2} - e_{i,0}\psi^+_{i,1}. \tag{v11}$$

Combining the formulas $\psi^+_{i,1} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})\psi^+_{i,0}h_{i,1},\ \psi^+_{i,2} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})\psi^+_{i,0}(h_{i,2} + \frac{\boldsymbol{v}_i - \boldsymbol{v}_i^{-1}}{2}h^2_{i,1})$ with relations (Û1, v1, v11), we finally get $[h_{i,2}, e_{i,-1}] = \frac{[4]_{\boldsymbol{v}_i}}{2}e_{i,1}$. Commuting this relation with $h_{i,\pm 1}$ and using (v1, v9), we obtain $[h_{i,2}, e_{i,r}] = \frac{[4]_{\boldsymbol{v}_i}}{2}e_{i,r+2}$ for any $r \in \mathbb{Z}$.

Likewise, starting from the relation $[e_{i,b_{2,i}}, e_{i,b_{2,i}-1}]_{\boldsymbol{v}_i^2} = 0$ and commuting it first with $h_{i,-1}$ and then with $f_{i,b_{1,i}}$, we recover $[h_{i,-2}, e_{i,b_{2,i}}] = \frac{[4]_{\boldsymbol{v}_i}}{2} e_{i,b_{2,i}-2}$. Commuting this further with $h_{i,\pm 1}$, we get $[h_{i,-2}, e_{i,r}] = \frac{[4]_{\boldsymbol{v}_i}}{2} e_{i,r-2}$ for any $r \in \mathbb{Z}$.

The proof of $[h_{i,\pm 2}, f_{i,r}] = -\frac{[4]_{\boldsymbol{v}_i}}{2} \cdot f_{i,r\pm 2}$ is completely analogous. □

A(ii) Verification of Relations (U1–U6) with $i = j$ for $\hat{\mathcal{U}}_{\mu_1,\mu_2}$

A(ii).a Verification of (U2)

We need to prove $X^+(i;r,s) = 0$ for any $r, s \in \mathbb{Z}$, where

$$X^+(i;r,s) := [e_{i,r+1}, e_{i,s}]_{\boldsymbol{v}_i^2} + [e_{i,s+1}, e_{i,r}]_{\boldsymbol{v}_i^2}.$$

Note that $X^+(i;r,s) = X^+(i;s,r)$, and $X^+(i;-1,-1) = 0$ due to relation ($\hat{\mathrm{U}}2$).

For $a \in \{\pm 1, \pm 2\}$, we define $L_{i,a} := a/[2a]_{\boldsymbol{v}_i} \cdot \mathrm{ad}(h_{i,a}) \in \mathrm{End}(\hat{\mathcal{U}}_{\mu_1,\mu_2})$. Then, we have $L_{i,a}(X^+(i;r,s)) = X^+(i;r+a,s) + X^+(i;r,s+a)$. Set $L_i^{\pm} := \frac{1}{2}(L_{i,\pm 1}^2 - L_{i,\pm 2})$. Then $L_i^{\pm}(X^+(i;r,s)) = X^+(i;r\pm 1, s\pm 1)$. Applying iteratively L_i^+ to the equality $X^+(i;-1,-1) = 0$, we get $X^+(i;r,r) = 0$ for any $r \geq -1$. Since $2X^+(i;-1,0) = L_{i,1}(X^+(i;-1,-1)) = 0$, we analogously get $X^+(i;r,r+1) = 0$ for $r \geq -1$. Fix $s \in \mathbb{Z}_{>0}$ and assume by induction that $X^+(i;r,r+N) = 0$ for any $r \geq -1, 0 \leq N \leq s$. Then $X^+(i;-1,s) = L_{i,1}(X^+(i;-1,s-1)) - X^+(i;0,s-1) = 0$, due to the above assumption. Applying $(L_i^+)^{r+1}$ to the latter equality, we get $X^+(i;r,r+s+1) = 0$ for $r \geq -1$. An induction in s completes the proof of $X^+(i;r,s) = 0$ for any $r, s \geq -1$. Finally, applying iteratively L_i^-, we obtain $X^+(i;r,s) = 0$ for any $r, s \in \mathbb{Z}$.

A(ii).b Verification of (U3)

This relation is verified completely analogously to (U2).

A(ii).c Verification of (U4)

We consider the case $\epsilon = +$ (the case $\epsilon = -$ is completely analogous). We need to prove $Y^+(i;r,s) = 0$ for any $r \in \mathbb{N}, s \in \mathbb{Z}$, where

$$Y^+(i;r,s) := [\psi_{i,r+1}^+, e_{i,s}]_{\boldsymbol{v}_i^2} + [e_{i,s+1}, \psi_{i,r}^+]_{\boldsymbol{v}_i^2}.$$

The $r = s = 0$ case is due to (v10) from our proof of Lemma A.1. Moreover, the same argument also yields $Y^+(i;0,s) = 0$ for any $s \in \mathbb{Z}$.

Note that $Y^+(i;r,s-1)+Y^+(i;s,r-1) = (\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})[X^+(i;r-1,s-1), f_{i,1}] = 0$ for $r,s \geq 0$. The first equality is due to (v1) and our definition of $\psi^+_{i,r}$, while the second equality follows from $X^+(i;r-1,s-1)=0$ proved above. In particular, $Y^+(i;r,-1)+Y^+(i;0,r-1)=0$ for $r \in \mathbb{N}$.

Combining the above two observations, we find

$$Y^+(i;r,-1) = 0 \text{ for any } r \in \mathbb{N}. \tag{v12}$$

Commuting iteratively the equality $Y^+(i;1,-1) = 0$ with $h_{i,\pm 1}$, we get $Y^+(i;1,s)=0$ for any $s \in \mathbb{Z}$, due to ($\hat{\text{U}}$1, v1, v9).

Next, we prove the following five statements by induction in $N \in \mathbb{Z}_+$:
(A_N) $[h_{i,1}, \psi^+_{i,r}] = 0$ for $0 \leq r \leq N+1$;
(B_N) $[h_{i,-1}, \psi^+_{i,r}] = 0$ for $0 \leq r \leq N+1$;
(C_N) $[e_{i,r}, f_{i,s}] = \psi^+_{i,r+s}/(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})$ for any $r,s \in \mathbb{N}$ with $1 \leq r+s \leq N+2$;
(D_N) $Y^+(i;r,s) = 0$ for any $0 \leq r \leq N$, $s \in \mathbb{Z}$;
(E_N) $[\psi^+_{i,r}, \psi^+_{i,s}] = 0$ for any $r,s \geq 0$ with $r+s \leq N+2$.

Base of Induction ($N = 1$) The assertions (A_1, B_1, D_1, E_1) have been already proved above, while (C_1) follows immediately from $[h_{i,1}, \psi^+_{i,2}] = 0$ (cf. (v7)) and (v1, v2, v4, v5).

Induction Step Assuming $(A_N$–$E_N)$ for a given $N \in \mathbb{Z}_{>0}$, we prove $(A_{N+1}$–$E_{N+1})$.

Proof of the Induction Step Consider a polynomial algebra $B := \mathbb{C}(\boldsymbol{v})[\{x_r\}_{r=1}^\infty]$, which is $\mathbb{N}$-graded via $\deg(x_r) = r$. Define elements $\{\mathsf{h}_r\}_{r=1}^\infty$ of B via $\exp\left((\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})\sum_{r=1}^\infty \mathsf{h}_r z^{-r}\right) = 1 + \sum_{r=1}^\infty x_r z^{-r}$. Then, $\mathsf{h}_r = \frac{x_r}{\boldsymbol{v}_i-\boldsymbol{v}_i^{-1}} + p_r(x_1,\ldots,x_{r-1})$ with polynomials p_r satisfying $\deg(p_r(x_1,\ldots,x_{r-1})) = r$.

Using the above polynomials p_r, we define $h_{i,1},\ldots,h_{i,N+1} \in \hat{\mathcal{U}}_{\mu_1,\mu_2}$ via

$$h_{i,r} := \frac{(\psi^+_{i,0})^{-1}\psi^+_{i,r}}{\boldsymbol{v}_i-\boldsymbol{v}_i^{-1}} + p_r((\psi^+_{i,0})^{-1}\psi^+_{i,1},\ldots,(\psi^+_{i,0})^{-1}\psi^+_{i,r-1}) \text{ for } 1 \leq r \leq N+1. \tag{A.2}$$

These $h_{i,r}$ are well-defined and are independent of the choice of $N > r-1$, due to the assumption (E_N) and the aforementioned degree condition on p_r. The following is straightforward:[10]

$$[h_{i,r}, e_{i,s}] = \frac{[2r]_{\boldsymbol{v}_i}}{r} \cdot e_{i,s+r} \text{ for } 1 \leq r \leq N+1,\ s \in \mathbb{Z}. \tag{v13}$$

[10]If we knew that $[\psi^+_{i,a}, \psi^+_{i,b}] = 0$ for any $0 \leq a,b \leq N+1$, then (v13) would immediately follow from (D_N) by the standard arguments. However, every monomial appearing in p_r involves only pairwise commuting $\psi^+_{i,a}$'s, due to the degree condition on p_r and the assumption (E_N). Hence, the equality (v13) follows formally from its validity in the aforementioned simpler case ($[\psi^+_{i,a}, \psi^+_{i,b}] = 0$ for any $0 \leq a,b \leq N+1$).

Validity of (A_{N+1}) We need to prove $[h_{i,1}, \psi^+_{i,N+2}] = 0$. According to (C_N), we have $\psi^+_{i,N+2} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})[e_{i,N+2-r}, f_{i,r}]$ for $0 \le r \le N+2$. Hence,

$$[h_{i,1}, \psi^+_{i,N+2}]/(\boldsymbol{v}_i^2 - \boldsymbol{v}_i^{-2}) = [e_{i,N+3-r}, f_{i,r}] - [e_{i,N+2-r}, f_{i,r+1}] \text{ for } 0 \le r \le N+2. \tag{v14}$$

Adding up these equalities for $r = 0, 1$ and using Lemma A.1 together with the assumption (C_N), we get

$$\frac{2[h_{i,1}, \psi^+_{i,N+2}]}{\boldsymbol{v}_i^2 - \boldsymbol{v}_i^{-2}} = [e_{i,N+3}, f_{i,0}] - [e_{i,N+1}, f_{i,2}]$$
$$= \frac{2}{[4]_{\boldsymbol{v}_i}} \cdot [h_{i,2}, [e_{i,N+1}, f_{i,0}]] = \frac{2[h_{i,2}, \psi^+_{i,N+1}]}{\boldsymbol{v}_i^4 - \boldsymbol{v}_i^{-4}}.$$

Likewise, adding up the equality (v14) for $r = 0, 1, \ldots, N$ and using (v13), we obtain

$$\frac{N+1}{\boldsymbol{v}_i^2 - \boldsymbol{v}_i^{-2}}[h_{i,1}, \psi^+_{i,N+2}] = \frac{N+1}{[2(N+1)]_{\boldsymbol{v}_i}} \cdot [h_{i,N+1}, [e_{i,2}, f_{i,0}]] = \frac{(N+1)[h_{i,N+1}, \psi^+_{i,2}]}{\boldsymbol{v}_i^{2(N+1)} - \boldsymbol{v}_i^{-2(N+1)}}.$$

Comparing the above two equalities, we find

$$[h_{i,1}, \psi^+_{i,N+2}] = \frac{\boldsymbol{v}_i^2 - \boldsymbol{v}_i^{-2}}{\boldsymbol{v}_i^4 - \boldsymbol{v}_i^{-4}}[h_{i,2}, \psi^+_{i,N+1}] = \frac{\boldsymbol{v}_i^2 - \boldsymbol{v}_i^{-2}}{\boldsymbol{v}_i^{2(N+1)} - \boldsymbol{v}_i^{-2(N+1)}}[h_{i,N+1}, \psi^+_{i,2}]. \tag{v15}$$

On the other hand, combining (A.2) with the assumption (E_N), we get

$$[h_{i,s}, \psi^+_{i,N+3-s}] = (\psi^+_{i,0})^{-1}[\psi^+_{i,s}, \psi^+_{i,N+3-s}]/(\boldsymbol{v}_i - \boldsymbol{v}_i^{-1}) \text{ for } 1 \le s \le N+1.$$

Hence,

$$[h_{i,1}, \psi^+_{i,N+2}] = \frac{(\psi^+_{i,0})^{-1}[\psi^+_{i,2}, \psi^+_{i,N+1}]}{(\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})[2]_{\boldsymbol{v}_i^2}} = \frac{(\psi^+_{i,0})^{-1}[\psi^+_{i,2}, \psi^+_{i,N+1}]}{(\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})[-N-1]_{\boldsymbol{v}_i^2}}. \tag{v16}$$

Since $[2]_{\boldsymbol{v}_i^2} \neq [-N-1]_{\boldsymbol{v}_i^2}$, the second equality of (v16) implies $[\psi^+_{i,2}, \psi^+_{i,N+1}] = 0$. Hence, $[h_{i,1}, \psi^+_{i,N+2}] = 0$, and (A_{N+1}) follows.

Validity of (B_{N+1}) We need to prove $[h_{i,-1}, \psi^+_{i,N+2}] = 0$. This follows from $[h_{i,-1}, \psi^+_{i,N+2}] = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})[2]_{\boldsymbol{v}_i} \cdot ([e_{i,N}, f_{i,1}] - [e_{i,N+1}, f_{i,0}]) = 0$, where we used (v1, v2) in the first equality and (C_N) in the second one. Hence, (B_{N+1}) holds.

Validity of (C_{N+1}) According to (C_N), we have $\psi^+_{i,N+2} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})[e_{i,r}, f_{i,N+2-r}]$ for any $0 \le r \le N+2$. Therefore, $[h_{i,1}, \psi^+_{i,N+2}] = (\boldsymbol{v}_i^2 - \boldsymbol{v}_i^{-2})([e_{i,r+1}, f_{i,N+2-r}] - [e_{i,r}, f_{i,N+3-r}])$ due to (v1, v2). The left-hand side is zero due to (A_{N+1}) established above, hence

$$[e_{i,N+3}, f_{i,0}] = [e_{i,N+2}, f_{i,1}] = \ldots = [e_{i,1}, f_{i,N+2}] = [e_{i,0}, f_{i,N+3}].$$

Combining this with our definition $\psi^+_{i,N+3} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})[e_{i,N+2}, f_{i,1}]$ yields (C_{N+1}).

Validity of (D_{N+1}) Due to (A_{N+1}) and (B_{N+1}) established above, we have $[h_{i,\pm 1}, Y^+(i; N+1, s)] = [2]_{\boldsymbol{v}_i} \cdot Y^+(i; N+1, s \pm 1)$. Combining this with (v12), we see that $Y^+(i; N+1, s) = 0$ for any $s \in \mathbb{Z}$. Hence, (D_{N+1}) holds.

Validity of (E_{N+1}) We need to prove $[\psi^+_{i,r}, \psi^+_{i,N+3-r}] = 0$ for any $1 \le r \le N+1$. Equivalently, it suffices to prove $[h_{i,r}, \psi^+_{i,N+3-r}] = 0$ for $1 \le r \le N+1$. According to (C_N), we have $\psi^+_{i,N+3-r} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})[e_{i,N+3-r}, f_{i,0}]$. Therefore, $[h_{i,r}, \psi^+_{i,N+3-r}] = \frac{\boldsymbol{v}_i^{2r} - \boldsymbol{v}_i^{-2r}}{r} \cdot ([e_{i,N+3}, f_{i,0}] - [e_{N+3-r}, f_{i,r}]) = 0$, due to (v13) and the assertion (C_{N+1}) proved above. □

The induction step is accomplished. In particular, (D_N) completes our verification of (U4) with $i = j$.

A(ii).d Verification of (U5)

This relation is verified completely analogously to (U4).

A(ii).e Verification of (U6)

We need to prove

$$[e_{i,r}, f_{i,N-r}] = \frac{1}{\boldsymbol{v}_i - \boldsymbol{v}_i^{-1}} \cdot \begin{cases} \psi^+_{i,N} - \delta_{N,0}\delta_{b_i,0}\psi^-_{i,b_i} & \text{if } N \ge 0, \\ -\psi^-_{i,N} + \delta_{N,0}\delta_{b_i,0}\psi^+_{i,0} & \text{if } N \le b_i, \\ 0 & \text{if } b_i < N < 0. \end{cases}$$

Note that given any value of $N \in \mathbb{Z}$, we know this equality for a certain value of $r \in \mathbb{Z}$.

Case $N > 0$ If $0 \le r \le N$, then $[e_{i,r}, f_{i,N-r}] = \psi^+_{i,N}/(\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})$, due to (C_N). For $r < 0$, we proceed by induction in $|r|$. Due to (v1, v2), we have $[e_{i,r}, f_{i,N-r}] = [2]^{-1}_{\boldsymbol{v}_i} \cdot [[h_{i,-1}, e_{i,r+1}], f_{i,N-r}] = [2]^{-1}_{\boldsymbol{v}_i} \cdot [h_{i,-1}, [e_{i,r+1}, f_{i,N-r}]] + [e_{i,r+1}, f_{i,N-r-1}] = \psi^+_{i,N}$, where in the last equality we used the induction assumption and the equality $[h_{i,-1}, \psi^+_{i,N+1}] = 0$, due to (B_N). The case $l := N - r < 0$ is treated in the same way.

Case $N \leq 0$ We proceed by induction in $|N|$. For any $r \in \mathbb{Z}$, we have

$$[e_{i,r}, f_{i,N-r}] = [2]^{-1}_{\boldsymbol{v}_i} \cdot [h_{i,-1}, [e_{i,r+1}, f_{i,N-r}]] + [e_{i,r+1}, f_{i,N-r-1}] = [e_{i,r+1}, f_{i,N-r-1}],$$

where we used the induction assumption together with ($\hat{\text{U}}$1, v1, v2) and $[h_{i,-1}, \psi^-_i(z)] = 0$ (the latter is proved completely analogously to (A_N)). Hence, the expression $[e_{i,r}, f_{i,N-r}]$ is independent of $r \in \mathbb{Z}$. The result follows since we know the equality holds for a certain value of r.

A(ii).f Verification of (U1)

We consider the case $\epsilon = +$ (the case $\epsilon = -$ is completely analogous). We need to prove $[\psi^+_{i,r}, \psi^+_{i,s^+_i}] = [\psi^+_{i,r}, \psi^-_{i,-s^-_i}] = 0$ for any $r, s^+_i \geq 0, s^-_i \geq -b_i$. This is clear for $r = 0$ or $s^+_i = 0$, or $s^-_i = -b_i$, due to (v3). Therefore, it remains to prove $[h_{i,r}, \psi^+_{i,s^+_i}] = 0$ and $[h_{i,r}, \psi^-_{i,-s^-_i}] = 0$ for $r > 0, s^+_i > 0, s^-_i > -b_i$.

For $s^+_i > 0$, we have $\psi^+_{i,s^+_i} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})[e_{i,s^+_i-1}, f_{i,1}]$, so that

$$[h_{i,r}, \psi^+_{i,s^+_i}] = \frac{[2r]_{\boldsymbol{v}_i}}{r}(\boldsymbol{v}_i - \boldsymbol{v}_i^{-1}) \cdot ([e_{i,s^+_i+r-1}, f_{i,1}] - [e_{i,s^+_i-1}, f_{i,r+1}]) = 0,$$

where the first equality is due to (v13), while the second equality is due to relation (U6) with $i = j$ proved above.

For $s^-_i > -b_i$, we have $\psi^-_{i,-s^-_i} = (\boldsymbol{v}_i^{-1} - \boldsymbol{v}_i)[e_{i,-b_{1,i}-s^-_i}, f_{i,b_{1,i}}]$, so that

$$[h_{i,r}, \psi^-_{i,-s^-_i}] = \frac{[2r]_{\boldsymbol{v}_i}}{r}(\boldsymbol{v}_i^{-1} - \boldsymbol{v}_i) \cdot ([e_{i,r-b_{1,i}-s^-_i}, f_{i,b_{1,i}}] - [e_{i,-b_{1,i}-s^-_i}, f_{i,r+b_{1,i}}]) = 0,$$

where the first equality is due to (v13), while the second equality is due to relation (U6) with $i = j$ proved above.

This completes our verification of relations (U1–U6) with $i = j$ for $\hat{\mathcal{U}}_{\mu_1,\mu_2}$.

A(iii) Verification of Relations (U1–U8) with $i \neq j$ *for* $\hat{\mathcal{U}}_{\mu_1,\mu_2}$

A(iii).a Verification of (U2)

We need to prove $X^+(i, j; r, s) = 0$ for any $r, s \in \mathbb{Z}$, where

$$X^+(i, j; r, s) := [e_{i,r+1}, e_{j,s}]_{\boldsymbol{v}_i^{c_{ij}}} + [e_{j,s+1}, e_{i,r}]_{\boldsymbol{v}_i^{c_{ij}}}.$$

First, the equality $X^+(i,j;-1,-1)=0$ follows from (Û2). Second, due to (v1) we have

$$[h_{i,1}, X^+(i,j;r,s)] = [c_{ii}]_{\boldsymbol{v}_i}\cdot X^+(i,j;r+1,s) + [c_{ij}]_{\boldsymbol{v}_i}\cdot X^+(i,j;r,s+1),$$
$$[h_{j,1}, X^+(i,j;r,s)] = [c_{ji}]_{\boldsymbol{v}_j}\cdot X^+(i,j;r+1,s) + [c_{jj}]_{\boldsymbol{v}_j}\cdot X^+(i,j;r,s+1).$$

Combining these equalities with nondegeneracy of the matrix $A_{ij} := \begin{bmatrix} [c_{ii}]_{\boldsymbol{v}_i} & [c_{ij}]_{\boldsymbol{v}_i} \\ [c_{ji}]_{\boldsymbol{v}_j} & [c_{jj}]_{\boldsymbol{v}_j} \end{bmatrix}$, we see that $X^+(i,j;r,s)=0 \Rightarrow X^+(i,j;r+1,s)=0$, $X^+(i,j;r,s+1)=0$. Since $X^+(i,j;-1,-1)=0$, we get $X^+(i,j;r,s)=0$ for $r,s\geq -1$ by induction in r,s.

A similar reasoning with $h_{i,-1},h_{j,-1}$ used instead of $h_{i,1},h_{j,1}$ yields the implication

$$X^+(i,j;r,s)=0 \Longrightarrow X^+(i,j;r-1,s)=0,\ \ X^+(i,j;r,s-1)=0.$$

Hence, an induction argument completes the proof of $X^+(i,j;r,s)=0$ for any $r,s\in\mathbb{Z}$.

A(iii).b Verification of (U3)

We need to prove $X^-(i,j;r,s)=0$ for any $r,s\in\mathbb{Z}$, where

$$X^-(i,j;r,s) := [f_{i,r+1}, f_{j,s}]_{\boldsymbol{v}_i^{-c_{ij}}} + [f_{j,s+1}, f_{i,r}]_{\boldsymbol{v}_i^{-c_{ij}}}.$$

The $r=s=0$ case follows from (Û3). The general case follows from

$$X^-(i,j;r,s)=0 \Longrightarrow X^-(i,j;r\pm 1,s)=0,\ \ X^-(i,j;r,s\pm 1)=0$$

applied iteratively to $X^-(i,j;0,0)=0$, in the same vein as in the above verification of (U2).

A(iii).c Verification of (U6)

We need to prove $X(i,j;r,s)=0$ for any $r,s\in\mathbb{Z}$, where

$$X(i,j;r,s) := [e_{i,r}, f_{j,s}].$$

First, the equality $X(i,j;0,0)=0$ follows from (Û6). Second, due to (v1, v2) we have

$$[h_{i,\pm 1}, X(i,j;r,s)] = [c_{ii}]_{\boldsymbol{v}_i}\cdot X(i,j;r\pm 1,s) - [c_{ij}]_{\boldsymbol{v}_i}\cdot X(i,j;r,s\pm 1),$$
$$[h_{j,\pm 1}, X(i,j;r,s)] = [c_{ji}]_{\boldsymbol{v}_j}\cdot X(i,j;r\pm 1,s) - [c_{jj}]_{\boldsymbol{v}_j}\cdot X(i,j;r,s\pm 1).$$

Combining these equalities with nondegeneracy of the matrix $B_{ij} := \begin{bmatrix} [c_{ii}]_{\boldsymbol{v}_i} & -[c_{ij}]_{\boldsymbol{v}_i} \\ [c_{ji}]_{\boldsymbol{v}_j} & -[c_{jj}]_{\boldsymbol{v}_j} \end{bmatrix}$, we see that $X(i,j;r,s) = 0 \Rightarrow X(i,j;r\pm 1,s) = 0, X(i,j;r,s\pm 1) = 0$. Hence, the equality $X(i,j;r,s)=0$ for any $r,s\in\mathbb{Z}$ follows from the $r=s=0$ case considered above.

A(iii).d Verification of (U4)

We consider the case $\epsilon=+$ (the case $\epsilon=-$ is completely analogous). We need to prove $Y^+(i,j;r,s)=0$ for any $r\in\mathbb{N}, s\in\mathbb{Z}$, where

$$Y^+(i,j;r,s) := [\psi^+_{i,r+1}, e_{j,s}]_{\boldsymbol{v}_i^{c_{ij}}} + [e_{j,s+1}, \psi^+_{i,r}]_{\boldsymbol{v}_i^{c_{ij}}}.$$

Due to relation (U6) (established already both for $i=j$ and $i\neq j$), we have

$$(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})[[e_{i,r+1}, e_{j,s}]_{\boldsymbol{v}_i^{c_{ij}}}, f_{i,0}] = [\psi^+_{i,r+1}, e_{j,s}]_{\boldsymbol{v}_i^{c_{ij}}},$$

$$(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})[[e_{j,s+1}, e_{i,r}]_{\boldsymbol{v}_i^{c_{ij}}}, f_{i,0}] = [e_{j,s+1}, \psi^+_{i,r} - \delta_{r,0}\delta_{b_i,0}\psi^-_{i,-b_i}]_{\boldsymbol{v}_i^{c_{ij}}} = [e_{j,s+1}, \psi^+_{i,r}]_{\boldsymbol{v}_i^{c_{ij}}}.$$

Therefore, $Y^+(i,j;r,s) = (\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})[X^+(i,j;r,s), f_{i,0}] = 0$, where the last equality follows from $X^+(i,j;r,s)=0$ proved above.

A(iii).e Verification of (U5)

We consider the case $\epsilon=+$ (the case $\epsilon=-$ is completely analogous). We need to prove $Y^-(i,j;r,s)=0$ for any $r\in\mathbb{N}, s\in\mathbb{Z}$, where

$$Y^-(i,j;r,s) := [\psi^+_{i,r+1}, f_{j,s}]_{\boldsymbol{v}_i^{-c_{ij}}} + [f_{j,s+1}, \psi^+_{i,r}]_{\boldsymbol{v}_i^{-c_{ij}}}.$$

Analogously to our verification of (U4), we have $Y^-(i,j;r,s) = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})[e_{i,0}, X^-(i,j;r,s)]$. Thus, the equality $Y^-(i,j;r,s)=0$ follows from $X^-(i,j;r,s)=0$ proved above.

A(iii).f Verification of (U1)

We consider the case $\epsilon=\epsilon'=+$ (other cases are completely analogous). Due to relation (v3), it suffices to prove $[h_{i,r}, \psi^+_{j,s}]=0$ for $r,s\in\mathbb{Z}_{>0}$, where the elements $\{h_{i,r}\}_{r=1}^{\infty}$ were defined in (A.2).

Analogously to (v13), relations (U4, U5) imply

$$[h_{i,r}, e_{j,s}] = \frac{[rc_{ij}]_{\boldsymbol{v}_i}}{r}\cdot e_{j,s+r},\ [h_{i,r}, f_{j,s}] = -\frac{[rc_{ij}]_{\boldsymbol{v}_i}}{r}\cdot f_{j,s+r} \text{ for any } r\in\mathbb{Z}_{>0}, s\in\mathbb{Z}.$$

Hence, we have

$$[h_{i,r},\psi^+_{j,s}]=(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})[h_{i,r},[e_{j,s},f_{j,0}]]=(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})\frac{[rc_{ij}]_{\boldsymbol{v}_i}}{r}\cdot([e_{j,s+r},f_{j,0}]-[e_{j,s},f_{j,r}])=0,$$

where the first and the last equalities follow from (U6) with $i=j$ established above.

A(iii).g Verification of (U7)

In the simplest case $c_{ij}=0$, we need to prove $[e_{i,r},e_{j,s}]=0$ for any $r,s\in\mathbb{Z}$. The equality $[e_{i,0},e_{j,0}]=0$ is due to ($\hat{\text{U}}7$), while commuting it iteratively with $h_{i,\pm1},h_{j,\pm1}$, we get $[e_{i,r},e_{j,s}]=0$, due to (v1, v2).

In general, we set $m:=1-c_{ij}$. For any $\vec{r}=(r_1,\ldots,r_m)\in\mathbb{Z}^m$ and $s\in\mathbb{Z}$, define

$$Z^+(i,j;\vec{r},s):=\sum_{\pi\in\mathfrak{S}_m}\sum_{t=0}^{m}(-1)^t\begin{bmatrix}m\\t\end{bmatrix}_{\boldsymbol{v}_i}e_{i,r_{\pi(1)}}\cdots e_{i,r_{\pi(t)}}e_{j,s}e_{i,r_{\pi(t+1)}}\cdots e_{i,r_{\pi(m)}}.$$

To check (U7), we need to prove $Z^+(i,j;\vec{r},s)=0$ for any $\vec{r}\in\mathbb{Z}^m$, $s\in\mathbb{Z}$.

Let $\vec{0}=(0,\ldots,0)\in\mathbb{Z}^m$. The equality $Z^+(i,j;\vec{0},0)=0$ follows from ($\hat{\text{U}}7$) (cf. Remark 5.4). Commuting $Z^+(i,j;\vec{0},s)$ with $h_{i,\pm1},h_{j,\pm1}$, and using nondegeneracy of the matrix A_{ij}, we get $Z^+(i,j;\vec{0},s)=0\Rightarrow Z^+(i,j;\vec{0},s\pm1)=0$. Therefore, $Z^+(i,j;\vec{0},s)=0$ for any $s\in\mathbb{Z}$.

Next, we prove that $Z^+(i,j;\vec{r},s)=0$ for any $\vec{r}=(r_1,\ldots,r_k,0,\ldots,0)\in\mathbb{Z}^m$, $s\in\mathbb{Z}$ by induction in $0\le k\le m$. The base case $k=0$ was just treated above. For the induction step, note that the commutator $[h_{i,r'},Z^+(i,j;\vec{r},s)]$ equals $\frac{(m-k)\cdot[2r']_{\boldsymbol{v}_i}}{r'}Z^+(i,j;(r_1,\ldots,r_k,r',0,\ldots,0),s)$ plus some other terms which are zero by the induction assumption. Hence, $Z^+(i,j;\vec{r},s)=0$ for any $\vec{r}\in\mathbb{Z}^m$, $s\in\mathbb{Z}$.

A(iii).h Verification of (U8)

Set $m:=1-c_{ij}$. For any $\vec{r}\in\mathbb{Z}^m$, $s\in\mathbb{Z}$, define

$$Z^-(i,j;\vec{r},s):=\sum_{\pi\in\mathfrak{S}_m}\sum_{t=0}^{m}(-1)^t\begin{bmatrix}m\\t\end{bmatrix}_{\boldsymbol{v}_i}f_{i,r_{\pi(1)}}\cdots f_{i,r_{\pi(t)}}f_{j,s}f_{i,r_{\pi(t+1)}}\cdots f_{i,r_{\pi(m)}}.$$

Then, we need to show $Z^-(i,j;\vec{r},s)=0$. This is proved completely analogously to (U7).

This completes our proof of Theorem 5.5.

Remark A.2

(a) Specializing $\boldsymbol{v} \mapsto v \in \mathbb{C}^\times$ from the beginning and viewing all algebras as $\mathbb{C}$-algebras, the statement of Theorem 5.5 still holds as long as v is not a root of unity.
(b) A slightly different proof can be obtained by following the arguments in [47].
(c) We note that both Theorem 5.5 and its proof are valid also for all affine Lie algebras, except for the type $A_1^{(1)}$.

A(iv) An Alternative Presentation of $\mathcal{U}^{\mathrm{sc}}_{0,\mu}$ for $\mu \in \Lambda^-$

Inspired by the recent result [33, Theorem 2.13], we provide another realization of $\mathcal{U}^{\mathrm{sc}}_{0,\mu}$ (with $\mu \in \Lambda^-$) without the defining relation ($\hat{\mathrm{U}}9$). Following the notations of Sect. 5.2, denote by $\widetilde{\mathcal{U}}_{\mu_1,\mu_2}$ the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by

$$\{e_{i,r}, f_{i,s}, (\psi^+_{i,0})^{\pm 1}, (\psi^-_{i,b_i})^{\pm 1}, h_{i,\pm 1} | i \in I, b_{2,i} - 1 \le r \le 1, b_{1,i} - 1 \le s \le 1\}$$

with the defining relations ($\hat{\mathrm{U}}1$–$\hat{\mathrm{U}}8$). Define inductively $e_{i,r}, f_{i,r}, \psi^\pm_{i,r}$ as it was done for $\hat{\mathcal{U}}_{\mu_1,\mu_2}$ right before Theorem 5.5.

Theorem A.3 *There is a unique $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $\widetilde{\mathcal{U}}_{\mu_1,\mu_2} \xrightarrow{\sim} \mathcal{U}^{\mathrm{sc}}_{0,\mu}$, such that*

$$e_{i,r} \mapsto e_{i,r},\ \ f_{i,r} \mapsto f_{i,r},\ \ \psi^\pm_{i,\pm s_i^\pm} \mapsto \psi^\pm_{i,\pm s_i^\pm} \ \text{ for } i \in I, r \in \mathbb{Z}, s_i^+ \ge 0, s_i^- \ge -b_i.$$

Proof Due to Theorem 5.5, it suffices to show that ($\hat{\mathrm{U}}9$) can be derived from ($\hat{\mathrm{U}}1$–$\hat{\mathrm{U}}8$). We will treat only the first relation of ($\hat{\mathrm{U}}9$) (the second is completely analogous).

First, we note that relations (v1–v5) and (U2, U3, U6) with $i \ne j$ hold in $\widetilde{\mathcal{U}}_{\mu_1,\mu_2}$, since their proofs for the algebra $\hat{\mathcal{U}}_{\mu_1,\mu_2}$ were solely based on relations ($\hat{\mathrm{U}}1$–$\hat{\mathrm{U}}6$). Likewise, the equalities $Y^\pm(i,j;r,s) = 0$ from our verifications of (U4, U5) for $i \ne j$ still hold for $r \in \{0,1\}, s \in \mathbb{Z}$.

Second, we have

$$[\psi^+_{i,2}, e_{i,0}]_{\boldsymbol{v}_i^2} + [e_{i,1}, \psi^+_{i,1}]_{\boldsymbol{v}_i^2} = 0,\ [\psi^+_{i,2}, f_{i,0}]_{\boldsymbol{v}_i^{-2}} + [f_{i,1}, \psi^+_{i,1}]_{\boldsymbol{v}_i^{-2}} = 0. \tag{v18}$$

These equalities are proved completely analogously to (v11) from our proof of Lemma A.1, but now we start from the equality $[e_{i,1}, e_{i,0}]_{\boldsymbol{v}_i^2} = 0$ rather than $[e_{i,0}, e_{i,-1}]_{\boldsymbol{v}_i^2} = 0$ (commuting it first with $h_{i,1}$ and then further with $f_{i,0}$).

Recall $h_{i,2}$ of (A.1). Analogously to Lemma A.1, we see that (v18) implies[11]

$$[h_{i,2}, e_{i,0}] = \frac{[4]_{\boldsymbol{v}_i}}{2} \cdot e_{i,2},\ [h_{i,2}, f_{i,0}] = -\frac{[4]_{\boldsymbol{v}_i}}{2} \cdot f_{i,2}. \tag{v19}$$

Likewise, the aforementioned equalities $Y^{\pm}(i, j; 1, s) = 0$ for $i \neq j, s \in \mathbb{Z}$, also imply

$$[h_{i,2}, e_{j,s}] = \frac{[2c_{ij}]_{\boldsymbol{v}_i}}{2} \cdot e_{j,s+2},\ [h_{i,2}, f_{j,s}] = -\frac{[2c_{ij}]_{\boldsymbol{v}_i}}{2} \cdot f_{j,s+2} \text{ for } i \neq j, s \in \mathbb{Z}. \tag{v20}$$

Finally, due to (Û7, Û8, v1, v2, v19, v20), we also get $[e_{i,r}, e_{j,s}] = [f_{i,r}, f_{j,s}] = 0$ if $c_{ij} = 0$ and $Z^{\pm}(i, j; r', 0, s) = Z^{\pm}(i, j; 1, 1, s) = 0$ if $c_{ij} = -1$ for $r, s \in \mathbb{Z}, r' \in \{0, 1, 2\}$.

In the simply-laced case, the rest of the proof follows from the next result.

Lemma A.4 *Let $i, j \in I$ be such that $c_{ij} = -1$. Then $[\psi^+_{i,1}, \psi^+_{i,2}] = 0$.*

Proof As just proved, we have $[f_{i,1}, [f_{i,1}, f_{j,0}]_{\boldsymbol{v}_i^{-1}}]_{\boldsymbol{v}_i} = 0$. Commuting this equality with $e_{j,1}$ and applying (v4) together with (U6) for $i \neq j$, we get $[f_{i,1}, [f_{i,1}, \psi^+_{j,1}]_{\boldsymbol{v}_i^{-1}}]_{\boldsymbol{v}_i} = 0$. Combining the latter equality with $\psi^+_{j,1} = (\boldsymbol{v}_j - \boldsymbol{v}_j^{-1})\psi^+_{j,0}h_{j,1} = (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})\psi^+_{j,0}h_{j,1}$ and using (v2), we find

$$[f_{i,1}, [f_{i,1}, h_{j,1}]]_{\boldsymbol{v}_i^2} = 0 \Longrightarrow [f_{i,1}, f_{i,2}]_{\boldsymbol{v}_i^2} = 0 \Longrightarrow [f_{i,2}, f_{i,1}]_{\boldsymbol{v}_i^{-2}} = 0.$$

Commuting this further with $e_{i,0}$, we obtain

$$[\psi^+_{i,2}, f_{i,1}]_{\boldsymbol{v}_i^{-2}} + [f_{i,2}, \psi^+_{i,1}]_{\boldsymbol{v}_i^{-2}} = 0.$$

Finally, we apply $[e_{i,0}, -]_{\boldsymbol{v}_i^{-2}}$ to the latter equality. In the left-hand side we get two summands computed below.

(1) We have $[e_{i,0}, [f_{i,2}, \psi^+_{i,1}]_{\boldsymbol{v}_i^{-2}}]_{\boldsymbol{v}_i^{-2}} = [[e_{i,0}, f_{i,2}], \psi^+_{i,1}]_{\boldsymbol{v}_i^{-4}} + [f_{i,2}, [e_{i,0}, \psi^+_{i,1}]_{\boldsymbol{v}_i^{-2}}]_{\boldsymbol{v}_i^{-2}}$. Due to (Û4), $[e_{i,0}, \psi^+_{i,1}]_{\boldsymbol{v}_i^{-2}} = (\boldsymbol{v}_i^{-2} - \boldsymbol{v}_i^2)e_{i,1}\psi^+_{i,0} \Rightarrow [f_{i,2}, [e_{i,0}, \psi^+_{i,1}]_{\boldsymbol{v}_i^{-2}}]_{\boldsymbol{v}_i^{-2}} = (\boldsymbol{v}_i^{-2} - \boldsymbol{v}_i^2)[f_{i,2}, e_{i,1}]_{\boldsymbol{v}_i^{-4}}\psi^+_{i,0}$. Combining this with (v5), we thus get

$$[e_{i,0}, [f_{i,2}, \psi^+_{i,1}]_{\boldsymbol{v}_i^{-2}}]_{\boldsymbol{v}_i^{-2}} = [\psi^+_{i,2}, \psi^+_{i,1}]_{\boldsymbol{v}_i^{-4}}/(\boldsymbol{v}_i - \boldsymbol{v}_i^{-1}) + (\boldsymbol{v}_i^{-2} - \boldsymbol{v}_i^2)[f_{i,2}, e_{i,1}]_{\boldsymbol{v}_i^{-4}}\psi^+_{i,0}. \tag{v21}$$

[11] Note that we cannot deduce the statement of Lemma A.1 due to the absence of (Û9).

(2) We have $[e_{i,0},[\psi^+_{i,2},f_{i,1}]_{\boldsymbol{v}_i^{-2}}]_{\boldsymbol{v}_i^{-2}} = [[e_{i,0},\psi^+_{i,2}]_{\boldsymbol{v}_i^{-2}},f_{i,1}]_{\boldsymbol{v}_i^{-2}} + \boldsymbol{v}_i^{-2}[\psi^+_{i,2},[e_{i,0},f_{i,1}]]$. By (v18): $[e_{i,0},\psi^+_{i,2}]_{\boldsymbol{v}_i^{-2}} = -\boldsymbol{v}_i^{-2}[\psi^+_{i,2},e_{i,0}]_{\boldsymbol{v}_i^2} = \boldsymbol{v}_i^{-2}[e_{i,1},\psi^+_{i,1}]_{\boldsymbol{v}_i^2} = \boldsymbol{v}_i^{-2}(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})[e_{i,1},h_{i,1}]_{\boldsymbol{v}_i^4}\psi^+_{i,0}$. Hence,

$$[[e_{i,0},\psi^+_{i,2}]_{\boldsymbol{v}_i^{-2}},f_{i,1}]_{\boldsymbol{v}_i^{-2}} = \boldsymbol{v}_i^{-4}(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})[e_{i,1}h_{i,1}-\boldsymbol{v}_i^4h_{i,1}e_{i,1},f_{i,1}]\psi^+_{i,0} =$$

$$\boldsymbol{v}_i^{-4}(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})([\psi^+_{i,2},h_{i,1}]_{\boldsymbol{v}_i^4}/(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})-(\boldsymbol{v}_i+\boldsymbol{v}_i^{-1})[e_{i,1},f_{i,2}]_{\boldsymbol{v}_i^4})\psi^+_{i,0}.$$

Therefore,

$$[e_{i,0},[\psi^+_{i,2},f_{i,1}]_{\boldsymbol{v}_i^{-2}}]_{\boldsymbol{v}_i^{-2}} = \frac{[\psi^+_{i,2},\psi^+_{i,1}]}{\boldsymbol{v}_i^2(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})} + (\boldsymbol{v}_i^2-\boldsymbol{v}_i^{-2})[f_{i,2},e_{i,1}]_{\boldsymbol{v}_i^{-4}}\psi^+_{i,0} + \frac{[\psi^+_{i,2},\psi^+_{i,1}]_{\boldsymbol{v}_i^4}}{\boldsymbol{v}_i^4(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})}. \tag{v22}$$

Substituting (v22) and (v21) into $[e_{i,0},[\psi^+_{i,2},f_{i,1}]_{\boldsymbol{v}_i^{-2}}+[f_{i,2},\psi^+_{i,1}]_{\boldsymbol{v}_i^{-2}}]_{\boldsymbol{v}_i^{-2}} = 0$, we find

$$[\psi^+_{i,2},\psi^+_{i,1}]_{\boldsymbol{v}_i^{-4}} + \boldsymbol{v}_i^{-2}[\psi^+_{i,2},\psi^+_{i,1}] + \boldsymbol{v}_i^{-4}[\psi^+_{i,2},\psi^+_{i,1}]_{\boldsymbol{v}_i^4} = 0.$$

The left-hand side of this equality equals $\frac{1-\boldsymbol{v}_i^{-6}}{1-\boldsymbol{v}_i^{-2}}\cdot[\psi^+_{i,2},\psi^+_{i,1}]$. Hence, $[\psi^+_{i,1},\psi^+_{i,2}]=0$.

□

Our next result completes the proof for non-simply-laced $\mathfrak{g}$.

Lemma A.5 *If $c_{ij}\neq 0$ and $[\psi^+_{i,1},\psi^+_{i,2}]=0$, then $[\psi^+_{j,1},\psi^+_{j,2}]=0$.*

Proof Due to (v1, v2): $[h_{i,1},e_{i,r}] = \frac{[2]_{\boldsymbol{v}_i}}{[c_{ji}]_{\boldsymbol{v}_j}}\cdot[h_{j,1},e_{i,r}]$, $[h_{i,1},f_{i,r}] = \frac{[2]_{\boldsymbol{v}_i}}{[c_{ji}]_{\boldsymbol{v}_j}}\cdot[h_{j,1},f_{i,r}]$. Hence $[h_{i,1},\psi^+_{i,2}] = (\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})([[h_{i,1},e_{i,1}],f_{i,1}]+[e_{i,1},[h_{i,1},f_{i,1}]]) = [2]_{\boldsymbol{v}_i}/[c_{ji}]_{\boldsymbol{v}_j}\cdot[h_{j,1},\psi^+_{i,2}]$. Therefore, $[\psi^+_{i,1},\psi^+_{i,2}]=0 \Rightarrow [h_{j,1},\psi^+_{i,2}]=0 \Rightarrow [h_{j,1},h_{i,2}]=0$ with the second implication due to (Û1). Commuting the latter equality with $f_{j,0}$, we get

$$0 = [f_{j,0},[h_{j,1},h_{i,2}]] = [c_{jj}]_{\boldsymbol{v}_j}\cdot[f_{j,1},h_{i,2}] + \frac{[2c_{ij}]_{\boldsymbol{v}_i}}{2}\cdot[h_{j,1},f_{j,2}].$$

Commuting this further with $e_{j,0}$, we obtain

$$[c_{jj}]_{\boldsymbol{v}_j}\cdot[e_{j,0},[f_{j,1},h_{i,2}]] + \frac{[2c_{ij}]_{\boldsymbol{v}_i}}{2}\cdot[e_{j,0},[h_{j,1},f_{j,2}]] = 0. \tag{v23}$$

Note that

$$[e_{j,0},[f_{j,1},h_{i,2}]]=[\psi^+_{j,1},h_{i,2}]/(\boldsymbol{v}_j-\boldsymbol{v}_j^{-1})-\frac{[2c_{ij}]_{\boldsymbol{v}_i}}{2}\cdot[f_{j,1},e_{j,2}]=-\frac{[2c_{ij}]_{\boldsymbol{v}_i}}{2}\cdot[f_{j,1},e_{j,2}],$$

$$[e_{j,0},[h_{j,1},f_{j,2}]]=-[c_{jj}]_{\boldsymbol{v}_j}\cdot[e_{j,1},f_{j,2}]+[h_{j,1},\psi^+_{j,2}]/(\boldsymbol{v}_j-\boldsymbol{v}_j^{-1}),$$

$$[e_{j,2},f_{j,1}]-[e_{j,1},f_{j,2}]=[c_{jj}]^{-1}_{\boldsymbol{v}_j}\cdot[h_{j,1},[e_{j,1},f_{j,1}]]=[c_{jj}]^{-1}_{\boldsymbol{v}_j}\cdot[h_{j,1},\psi^+_{j,2}]/(\boldsymbol{v}_j-\boldsymbol{v}_j^{-1}).$$

Substituting the last three equalities into (v23), we get $\frac{[2c_{ij}]_{\boldsymbol{v}_i}}{\boldsymbol{v}_j-\boldsymbol{v}_j^{-1}}\cdot[h_{j,1},\psi^+_{j,2}]=0$. Thus, $[h_{j,1},\psi^+_{j,2}]=0\Rightarrow[\psi^+_{j,1},\psi^+_{j,2}]=0$. □

This completes our proof of Theorem A.3. □

Appendix B Proof of Theorem 6.6

The proof of part (a) proceeds in two steps. First, we consider the simplest case $\mathfrak{g}=\mathfrak{sl}_2$. Then, we show how a general case can be easily reduced to the case of $\mathfrak{sl}_2$.

B(i) Proof of Theorem 6.6(a) for $\mathfrak{g}=\mathfrak{sl}_2$

First, let us derive an explicit formula for $A^\pm(z)$. Recall the elements $\{h_{\pm r}\}_{r=1}^\infty$ of Sect. 5, such that $z^{\mp b^\pm}(\psi^\pm_{\mp b^\pm})^{-1}\psi^\pm(z)=\exp\left(\pm(\boldsymbol{v}-\boldsymbol{v}^{-1})\sum_{r>0}h_{\pm r}z^{\mp r}\right)$. For $r\neq 0$, define $t_r:=-h_r/(1+\boldsymbol{v}^{2r})$, and set

$$A^\pm(z):=(\phi^\pm)^{-1}\cdot\exp\left(\pm(\boldsymbol{v}-\boldsymbol{v}^{-1})\sum_{r>0}t_{\pm r}z^{\mp r}\right). \tag{B.1}$$

Then, $z^{\mp b^\pm}\psi^\pm(z)=\frac{1}{A^\pm(z)A^\pm(\boldsymbol{v}^{-2}z)}$ and $A^\pm(z)$ is the unique solution with $A_0^\pm:=(\phi^\pm)^{-1}$.

Relations (6.6) and (6.7) follow immediately from (U10) and (U1), respectively, while the verification of (6.9–6.16) is based on the following result.

Lemma B.1 *For any $\epsilon,\epsilon'\in\{\pm\}$, we have:*

(a1) $(\boldsymbol{v}z-\boldsymbol{v}^{-1}w)A^\epsilon(z)e(w)=(z-w)e(w)A^\epsilon(z)$.
(a2) $(\boldsymbol{v}z-\boldsymbol{v}^{-1}w)A^\epsilon(z)e^{\epsilon'}(w)-(z-w)e^{\epsilon'}(w)A^\epsilon(z)=(\boldsymbol{v}-\boldsymbol{v}^{-1})wA^\epsilon(z)e^\epsilon(z)$.
(a3) $(\boldsymbol{v}z-\boldsymbol{v}^{-1}w)A^\epsilon(z)e^{\epsilon'}(w)-(z-w)e^{\epsilon'}(w)A^\epsilon(z)=(1-\boldsymbol{v}^{-2})we^\epsilon(\boldsymbol{v}^2z)A^\epsilon(z)$.
(b1) $(z-w)A^\epsilon(z)f(w)=(\boldsymbol{v}z-\boldsymbol{v}^{-1}w)f(w)A^\epsilon(z)$.
(b2) $(z-w)A^\epsilon(z)f^{\epsilon'}(w)-(\boldsymbol{v}z-\boldsymbol{v}^{-1}w)f^{\epsilon'}(w)A^\epsilon(z)=(\boldsymbol{v}^{-1}-\boldsymbol{v})zf^\epsilon(z)A^\epsilon(z)$.
(b3) $(z-w)A^\epsilon(z)f^{\epsilon'}(w)-(\boldsymbol{v}z-\boldsymbol{v}^{-1}w)f^{\epsilon'}(w)A^\epsilon(z)=(1-\boldsymbol{v}^2)zA^\epsilon(z)f^\epsilon(\boldsymbol{v}^2z)$.

(c) $(z-w)[e^\epsilon(z), f^{\epsilon'}(w)] = z(\psi^{\epsilon'}(w) - \psi^\epsilon(z))/(\boldsymbol{v}-\boldsymbol{v}^{-1})$.

(d1) $(z-\boldsymbol{v}^2w)e^\epsilon(z)e^{\epsilon'}(w) - (\boldsymbol{v}^2z-w)e^{\epsilon'}(w)e^\epsilon(z) = z[e_0, e^{\epsilon'}(w)]_{\boldsymbol{v}^2} + w[e_0, e^\epsilon(z)]_{\boldsymbol{v}^2}$.

(d2) $(z-\boldsymbol{v}^2w)e^\epsilon(z)e^{\epsilon'}(w) - (\boldsymbol{v}^2z-w)e^{\epsilon'}(w)e^\epsilon(z) = (1-\boldsymbol{v}^2)(we^\epsilon(z)^2 + ze^{\epsilon'}(w)^2)$.

(e1) $(\boldsymbol{v}^2z-w)f^\epsilon(z)f^{\epsilon'}(w) - (z-\boldsymbol{v}^2w)f^{\epsilon'}(w)f^\epsilon(z) = \boldsymbol{v}^2[f_1, f^{\epsilon'}(w)]_{\boldsymbol{v}^{-2}} + \boldsymbol{v}^2[f_1, f^\epsilon(z)]_{\boldsymbol{v}^{-2}}$.

(e2) $(\boldsymbol{v}^2z-w)f^\epsilon(z)f^{\epsilon'}(w) - (z-\boldsymbol{v}^2w)f^{\epsilon'}(w)f^\epsilon(z) = (\boldsymbol{v}^2-1)(zf^\epsilon(z)^2 + wf^{\epsilon'}(w)^2)$.

(f1) $(z-\boldsymbol{v}^2w)\psi^\epsilon(z)e^{\epsilon'}(w) - (\boldsymbol{v}^2z-w)e^{\epsilon'}(w)\psi^\epsilon(z) = (\boldsymbol{v}^{-2}-\boldsymbol{v}^2)w\psi^\epsilon(z)e^\epsilon(\boldsymbol{v}^2z)$.

(f2) $(z-\boldsymbol{v}^2w)\psi^\epsilon(z)e^{\epsilon'}(w) - (\boldsymbol{v}^2z-w)e^{\epsilon'}(w)\psi^\epsilon(z) = (1-\boldsymbol{v}^4)we^\epsilon(\boldsymbol{v}^{-2}z)\psi^\epsilon(z)$.

(g1) $(\boldsymbol{v}^2z-w)\psi^\epsilon(z)f^{\epsilon'}(w) - (z-\boldsymbol{v}^2w)f^{\epsilon'}(w)\psi^\epsilon(z) = (\boldsymbol{v}^2-\boldsymbol{v}^{-2})z\psi^\epsilon(z)f^\epsilon(\boldsymbol{v}^{-2}z)$.

(g2) $(\boldsymbol{v}^2z-w)\psi^\epsilon(z)f^{\epsilon'}(w) - (z-\boldsymbol{v}^2w)f^{\epsilon'}(w)\psi^\epsilon(z) = (\boldsymbol{v}^4-1)zf^\epsilon(\boldsymbol{v}^2z)\psi^\epsilon(z)$.

Proof

(a1) According to (U4′), we have $[t_r, e_s] = \frac{\boldsymbol{v}^{-2r}-1}{r(\boldsymbol{v}-\boldsymbol{v}^{-1})}e_{s+r}$ for $r \neq 0$, $s \in \mathbb{Z}$. Combining this with (B.1), we find $A^\pm(z)e(w) = e(w)A^\pm(z)\boldsymbol{v}^{\mp 1}\exp\left(\sum_{r>0}\frac{\boldsymbol{v}^{\mp 2r}-1}{r}(w/z)^{\pm r}\right)$. The latter exponent equals $\frac{z-w}{z-\boldsymbol{v}^{-2}w}$ (in the "+" case) or $\frac{z-w}{\boldsymbol{v}^2z-w}$ (in the "−" case), hence, (a1).

(a2, a3) First, we consider the case $\epsilon = \epsilon' = +$. Due to (a1), we have $\boldsymbol{v}A^+_{r+1}e_s - \boldsymbol{v}^{-1}A^+_r e_{s+1} = e_sA^+_{r+1} - e_{s+1}A^+_r$ for any $r \in \mathbb{N}$, $s \in \mathbb{Z}$. Multiplying this equality by $z^{-r}w^{-s-1}$ and summing over all $r, s \in \mathbb{N}$, we find $w^{-1}((\boldsymbol{v}z - \boldsymbol{v}^{-1}w)A^+(z)e^+(w) - (z-w)e^+(w)A^+(z)) = [e_0, A^+(z)]_{\boldsymbol{v}^{-1}}$. Note that the right-hand side is independent of w. Substituting either $w = z$ or $w = \boldsymbol{v}^2z$ into the left-hand side, we get the equalities (a2) and (a3) for $\epsilon = \epsilon' = +$, respectively.

Next, we consider the case $\epsilon = \epsilon' = -$. Due to (a1), we have $\boldsymbol{v}A^-_{-r+1}e_{-s} - \boldsymbol{v}^{-1}A^-_r e_{-s+1} = e_{-s}A^-_{-r+1} - e_{-s+1}A^-_{-r}$ for any $r \in \mathbb{N}$, $s \in \mathbb{Z}$, where we set $A^-_1 := 0$. Multiplying this equality by $-z^rw^{s-1}$ and summing over all $r \in \mathbb{N}$, $s \in \mathbb{Z}_{>0}$, we find $w^{-1}((\boldsymbol{v}z - \boldsymbol{v}^{-1}w)A^-(z)e^-(w) - (z-w)e^-(w)A^-(z)) = [e_0, A^-(z)]_{\boldsymbol{v}^{-1}}$. Note that the right-hand side is independent of w. Substituting either $w = z$ or $w = \boldsymbol{v}^2z$ into the left-hand side, we get the equalities (a2) and (a3) for $\epsilon = \epsilon' = -$, respectively.

The case $\epsilon' \neq \epsilon$ follows by combining the formula $e^{\epsilon'}(w) = e^\epsilon(w) + \epsilon' e(w)$ with part (a1) and the cases $\epsilon = \epsilon'$ of parts (a2, a3), established above.

(b1–b3) Parts (b1, b2, b3) are proved completely analogously to (a1, a2, a3), respectively.

(c) First, we consider the case $\epsilon = \epsilon'$. According to (U6), we have $[e_r, f_s] = \frac{\psi^+_{r+s}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$ for $r \geq 0, s > 0$. For $N > 0$, we have $(z-w)\sum_{s=1}^N w^{-s}z^{s-N} = z(w^{-N} - z^{-N})$. Hence, $(z-w)[e^+(z), f^+(w)] =$

$\sum_{N>0} z(w^{-N} - z^{-N})\frac{\psi^+_N}{\boldsymbol{v}-\boldsymbol{v}^{-1}} = z\frac{\psi^+(w)-\psi^+(z)}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$. Likewise, we have $[e_{-r}, f_{-s}] = -\frac{\psi^-_{-r-s}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$ for $r > 0, s \geq 0$. For $N > 0$, we have $(z-w)\sum_{s=1}^N z^s w^{N-s} = z(z^N - w^N)$. Hence, $(z-w)[e^-(z), f^-(w)] = -\sum_{N>0} z(z^N - w^N)\frac{\psi^-_{-N}}{\boldsymbol{v}-\boldsymbol{v}^{-1}} = z\frac{\psi^-(w)-\psi^-(z)}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$.

Next, we consider the case $\epsilon \neq \epsilon'$. According to (U6), we have $[e(z), f(w)] = \frac{\delta(z/w)}{\boldsymbol{v}-\boldsymbol{v}^{-1}}(\psi^+(z) - \psi^-(z)) = \frac{\delta(z/w)}{\boldsymbol{v}-\boldsymbol{v}^{-1}}(\psi^+(w) - \psi^-(w))$. Taking the terms with negative powers of w, we find $[e(z), f^+(w)] = \frac{z/w}{1-z/w}\frac{\psi^+(z)-\psi^-(z)}{\boldsymbol{v}-\boldsymbol{v}^{-1}} \Rightarrow (z-w)[e(z), f^+(w)] = z\frac{\psi^-(z)-\psi^+(z)}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$, while taking the terms with nonpositive powers of z, we find $[e^+(z), f(w)] = \frac{1}{1-w/z}\frac{\psi^+(w)-\psi^-(w)}{\boldsymbol{v}-\boldsymbol{v}^{-1}} \Rightarrow (z-w)[e^+(z), f(w)] = z\frac{\psi^+(w)-\psi^-(w)}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$. Combining these equalities with $(z-w)[e^+(z), f^+(w)] = z\frac{\psi^+(w)-\psi^+(z)}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$ from above and $e^-(z) = e^+(z) - e(z)$, $f^-(z) = f^+(z) - f(z)$, we obtain the $\epsilon \neq \epsilon'$ cases of part (c).

(d1) Comparing the coefficients of $z^{-\epsilon r} w^{-\epsilon' s}$ in both sides of relation (U2), we find $e_{\epsilon r+1}e_{\epsilon' s} - \boldsymbol{v}^2 e_{\epsilon r}e_{\epsilon' s+1} = \boldsymbol{v}^2 e_{\epsilon' s}e_{\epsilon r+1} - e_{\epsilon' s+1}e_{\epsilon r}$ for any $r, s \in \mathbb{Z}$. Multiplying this equality by $\epsilon\epsilon' \cdot z^{-\epsilon r}w^{-\epsilon' s}$ and summing over $r \geq \delta_{\epsilon,-},\ s \geq \delta_{\epsilon',-}$, we get (d1).

(d2) Substituting $w = z$ into the $\epsilon = \epsilon'$ case of (d1), we find $[e_0, e^\pm(z)]_{\boldsymbol{v}^2} = (1-\boldsymbol{v}^2)e^\pm(z)^2$. Replacing accordingly the right-hand side of (d1), we obtain (d2).

(e1, e2) Parts (e1, e2) are proved completely analogously to (d1, d2), respectively.

(f1, f2) Parts (f1, f2) are deduced from relation (U4) in the same way as we deduced parts (a2, a3) from (a1).

(g1, g2) Parts (g1, g2) are proved completely analogously to (f1, f2), respectively.

□

Now let us verify relations (6.9–6.16) using Lemma B.1. The idea is first to use parts (a3, b2) of Lemma B.1 (resp. parts (a2, b3)) to move all the series $A^\bullet(\cdot)$ to the right (resp. to the left), and then to use Lemma B.1(c–g2) to simplify the remaining part. Since $\mathfrak{g} = \mathfrak{sl}_2$ we will drop the index i from our notation.

B(i).a Verification of the First Relation in (6.9)

We need to prove $[B^\epsilon(z), B^{\epsilon'}(w)] = 0$, or equivalently, $(z-w)[B^\epsilon(z), B^{\epsilon'}(w)] = 0$. By definition, $B^\epsilon(z)B^{\epsilon'}(w) = (\boldsymbol{v}-\boldsymbol{v}^{-1})^2 A^\epsilon(z)e^\epsilon(z)A^{\epsilon'}(w)e^{\epsilon'}(w)$. Applying Lemma B.1(a2), we see that

$$(z-w)B^\epsilon(z)B^{\epsilon'}(w) = (\boldsymbol{v}-\boldsymbol{v}^{-1})^2 A^\epsilon(z)A^{\epsilon'}(w)((\boldsymbol{v}^{-1}z-\boldsymbol{v}w)e^\epsilon(z)e^{\epsilon'}(w)+(\boldsymbol{v}-\boldsymbol{v}^{-1})ze^{\epsilon'}(w)^2).$$

Hence, the equality $(z-w)[B^{\epsilon}(z), B^{\epsilon'}(w)]=0$ boils down to the vanishing of

$$(\boldsymbol{v}^{-1}z-\boldsymbol{v}w)e^{\epsilon}(z)e^{\epsilon'}(w)+(\boldsymbol{v}-\boldsymbol{v}^{-1})ze^{\epsilon'}(w)^2+(\boldsymbol{v}^{-1}w-\boldsymbol{v}z)e^{\epsilon'}(w)e^{\epsilon}(z)+(\boldsymbol{v}-\boldsymbol{v}^{-1})we^{\epsilon}(z)^2,$$

which is exactly the statement of Lemma B.1(d2).

B(i).b Verification of the Second Relation in (6.9)

We need to prove $[C^{\epsilon}(z), C^{\epsilon'}(w)]=0$, or equivalently, $(z-w)[C^{\epsilon}(z), C^{\epsilon'}(w)]=0$. By definition, $C^{\epsilon}(z)C^{\epsilon'}(w)=(\boldsymbol{v}-\boldsymbol{v}^{-1})^2 f^{\epsilon}(z)A^{\epsilon}(z)f^{\epsilon'}(w)A^{\epsilon'}(w)$. Applying Lemma B.1(b2), we see that

$$(z-w)C^{\epsilon}(z)C^{\epsilon'}(w)=(\boldsymbol{v}-\boldsymbol{v}^{-1})^2((\boldsymbol{v}z-\boldsymbol{v}^{-1}w)f^{\epsilon}(z)f^{\epsilon'}(w)+(\boldsymbol{v}^{-1}-\boldsymbol{v})zf^{\epsilon}(z)^2)A^{\epsilon}(z)A^{\epsilon'}(w).$$

Hence, the equality $(z-w)[C^{\epsilon}(z), C^{\epsilon'}(w)]=0$ boils down to the vanishing of

$$(\boldsymbol{v}z-\boldsymbol{v}^{-1}w)f^{\epsilon}(z)f^{\epsilon'}(w)+(\boldsymbol{v}^{-1}-\boldsymbol{v})zf^{\epsilon}(z)^2+(\boldsymbol{v}w-\boldsymbol{v}^{-1}z)f^{\epsilon'}(w)f^{\epsilon}(z)+(\boldsymbol{v}^{-1}-\boldsymbol{v})wf^{\epsilon'}(w)^2,$$

which is exactly the statement of Lemma B.1(e2).

B(i).c Verification of the Third Relation in (6.9)

The verification of the equality $[D^{\epsilon}(z), D^{\epsilon'}(w)]=0$ is much more cumbersome and is left to the interested reader.

B(i).d Verification of (6.10)

We need to prove $(z-w)[B^{\epsilon'}(w), A^{\epsilon}(z)]_{\boldsymbol{v}^{-1}}=(\boldsymbol{v}-\boldsymbol{v}^{-1})(zA^{\epsilon}(z)B^{\epsilon'}(w)-wA^{\epsilon'}(w)B^{\epsilon}(z))$. By definition and (6.7), the RHS equals $(\boldsymbol{v}-\boldsymbol{v}^{-1})^2A^{\epsilon}(z)A^{\epsilon'}(w)(ze^{\epsilon'}(w)-we^{\epsilon}(z))$. Meanwhile, the LHS equals $(\boldsymbol{v}-\boldsymbol{v}^{-1})(z-w)(A^{\epsilon'}(w)e^{\epsilon'}(w)A^{\epsilon}(z)-\boldsymbol{v}^{-1}A^{\epsilon}(z)A^{\epsilon'}(w)e^{\epsilon'}(w))$. We use Lemma B.1(a2) to replace the first term, so that the LHS equals

$$(\boldsymbol{v}-\boldsymbol{v}^{-1})A^{\epsilon}(z)A^{\epsilon'}(w)\left((\boldsymbol{v}z-\boldsymbol{v}^{-1}w)e^{\epsilon'}(w)-(\boldsymbol{v}-\boldsymbol{v}^{-1})we^{\epsilon}(z)-\boldsymbol{v}^{-1}(z-w)e^{\epsilon'}(w)\right),$$

which exactly coincides with the above formula for the RHS.

B(i).e Verification of (6.11)

We need to prove $(z-w)[A^{\epsilon}(z), C^{\epsilon'}(w)]_{\boldsymbol{v}} = (\boldsymbol{v}-\boldsymbol{v}^{-1})(wC^{\epsilon'}(w)A^{\epsilon}(z) - zC^{\epsilon}(z)A^{\epsilon'}(w))$. By definition and (6.7), the RHS equals $(\boldsymbol{v}-\boldsymbol{v}^{-1})^2(wf^{\epsilon'}(w) - zf^{\epsilon}(z))A^{\epsilon}(z)A^{\epsilon'}(w)$. Meanwhile, the LHS equals $(\boldsymbol{v}-\boldsymbol{v}^{-1})(z-w)(A^{\epsilon}(z)f^{\epsilon'}(w)A^{\epsilon'}(w) - \boldsymbol{v}f^{\epsilon'}(w)A^{\epsilon'}(w)A^{\epsilon}(z))$. We use Lemma B.1(b2) to replace the first term, so that the LHS equals

$$(\boldsymbol{v}-\boldsymbol{v}^{-1})\left((\boldsymbol{v}z-\boldsymbol{v}^{-1}w)f^{\epsilon'}(w) + (\boldsymbol{v}^{-1}-\boldsymbol{v})zf^{\epsilon}(z) - \boldsymbol{v}(z-w)f^{\epsilon'}(w)\right)A^{\epsilon}(z)A^{\epsilon'}(w),$$

which exactly coincides with the above formula for the RHS.

B(i).f Verification of (6.12)

We need to prove $(z-w)[B^{\epsilon}(z), C^{\epsilon'}(w)] = (\boldsymbol{v}-\boldsymbol{v}^{-1})z(D^{\epsilon'}(w)A^{\epsilon}(z) - D^{\epsilon}(z)A^{\epsilon'}(w))$. Applying the equality $A^{\epsilon}(z)e^{\epsilon}(z) = \boldsymbol{v}^{-1}e^{\epsilon}(\boldsymbol{v}^2z)A^{\epsilon}(z)$, which follows from Lemma B.1(a2), we see that the LHS equals

$$\boldsymbol{v}^{-1}(\boldsymbol{v}-\boldsymbol{v}^{-1})^2(z-w)\left(e^{\epsilon}(\boldsymbol{v}^2z)A^{\epsilon}(z)f^{\epsilon'}(w)A^{\epsilon'}(w) - f^{\epsilon'}(w)A^{\epsilon'}(w)e^{\epsilon}(\boldsymbol{v}^2z)A^{\epsilon}(z)\right).$$

Applying Lemma B.1(a3, b2) to move both $A^{\epsilon}(z)$, $A^{\epsilon'}(w)$ to the right and simplifying the resulting expression, we find that the LHS equals

$$\boldsymbol{v}^{-1}(\boldsymbol{v}-\boldsymbol{v}^{-1})^2\Big((\boldsymbol{v}z-\boldsymbol{v}^{-1}w)[e^{\epsilon}(\boldsymbol{v}^2z), f^{\epsilon'}(w)]+$$
$$(\boldsymbol{v}-\boldsymbol{v}^{-1})(zf^{\epsilon'}(w)e^{\epsilon'}(\boldsymbol{v}^2w) - (\boldsymbol{v}-\boldsymbol{v}^{-1})ze^{\epsilon}(\boldsymbol{v}^2z)f^{\epsilon}(z))\Big)A^{\epsilon}(z)A^{\epsilon'}(w).$$

Meanwhile, $D^{\epsilon}(z) = \psi^{\epsilon}(z)A^{\epsilon}(z) + \boldsymbol{v}^{-1}(\boldsymbol{v}-\boldsymbol{v}^{-1})^2f^{\epsilon}(z)e^{\epsilon}(\boldsymbol{v}^2z)A^{\epsilon}(z)$, so that the RHS equals

$$(\boldsymbol{v}-\boldsymbol{v}^{-1})\left(z(\psi^{\epsilon'}(w)-\psi^{\epsilon}(z)) + \boldsymbol{v}^{-1}(\boldsymbol{v}-\boldsymbol{v}^{-1})^2z(f^{\epsilon'}(w)e^{\epsilon'}(\boldsymbol{v}^2w) - f^{\epsilon}(z)e^{\epsilon}(\boldsymbol{v}^2z))\right)A^{\epsilon}(z)A^{\epsilon'}(w).$$

Thus, the equality LHS = RHS boils down to proving

$$\boldsymbol{v}^{-2}(\boldsymbol{v}^2z-w)[e^{\epsilon}(\boldsymbol{v}^2z), f^{\epsilon'}(w)] - (1-\boldsymbol{v}^{-2})z[e^{\epsilon}(\boldsymbol{v}^2z), f^{\epsilon}(z)] = \frac{z}{\boldsymbol{v}-\boldsymbol{v}^{-1}}(\psi^{\epsilon'}(w)-\psi^{\epsilon}(z)),$$

which immediately follows by applying Lemma B.1(c) to both terms on the left.

B(i).g Verification of (6.13)

We need to prove $(z-w)[B^\epsilon(z),D^{\epsilon'}(w)]_{\boldsymbol{v}} = (\boldsymbol{v}-\boldsymbol{v}^{-1})(wD^{\epsilon'}(w)B^\epsilon(z) - zD^\epsilon(z)B^{\epsilon'}(w))$. Combining the aforementioned equality $A^\epsilon(z)e^\epsilon(z) = \boldsymbol{v}^{-1}e^\epsilon(\boldsymbol{v}^2z)A^\epsilon(z)$ with Lemma B.1(a3), we find that $(w-z)\cdot$ RHS equals

$$\Big(\boldsymbol{v}^{-1}(\boldsymbol{v}-\boldsymbol{v}^{-1})^2(w(\boldsymbol{v}^{-1}w-\boldsymbol{v}z)\psi^{\epsilon'}(w)e^\epsilon(\boldsymbol{v}^2z)+z(\boldsymbol{v}^{-1}z-\boldsymbol{v}w)\psi^\epsilon(z)e^{\epsilon'}(\boldsymbol{v}^2w))+$$
$$\boldsymbol{v}^{-1}(\boldsymbol{v}-\boldsymbol{v}^{-1})^3zw(\psi^{\epsilon'}(w)e^{\epsilon'}(\boldsymbol{v}^2w)+\psi^\epsilon(z)e^\epsilon(\boldsymbol{v}^2z))+$$
$$\boldsymbol{v}^{-2}(\boldsymbol{v}-\boldsymbol{v}^{-1})^4(w(\boldsymbol{v}^{-1}w-\boldsymbol{v}z)f^{\epsilon'}(w)e^{\epsilon'}(\boldsymbol{v}^2w)e^\epsilon(\boldsymbol{v}^2z)+z(\boldsymbol{v}^{-1}z-\boldsymbol{v}w)f^\epsilon(z)e^\epsilon(\boldsymbol{v}^2z)e^{\epsilon'}(\boldsymbol{v}^2w))+$$
$$\boldsymbol{v}^{-2}(\boldsymbol{v}-\boldsymbol{v}^{-1})^5zw(f^{\epsilon'}(w)e^{\epsilon'}(\boldsymbol{v}^2w)^2+f^\epsilon(z)e^\epsilon(\boldsymbol{v}^2z)^2)\Big)A^\epsilon(z)A^{\epsilon'}(w).$$

Meanwhile, using Lemma B.1(a3, b2) to move $A^\epsilon(z)$ to the right of $f^{\epsilon'}(w)e^{\epsilon'}(\boldsymbol{v}^2w)$, we find that $(w-z)\cdot$ LHS equals

$$\boldsymbol{v}^{-1}(\boldsymbol{v}-\boldsymbol{v}^{-1})(w-z)\cdot\Big((w-\boldsymbol{v}^2z)\psi^{\epsilon'}(w)e^\epsilon(\boldsymbol{v}^2z)+(\boldsymbol{v}^2-1)z\psi^{\epsilon'}(w)e^{\epsilon'}(\boldsymbol{v}^2w)+$$
$$(z-w)e^\epsilon(\boldsymbol{v}^2z)\psi^{\epsilon'}(w))\Big)A^\epsilon(z)A^{\epsilon'}(w)+$$
$$\boldsymbol{v}^{-2}(\boldsymbol{v}-\boldsymbol{v}^{-1})^3\cdot\Big((w-\boldsymbol{v}^2z)(w-z)f^{\epsilon'}(w)e^{\epsilon'}(\boldsymbol{v}^2w)e^\epsilon(\boldsymbol{v}^2z)+(\boldsymbol{v}^2-1)z(w-z)f^{\epsilon'}(w)e^{\epsilon'}(\boldsymbol{v}^2w)^2-$$
$$(\boldsymbol{v}z-\boldsymbol{v}^{-1}w)(\boldsymbol{v}^{-1}z-\boldsymbol{v}w)e^\epsilon(\boldsymbol{v}^2z)f^{\epsilon'}(w)e^{\epsilon'}(\boldsymbol{v}^2w)-(\boldsymbol{v}^{-1}-\boldsymbol{v})z(\boldsymbol{v}^{-1}z-\boldsymbol{v}w)e^\epsilon(\boldsymbol{v}^2z)f^\epsilon(z)e^{\epsilon'}(\boldsymbol{v}^2w)-$$
$$(\boldsymbol{v}z-\boldsymbol{v}^{-1}w)(\boldsymbol{v}-\boldsymbol{v}^{-1})we^\epsilon(\boldsymbol{v}^2z)f^{\epsilon'}(w)e^\epsilon(\boldsymbol{v}^2z)-(\boldsymbol{v}^{-1}-\boldsymbol{v})(\boldsymbol{v}-\boldsymbol{v}^{-1})zwe^\epsilon(\boldsymbol{v}^2z)f^\epsilon(z)e^\epsilon(\boldsymbol{v}^2z)\Big)\times$$
$$A^\epsilon(z)A^{\epsilon'}(w).$$

To check that the above two big expressions coincide, we first reorder some of the terms. We use Lemma B.1(f1) to move $\psi^{\epsilon'}(w)$ to the left of $e^\epsilon(\boldsymbol{v}^2z)$ via

$$(w-z)e^\epsilon(\boldsymbol{v}^2z)\psi^{\epsilon'}(w)=\psi^{\epsilon'}(w)\Big((\boldsymbol{v}^{-2}w-\boldsymbol{v}^2z)e^\epsilon(\boldsymbol{v}^2z)-(\boldsymbol{v}^{-2}-\boldsymbol{v}^2)ze^{\epsilon'}(\boldsymbol{v}^2w)\Big).$$

We also use Lemma B.1(c) to move $f^\bullet(\cdot)$ to the left of $e^\bullet(\cdot)$. After obvious cancelations, everything boils down to proving

$$(\boldsymbol{v}^{-1}z-\boldsymbol{v}w)e^\epsilon(\boldsymbol{v}^2z)e^{\epsilon'}(\boldsymbol{v}^2w)-(\boldsymbol{v}z-\boldsymbol{v}^{-1}w)e^{\epsilon'}(\boldsymbol{v}^2w)e^\epsilon(\boldsymbol{v}^2z)=(\boldsymbol{v}^{-1}-\boldsymbol{v})(ze^{\epsilon'}(\boldsymbol{v}^2w)^2+we^\epsilon(\boldsymbol{v}^2z)^2),$$

which is exactly the statement of Lemma B.1(d2).

B(i).h Verification of (6.14)

This verification is completely analogous to the above verification of (6.13) and is left to the interested reader.

B(i).i Verification of (6.15)

We need to prove $(z-w)[A^\epsilon(z), D^{\epsilon'}(w)] = (\boldsymbol{v}-\boldsymbol{v}^{-1})(wC^{\epsilon'}(w)B^\epsilon(z) - zC^\epsilon(z)B^{\epsilon'}(w))$. The LHS equals $(\boldsymbol{v}-\boldsymbol{v}^{-1})^2(z-w)(A^\epsilon(z)f^{\epsilon'}(w)A^{\epsilon'}(w)e^{\epsilon'}(w) - f^{\epsilon'}(w)A^{\epsilon'}(w)e^{\epsilon'}(w)A^\epsilon(z))$. Applying Lemma B.1(b2) to the first summand and Lemma B.1(a2) to the second summand, we see that the LHS equals

$$(\boldsymbol{v}-\boldsymbol{v}^{-1})^2((\boldsymbol{v}z-\boldsymbol{v}^{-1}w)f^{\epsilon'}(w)+(\boldsymbol{v}^{-1}-v)zf^\epsilon(z))A^\epsilon(z)A^{\epsilon'}(w)e^{\epsilon'}(w)-$$
$$(\boldsymbol{v}-\boldsymbol{v}^{-1})^2f^{\epsilon'}(w)A^\epsilon(z)A^{\epsilon'}(w)((\boldsymbol{v}z-\boldsymbol{v}^{-1}w)e^{\epsilon'}(w)-(\boldsymbol{v}-\boldsymbol{v}^{-1})we^\epsilon(z))=$$
$$(\boldsymbol{v}-\boldsymbol{v}^{-1})^3(wf^{\epsilon'}(w)A^{\epsilon'}(w)A^\epsilon(z)e^\epsilon(z)-zf^\epsilon(z)A^\epsilon(z)A^{\epsilon'}(w)e^{\epsilon'}(w)),$$

which obviously coincides with the RHS.

B(i).j Verification of (6.16)

We need to prove $A^\epsilon(z)D^\epsilon(\boldsymbol{v}^{-2}z) - \boldsymbol{v}^{-1}B^\epsilon(z)C^\epsilon(\boldsymbol{v}^{-2}z) = z^{\epsilon b^\epsilon}$. Due to Lemma B.1(b3), we have $f^\epsilon(\boldsymbol{v}^{-2}z)A^\epsilon(\boldsymbol{v}^{-2}z) = \boldsymbol{v}A^\epsilon(\boldsymbol{v}^{-2}z)f^\epsilon(z)$. Thus,

$$A^\epsilon(z)D^\epsilon(\boldsymbol{v}^{-2}z) = A^\epsilon(z)A^\epsilon(\boldsymbol{v}^{-2}z)(\psi^\epsilon(\boldsymbol{v}^{-2}z)+\boldsymbol{v}(\boldsymbol{v}-\boldsymbol{v}^{-1})^2f^\epsilon(z)e^\epsilon(\boldsymbol{v}^{-2}z)),$$
$$B^\epsilon(z)C^\epsilon(\boldsymbol{v}^{-2}z) = \boldsymbol{v}(\boldsymbol{v}-\boldsymbol{v}^{-1})^2A^\epsilon(z)e^\epsilon(z)A^\epsilon(\boldsymbol{v}^{-2}z)f^\epsilon(z).$$

According to Lemma B.1(a2), we have $e^\epsilon(z)A^\epsilon(\boldsymbol{v}^{-2}z) = \boldsymbol{v}A^\epsilon(\boldsymbol{v}^{-2}z)e^\epsilon(\boldsymbol{v}^{-2}z)$. Hence,

$$B^\epsilon(z)C^\epsilon(\boldsymbol{v}^{-2}z) = \boldsymbol{v}^2(\boldsymbol{v}-\boldsymbol{v}^{-1})^2A^\epsilon(z)A^\epsilon(\boldsymbol{v}^{-2}z)e^\epsilon(\boldsymbol{v}^{-2}z)f^\epsilon(z).$$

Due to Lemma B.1(c), we have $-\boldsymbol{v}(\boldsymbol{v}-\boldsymbol{v}^{-1})^2[e^\epsilon(\boldsymbol{v}^{-2}z), f^\epsilon(z)] = \psi^\epsilon(z) - \psi^\epsilon(\boldsymbol{v}^{-2}z)$. Therefore, we finally get

$$A^\epsilon(z)D^\epsilon(\boldsymbol{v}^{-2}z)-\boldsymbol{v}^{-1}B^\epsilon(z)C^\epsilon(\boldsymbol{v}^{-2}z) = A^\epsilon(z)A^\epsilon(\boldsymbol{v}^{-2}z)\psi^\epsilon(\boldsymbol{v}^{-2}z)-$$
$$\boldsymbol{v}(\boldsymbol{v}-\boldsymbol{v}^{-1})^2A^\epsilon(z)A^\epsilon(\boldsymbol{v}^{-2}z)[e^\epsilon(\boldsymbol{v}^{-2}z), f^\epsilon(z)] = A^\epsilon(z)A^\epsilon(\boldsymbol{v}^{-2}z)\psi^\epsilon(z) = z^{\epsilon b^\epsilon},$$

which completes our verification of (6.16).

B(ii) Proof of Theorem 6.6(a) for a General $\mathfrak{g}$

First, let us derive an explicit formula for $A_i^\pm(z)$. Recall the elements $\{h_{i,\pm r}\}_{i\in I}^{r>0}$ of Sect. 5, such that $z^{\mp b_i^\pm}(\psi^\pm_{i,\mp b_i^\pm})^{-1}\psi_i^\pm(z) = \exp\left(\pm(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})\sum_{r>0}h_{i,\pm r}z^{\mp r}\right)$.

For $r \neq 0$, consider the following $I \times I$ matrix $C_{\boldsymbol{v}}(r)$:

$$C_{\boldsymbol{v}}(r)_{ij} = \begin{cases} 0 & \text{if } c_{ij} = 0, \\ -1 - \boldsymbol{v}_i^{2r} & \text{if } j = i, \\ \frac{\boldsymbol{v}_j - \boldsymbol{v}_j^{-1}}{\boldsymbol{v}_i - \boldsymbol{v}_i^{-1}} \sum_{p=1}^{-c_{ji}} \boldsymbol{v}_j^{r(c_{ji}+2p)} & \text{if } j - i. \end{cases}$$

Set $t_{i,r} := \sum_{j \in I} (C_{\boldsymbol{v}}(r)^{-1})_{ij} h_{j,r}$ (matrix $C_{\boldsymbol{v}}(r)$ is invertible, due to Lemma B.3 below). Define

$$A_i^{\pm}(z) := (\phi_i^{\pm})^{-1} \cdot \exp\left(\pm(\boldsymbol{v}_i - \boldsymbol{v}_i^{-1}) \sum_{r>0} t_{i,\pm r} z^{\mp r}\right). \tag{B.2}$$

These $A_i^{\pm}(z)$ satisfy $z^{\mp b_i^{\pm}} \psi_i^{\pm}(z) = \frac{\prod_{j-i} \prod_{p=1}^{-c_{ji}} A_j^{\pm}(\boldsymbol{v}_j^{-c_{ji}-2p} z)}{A_i^{\pm}(z) A_i^{\pm}(\boldsymbol{v}_i^{-2} z)}$ as well as $A_{i,0}^{\pm} = (\phi_i^{\pm})^{-1}$. This provides an explicit formula for $A_i^{\pm}(z)$, which we referred to in Sect. 6.

Remark B.2 Comparing the coefficients of $z^{\mp r}$ $(r > 0)$ in the system of equations (6.1) for all i, we see that $A_{i,\pm r}$ are recovered uniquely modulo the values of $A_{i,\pm s} (0 \leq s < r)$, due to invertibility of $C_{\boldsymbol{v}}(r)$. Therefore, an induction in r implies that the system of equations (6.1) has a unique solution $\{A_i^{\pm}(z)\}_{i \in I}$, hence, given by (B.2).

Define auxiliary $I \times I$ matrices $B_{\boldsymbol{v}}(r)$, $D_{\boldsymbol{v}}(r)$ via $B_{\boldsymbol{v}}(r)_{ij} = \frac{[rc_{ij}]_{\boldsymbol{v}_i}}{r}$, $D_{\boldsymbol{v}}(r)_{ij} = \delta_{ij} \frac{\boldsymbol{v}_j^{-2r} - 1}{r(\boldsymbol{v}_j - \boldsymbol{v}_j^{-1})}$. The matrix $B_{\boldsymbol{v}}(r)$ is a $\boldsymbol{v}$-version of the Cartan matrix of $\mathfrak{g}$ and it is known to be invertible for any $r \neq 0$. The following is straightforward.

Lemma B.3 *For $r \neq 0$, we have $B_{\boldsymbol{v}}(r) = C_{\boldsymbol{v}}(r) D_{\boldsymbol{v}}(r)$. In particular, $C_{\boldsymbol{v}}(r)$ is invertible.*

The following result is an immediate corollary of Lemma B.3 and relations (U4′, U5′).

Lemma B.4 *For $\epsilon \in \{\pm\}$, we have:*

(a) $(\boldsymbol{v}_i z - \boldsymbol{v}_i^{-1} w) A_i^{\epsilon}(z) e_i(w) = (z - w) e_i(w) A_i^{\epsilon}(z)$, *while* $A_i^{\epsilon}(z) e_j(w) = e_j(w) A_i^{\epsilon}(z)$ *for* $j \neq i$.

(b) $(z - w) A_i^{\epsilon}(z) f_i(w) = (\boldsymbol{v}_i z - \boldsymbol{v}_i^{-1} w) f_i(w) A_i^{\epsilon}(z)$, *while* $A_i^{\epsilon}(z) f_j(w) = f_j(w) A_i^{\epsilon}(z)$ *for* $j \neq i$.

Now we are ready to sketch the proof of Theorem 6.6(a) for a general $\mathfrak{g}$.

B(ii).a Verification of (6.7) and (6.8)

Relations (6.7, 6.8) follow from Lemma B.4 and relations (U1, U6).

B(ii).b Verification of (6.9–6.16)

Let us introduce the series $\bar{A}_i^{\pm}(z)$ via $z^{\mp b_i^{\pm}}\psi_i^{\pm}(z) = \frac{1}{\bar{A}_i^{\pm}(z)\bar{A}_i^{\pm}(\boldsymbol{v}_i^{-2}z)}$, and define the generating series $\bar{B}_i^{\pm}(z)$, $\bar{C}_i^{\pm}(z)$, $\bar{D}_i^{\pm}(z)$ by using formulas (6.2–6.4) but with $\bar{A}_i^{\pm}(z)$ instead of $A_i^{\pm}(z)$. For a fixed i, these series satisfy the corresponding relations (6.9–6.16) of the $\mathfrak{sl}_2$ case. However, $A_i^{\pm}(z)\bar{A}_i^{\pm}(z)^{-1}$ is expressed through $\{A_j^{\pm}(z)\}_{j\neq i}$, hence, commutes with $e_i^{\epsilon}(z)$, $f_i^{\epsilon}(z)$, $A_i^{\epsilon}(z)$, due to Lemma B.4. Relations (6.9–6.16) follow (this also explains the RHS of (6.16)).

B(ii).c Verification of (6.17)

Analogously to Lemma B.1(d1), relation (U2) implies the following equality:

$$(z-\boldsymbol{v}_i^{c_{ij}}w)e_i^{\epsilon}(z)e_j^{\epsilon'}(w)-(\boldsymbol{v}_i^{c_{ij}}z-w)e_j^{\epsilon'}(w)e_i^{\epsilon}(z)=z[e_{i,0},e_j^{\epsilon'}(w)]_{\boldsymbol{v}_i^{c_{ij}}}+w[e_{j,0},e_i^{\epsilon}(z)]_{\boldsymbol{v}_i^{c_{ij}}}$$

for any $\epsilon,\epsilon' \in \{\pm\}$ (we also note that these equalities for all possible ϵ,ϵ' imply (U2)). Multiplying the above equality by $(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})(\boldsymbol{v}_j-\boldsymbol{v}_j^{-1})A_i^{\epsilon}(z)A_j^{\epsilon'}(w)$ on the left and using Lemma B.4(a), relation (6.7), and an equality $(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})e_{i,0} = \phi_i^{+}B_{i,0}^{+}$, we obtain (6.17).

B(ii).d Verification of (6.18)

Analogously to Lemma B.1(e1), relation (U3) implies the following equality:

$$(\boldsymbol{v}_i^{c_{ij}}z-w)f_i^{\epsilon}(z)f_j^{\epsilon'}(w)-(z-\boldsymbol{v}_i^{c_{ij}}w)f_j^{\epsilon'}(w)f_i^{\epsilon}(z)=-[f_j^{\epsilon'}(w),f_{i,1}]_{\boldsymbol{v}_i^{c_{ij}}}-[f_i^{\epsilon}(z),f_{j,1}]_{\boldsymbol{v}_i^{c_{ij}}}$$

for any $\epsilon,\epsilon' \in \{\pm\}$ (we also note that these equalities for all possible ϵ,ϵ' imply (U3)). Multiplying the above equality by $(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})(\boldsymbol{v}_j-\boldsymbol{v}_j^{-1})A_i^{\epsilon}(z)A_j^{\epsilon'}(w)$ on the right and using Lemma B.4(b), relation (6.7), and an equality $(\boldsymbol{v}_i-\boldsymbol{v}_i^{-1})f_{i,1} = C_{i,1}^{+}\phi_i^{+}$, we obtain (6.18).

B(ii).e Verification of (6.19)

Case $c_{ij}=0$ The equality $[B_i^{\epsilon}(z),B_j^{\epsilon'}(w)] = 0$ follows immediately from Lemma B.4(a) and $[e_i^{\epsilon}(z),e_j^{\epsilon'}(w)]=0$, which is a consequence of the corresponding Serre relation (U7).

Case $c_{ij} = -1$ The corresponding Serre relation (U7) is equivalent to

$$\{e_i^{\epsilon_1}(z_1)e_i^{\epsilon_2}(z_2)e_j^{\epsilon'}(w)-(\boldsymbol{v}_i+\boldsymbol{v}_i^{-1})e_i^{\epsilon_1}(z_1)e_j^{\epsilon'}(w)e_i^{\epsilon_2}(z_2)+e_j^{\epsilon'}(w)e_i^{\epsilon_1}(z_1)e_i^{\epsilon_2}(z_2)\}+\{z_1 \leftrightarrow z_2\}=0$$

for any $\epsilon_1, \epsilon_2, \epsilon' \in \{\pm\}$. Let us denote the first $\{\cdots\}$ in the LHS by $J^{\epsilon_1,\epsilon_2,\epsilon'}(z_1, z_2, w)$. Set

$$M := (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})^2(\boldsymbol{v}_j - \boldsymbol{v}_j^{-1})(\boldsymbol{v}_i z_1 - \boldsymbol{v}_i^{-1}z_2)(\boldsymbol{v}_i z_2 - \boldsymbol{v}_i^{-1}z_1)A_i^{\epsilon_1}(z_1)A_i^{\epsilon_2}(z_2)A_j^{\epsilon'}(w).$$

Combining the equality

$$(\boldsymbol{v}_i z_2 - \boldsymbol{v}_i^{-1}z_1)A_i^{\epsilon_2}(z_2)e_i^{\epsilon_1}(z_1) = (z_2 - z_1)e_i^{\epsilon_1}(z_1)A_i^{\epsilon_2}(z_2) + (\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})z_1 A_i^{\epsilon_2}(z_2)e_i^{\epsilon_2}(z_2)$$

(see Lemma B.1(a2)) with Lemma B.4(a), we find

$$M \cdot J^{\epsilon_1,\epsilon_2,\epsilon'}(z_1, z_2, w) = \frac{(\boldsymbol{v}_i - \boldsymbol{v}_i^{-1})z_1}{\boldsymbol{v}_i z_2 - \boldsymbol{v}_i^{-1}z_1} M \cdot J^{\epsilon_2,\epsilon_2,\epsilon'}(z_2, z_2, w) + (z_2 - z_1)(\boldsymbol{v}_i z_1 - \boldsymbol{v}_i^{-1}z_2)\times$$

$$\{B_i^{\epsilon_1}(z_1)B_i^{\epsilon_2}(z_2)B_j^{\epsilon'}(w) - (\boldsymbol{v}_i + \boldsymbol{v}_i^{-1})B_i^{\epsilon_1}(z_1)B_j^{\epsilon'}(w)B_i^{\epsilon_2}(z_2) + B_j^{\epsilon'}(w)B_i^{\epsilon_1}(z_1)B_i^{\epsilon_2}(z_2)\}.$$

The first summand in the RHS is zero as $J^{\epsilon_2,\epsilon_2,\epsilon'}(z_2, z_2, w) = 0$. Therefore, multiplying $J^{\epsilon_1,\epsilon_2,\epsilon'}(z_1, z_2, w) + J^{\epsilon_2,\epsilon_1,\epsilon'}(z_2, z_1, w) = 0$ by M on the left, we obtain (6.19).

Case $c_{ij} = -2, -3$ These cases are treated similarly to $c_{ij} = -1$, but the corresponding computations become more cumbersome. We verified these cases using **MATLAB**.

B(ii).f Verification of (6.20)

This verification is analogous to that of (6.19) and is left to the interested reader.

B(iii) Proof of Theorem 6.6(b)

Part (b) of Theorem 6.6 can be obtained by reversing the above arguments. In other words, starting from the algebra generated by $(A_{i,0}^{\pm})^{-1}$ and the coefficients of the currents $A_i^{\pm}(z)$, $B_i^{\pm}(z)$, $C_i^{\pm}(z)$, $D_i^{\pm}(z)$ with the defining relations (6.6–6.20), we need to show that the elements $\phi_i^{\pm}$ and currents $e_i(z)$, $f_i(z)$, $\psi_i^{\pm}(z)$, defined via (6.1–6.4), satisfy relations (U1–U10).

This completes our proof of Theorem 6.6.

Appendix C Proof of Theorem 7.1

We denote the images of $e_i(z)$, $f_i(z)$, $\psi_i^{\pm}(z)$ under $\widetilde{\Phi}_{\mu}^{\lambda}$ by $E_i(z)$, $F_i(z)$, $\Psi_i(z)^{\pm}$. It suffices to prove that they satisfy relations (U1–U8), since relations (U9, U10) are obviously preserved by $\widetilde{\Phi}_{\mu}^{\lambda}$. While checking these relations, we will use LHS and RHS when referring to their left-hand and right-hand sides. Set $\rho_i^+ := \frac{-\boldsymbol{v}_i}{1-\boldsymbol{v}_i^2}$, $\rho_i^- := \frac{1}{1-\boldsymbol{v}_i^2}$, $W_{i,rs}(z) := \prod_{1\leq t\leq a_i}^{r\neq t\neq s}(1-\frac{\mathsf{w}_{i,t}}{z})$.

C(i) Compatibility with (U1)

First, we check that the range of powers of z in $\psi_i^{\pm}(z)$ and $\Psi_i(z)^{\pm}$ agree. Note that

$$(1-\nu/z)^+ = 1-\nu\cdot z^{-1} \in \mathbb{C}[[z^{-1}]],\ (1/(1-\nu/z))^+ = 1+\nu z^{-1}+\nu^2z^{-2}+\ldots \in \mathbb{C}[[z^{-1}]],$$

$$(1-\nu/z)^- = -\nu\cdot z^{-1}(1-z/\nu) \in z^{-1}\mathbb{C}[[z]],\ (1/(1-\nu/z))^- = -z/\nu-z^2/\nu^2-\ldots \in z\mathbb{C}[[z]].$$

Therefore, $\Psi_i(z)^+$ contains only nonpositive powers of z, while $\Psi_i(z)^-$ contains only powers of z bigger or equal to

$$-\#\{s : i_s = i\} + 2a_i - \sum_{j-i} a_j(-c_{ji}) = -\alpha_i^\vee(\lambda) + \alpha_i^\vee(\lambda-\mu) = -\alpha_i^\vee(\mu) = -\alpha_i^\vee(\mu^-) = -b_i^-.$$

Moreover, the coefficients of z^0 in $\Psi_i(z)^+$ and of $z^{-b_i^-}$ in $\Psi_i(z)^-$ are invertible.

The equality $[\Psi_i(z)^\epsilon, \Psi_j(w)^{\epsilon'}] = 0$ follows from the commutativity of $\{\mathsf{w}_{i,r}^{\pm 1/2}\}_{i\in I}^{1\leq r\leq a_i}$.

C(ii) Compatibility with (U2)

Case $c_{ij} = 0$ The equality $[E_i(z), E_j(w)] = 0$ is obvious in this case, since $D_{i,r}^{-1}$ commute with $\mathsf{w}_{k,s}^{\pm 1/2}$ for $k = j$ or $k \to j$, while $D_{j,s}^{-1}$ commute with $\mathsf{w}_{k,r}^{\pm 1/2}$ for $k = i$ or $k \to i$.

Case $c_{ij} = 2$ We may assume $\mathfrak{g} = \mathfrak{sl}_2$ and we will drop the index i from our notation. We need to prove $(z-\boldsymbol{v}^2w)E(z)E(w)/(\rho^+)^2 = -(w-\boldsymbol{v}^2z)E(w)E(z)/(\rho^+)^2$.

The LHS equals

$$\boldsymbol{v}^{-2}\prod_{t=1}^{a}\mathsf{w}_t^2\cdot(z-\boldsymbol{v}^2w)\cdot\sum_{r=1}^{a}\delta\left(\frac{\mathsf{w}_r}{z}\right)\delta\left(\frac{\boldsymbol{v}^{-2}\mathsf{w}_r}{w}\right)\frac{Z(\mathsf{w}_r)Z(\boldsymbol{v}^{-2}\mathsf{w}_r)}{W_r(\mathsf{w}_r)W_r(\boldsymbol{v}^{-2}\mathsf{w}_r)}D_r^{-2}+$$

$$\boldsymbol{v}^{-2}\prod_{t=1}^{a}\mathsf{w}_t^2\cdot(z-\boldsymbol{v}^2w)\cdot\sum_{1\le r\ne s\le a}\delta\left(\frac{\mathsf{w}_r}{z}\right)\delta\left(\frac{\mathsf{w}_s}{w}\right)\frac{Z(\mathsf{w}_r)Z(\mathsf{w}_s)}{W_r(\mathsf{w}_r)W_{rs}(\mathsf{w}_s)(1-\boldsymbol{v}^{-2}\mathsf{w}_r/\mathsf{w}_s)}D_r^{-1}D_s^{-1}.$$

Using the equality

$$G(z,w)\delta\left(\frac{\nu_1}{z}\right)\delta\left(\frac{\nu_2}{w}\right)=G(\nu_1,\nu_2)\delta\left(\frac{\nu_1}{z}\right)\delta\left(\frac{\nu_2}{w}\right),\tag{C.1}$$

we see that the first sum is zero, while the second sum equals

$$\prod_{t=1}^{a}\mathsf{w}_t^2\cdot\sum_{1\le r\ne s\le a}\delta\left(\frac{\mathsf{w}_r}{z}\right)\delta\left(\frac{\mathsf{w}_s}{w}\right)\frac{Z(\mathsf{w}_r)Z(\mathsf{w}_s)}{W_{rs}(\mathsf{w}_r)W_{rs}(\mathsf{w}_s)}\frac{\boldsymbol{v}^{-2}(\mathsf{w}_r-\boldsymbol{v}^2\mathsf{w}_s)}{(1-\mathsf{w}_s/\mathsf{w}_r)(1-\boldsymbol{v}^{-2}\mathsf{w}_r/\mathsf{w}_s)}D_r^{-1}D_s^{-1}=$$

$$\prod_{t=1}^{a}\mathsf{w}_t^2\cdot\sum_{1\le r\ne s\le a}\delta\left(\frac{\mathsf{w}_r}{z}\right)\delta\left(\frac{\mathsf{w}_s}{w}\right)\frac{Z(\mathsf{w}_r)Z(\mathsf{w}_s)}{W_{rs}(\mathsf{w}_r)W_{rs}(\mathsf{w}_s)}\frac{\mathsf{w}_r\mathsf{w}_s}{\mathsf{w}_s-\mathsf{w}_r}D_r^{-1}D_s^{-1}.$$

Swapping z and w, we see that $-(w-\boldsymbol{v}^2z)E(w)E(z)/(\rho^+)^2$ equals

$$-\prod_{t=1}^{a}\mathsf{w}_t^2\cdot\sum_{1\le r\ne s\le a}\delta\left(\frac{\mathsf{w}_r}{w}\right)\delta\left(\frac{\mathsf{w}_s}{z}\right)\frac{Z(\mathsf{w}_r)Z(\mathsf{w}_s)}{W_{rs}(\mathsf{w}_r)W_{rs}(\mathsf{w}_s)}\frac{\mathsf{w}_r\mathsf{w}_s}{\mathsf{w}_s-\mathsf{w}_r}D_r^{-1}D_s^{-1}.$$

Swapping r and s in the latter sum, we get exactly the same expression as for the LHS.

Case $c_{ij}<0$ In this case, we can assume $I=\{i,j\}$ and $i\to j$. We need to prove $(z-\boldsymbol{v}_i^{c_{ij}}w)E_i(z)E_j(w)/(\rho_i^+\rho_j^+)=(\boldsymbol{v}_i^{c_{ij}}z-w)E_j(w)E_i(z)/(\rho_i^+\rho_j^+)$. The LHS equals

$$\boldsymbol{v}_i^{-c_{ij}}\prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{1+c_{ij}/2}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}\cdot(z-\boldsymbol{v}_i^{c_{ij}}w)\times$$

$$\sum_{1\le r\le a_i}^{1\le s\le a_j}\delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\delta\left(\frac{\mathsf{w}_{j,s}}{w}\right)\frac{Z_i(\mathsf{w}_{i,r})}{W_{i,r}(\mathsf{w}_{i,r})}D_{i,r}^{-1}\frac{Z_j(\mathsf{w}_{j,s})}{W_{j,s}(\mathsf{w}_{j,s})}\prod_{p=1}^{-c_{ij}}W_i(\boldsymbol{v}_i^{-c_{ij}-2p}w)D_{j,s}^{-1}=$$

$$\prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{1+c_{ij}/2}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}\cdot A(z,w)\times$$

$$\sum_{1\le r\le a_i}^{1\le s\le a_j}\delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\delta\left(\frac{\mathsf{w}_{j,s}}{w}\right)\frac{Z_i(\mathsf{w}_{i,r})Z_j(\mathsf{w}_{j,s})\prod_{p=1}^{-c_{ij}}W_{i,r}(\boldsymbol{v}_i^{-c_{ij}-2p}\mathsf{w}_{j,s})}{W_{i,r}(\mathsf{w}_{i,r})W_{j,s}(\mathsf{w}_{j,s})}D_{i,r}^{-1}D_{j,s}^{-1},$$

where $A(z,w) = \boldsymbol{v}_i^{-c_{ij}}(z - \boldsymbol{v}_i^{c_{ij}} w)\prod_{p=1}^{-c_{ij}}\left(1 - \frac{\boldsymbol{v}_i^{-2} z}{\boldsymbol{v}_i^{-c_{ij}-2p} w}\right)$, due to (C.1). Likewise, the RHS equals

$$\prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{1+c_{ij}/2} \prod_{t=1}^{a_j} \mathsf{w}_{j,t} \cdot (\boldsymbol{v}_i^{c_{ij}} z - w)\times$$

$$\sum_{1\le r\le a_i}^{1\le s\le a_j} \delta\left(\frac{\mathsf{w}_{j,s}}{w}\right)\delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\frac{Z_j(\mathsf{w}_{j,s})}{W_{j,s}(\mathsf{w}_{j,s})}\prod_{p=1}^{-c_{ij}} W_i(\boldsymbol{v}_i^{-c_{ij}-2p} w) D_{j,s}^{-1}\frac{Z_i(\mathsf{w}_{i,r})}{W_{i,r}(\mathsf{w}_{i,r})} D_{i,r}^{-1} =$$

$$\prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{1+c_{ij}/2} \prod_{t=1}^{a_j} \mathsf{w}_{j,t} \cdot B(z,w)\times$$

$$\sum_{1\le r\le a_i}^{1\le s\le a_j} \delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\delta\left(\frac{\mathsf{w}_{j,s}}{w}\right)\frac{Z_i(\mathsf{w}_{i,r})Z_j(\mathsf{w}_{j,s})\prod_{p=1}^{-c_{ij}} W_{i,r}(\boldsymbol{v}_i^{-c_{ij}-2p}\mathsf{w}_{j,s})}{W_{i,r}(\mathsf{w}_{i,r})W_{j,s}(\mathsf{w}_{j,s})} D_{i,r}^{-1} D_{j,s}^{-1},$$

where $B(z,w) = (\boldsymbol{v}_i^{c_{ij}} z - w)\prod_{p=1}^{-c_{ij}}\left(1 - \frac{z}{\boldsymbol{v}_i^{-c_{ij}-2p} w}\right)$, due to (C.1).

The equality LHS = RHS follows from $A(z,w) = B(z,w)$.

C(iii) Compatibility with (U3)

Case $c_{ij} = 0$ The equality $[F_i(z), F_j(w)] = 0$ is obvious in this case, since $D_{i,r}$ commute with $\mathsf{w}_{k,s}^{\pm 1/2}$ for $k = j$ or $k \leftarrow j$, while $D_{j,s}$ commute with $\mathsf{w}_{k,r}^{\pm 1/2}$ for $k = i$ or $k \leftarrow i$.

Case $c_{ij} = 2$ We may assume $\mathfrak{g} = \mathfrak{sl}_2$ and we will drop the index i from our notation. We need to prove $(\boldsymbol{v}^2 z - w)F(z)F(w)/(\rho^-)^2 = -(\boldsymbol{v}^2 w - z)F(w)F(z)/(\rho^-)^2$. The LHS equals

$$(\boldsymbol{v}^2 z - w)\cdot \sum_{r=1}^{a} \delta\left(\frac{\boldsymbol{v}^2\mathsf{w}_r}{z}\right)\delta\left(\frac{\boldsymbol{v}^4\mathsf{w}_r}{w}\right)\frac{1}{W_r(\mathsf{w}_r)W_r(\boldsymbol{v}^2\mathsf{w}_r)} D_r^2 +$$

$$(\boldsymbol{v}^2 z - w)\cdot \sum_{1\le r\ne s\le a} \delta\left(\frac{\boldsymbol{v}^2\mathsf{w}_r}{z}\right)\delta\left(\frac{\boldsymbol{v}^2\mathsf{w}_s}{w}\right)\frac{1}{W_r(\mathsf{w}_r)W_{rs}(\mathsf{w}_s)(1 - \boldsymbol{v}^2\mathsf{w}_r/\mathsf{w}_s)} D_r D_s.$$

Using equality (C.1), we see that the first sum is zero, while the second sum equals

$$\sum_{1\le r\neq s\le a}\delta\left(\frac{\boldsymbol{v}^2\mathsf{w}_r}{z}\right)\delta\left(\frac{\boldsymbol{v}^2\mathsf{w}_s}{w}\right)\frac{1}{W_{rs}(\mathsf{w}_r)W_{rs}(\mathsf{w}_s)}\frac{\boldsymbol{v}^4\mathsf{w}_r-\boldsymbol{v}^2\mathsf{w}_s}{(1-\mathsf{w}_s/\mathsf{w}_r)(1-\boldsymbol{v}^2\mathsf{w}_r/\mathsf{w}_s)}D_rD_s=$$

$$\sum_{1\le r\neq s\le a}\delta\left(\frac{\boldsymbol{v}^2\mathsf{w}_r}{z}\right)\delta\left(\frac{\boldsymbol{v}^2\mathsf{w}_s}{w}\right)\frac{1}{W_{rs}(\mathsf{w}_r)W_{rs}(\mathsf{w}_s)}\frac{\boldsymbol{v}^2\mathsf{w}_r\mathsf{w}_s}{\mathsf{w}_s-\mathsf{w}_r}D_rD_s.$$

Swapping z and w, we see that $-(\boldsymbol{v}^2w-z)F(w)F(z)/(\rho^-)^2$ equals

$$-\sum_{1\le r\neq s\le a}\delta\left(\frac{\boldsymbol{v}^2\mathsf{w}_r}{w}\right)\delta\left(\frac{\boldsymbol{v}^2\mathsf{w}_s}{z}\right)\frac{1}{W_{rs}(\mathsf{w}_r)W_{rs}(\mathsf{w}_s)}\frac{\boldsymbol{v}^2\mathsf{w}_r\mathsf{w}_s}{\mathsf{w}_s-\mathsf{w}_r}D_rD_s.$$

Swapping r and s in this sum, we get exactly the same expression as for the LHS.

Case $c_{ij}<0$ In this case, we can assume $I=\{i,j\}$ and $i\to j$. Recall that $\boldsymbol{v}_i^{c_{ij}}=\boldsymbol{v}_j^{c_{ji}}$. We need to prove $(\boldsymbol{v}_j^{c_{ji}}z-w)F_i(z)F_j(w)/(\rho_i^-\rho_j^-)=(z-\boldsymbol{v}_j^{c_{ji}}w)F_j(w)F_i(z)/(\rho_i^-\rho_j^-)$. The LHS equals

$$\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{c_{ji}/2}\cdot(\boldsymbol{v}_j^{c_{ji}}z-w)\times$$

$$\sum_{1\le r\le a_i}^{1\le s\le a_j}\delta\left(\frac{\boldsymbol{v}_i^2\mathsf{w}_{i,r}}{z}\right)\delta\left(\frac{\boldsymbol{v}_j^2\mathsf{w}_{j,s}}{w}\right)\frac{1}{W_{i,r}(\mathsf{w}_{i,r})}\prod_{p=1}^{-c_{ji}}W_j(\boldsymbol{v}_j^{-c_{ji}-2p}z)D_{i,r}\frac{1}{W_{j,s}(\mathsf{w}_{j,s})}D_{j,s}=$$

$$\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{c_{ji}/2}\cdot A(z,w)\cdot\sum_{1\le r\le a_i}^{1\le s\le a_j}\delta\left(\frac{\boldsymbol{v}_i^2\mathsf{w}_{i,r}}{z}\right)\delta\left(\frac{\boldsymbol{v}_j^2\mathsf{w}_{j,s}}{w}\right)\frac{\prod_{p=1}^{-c_{ji}}W_{j,s}(\boldsymbol{v}_j^{-c_{ji}-2p}z)}{W_{i,r}(\mathsf{w}_{i,r})W_{j,s}(\mathsf{w}_{j,s})}D_{i,r}D_{j,s},$$

where $A(z,w)=(\boldsymbol{v}_j^{c_{ji}}z-w)\prod_{p=1}^{-c_{ji}}\left(1-\frac{\boldsymbol{v}_j^{-2}w}{\boldsymbol{v}_j^{-c_{ji}-2p}z}\right)$, due to (C.1). Likewise, the RHS equals

$$\boldsymbol{v}_j^{c_{ji}}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{c_{ji}/2}\cdot(z-\boldsymbol{v}_j^{c_{ji}}w)\times$$

$$\sum_{1\le r\le a_i}^{1\le s\le a_j}\delta\left(\frac{\boldsymbol{v}_j^2\mathsf{w}_{j,s}}{w}\right)\delta\left(\frac{\boldsymbol{v}_i^2\mathsf{w}_{i,r}}{z}\right)\frac{1}{W_{j,s}(\mathsf{w}_{j,s})}D_{j,s}\frac{1}{W_{i,r}(\mathsf{w}_{i,r})}\prod_{p=1}^{-c_{ji}}W_j(\boldsymbol{v}_j^{-c_{ji}-2p}z)D_{i,r}=$$

$$\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{c_{ji}/2}\cdot B(z,w)\cdot\sum_{1\le r\le a_i}^{1\le s\le a_j}\delta\left(\frac{\boldsymbol{v}_i^2\mathsf{w}_{i,r}}{z}\right)\delta\left(\frac{\boldsymbol{v}_j^2\mathsf{w}_{j,s}}{w}\right)\frac{\prod_{p=1}^{-c_{ji}}W_{j,s}(\boldsymbol{v}_j^{-c_{ji}-2p}z)}{W_{i,r}(\mathsf{w}_{i,r})W_{j,s}(\mathsf{w}_{j,s})}D_{i,r}D_{j,s},$$

where $B(z,w)=\boldsymbol{v}_j^{c_{ji}}(z-\boldsymbol{v}_j^{c_{ji}}w)\prod_{p=1}^{-c_{ji}}\left(1-\frac{w}{\boldsymbol{v}_j^{-c_{ji}-2p}z}\right)$, due to (C.1).

The equality LHS = RHS follows from $A(z,w)=B(z,w)$.

C(iv) Compatibility with (U4)

Case $c_{ij} = 0$ The equality $[\Psi_i(z), E_j(w)] = 0$ is obvious in this case, since $D_{j,s}^{-1}$ commute with $\mathsf{w}_{k,r}^{\pm 1/2}$ for $k = i$ or $k - i$.

Case $c_{ij} = 2$ We may assume $\mathfrak{g} = \mathfrak{sl}_2$ and we will drop the index i from our notation. We need to prove $(z - \boldsymbol{v}^2 w)\Psi(z)E(w)/\rho^+ = (\boldsymbol{v}^2 z - w)E(w)\Psi(z)/\rho^+$. The LHS equals

$$\prod_{t=1}^{a} \mathsf{w}_t^2 \cdot (z - \boldsymbol{v}^2 w) \cdot \frac{Z(z)}{W(z)W(\boldsymbol{v}^{-2}z)} \sum_{r=1}^{a} \delta\left(\frac{\mathsf{w}_r}{w}\right) \frac{Z(\mathsf{w}_r)}{W_r(\mathsf{w}_r)} D_r^{-1} =$$

$$\prod_{t=1}^{a} \mathsf{w}_t^2 \cdot \sum_{r=1}^{a} \delta\left(\frac{\mathsf{w}_r}{w}\right) \frac{Z(z)Z(\mathsf{w}_r)}{W_r(\mathsf{w}_r)W_r(z)W_r(\boldsymbol{v}^{-2}z)} \frac{z - \boldsymbol{v}^2 w}{(1 - w/z)(1 - w/\boldsymbol{v}^{-2}z)} D_r^{-1},$$

due to (C.1). Likewise, the RHS equals

$$\boldsymbol{v}^{-2} \prod_{t=1}^{a} \mathsf{w}_t^2 \cdot (\boldsymbol{v}^2 z - w) \cdot \sum_{r=1}^{a} \delta\left(\frac{\mathsf{w}_r}{w}\right) \frac{Z(\mathsf{w}_r)}{W_r(\mathsf{w}_r)} D_r^{-1} \frac{Z(z)}{W(z)W(\boldsymbol{v}^{-2}z)} =$$

$$\prod_{t=1}^{a} \mathsf{w}_t^2 \cdot \sum_{r=1}^{a} \delta\left(\frac{\mathsf{w}_r}{w}\right) \frac{Z(z)Z(\mathsf{w}_r)}{W_r(\mathsf{w}_r)W_r(z)W_r(\boldsymbol{v}^{-2}z)} \frac{\boldsymbol{v}^{-2}(\boldsymbol{v}^2 z - w)}{(1 - \boldsymbol{v}^{-2}w/z)(1 - \boldsymbol{v}^{-2}w/\boldsymbol{v}^{-2}z)} D_r^{-1}.$$

The equality LHS = RHS follows.

Case $c_{ij} < 0$ In this case, we can assume $I = \{i, j\}$. There are two situations to consider: $i \to j$ and $i \leftarrow j$. Let us first treat the former case. Since $\boldsymbol{v}_i^{c_{ij}} = \boldsymbol{v}_j^{c_{ji}}$, we need to prove $(z - \boldsymbol{v}_j^{c_{ji}} w)\Psi_i(z)E_j(w)/\rho_j^+ = (\boldsymbol{v}_j^{c_{ji}} z - w)E_j(w)\Psi_i(z)/\rho_j^+$. The LHS equals

$$\prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{1+c_{ij}/2} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{1+c_{ji}/2} \cdot (z - \boldsymbol{v}_j^{c_{ji}} w) \times$$

$$\frac{Z_i(z)}{W_i(z)W_i(\boldsymbol{v}_i^{-2}z)} \prod_{p=1}^{-c_{ji}} W_j(\boldsymbol{v}_j^{-c_{ji}-2p} z) \sum_{s=1}^{a_j} \delta\left(\frac{\mathsf{w}_{j,s}}{w}\right) \frac{Z_j(\mathsf{w}_{j,s})}{W_{j,s}(\mathsf{w}_{j,s})} \prod_{p'=1}^{-c_{ij}} W_i(\boldsymbol{v}_i^{-c_{ij}-2p'} w) D_{j,s}^{-1} =$$

$$\prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{1+c_{ij}/2} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{1+c_{ji}/2} \cdot A(z, w) \times$$

$$\sum_{s=1}^{a_j} \delta\left(\frac{\mathsf{w}_{j,s}}{w}\right) \frac{Z_i(z)Z_j(\mathsf{w}_{j,s}) \prod_{p'=1}^{-c_{ij}} W_i(\boldsymbol{v}_i^{-c_{ij}-2p'} w) \prod_{p=1}^{-c_{ji}} W_{j,s}(\boldsymbol{v}_j^{-c_{ji}-2p} z)}{W_i(z)W_i(\boldsymbol{v}_i^{-2}z)W_{j,s}(\mathsf{w}_{j,s})} D_{j,s}^{-1},$$

where $A(z,w) = (z - \boldsymbol{v}_j^{c_{ji}} w) \prod_{p=1}^{-c_{ji}} \left(1 - \frac{w}{\boldsymbol{v}_j^{-c_{ji}-2p} z}\right)$. Likewise, the RHS equals

$$\boldsymbol{v}_j^{-c_{ji}} \prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{1+c_{ij}/2} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{1+c_{ji}/2} \cdot (\boldsymbol{v}_j^{c_{ji}} z - w) \times$$

$$\sum_{s=1}^{a_j} \delta\left(\frac{\mathsf{w}_{j,s}}{w}\right) \frac{Z_j(\mathsf{w}_{j,s})}{W_{j,s}(\mathsf{w}_{j,s})} \prod_{p'=1}^{-c_{ij}} W_i(\boldsymbol{v}_i^{-c_{ij}-2p'} w) D_{j,s}^{-1} \frac{Z_i(z)}{W_i(z) W_i(\boldsymbol{v}_i^{-2} z)} \prod_{p=1}^{-c_{ji}} W_j(\boldsymbol{v}_j^{-c_{ji}-2p} z) =$$

$$\prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{1+c_{ij}/2} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{1+c_{ji}/2} \cdot B(z,w) \times$$

$$\sum_{s=1}^{a_j} \delta\left(\frac{\mathsf{w}_{j,s}}{w}\right) \frac{Z_i(z) Z_j(\mathsf{w}_{j,s}) \prod_{p'=1}^{-c_{ij}} W_i(\boldsymbol{v}_i^{-c_{ij}-2p'} w) \prod_{p=1}^{-c_{ji}} W_{j,s}(\boldsymbol{v}_j^{-c_{ji}-2p} z)}{W_i(z) W_i(\boldsymbol{v}_i^{-2} z) W_{j,s}(\mathsf{w}_{j,s})} D_{j,s}^{-1},$$

where $B(z,w) = \boldsymbol{v}_j^{-c_{ji}} (\boldsymbol{v}_j^{c_{ji}} z - w) \prod_{p=1}^{-c_{ji}} \left(1 - \frac{\boldsymbol{v}_j^{-2} w}{\boldsymbol{v}_j^{-c_{ji}-2p} z}\right)$.

The equality LHS = RHS follows from $A(z,w) = B(z,w)$.

The case $i \leftarrow j$ is analogous: $\Psi_i(z)$ is given by the same formula, while $E_j(w)$ differs by an absence of the factor $\prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{c_{ij}/2} \cdot \prod_{p'=1}^{-c_{ij}} W_i(\boldsymbol{v}_i^{-c_{ij}-2p'} w)$. Tracing back the above calculations, it is clear that the equality still holds when this factor is dropped out.

C(v) Compatibility with (U5)

Case $c_{ij} = 0$ The equality $[\Psi_i(z), F_j(w)] = 0$ is obvious in this case, since $D_{j,s}$ commute with $\mathsf{w}_{k,r}^{\pm 1/2}$ for $k = i$ or $k - i$.

Case $c_{ij} = 2$ We may assume $\mathfrak{g} = \mathfrak{sl}_2$ and we will drop the index i from our notation. We need to prove $(\boldsymbol{v}^2 z - w)\Psi(z)F(w)/\rho^- = (z - \boldsymbol{v}^2 w)F(w)\Psi(z)/\rho^-$. The LHS equals

$$\prod_{t=1}^{a} \mathsf{w}_t \cdot (\boldsymbol{v}^2 z - w) \cdot \frac{Z(z)}{W(z) W(\boldsymbol{v}^{-2} z)} \sum_{r=1}^{a} \delta\left(\frac{\boldsymbol{v}^2 \mathsf{w}_r}{w}\right) \frac{1}{W_r(\mathsf{w}_r)} D_r =$$

$$\prod_{t=1}^{a} \mathsf{w}_t \cdot \sum_{r=1}^{a} \delta\left(\frac{\boldsymbol{v}^2 \mathsf{w}_r}{w}\right) \frac{Z(z)}{W_r(\mathsf{w}_r) W_r(z) W_r(\boldsymbol{v}^{-2} z)} \frac{\boldsymbol{v}^2 z - w}{(1 - \boldsymbol{v}^{-2} w/z)(1 - \boldsymbol{v}^{-2} w/\boldsymbol{v}^{-2} z)} D_r,$$

due to (C.1). Likewise, the RHS equals

$$\boldsymbol{v}^2 \prod_{t=1}^{a} \mathsf{w}_t \cdot (z - \boldsymbol{v}^2 w) \cdot \sum_{r=1}^{a} \delta\left(\frac{\boldsymbol{v}^2 \mathsf{w}_r}{w}\right) \frac{1}{W_r(\mathsf{w}_r)} D_r \frac{Z(z)}{W(z)W(\boldsymbol{v}^{-2}z)} =$$

$$\prod_{t=1}^{a} \mathsf{w}_t \cdot \sum_{r=1}^{a} \delta\left(\frac{\boldsymbol{v}^2 \mathsf{w}_r}{w}\right) \frac{Z(z)}{W_r(\mathsf{w}_r)W_r(z)W_r(\boldsymbol{v}^{-2}z)} \frac{\boldsymbol{v}^2(z - \boldsymbol{v}^2 w)}{(1 - w/z)(1 - w/\boldsymbol{v}^{-2}z)} D_r.$$

The equality LHS = RHS follows.

Case $c_{ij} < 0$ In this case, we can assume $I = \{i, j\}$. There are two situations to consider: $i \to j$ and $i \leftarrow j$. Let us first treat the former case. Since $\boldsymbol{v}_i^{c_{ij}} = \boldsymbol{v}_j^{c_{ji}}$, we need to prove $(\boldsymbol{v}_j^{c_{ji}} z - w)\Psi_i(z)F_j(w)/\rho_j^- = (z - \boldsymbol{v}_j^{c_{ji}} w)F_j(w)\Psi_i(z)/\rho_j^-$. The LHS equals

$$\prod_{t=1}^{a_i} \mathsf{w}_{i,t} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{c_{ji}/2} \cdot (\boldsymbol{v}_j^{c_{ji}} z - w) \times$$

$$\frac{Z_i(z)}{W_i(z)W_i(\boldsymbol{v}_i^{-2}z)} \prod_{p=1}^{-c_{ji}} W_j(\boldsymbol{v}_j^{-c_{ji}-2p} z) \sum_{s=1}^{a_j} \delta\left(\frac{\boldsymbol{v}_j^2 \mathsf{w}_{j,s}}{w}\right) \frac{1}{W_{j,s}(\mathsf{w}_{j,s})} D_{j,s} =$$

$$\prod_{t=1}^{a_i} \mathsf{w}_{i,t} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{c_{ji}/2} \cdot A(z, w) \cdot \sum_{s=1}^{a_j} \delta\left(\frac{\boldsymbol{v}_j^2 \mathsf{w}_{j,s}}{w}\right) \frac{Z_i(z) \prod_{p=1}^{-c_{ji}} W_{j,s}(\boldsymbol{v}_j^{-c_{ji}-2p} z)}{W_i(z)W_i(\boldsymbol{v}_i^{-2}z)W_{j,s}(\mathsf{w}_{j,s})} D_{j,s},$$

where $A(z, w) = (\boldsymbol{v}_j^{c_{ji}} z - w) \prod_{p=1}^{-c_{ji}} \left(1 - \frac{\boldsymbol{v}_j^{-2} w}{\boldsymbol{v}_j^{-c_{ji}-2p} z}\right)$. Likewise, the RHS equals

$$\boldsymbol{v}_j^{c_{ji}} \prod_{t=1}^{a_i} \mathsf{w}_{i,t} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{c_{ji}/2} \cdot (z - \boldsymbol{v}_j^{c_{ji}} w) \times$$

$$\sum_{s=1}^{a_j} \delta\left(\frac{\boldsymbol{v}_j^2 \mathsf{w}_{j,s}}{w}\right) \frac{1}{W_{j,s}(\mathsf{w}_{j,s})} D_{j,s} \frac{Z_i(z)}{W_i(z)W_i(\boldsymbol{v}_i^{-2}z)} \prod_{p=1}^{-c_{ji}} W_j(\boldsymbol{v}_j^{-c_{ji}-2p} z) =$$

$$\prod_{t=1}^{a_i} \mathsf{w}_{i,t} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{c_{ji}/2} \cdot B(z, w) \cdot \sum_{s=1}^{a_j} \delta\left(\frac{\boldsymbol{v}_j^2 \mathsf{w}_{j,s}}{w}\right) \frac{Z_i(z) \prod_{p=1}^{-c_{ji}} W_{j,s}(\boldsymbol{v}_j^{-c_{ji}-2p} z)}{W_i(z)W_i(\boldsymbol{v}_i^{-2}z)W_{j,s}(\mathsf{w}_{j,s})} D_{j,s},$$

where $B(z, w) = \boldsymbol{v}_j^{c_{ji}}(z - \boldsymbol{v}_j^{c_{ji}} w) \prod_{p=1}^{-c_{ji}} \left(1 - \frac{w}{\boldsymbol{v}_j^{-c_{ji}-2p} z}\right)$.

The equality LHS = RHS follows from $A(z, w) = B(z, w)$.

The case $i \leftarrow j$ is analogous: $\Psi_i(z)$ is given by the same formula, while $F_j(w)$ has an extra factor $\prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{c_{ij}/2} \cdot \prod_{p'=1}^{-c_{ij}} W_i(\boldsymbol{v}_i^{-c_{ij}-2p'} w)$. The contributions of this factor into the LHS and the RHS are the same, hence, the equality still holds.

C(vi) Compatibility with (U6)

Case $c_{ij} = 0$ The equality $[E_i(z), F_j(w)] = 0$ is obvious in this case, since $D_{i,r}^{-1}$ commute with $\mathsf{w}_{k,s}^{\pm 1/2}$ for $k = i$ or $k \leftarrow j$, while $D_{j,s}$ commute with $\mathsf{w}_{k,r}^{\pm 1/2}$ for $k = i$ or $k \to i$.

Case $c_{ij} = 2$ We may assume $\mathfrak{g} = \mathfrak{sl}_2$, and we will drop the index i from our notation. We need to prove $[E(z), F(w)] = \frac{1}{\boldsymbol{v}-\boldsymbol{v}^{-1}} \delta\left(\frac{z}{w}\right)\left(\Psi(z)^+ - \Psi(z)^-\right)$. The LHS equals

$$\rho^+\rho^- \left[\prod_{t=1}^{a} \mathsf{w}_t \cdot \sum_{r=1}^{a} \delta\left(\frac{\mathsf{w}_r}{z}\right) \frac{Z(\mathsf{w}_r)}{W_r(\mathsf{w}_r)} D_r^{-1}, \sum_{s=1}^{a} \delta\left(\frac{\boldsymbol{v}^2 \mathsf{w}_s}{w}\right) \frac{1}{W_s(\mathsf{w}_s)} D_s\right] = \frac{-\boldsymbol{v}}{(1-\boldsymbol{v}^2)^2} \prod_{t=1}^{a} \mathsf{w}_t \times$$

$$\left\{\sum_{r=1}^{a} \left(\delta\left(\frac{\mathsf{w}_r}{z}\right)\delta\left(\frac{\mathsf{w}_r}{w}\right) \frac{Z(\mathsf{w}_r)}{W_r(\mathsf{w}_r) W_r(\boldsymbol{v}^{-2}\mathsf{w}_r)} - \boldsymbol{v}^2 \delta\left(\frac{\boldsymbol{v}^2 \mathsf{w}_r}{z}\right)\delta\left(\frac{\boldsymbol{v}^2 \mathsf{w}_r}{w}\right) \frac{Z(\boldsymbol{v}^2 \mathsf{w}_r)}{W_r(\mathsf{w}_r) W_r(\boldsymbol{v}^2 \mathsf{w}_r)}\right) + \right.$$

$$\left.\sum_{1 \le r \ne s \le a} \delta\left(\frac{\mathsf{w}_r}{z}\right)\delta\left(\frac{\boldsymbol{v}^2 \mathsf{w}_s}{w}\right) \frac{Z(\mathsf{w}_r)}{W_{rs}(\mathsf{w}_r) W_{rs}(\mathsf{w}_s)} \left(\frac{1}{A(z,w)} - \frac{\boldsymbol{v}^2}{B(z,w)}\right) D_r^{-1} D_s \right\},$$

where $A(z, w) = (1-\boldsymbol{v}^{-2}w/z)(1-\boldsymbol{v}^{-2}z/\boldsymbol{v}^{-2}w)$ and $B(z, w) = (1-z/\boldsymbol{v}^{-2}w)(1-w/z)$. The second sum is zero as $A(z, w) = \boldsymbol{v}^{-2}B(z, w)$.

To evaluate the RHS, we need the following standard result.

Lemma C.1 *For any rational function* $\gamma(z)$ *with simple poles* $\{x_t\} \subset \mathbb{C}^\times$ *and possibly poles of higher order at* $z = 0, \infty$*, the following equality holds:*

$$\gamma(z)^+ - \gamma(z)^- = \sum_t \delta\left(\frac{z}{x_t}\right) \mathrm{Res}_{z=x_t} \gamma(z) \frac{dz}{z}. \tag{C.2}$$

Proof Consider the partial fraction decomposition of $\gamma(z)$:

$$\gamma(z) = P(z) + \sum_t \frac{\nu_t}{z - x_t},$$

where $P(z)$ is a Laurent polynomial. Then $P(z)^{\pm} = P(z) \Rightarrow P(z)^{+} - P(z)^{-} = 0$. Meanwhile:

$$\left(\frac{\nu_t}{z-x_t}\right)^{+} = \frac{\nu_t}{z} + \frac{\nu_t x_t}{z^2} + \frac{\nu_t x_t^2}{z^3} + \ldots \text{ and } \left(\frac{\nu_t}{z-x_t}\right)^{-} = -\frac{\nu_t}{x_t} - \frac{\nu_t z}{x_t^2} - \frac{\nu_t z^2}{x_t^3} - \ldots,$$

so that

$$\left(\frac{\nu_t}{z-x_t}\right)^{+} - \left(\frac{\nu_t}{z-x_t}\right)^{-} = \frac{\nu_t}{x_t}\delta\left(\frac{z}{x_t}\right) = \delta\left(\frac{z}{x_t}\right)\cdot \mathrm{Res}_{z=x_t}\frac{\nu_t}{z-x_t}\frac{dz}{z}.$$

The lemma is proved. □

Since $\Psi(z)$ is a rational function in z, which has (simple) poles only at $\{\mathsf{w}_r, \boldsymbol{v}^2\mathsf{w}_r\}_{r=1}^{a}$ and possibly poles of higher order at $z = 0, \infty$, we can apply Lemma C.1 to evaluate $\Psi(z)^{+} - \Psi(z)^{-}$:

$$\Psi(z)^{+} - \Psi(z)^{-} = \prod_{t=1}^{a}\mathsf{w}_t \cdot \sum_{r=1}^{a}\left(\delta\left(\frac{z}{\mathsf{w}_r}\right)\frac{Z(\mathsf{w}_r)}{W_r(\mathsf{w}_r)W(\boldsymbol{v}^{-2}\mathsf{w}_r)} + \delta\left(\frac{z}{\boldsymbol{v}^2\mathsf{w}_r}\right)\frac{Z(\boldsymbol{v}^2\mathsf{w}_r)}{W_r(\mathsf{w}_r)W(\boldsymbol{v}^2\mathsf{w}_r)}\right) =$$

$$\frac{1}{1-\boldsymbol{v}^2}\prod_{t=1}^{a}\mathsf{w}_t \cdot \sum_{r=1}^{a}\left(\delta\left(\frac{\mathsf{w}_r}{z}\right)\frac{Z(\mathsf{w}_r)}{W_r(\mathsf{w}_r)W_r(\boldsymbol{v}^{-2}\mathsf{w}_r)} - \boldsymbol{v}^2\delta\left(\frac{\boldsymbol{v}^2\mathsf{w}_r}{z}\right)\frac{Z(\boldsymbol{v}^2\mathsf{w}_r)}{W_r(\mathsf{w}_r)W_r(\boldsymbol{v}^2\mathsf{w}_r)}\right).$$

Hence, the RHS equals

$$\frac{1}{(\boldsymbol{v}-\boldsymbol{v}^{-1})(1-\boldsymbol{v}^2)}\prod_{t=1}^{a}\mathsf{w}_t \times$$

$$\sum_{r=1}^{a}\left(\delta\left(\frac{\mathsf{w}_r}{z}\right)\delta\left(\frac{\mathsf{w}_r}{w}\right)\frac{Z(\mathsf{w}_r)}{W_r(\mathsf{w}_r)W_r(\boldsymbol{v}^{-2}\mathsf{w}_r)} - \delta\left(\frac{\boldsymbol{v}^2\mathsf{w}_r}{z}\right)\delta\left(\frac{\boldsymbol{v}^2\mathsf{w}_r}{w}\right)\frac{\boldsymbol{v}^2 Z(\mathsf{w}_r)}{W_r(\mathsf{w}_r)W_r(\boldsymbol{v}^2\mathsf{w}_r)}\right).$$

As a result, we finally get LHS = RHS.

Case $c_{ij} < 0, i \to j$ We may assume $I = \{i, j\}$, and we need to check $[E_i(z), F_j(w)] = 0$. We have

$$\frac{[E_i(z), F_j(w)]}{\rho_i^{+}\rho_j^{-}} = \prod_{t=1}^{a_i}\mathsf{w}_{i,t}\cdot\left[\sum_{r=1}^{a_i}\delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\frac{Z_i(\mathsf{w}_{i,r})}{W_{i,r}(\mathsf{w}_{i,r})}D_{i,r}^{-1}, \sum_{s=1}^{a_j}\delta\left(\frac{\boldsymbol{v}_j^2\mathsf{w}_{j,s}}{w}\right)\frac{1}{W_{j,s}(\mathsf{w}_{j,s})}D_{j,s}\right].$$

The latter is obviously zero, since $[D_{i,r}^{-1}, \mathsf{w}_{j,s}] = 0 = [D_{j,s}, \mathsf{w}_{i,r}]$.

Case $c_{ij} < 0, i \leftarrow j$ We may assume $I = \{i, j\}$, and we need to check $E_i(z)F_j(w)/(\rho_i^+\rho_j^-) = F_j(w)E_i(z)/(\rho_i^+\rho_j^-)$. The LHS equals

$$\boldsymbol{v}_i^{-c_{ij}} \prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{1+c_{ij}/2} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{c_{ji}/2} \times$$

$$\sum_{1\le r\le a_i}^{1\le s\le a_j} \delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\delta\left(\frac{\boldsymbol{v}_j^2\mathsf{w}_{j,s}}{w}\right)\frac{Z_i(\mathsf{w}_{i,r})\prod_{p=1}^{-c_{ji}} W_j(\boldsymbol{v}_j^{-c_{ji}-2p}z)}{W_{i,r}(\mathsf{w}_{i,r})} D_{i,r}^{-1} \frac{\prod_{p'=1}^{-c_{ij}} W_i(\boldsymbol{v}_i^{-c_{ij}-2p'}w)}{W_{j,s}(\mathsf{w}_{j,s})} D_{j,s} =$$

$$\prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{1+c_{ij}/2} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{c_{ji}/2} \cdot A(z,w)\times$$

$$\sum_{1\le r\le a_i}^{1\le s\le a_j} \delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\delta\left(\frac{\boldsymbol{v}_j^2\mathsf{w}_{j,s}}{w}\right)\frac{Z_i(\mathsf{w}_{i,r})\prod_{p=1}^{-c_{ji}} W_{j,s}(\boldsymbol{v}_j^{-c_{ji}-2p}z)\prod_{p'=1}^{-c_{ij}} W_{i,r}(\boldsymbol{v}_i^{-c_{ij}-2p'}w)}{W_{i,r}(\mathsf{w}_{i,r})W_{j,s}(\mathsf{w}_{j,s})} D_{i,r}^{-1} D_{j,s},$$

where $A(z,w) = \boldsymbol{v}_i^{-c_{ij}} \prod_{p=1}^{-c_{ji}} \left(1 - \frac{\boldsymbol{v}_j^{-2}w}{\boldsymbol{v}_j^{-c_{ji}-2p}z}\right) \prod_{p'=1}^{-c_{ij}} \left(1 - \frac{\boldsymbol{v}_i^{-2}z}{\boldsymbol{v}_i^{-c_{ij}-2p'}w}\right)$.

Likewise, the RHS equals

$$\boldsymbol{v}_j^{c_{ji}} \prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{1+c_{ij}/2} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{c_{ji}/2} \times$$

$$\sum_{1\le r\le a_i}^{1\le s\le a_j} \delta\left(\frac{\boldsymbol{v}_j^2\mathsf{w}_{j,s}}{w}\right)\delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\frac{\prod_{p'=1}^{-c_{ij}} W_i(\boldsymbol{v}_i^{-c_{ij}-2p'}w)}{W_{j,s}(\mathsf{w}_{j,s})} D_{j,s} \frac{Z_i(\mathsf{w}_{i,r})\prod_{p=1}^{-c_{ji}} W_j(\boldsymbol{v}_j^{-c_{ji}-2p}z)}{W_{i,r}(\mathsf{w}_{i,r})} D_{i,r}^{-1} =$$

$$\prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{1+c_{ij}/2} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{c_{ji}/2} \cdot B(z,w)\times$$

$$\sum_{1\le r\le a_i}^{1\le s\le a_j} \delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\delta\left(\frac{\boldsymbol{v}_j^2\mathsf{w}_{j,s}}{w}\right)\frac{Z_i(\mathsf{w}_{i,r})\prod_{p=1}^{-c_{ji}} W_{j,s}(\boldsymbol{v}_j^{-c_{ji}-2p}z)\prod_{p'=1}^{-c_{ij}} W_{i,r}(\boldsymbol{v}_i^{-c_{ij}-2p'}w)}{W_{i,r}(\mathsf{w}_{i,r})W_{j,s}(\mathsf{w}_{j,s})} D_{i,r}^{-1} D_{j,s},$$

where $B(z,w) = \boldsymbol{v}_j^{c_{ji}} \prod_{p=1}^{-c_{ji}} \left(1 - \frac{w}{\boldsymbol{v}_j^{-c_{ji}-2p}z}\right) \prod_{p'=1}^{-c_{ij}} \left(1 - \frac{z}{\boldsymbol{v}_i^{-c_{ij}-2p'}w}\right)$.

The equality LHS = RHS follows from $A(z,w) = B(z,w)$.

C(vii) Compatibility with (U7)

Case $c_{ij} = 0$ In this case, $[E_i(z), E_j(w)] = 0$, due to our verification of (U2).

Case $c_{ij} < 0$ To simplify our calculations, we introduce

$$\chi_{i',r} := \prod_{t=1}^{a_{i'}} \mathsf{w}_{i',t} \cdot \prod_{j'\to i'} \prod_{t=1}^{a_{j'}} \mathsf{w}_{j',t}^{c_{j'i'}/2} \cdot \frac{Z_{i'}(\mathsf{w}_{i',r})}{W_{i',r}(\mathsf{w}_{i',r})} \prod_{j'\to i'} \prod_{p=1}^{-c_{j'i'}} W_{j'}(\boldsymbol{v}_{j'}^{-c_{j'i'}-2p} \mathsf{w}_{i',r}) D_{i',r}^{-1},$$

so that $E_{i'}(z) = \rho_{i'}^{+} \sum_{r=1}^{a_{i'}} \delta\left(\frac{\mathsf{w}_{i',r}}{z}\right) \chi_{i',r}$.

The verification of (U7) is based on the following result.

Lemma C.2 *The following relations hold:*

$$\chi_{i,r}\mathsf{w}_{j,s} = \boldsymbol{v}_i^{-2\delta_{ij}\delta_{rs}} \mathsf{w}_{j,s}\chi_{i,r} \text{ for } 1 \le r \le a_i,\ 1 \le s \le a_j,$$

$$(\mathsf{w}_{i,r_1} - \boldsymbol{v}_i^2 \mathsf{w}_{i,r_2})\chi_{i,r_1}\chi_{i,r_2} = (\boldsymbol{v}_i^2 \mathsf{w}_{i,r_1} - \mathsf{w}_{i,r_2})\chi_{i,r_2}\chi_{i,r_1} \text{ for } 1 \le r_1 \ne r_2 \le a_i,$$

$$(\mathsf{w}_{i,r} - \boldsymbol{v}_i^{c_{ij}} \mathsf{w}_{j,s})\chi_{i,r}\chi_{j,s} = (\boldsymbol{v}_i^{c_{ij}} \mathsf{w}_{i,r} - \mathsf{w}_{j,s})\chi_{j,s}\chi_{i,r} \text{ for } 1 \le r \le a_i,\ 1 \le s \le a_j.$$

Proof Follows from straightforward computations. □

With the help of this lemma, let us verify (U7) for $c_{ij} = -1$. The latter amounts to proving $[E_i(z_1), [E_i(z_2), E_j(w)]_{\boldsymbol{v}}]_{\boldsymbol{v}^{-1}}/((\rho_i^+)^2\rho_j^+) = -[E_i(z_2), [E_i(z_1), E_j(w)]_{\boldsymbol{v}}]_{\boldsymbol{v}^{-1}}/((\rho_i^+)^2\rho_j^+)$. The LHS equals

$$(1-\boldsymbol{v}^2)\left[\sum_{r_1=1}^{a_i} \delta\left(\frac{\mathsf{w}_{i,r_1}}{z_1}\right)\chi_{i,r_1}, \sum_{1\le r_2\le a_i}^{1\le s\le a_j} \delta\left(\frac{\mathsf{w}_{i,r_2}}{z_2}\right)\delta\left(\frac{\mathsf{w}_{j,s}}{w}\right)\frac{\mathsf{w}_{i,r_2}}{\mathsf{w}_{i,r_2}-\boldsymbol{v}\mathsf{w}_{j,s}}\chi_{i,r_2}\chi_{j,s}\right]_{\boldsymbol{v}^{-1}} =$$

$$\sum_{1\le r\le a_i}^{1\le s\le a_j} \delta\left(\frac{\mathsf{w}_{j,s}}{w}\right)\left\{\delta\left(\frac{\mathsf{w}_{i,r}}{z_1}\right)\delta\left(\frac{\boldsymbol{v}^{-2}\mathsf{w}_{i,r}}{z_2}\right) - \delta\left(\frac{\mathsf{w}_{i,r}}{z_2}\right)\delta\left(\frac{\boldsymbol{v}^{-2}\mathsf{w}_{i,r}}{z_1}\right)\right\}\frac{(\boldsymbol{v}^2-1)\mathsf{w}_{i,r}}{\mathsf{w}_{i,r}-\boldsymbol{v}^3\mathsf{w}_{j,s}}\chi_{i,r}^2\chi_{j,s} -$$

$$(\boldsymbol{v}^2-1)^2 \sum_{1\le r_1\ne r_2\le a_i}^{1\le s\le a_j} \delta\left(\frac{\mathsf{w}_{i,r_1}}{z_1}\right)\delta\left(\frac{\mathsf{w}_{i,r_2}}{z_2}\right)\delta\left(\frac{\mathsf{w}_{j,s}}{w}\right)\frac{A(z_1,z_2,w)}{\boldsymbol{v}^2\mathsf{w}_{i,r_1}-\mathsf{w}_{i,r_2}}\chi_{i,r_1}\chi_{i,r_2}\chi_{j,s},$$

where $A(z_1, z_2, w) = \frac{z_1z_2(z_1+z_2-(\boldsymbol{v}+\boldsymbol{v}^{-1})w)}{(z_1-\boldsymbol{v}w)(z_2-\boldsymbol{v}w)}$ and the last equality is obtained by treating separately $r_1 = r_2$ and $r_1 \ne r_2$ cases. The first sum is obviously skew-symmetric in z_1, z_2. The second sum is also skew-symmetric, due to the above relations on $\chi_{i,r}$.

The cases $c_{ij} = -2, -3$ can be treated similarly, but the corresponding computations become more cumbersome. We verified these cases using **MATLAB**.

C(viii) Compatibility with (U8)

The case $c_{ij} = 0$ is obvious. The case $c_{ij} = -1$ can be treated analogously to the above verification of (U7). The verification for the cases $c_{ij} = -2, -3$ is more cumbersome and can be performed as outlined in the verification of (U7). Our verification involved a simple computation in **MATLAB**.

This completes our proof of Theorem 7.1.

Remark C.3 Theorem 7.1 admits the following straightforward generalization. For every $i \in I$, pick two polynomials $Z_i^{(1)}(z), Z_i^{(2)}(z)$ in z^{-1} such that $Z_i(z) = Z_i^{(1)}(z)Z_i^{(2)}(z)$. There is a unique $\mathbb{C}(\boldsymbol{v})[\mathsf{z}_1^{\pm1}, \ldots, \mathsf{z}_N^{\pm1}]$-algebra homomorphism $\mathcal{U}^{\mathrm{ad}}_{0,\mu}[\mathsf{z}_1^{\pm1}, \ldots, \mathsf{z}_N^{\pm1}] \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}[\mathsf{z}_1^{\pm1}, \ldots, \mathsf{z}_N^{\pm1}]$, such that

$$e_i(z) \mapsto \frac{-\boldsymbol{v}_i}{1-\boldsymbol{v}_i^2}\prod_{t=1}^{a_i}\mathsf{w}_{i,t}\prod_{j\to i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{c_{ji}/2}\cdot\sum_{r=1}^{a_i}\delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\frac{Z_i^{(1)}(\mathsf{w}_{i,r})}{W_{i,r}(\mathsf{w}_{i,r})}\prod_{j\to i}\prod_{p=1}^{-c_{ji}}W_j(\boldsymbol{v}_j^{-c_{ji}-2p}z)D_{i,r}^{-1},$$

$$f_i(z) \mapsto \frac{1}{1-\boldsymbol{v}_i^2}\prod_{j\leftarrow i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{c_{ji}/2}\cdot\sum_{r=1}^{a_i}\delta\left(\frac{\boldsymbol{v}_i^2\mathsf{w}_{i,r}}{z}\right)\frac{Z_i^{(2)}(\boldsymbol{v}_i^2\mathsf{w}_{i,r})}{W_{i,r}(\mathsf{w}_{i,r})}\prod_{j\leftarrow i}\prod_{p=1}^{-c_{ji}}W_j(\boldsymbol{v}_j^{-c_{ji}-2p}z)D_{i,r},$$

$$\psi_i^{\pm}(z) \mapsto \Psi_i(z)^{\pm},\ (\phi_i^+)^{\pm1} \mapsto \prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{\pm1/2},\ (\phi_i^-)^{\pm1} \mapsto (-\boldsymbol{v}_i)^{\mp a_i}\prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{\mp1/2}.$$

Appendix D Proof of Theorem 10.5

Due to Theorem 5.5, it suffices to check that the assignment Δ of Theorem 10.5 preserves defining relations (Û1–Û6, Û9). To simplify our exposition, we will assume that $b_1, b_2 < 0$, while the case when one of them is zero is left to the interested reader (note that the case $b_1 = b_2 = 0$ has been treated in Remark 10.4). We will also work with $\bar{h}_{\pm1} := [2]_{\boldsymbol{v}}^{-1}h_{\pm1}$ instead of $h_{\pm1}$, so that $[\bar{h}_{\pm1}, e_r] = e_{r\pm1}, [\bar{h}_{\pm1}, f_r] = -f_{r\pm1}$.

D(i) Compatibility with (Û1)

The equalities $\Delta((\psi_0^+)^{\pm1})\Delta((\psi_0^+)^{\mp1}) = 1$ and $\Delta((\psi_b^-)^{\pm1})\Delta((\psi_b^-)^{\mp1}) = 1$ follow immediately from relation (Û1) for both $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$ and $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

The commutativity of $\Delta((\psi_0^+)^{\pm1}), \Delta((\psi_b^-)^{\pm1})$ between themselves and with each of $\Delta(\bar{h}_{\pm1})$ is due to relations (Û1, Û4, Û5) for both $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$ and $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

It remains to prove $[\Delta(\bar{h}_1), \Delta(\bar{h}_{-1})] = 0$. The LHS is equal to $[\bar{h}_1 \otimes 1 + 1 \otimes \bar{h}_1 - (\boldsymbol{v} - \boldsymbol{v}^{-1})e_0 \otimes f_1, \bar{h}_{-1} \otimes 1 + 1 \otimes \bar{h}_{-1} + (\boldsymbol{v} - \boldsymbol{v}^{-1})e_{-1} \otimes f_0] = (\boldsymbol{v}-\boldsymbol{v}^{-1})(e_0 \otimes f_0 - e_{-1} \otimes f_1 + e_{-1} \otimes f_1 - e_0 \otimes f_0) - (\boldsymbol{v}-\boldsymbol{v}^{-1})^2[e_0 \otimes f_1, e_{-1} \otimes f_0] = -(\boldsymbol{v} - \boldsymbol{v}^{-1})^2(e_0 e_{-1} \otimes f_1 f_0 - e_{-1} e_0 \otimes f_0 f_1) = 0$.

Here we used ($\hat{\mathrm{U}}1$, $\hat{\mathrm{U}}4$, $\hat{\mathrm{U}}5$) for both $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$, $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$ in the first equality, while the second equality follows from $e_0 e_{-1} = \boldsymbol{v}^2 e_{-1} e_0$, $f_1 f_0 = \boldsymbol{v}^{-2} f_0 f_1$, due to ($\hat{\mathrm{U}}2$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$ and ($\hat{\mathrm{U}}3$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

D(ii) Compatibility with ($\hat{\mathrm{U}}2$)

We need to prove $[\Delta(e_{r+1}), \Delta(e_s)]_{\boldsymbol{v}^2} + [\Delta(e_{s+1}), \Delta(e_r)]_{\boldsymbol{v}^2} = 0$ for $b_2 - 1 \le r, s \le -1$.

Case $b_2 - 1 < r, s < -1$ Then, $[\Delta(e_{r+1}), \Delta(e_s)]_{\boldsymbol{v}^2} + [\Delta(e_{s+1}), \Delta(e_r)]_{\boldsymbol{v}^2} = 1 \otimes ([e_{r+1}, e_s]_{\boldsymbol{v}^2} + [e_{s+1}, e_r]_{\boldsymbol{v}^2}) = 0$ as the second term is zero in $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$ by ($\hat{\mathrm{U}}2$).

Case $r = s = b_2 - 1$ It suffices to show that $[\Delta(e_{b_2}), \Delta(e_{b_2-1})]_{\boldsymbol{v}^2} = 0$, which follows from $[\Delta(e_{b_2}), \Delta(e_{b_2-1})]_{\boldsymbol{v}^2} = [1 \otimes e_{b_2}, e_{-1} \otimes \psi^-_{b_2} + 1 \otimes e_{b_2-1}]_{\boldsymbol{v}^2} = e_{-1} \otimes [e_{b_2}, \psi^-_{b_2}]_{\boldsymbol{v}^2} + 1 \otimes [e_{b_2}, e_{b_2-1}]_{\boldsymbol{v}^2} = 0$. The last equality follows from $[e_{b_2}, \psi^-_{b_2}]_{\boldsymbol{v}^2} = 0$ and $[e_{b_2}, e_{b_2-1}]_{\boldsymbol{v}^2} = 0$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$, due to ($\hat{\mathrm{U}}2$) and ($\hat{\mathrm{U}}4$), respectively.

Case $r = b_2 - 1, b_2 - 1 < s < -1$ Then, $[\Delta(e_{b_2}), \Delta(e_s)]_{\boldsymbol{v}^2} + [\Delta(e_{s+1}), \Delta(e_{b_2-1})]_{\boldsymbol{v}^2} = 1 \otimes ([e_{b_2}, e_s]_{\boldsymbol{v}^2} + [e_{s+1}, e_{b_2-1}]_{\boldsymbol{v}^2}) + e_{-1} \otimes [e_{s+1}, \psi^-_{b_2}]_{\boldsymbol{v}^2} = 0$. The last equality follows again from ($\hat{\mathrm{U}}2$) and ($\hat{\mathrm{U}}4$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

Case $r = b_2 - 1, s = -1$ Then $[\Delta(e_{b_2}), \Delta(e_{-1})]_{\boldsymbol{v}^2} = 1 \otimes [e_{b_2}, e_{-1}]_{\boldsymbol{v}^2}$ and $[\Delta(e_0), \Delta(e_{b_2-1})]_{\boldsymbol{v}^2} = [e_0 \otimes \psi^+_0 + 1 \otimes e_0, e_{-1} \otimes \psi^-_{b_2} + 1 \otimes e_{b_2-1}]_{\boldsymbol{v}^2} = e_0 \otimes [\psi^+_0, e_{b_2-1}]_{\boldsymbol{v}^2} + [e_0, e_{-1}]_{\boldsymbol{v}^2} \otimes \psi^+_0 \psi^-_{b_2} + e_{-1} \otimes [e_0, \psi^-_{b_2}]_{\boldsymbol{v}^2} + 1 \otimes [e_0, e_{b_2-1}]_{\boldsymbol{v}^2} = 1 \otimes [e_0, e_{b_2-1}]_{\boldsymbol{v}^2}$ as the first three terms are zero, due to ($\hat{\mathrm{U}}2$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$ and ($\hat{\mathrm{U}}4$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$. The result follows from ($\hat{\mathrm{U}}2$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

Case $r = s = -1$ It suffices to show that $[\Delta(e_0), \Delta(e_{-1})]_{\boldsymbol{v}^2} = 0$, which follows from $[\Delta(e_0), \Delta(e_{-1})]_{\boldsymbol{v}^2} = [e_0 \otimes \psi^+_0 + 1 \otimes e_0, 1 \otimes e_{-1}]_{\boldsymbol{v}^2} = e_0 \otimes [\psi^+_0, e_{-1}]_{\boldsymbol{v}^2} + 1 \otimes [e_0, e_{-1}]_{\boldsymbol{v}^2} = 0$. The last equality follows again from relations ($\hat{\mathrm{U}}2$, $\hat{\mathrm{U}}4$) for the algebra $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

Case $r = -1, b_2 - 1 < s < -1$ Then, $[\Delta(e_0), \Delta(e_s)]_{\boldsymbol{v}^2} = [e_0 \otimes \psi^+_0 + 1 \otimes e_0, 1 \otimes e_s]_{\boldsymbol{v}^2} = 1 \otimes [e_0, e_s]_{\boldsymbol{v}^2}$, while $[\Delta(e_{s+1}), \Delta(e_{-1})]_{\boldsymbol{v}^2} = 1 \otimes [e_{s+1}, e_{-1}]_{\boldsymbol{v}^2}$. The sum of these two terms is zero, due to ($\hat{\mathrm{U}}2$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

D(iii) Compatibility with ($\hat{\mathrm{U}}3$)

We need to prove $[\Delta(f_r), \Delta(f_{s+1})]_{\boldsymbol{v}^2} + [\Delta(f_s), \Delta(f_{r+1})]_{\boldsymbol{v}^2} = 0$ for $b_1 \leq r, s \leq 0$.

Case $b_1 < r, s < 0$ Then, $[\Delta(f_r), \Delta(f_{s+1})]_{\boldsymbol{v}^2} + [\Delta(f_s), \Delta(f_{r+1})]_{\boldsymbol{v}^2} = ([f_r, f_{s+1}]_{\boldsymbol{v}^2} + [f_s, f_{r+1}]_{\boldsymbol{v}^2}) \otimes 1 = 0$ as the first term is zero in $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$ by ($\hat{\mathrm{U}}3$).

Case $r = s = b_1$ It suffices to show that $[\Delta(f_{b_1}), \Delta(f_{1+b_1})]_{\boldsymbol{v}^2} = 0$, which follows from $[\Delta(f_{b_1}), \Delta(f_{1+b_1})]_{\boldsymbol{v}^2} = [f_{b_1} \otimes 1 + \psi^-_{b_1} \otimes f_0, f_{1+b_1} \otimes 1]_{\boldsymbol{v}^2} = [f_{b_1}, f_{1+b_1}]_{\boldsymbol{v}^2} \otimes 1 + [\psi^-_{b_1}, f_{1+b_1}]_{\boldsymbol{v}^2} \otimes f_0 = 0$. The last equality follows from $[f_{b_1}, f_{1+b_1}]_{\boldsymbol{v}^2} = 0 = [\psi^-_{b_1}, f_{1+b_1}]_{\boldsymbol{v}^2}$, due to ($\hat{\mathrm{U}}3$, $\hat{\mathrm{U}}5$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$.

Case $r = b_1 < s < 0$ Then, $[\Delta(f_s), \Delta(f_{1+b_1})]_{\boldsymbol{v}^2} = [f_s, f_{1+b_1}]_{\boldsymbol{v}^2} \otimes 1$ and $[\Delta(f_{b_1}), \Delta(f_{s+1})]_{\boldsymbol{v}^2} = [f_{b_1}, f_{s+1}]_{\boldsymbol{v}^2} \otimes 1$ as $[\psi^-_{b_1}, f_{s+1}]_{\boldsymbol{v}^2} = 0$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$ by ($\hat{\mathrm{U}}5$). It remains to use ($\hat{\mathrm{U}}3$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$.

Case $r = b_1, s = 0$ Then $[\Delta(f_{b_1}), \Delta(f_1)]_{\boldsymbol{v}^2} = [f_{b_1}, f_1]_{\boldsymbol{v}^2} \otimes 1 + [f_{b_1}, \psi^+_0]_{\boldsymbol{v}^2} \otimes f_1 + [\psi^-_{b_1}, f_1]_{\boldsymbol{v}^2} \otimes f_0 + \psi^-_{b_1}\psi^+_0 \otimes [f_0, f_1]_{\boldsymbol{v}^2}$, and $[\Delta(f_0), \Delta(f_{1+b_1})]_{\boldsymbol{v}^2} = [f_0, f_{1+b_1}]_{\boldsymbol{v}^2} \otimes 1$. It remains to use $[f_{b_1}, f_1]_{\boldsymbol{v}^2} + [f_0, f_{1+b_1}]_{\boldsymbol{v}^2} = [f_{b_1}, \psi^+_0]_{\boldsymbol{v}^2} = [\psi^-_{b_1}, f_1]_{\boldsymbol{v}^2} = 0$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$, due ($\hat{\mathrm{U}}3$) and ($\hat{\mathrm{U}}5$), and $[f_0, f_1]_{\boldsymbol{v}^2} = 0$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$, due to ($\hat{\mathrm{U}}3$).

Case $r = s = 0$ It suffices to show that $[\Delta(f_0), \Delta(f_1)]_{\boldsymbol{v}^2} = 0$, which follows from $[\Delta(f_0), \Delta(f_1)]_{\boldsymbol{v}^2} = [f_0 \otimes 1, f_1 \otimes 1 + \psi^+_0 \otimes f_1]_{\boldsymbol{v}^2} = [f_0, f_1]_{\boldsymbol{v}^2} \otimes 1 + [f_0, \psi^+_0]_{\boldsymbol{v}^2} \otimes f_1 = 0$, due to ($\hat{\mathrm{U}}3$, $\hat{\mathrm{U}}5$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$.

Case $r = 0, b_1 < s < 0$ Then $[\Delta(f_0), \Delta(f_{s+1})]_{\boldsymbol{v}^2} = [f_0, f_{s+1}]_{\boldsymbol{v}^2} \otimes 1$, and $[\Delta(f_s), \Delta(f_1)]_{\boldsymbol{v}^2} = [f_s \otimes 1, f_1 \otimes 1 + \psi^+_0 \otimes f_1]_{\boldsymbol{v}^2} = [f_s, f_1]_{\boldsymbol{v}^2} \otimes 1 + [f_s, \psi^+_0]_{\boldsymbol{v}^2} \otimes f_1$. It remains to apply the equalities $[f_0, f_{s+1}]_{\boldsymbol{v}^2} + [f_s, f_1]_{\boldsymbol{v}^2} = 0$ and $[f_s, \psi^+_0]_{\boldsymbol{v}^2} = 0$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$, due to ($\hat{\mathrm{U}}3$) and ($\hat{\mathrm{U}}5$).

D(iv) Compatibility with ($\hat{\mathrm{U}}4$)

The equalities $\Delta(\psi^+_0)\Delta(e_r) = \boldsymbol{v}^2\Delta(e_r)\Delta(\psi^+_0)$ and $\Delta(\psi^-_b)\Delta(e_r) = \boldsymbol{v}^{-2}\Delta(e_r)\Delta(\psi^-_b)$ for $b_2 - 1 \leq r \leq 0$ are obvious, due to relations ($\hat{\mathrm{U}}1$) and ($\hat{\mathrm{U}}4$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_1}, \mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

Let us now verify the equality $[\Delta(\bar{h}_1), \Delta(e_r)] = \Delta(e_{r+1})$ for $b_2 - 1 \leq r \leq -1$.

Case $b_2 \leq r \leq -2$ We have $[\Delta(\bar{h}_1), \Delta(e_r)] = [\bar{h}_1 \otimes 1 + 1 \otimes \bar{h}_1 - (\boldsymbol{v} - \boldsymbol{v}^{-1})e_0 \otimes f_1, 1 \otimes e_r] = 1 \otimes e_{r+1} - (\boldsymbol{v} - \boldsymbol{v}^{-1})e_0 \otimes [f_1, e_r] = 1 \otimes e_{r+1} = \Delta(e_{r+1})$, due to ($\hat{\mathrm{U}}4$, $\hat{\mathrm{U}}6$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

Case $r=-1$ As above, we get $[\Delta(\bar{h}_1),\Delta(e_{-1})]=[\bar{h}_1\otimes 1+1\otimes\bar{h}_1-(\boldsymbol{v}-\boldsymbol{v}^{-1})e_0\otimes f_1, 1\otimes e_{-1}]=1\otimes e_0-(\boldsymbol{v}-\boldsymbol{v}^{-1})e_0\otimes[f_1,e_{-1}]=1\otimes e_0+(\boldsymbol{v}-\boldsymbol{v}^{-1})e_0\otimes\frac{\psi_0^+}{\boldsymbol{v}-\boldsymbol{v}^{-1}}=\Delta(e_0)$.

Case $r=b_2-1$ We have $[\Delta(\bar{h}_1),\Delta(e_{b_2-1})]=[\bar{h}_1\otimes 1+1\otimes\bar{h}_1-(\boldsymbol{v}-\boldsymbol{v}^{-1})e_0\otimes f_1, e_{-1}\otimes\psi_{b_2}^-+1\otimes e_{b_2-1}]=e_0\otimes\psi_{b_2}^-+1\otimes e_{b_2}-e_0\otimes\psi_{b_2}^--(\boldsymbol{v}-\boldsymbol{v}^{-1})[e_0\otimes f_1,e_{-1}\otimes\psi_{b_2}^-]=1\otimes e_{b_2}=\Delta(e_{b_2})$, where we used $[e_0\otimes f_1, e_{-1}\otimes\psi_{b_2}^-]=0$ as $e_0e_{-1}=\boldsymbol{v}^2e_{-1}e_0$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$, due to (Û2), and $\psi_{b_2}^-f_1=\boldsymbol{v}^2f_1\psi_{b_2}^-$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$, due to (Û5).

Let us now verify the equality $[\Delta(\bar{h}_{-1}),\Delta(e_r)]=\Delta(e_{r-1})$ for $b_2\le r\le 0$.

Case $b_2<r<0$ We have $[\Delta(\bar{h}_{-1}),\Delta(e_r)]=[\bar{h}_{-1}\otimes 1+1\otimes\bar{h}_{-1}+(\boldsymbol{v}-\boldsymbol{v}^{-1})e_{-1}\otimes f_0, 1\otimes e_r]=1\otimes e_{r-1}+(\boldsymbol{v}-\boldsymbol{v}^{-1})e_{-1}\otimes[f_0,e_r]=1\otimes e_{r-1}=\Delta(e_{r-1})$, due to (Û4, Û6) for $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

Case $r=0$ We have $[\Delta(\bar{h}_{-1}),\Delta(e_0)]=[\bar{h}_{-1}\otimes 1+1\otimes\bar{h}_{-1}+(\boldsymbol{v}-\boldsymbol{v}^{-1})e_{-1}\otimes f_0, e_0\otimes\psi_0^++1\otimes e_0]=e_{-1}\otimes\psi_0^++1\otimes e_{-1}+(\boldsymbol{v}-\boldsymbol{v}^{-1})e_{-1}\otimes[f_0,e_0]+(\boldsymbol{v}-\boldsymbol{v}^{-1})[e_{-1}\otimes f_0, e_0\otimes\psi_0^+]=1\otimes e_{-1}=\Delta(e_{-1})$, where we used $[e_{-1}\otimes f_0, e_0\otimes\psi_0^+]=0$ as $e_0e_{-1}=\boldsymbol{v}^2e_0e_{-1}$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$, due to (Û2), and $f_0\psi_0^+=\boldsymbol{v}^2\psi_0^+f_0$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$, due to (Û5).

Case $r=b_2$ We have $[\Delta(\bar{h}_{-1}),\Delta(e_{b_2})]=[\bar{h}_{-1}\otimes 1+1\otimes\bar{h}_{-1}+(\boldsymbol{v}-\boldsymbol{v}^{-1})e_{-1}\otimes f_0, 1\otimes e_{b_2}]=1\otimes e_{b_2-1}+(\boldsymbol{v}-\boldsymbol{v}^{-1})e_{-1}\otimes\frac{\psi_{b_2}^-}{\boldsymbol{v}-\boldsymbol{v}^{-1}}=\Delta(e_{b_2-1})$, due to (Û4, Û6) for $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

D(v) Compatibility with (Û5)

The equalities $\Delta(\psi_0^+)\Delta(f_r)=\boldsymbol{v}^{-2}\Delta(f_r)\Delta(\psi_0^+)$ and $\Delta(\psi_b^-)\Delta(f_r)=\boldsymbol{v}^2\Delta(f_r)\Delta(\psi_b^-)$ for $b_1\le r\le 1$ are obvious, due to relations (Û1) and (Û5) for $\mathcal{U}^{\mathrm{sc}}_{0,b_1},\mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

Let us now verify the equality $[\Delta(\bar{h}_1),\Delta(f_r)]=-\Delta(f_{r+1})$ for $b_1\le r\le 0$.

Case $b_1<r<0$ We have $[\Delta(\bar{h}_1),\Delta(f_r)]=[\bar{h}_1\otimes 1+1\otimes\bar{h}_1-(\boldsymbol{v}-\boldsymbol{v}^{-1})e_0\otimes f_1, f_r\otimes 1]=-f_{r+1}\otimes 1-(\boldsymbol{v}-\boldsymbol{v}^{-1})[e_0,f_r]\otimes f_1=-f_{r+1}\otimes 1=-\Delta(f_{r+1})$, due to (Û5, Û6) for $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$.

Case $r=0$ As above, we get $[\Delta(\bar{h}_1),\Delta(f_0)]=[\bar{h}_1\otimes 1+1\otimes\bar{h}_1-(\boldsymbol{v}-\boldsymbol{v}^{-1})e_0\otimes f_1, f_0\otimes 1]=-f_1\otimes 1-(\boldsymbol{v}-\boldsymbol{v}^{-1})[e_0,f_0]\otimes f_1=-f_1\otimes 1-\psi_0^+\otimes f_1=-\Delta(f_1)$.

Case $r=b_1$ We have $[\Delta(\bar{h}_1),\Delta(f_{b_1})]=[\bar{h}_1\otimes 1+1\otimes\bar{h}_1-(\boldsymbol{v}-\boldsymbol{v}^{-1})e_0\otimes f_1, f_{b_1}\otimes 1+\psi_{b_1}^-\otimes f_0]=-f_{1+b_1}\otimes 1-\psi_{b_1}^-\otimes f_1+\psi_{b_1}^-\otimes f_1-(\boldsymbol{v}-\boldsymbol{v}^{-1})[e_0\otimes f_1,\psi_{b_1}^-\otimes f_0]=-f_{1+b_1}\otimes 1=-\Delta(f_{1+b_1})$, where we used $[e_0\otimes f_1,\psi_{b_1}^-\otimes f_0]=0$ as $f_1f_0=\boldsymbol{v}^{-2}f_0f_1$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$, due to (Û3), and $\psi_{b_1}^-e_0=\boldsymbol{v}^{-2}e_0\psi_{b_1}^-$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$, due to (Û4).

Let us now verify the equality $[\Delta(\bar{h}_{-1}), \Delta(f_r)] = -\Delta(f_{r-1})$ for $1+b_1 \le r \le 1$.

Case $1 + b_1 < r < 1$ We have $[\Delta(\bar{h}_{-1}), \Delta(f_r)] = [\bar{h}_{-1} \otimes 1 + 1 \otimes \bar{h}_{-1} + (\boldsymbol{v} - \boldsymbol{v}^{-1})e_{-1} \otimes f_0, f_r \otimes 1] = -f_{r-1} \otimes 1 + (\boldsymbol{v} - \boldsymbol{v}^{-1})[e_{-1}, f_r] \otimes f_0 = -f_{r-1} \otimes 1 = -\Delta(f_{r-1})$, due to (Û5, Û6) for $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$.

Case $r = 1$ We have $[\Delta(\bar{h}_{-1}), \Delta(f_1)] = [\bar{h}_{-1} \otimes 1 + 1 \otimes \bar{h}_{-1} + (\boldsymbol{v} - \boldsymbol{v}^{-1})e_{-1} \otimes f_0, f_1 \otimes 1 + \psi_0^+ \otimes f_1] = -f_0 \otimes 1 - \psi_0^+ \otimes f_0 + \psi_0^+ \otimes f_0 + (\boldsymbol{v} - \boldsymbol{v}^{-1})[e_{-1} \otimes f_0, \psi_0^+ \otimes f_1] = -f_0 \otimes 1 = -\Delta(f_0)$, where we used $[e_{-1} \otimes f_0, \psi_0^+ \otimes f_1] = 0$ as $f_0 f_1 = \boldsymbol{v}^2 f_1 f_0$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$ and $\psi_0^+ e_{-1} = \boldsymbol{v}^2 e_{-1} \psi_0^+$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$.

Case $r = 1 + b_1$ We have $[\Delta(\bar{h}_{-1}), \Delta(f_{1+b_1})] = [\bar{h}_{-1} \otimes 1 + 1 \otimes \bar{h}_{-1} + (\boldsymbol{v} - \boldsymbol{v}^{-1})e_{-1} \otimes f_0, f_{1+b_1} \otimes 1] = -f_{b_1} \otimes 1 - \psi_{b_1}^- \otimes f_0 = -\Delta(f_{b_1})$, due to (Û5, Û6) for $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$.

D(vi) Compatibility with (Û6)

Case $b_2 \le r < 0, b_1 < s \le 0$ The equality $[\Delta(e_r), \Delta(f_s)] = 0$ is obvious.

Case $r = s = 0$ We need to prove $[\Delta(e_0), \Delta(f_0)] = \frac{1}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\Delta(\psi_0^+)$. This follows from $[\Delta(e_0), \Delta(f_0)] = [e_0 \otimes \psi_0^+ + 1 \otimes e_0, f_0 \otimes 1] = [e_0, f_0] \otimes \psi_0^+ = \frac{\psi_0^+ \otimes \psi_0^+}{\boldsymbol{v}-\boldsymbol{v}^{-1}} = \frac{\Delta(\psi_0^+)}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$, due to (Û6) for $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$.

Case $r = 0, s = 1$ We need to prove $[\Delta_{b_1,b_2}(e_0), \Delta_{b_1,b_2}(f_1)] = \Delta_{b_1,b_2}(\psi_0^+)\Delta_{b_1,b_2}(h_1)$. This can be easily deduced from the unshifted case $b_1 = b_2 = 0$ by applying Remark 10.6. Indeed, $[\Delta_{b_1,b_2}(e_0), \Delta_{b_1,b_2}(f_1)] = [\jmath^+_{b_1,0} \otimes \jmath^+_{0,b_2}(\Delta(e_0)), \jmath^+_{b_1,0} \otimes \jmath^+_{0,b_2}(\Delta(f_1))] = \jmath^+_{b_1,0} \otimes \jmath^+_{0,b_2}(\Delta([e_0, f_1])) = \jmath^+_{b_1,0} \otimes \jmath^+_{0,b_2}(\Delta(\psi_0^+)\Delta(h_1)) = \Delta_{b_1,b_2}(\psi_0^+)\Delta_{b_1,b_2}(h_1)$, where the subscripts in Δ_{b_1,b_2} are used this time to distinguish it from the Drinfeld-Jimbo coproduct Δ.

Case $r = 0, b_1 < s < 0$ We need to prove $[\Delta(e_0), \Delta(f_s)] = 0$. This follows from $[\Delta(e_0), \Delta(f_s)] = [e_0 \otimes \psi_0^+ + 1 \otimes e_0, f_s \otimes 1] = [e_0, f_s] \otimes \psi_0^+ = 0$ as $[e_0, f_s] = 0$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$ by (Û6).

Case $r = 0, s = b_1$ We need to prove $[\Delta(e_0), \Delta(f_{b_1})] = 0$. This follows from $[\Delta(e_0), \Delta(f_{b_1})] = [e_0 \otimes \psi_0^+ + 1 \otimes e_0, f_{b_1} \otimes 1 + \psi_{b_1}^- \otimes f_0] = [e_0, f_{b_1}] \otimes \psi_0^+ + \psi_{b_1}^- \otimes [e_0, f_0] = -\frac{\psi_{b_1}^- \otimes \psi_0^+}{\boldsymbol{v}-\boldsymbol{v}^{-1}} + \frac{\psi_{b_1}^- \otimes \psi_0^+}{\boldsymbol{v}-\boldsymbol{v}^{-1}} = 0$, where we used $[e_0 \otimes \psi_0^+, \psi_{b_1}^- \otimes f_0] = 0$ as $\psi_0^+ f_0 = \boldsymbol{v}^{-2} f_0 \psi_0^+$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$, $\psi_{b_1}^- e_0 = \boldsymbol{v}^{-2} e_0 \psi_{b_1}^-$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$.

Case $r = -1, s = 1$ We need to prove $[\Delta(e_{-1}), \Delta(f_1)] = \frac{1}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\Delta(\psi_0^+)$. This follows from $[\Delta(e_{-1}), \Delta(f_1)] = [1 \otimes e_{-1}, f_1 \otimes 1 + \psi_0^+ \otimes f_1] = \psi_0^+ \otimes [e_{-1}, f_1] = \frac{\psi_0^+ \otimes \psi_0^+}{\boldsymbol{v}-\boldsymbol{v}^{-1}} = \frac{\Delta(\psi_0^+)}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$, due to (Û6) for $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

Case $b_2 \leq r < -1, s = 1$ We need to prove $[\Delta(e_r), \Delta(f_1)] = 0$. This follows from $[\Delta(e_r), \Delta(f_1)] = [1 \otimes e_r, f_1 \otimes 1 + \psi_0^+ \otimes f_1] = \psi_0^+ \otimes [e_r, f_1] = 0$ as $[e_r, f_1] = 0$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$ by ($\hat{\mathrm{U}}6$).

Case $r = b_2 - 1, s = 1$ We need to prove $[\Delta(e_{b_2-1}), \Delta(f_1)] = 0$. This follows from $[\Delta(e_{b_2-1}), \Delta(f_1)] = [e_{-1} \otimes \psi^-_{b_2} + 1 \otimes e_{b_2-1}, f_1 \otimes 1 + \psi_0^+ \otimes f_1] = [e_{-1}, f_1] \otimes \psi^-_{b_2} + \psi_0^+ \otimes [e_{b_2-1}, f_1] + [e_{-1} \otimes \psi^-_{b_2}, \psi_0^+ \otimes f_1] = \frac{\psi_0^+ \otimes \psi^-_{b_2}}{\boldsymbol{v}-\boldsymbol{v}^{-1}} - \frac{\psi_0^+ \otimes \psi^-_{b_2}}{\boldsymbol{v}-\boldsymbol{v}^{-1}} = 0$. Here we used $[e_{-1} \otimes \psi^-_{b_2}, \psi_0^+ \otimes f_1] = 0$ as $\psi^-_{b_2} f_1 = \boldsymbol{v}^2 f_1 \psi^-_{b_2}$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$, due to ($\hat{\mathrm{U}}5$), and $\psi_0^+ e_{-1} = \boldsymbol{v}^2 e_{-1} \psi_0^+$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$, due to ($\hat{\mathrm{U}}4$).

Case $r = b_2 - 1, s = b_1$ The proof of $[\Delta_{b_1,b_2}(e_{b_2-1}), \Delta_{b_1,b_2}(f_{b_1})] = \Delta_{b_1,b_2}(\psi^-_b)\Delta_{b_1,b_2}(h_{-1})$ can be deduced by applying Remark 10.6 analogously to the case $r = 0, s = 1$. Indeed, $[\Delta_{b_1,b_2}(e_{b_2-1}), \Delta_{b_1,b_2}(f_{b_1})] = [\jmath^-_{b_1,0} \otimes \jmath^-_{0,b_2}(\Delta(e_{-1})), \jmath^-_{b_1,0} \otimes \jmath^-_{0,b_2}(\Delta(f_0))] = \jmath^-_{b_1,0} \otimes \jmath^-_{0,b_2}(\Delta([e_{-1}, f_0])) = \jmath^-_{b_1,0} \otimes \jmath^-_{0,b_2}(\Delta(\psi^-_0)\Delta(h_{-1})) = \Delta_{b_1,b_2}(\psi^-_b)\Delta_{b_1,b_2}(h_{-1})$.

Case $r = b_2, s = b_1$ We need to prove $[\Delta(e_{b_2}), \Delta(f_{b_1})] = -\frac{1}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\Delta(\psi^-_b)$. This follows from $[\Delta(e_{b_2}), \Delta(f_{b_1})] = [1 \otimes e_{b_2}, f_{b_1} \otimes 1 + \psi^-_{b_1} \otimes f_0] = \psi^-_{b_1} \otimes [e_{b_2}, f_0] = -\frac{\psi^-_{b_1} \otimes \psi^-_{b_2}}{\boldsymbol{v}-\boldsymbol{v}^{-1}} = -\frac{\Delta(\psi^-_b)}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$, due to ($\hat{\mathrm{U}}6$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$.

Case $b_2 < r < 0, s = b_1$ We need to prove $[\Delta(e_r), \Delta(f_{b_1})] = 0$. This follows from $[\Delta(e_r), \Delta(f_{b_1})] = [1 \otimes e_r, f_{b_1} \otimes 1 + \psi^-_{b_1} \otimes f_0] = \psi^-_{b_1} \otimes [e_r, f_0] = 0$ as $[e_r, f_0] = 0$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_2}$ by ($\hat{\mathrm{U}}6$).

Case $r = b_2 - 1, 1 + b_1 < s \leq 0$ We need to prove $[\Delta(e_{b_2-1}), \Delta(f_s)] = 0$. This follows from $[\Delta(e_{b_2-1}), \Delta(f_s)] = [e_{-1} \otimes \psi^-_{b_2} + 1 \otimes e_{b_2-1}, f_s \otimes 1] = [e_{-1}, f_s] \otimes \psi^-_{b_2} = 0$ as $[e_{-1}, f_s] = 0$ in $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$.

Case $r = b_2 - 1, s = 1 + b_1$ We need to prove $[\Delta(e_{b_2-1}), \Delta(f_{1+b_1})] = -\frac{1}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\Delta(\psi^-_b)$. This follows from $[\Delta(e_{b_2-1}), \Delta(f_{1+b_1})] = [e_{-1} \otimes \psi^-_{b_2} + 1 \otimes e_{b_2-1}, f_{1+b_1} \otimes 1] = [e_{-1}, f_{1+b_1}] \otimes \psi^-_{b_2} = -\frac{\psi^-_{b_1} \otimes \psi^-_{b_2}}{\boldsymbol{v}-\boldsymbol{v}^{-1}} = -\frac{\Delta(\psi^-_b)}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$, due to ($\hat{\mathrm{U}}6$) for $\mathcal{U}^{\mathrm{sc}}_{0,b_1}$.

D(vii) Compatibility with ($\hat{\mathrm{U}}9$)

Applying Remark 10.6 as we did above, we see that the equalities

$$[\Delta(h_1), [\Delta(f_1), [\Delta(h_1), \Delta(e_0)]]] = 0 \text{ and } [\Delta(h_{-1}), [\Delta(e_{b_2-1}), [\Delta(h_{-1}), \Delta(f_{b_1})]]] = 0$$

follow from the equalities $[h_1,[f_1,[h_1,e_0]]] = [2]_{\boldsymbol{v}} \cdot [h_1,[f_1,e_1]] = [2]_{\boldsymbol{v}} \cdot [h_1, \frac{-\psi_2^+}{\boldsymbol{v}-\boldsymbol{v}^{-1}}] = 0$ in $U_{\boldsymbol{v}}^+$ and $[h_{-1},[e_{-1},[h_{-1},f_0]]] = -[2]_{\boldsymbol{v}} \cdot [h_{-1},[e_{-1},f_{-1}]] = [2]_{\boldsymbol{v}} \cdot [h_{-1}, \frac{\psi_{-2}^-}{\boldsymbol{v}-\boldsymbol{v}^{-1}}] = 0$ in $U_{\boldsymbol{v}}^-$, respectively.

This completes our proof of Theorem 10.5.

Appendix E Proof of Lemma 10.9(b)

E(i) PBW Property for $\mathcal{U}^{\mathrm{sc}}_{0,n}$

For $\mathcal{U}^{\mathrm{sc}}_{0,n}$, the simply-connected shifted quantum affine algebra of $\mathfrak{sl}_2$, define the PBW variables to be $\{e_s\}_{s\in\mathbb{Z}} \cup \{f_s\}_{s\in\mathbb{Z}} \cup \{\psi_r^+\}_{r>0} \cup \{\psi_{n-r}^-\}_{r>0} \cup \{(\psi_0^+)^{\pm 1}\} \cup \{(\psi_n^-)^{\pm 1}\}$. We order the elements in each group according to the decreasing order of s, r. Any expression of the form

$$e_{s_1^+}\cdots e_{s_a^+} f_{s_1^-}\cdots f_{s_b^-} \psi^+_{r_1^+}\cdots\psi^+_{r^+_{c^+}} \psi^-_{r_1^-}\cdots\psi^-_{r^-_{c^-}} (\psi_0^+)^{\gamma^+}(\psi_n^-)^{\gamma^-}$$

with $s_1^+ \geq \cdots \geq s_a^+,\ s_1^- \geq \cdots \geq s_b^-,\ r_1^+ \geq \cdots \geq r_{c^+}^+ > 0,\ r_1^- \leq \cdots \leq r_{c^-}^- < n,\ \gamma^\pm \in \mathbb{Z},\ a, b, c^\pm \in \mathbb{N}$, will be referred to as the ordered monomial in the PBW variables.

The following result is easy to check using defining relations (U1–U6).

Lemma E.1 *The algebra $\mathcal{U}^{\mathrm{sc}}_{0,n}$ is spanned by the ordered monomials in the PBW variables.*

The key result of this section is a refinement of the previous statement.

Theorem E.2 *For any $n \in \mathbb{Z}$, the algebra $\mathcal{U}^{\mathrm{sc}}_{0,n}$ satisfies the PBW property, that is, the set of the ordered monomials in the PBW variables forms a $\mathbb{C}(\boldsymbol{v})$-basis of $\mathcal{U}^{\mathrm{sc}}_{0,n}$.*

E(ii) Proof of Theorem E.2

We will prove this result in four steps.

Step 1 Reduction to $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n}$.

Consider the associative $\mathbb{C}(\boldsymbol{v})$-algebra $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n}$, defined in the same way as $\mathcal{U}^{\mathrm{sc}}_{0,n}$ but without the generators $(\psi_0^+)^{-1}, (\psi_n^-)^{-1}$. Note that $\mathcal{U}^{\mathrm{sc}}_{0,n}$ is the localization of $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n}$ by the multiplicative set generated by ψ_0^+, ψ_n^-. Since these generators are among the PBW variables, the PBW property for $\mathcal{U}^{\mathrm{sc}}_{0,n}$ follows from the PBW property for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n}$.

Step 2 PBW property for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,0}$.

It is well-known that the algebra $U_{\boldsymbol{v}}(L\mathfrak{sl}_2)$ satisfies the PBW property with the PBW variables chosen as $\{e_s\}_{s\in\mathbb{Z}}\cup\{f_s\}_{s\in\mathbb{Z}}\cup\{\psi^+_r\}_{r>0}\cup\{\psi^-_{-r}\}_{r>0}\cup\{(\psi^+_0)^{\pm1}\}$. Here the elements in each group are ordered according to the decreasing order of r, s.

Lemma E.3 *There is an embedding of algebras $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,0} \hookrightarrow U_{\boldsymbol{v}}(L\mathfrak{sl}_2) \otimes_{\mathbb{C}(\boldsymbol{v})} \mathbb{C}(\boldsymbol{v})[t]$, such that*

$$e_s \mapsto e_s \otimes t,\;\; f_s \mapsto f_s \otimes 1,\;\; \psi^{\pm}_{\pm r} \mapsto \psi^{\pm}_{\pm r} \otimes t.$$

Proof The above assignment obviously preserves all the defining relations of $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,0}$. Hence, it gives rise to a homomorphism $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,0} \to U_{\boldsymbol{v}}(L\mathfrak{sl}_2) \otimes_{\mathbb{C}(\boldsymbol{v})} \mathbb{C}(\boldsymbol{v})[t]$.

To prove the injectivity of this homomorphism, let us first note that $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,0}$ is spanned by the ordered monomials in the PBW variables, cf. Lemma E.1. The above homomorphism maps these monomials to a subset of the basis for $U_{\boldsymbol{v}}(L\mathfrak{sl}_2) \otimes_{\mathbb{C}(\boldsymbol{v})} \mathbb{C}(\boldsymbol{v})[t]$, where we used the PBW property for $U_{\boldsymbol{v}}(L\mathfrak{sl}_2)$. Hence, the ordered monomials in the PBW variable for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,0}$ are linearly independent and the above homomorphism is injective. □

Our proof of Lemma E.3 implies the PBW property for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,0}$.

Step 3 PBW property for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n},\ n < 0$.

For $n < 0$, the algebra $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n}$ is obviously a quotient of $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,0}$ by the 2-sided ideal

$$I_n := \langle \psi^-_0, \psi^-_{-1}, \ldots, \psi^-_{1+n}\rangle_{2-\mathrm{sided}}.$$

Let I^l_n be the left ideal generated by the same elements

$$I^l_n := \langle \psi^-_0, \psi^-_{-1}, \ldots, \psi^-_{1+n}\rangle_{\mathrm{left}}.$$

Lemma E.4 *We have $I^l_n = I_n$.*

Proof It suffices to show that I^l_n is also a right ideal. According to (U4), we have

$$\psi^-_{-r}e_s = \boldsymbol{v}^{-2}\psi^-_{-r+1}e_{s-1} - e_{s-1}\psi^-_{-r+1} + \boldsymbol{v}^{-2}e_s\psi^-_{-r},\;\; \psi^-_0 e_s = \boldsymbol{v}^{-2}e_s\psi^-_0,$$

so that the right multiplication by e_s preserves I^l_n. Similarly for f_s (need to apply (U5)), while for ψ^+_r, ψ^-_{-r} this is obvious. These elements generate $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,0}$, hence, the claim. □

Combining the PBW property for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,0}$ (established in Step 2) with Lemma E.4 and $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n} \simeq \widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,0}/I_n$, we get the PBW property for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n}$.

Step 4 PBW property for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n},\ n > 0$.

The proof proceeds by induction in n. We assume that the PBW property holds for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,m}$ with $m < n$ and want to deduce the PBW property for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n}$. Consider the homomorphism $\widetilde{\iota}_{n,-1,0}\colon \widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n} \to \widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n-1}$ defined analogously to $\iota_{n,-1,0}$ of Proposition 10.8. Explicitly,

$$\widetilde{\iota}_{n,-1,0}\colon e_s \mapsto e_s - e_{s-1},\ f_s \mapsto f_s,\ \psi^+_r \mapsto \psi^+_r - \psi^+_{r-1},\ \psi^-_r \mapsto \psi^-_r - \psi^-_{r-1},$$

where we set $\psi^+_{-1} := 0,\ \psi^-_n := 0$ in the right-hand sides. The image of an ordered monomial in the PBW variables for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n}$ under $\widetilde{\iota}_{n,-1,0}$ is a linear combination of the same ordered monomial in the PBW variables for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n-1}$ with all ψ^-_r replaced by $(-\psi^-_{r-1})$, called the leading monomial, and several other (not necessarily ordered) monomials in the PBW variables. Based on the equality $e_s e_{s-1} = \boldsymbol{v}^2 e_{s-1} e_s$ $(s \in \mathbb{Z})$, we see that rewriting these extra monomials as linear combinations of the ordered monomials in the PBW variables, all of them are actually lexicographically smaller than the leading monomial. Hence, the PBW property for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n-1}$ implies the PBW property for $\widetilde{\mathcal{U}}^{\mathrm{sc}}_{0,n}$. Moreover, we immediately get the injectivity of $\widetilde{\iota}_{n,-1,0}$.

This completes our proof of Theorem E.2.

E(iii) Proof of Lemma 10.9(b)

Now we are ready to prove Lemma 10.9(b). Due to Lemma 10.9(a), it suffices to verify the injectivity of the homomorphisms $\iota_{n,-1,0}$ and $\iota_{n,0,-1}$. The former follows from the injectivity of $\widetilde{\iota}_{n,-1,0}$ from Step 4 above, while the latter can be deduced in the same way.

Appendix F Proof of Theorem 10.10

The proof of Theorem 10.10 proceeds in three steps. First, we construct Δ_{b_1,b_2} (this construction depends on a choice of sufficiently small $m_1, m_2 \leq 0$). Then, we verify that this construction is independent of the choice made. Finally, we prove the commutativity of the diagram of Theorem 10.10 for any $m_1, m_2 \in \mathbb{Z}_{\leq 0}$.

F(i) Construction of Δ_{b_1,b_2}

Fix any $m_1, m_2 \in \mathbb{Z}_{\leq 0}$ such that $b_1+m_1, b_2+m_2 \in \mathbb{Z}_{\leq 0}$. Consider the diagram

$$\begin{array}{ccc}
\mathcal{U}^{\mathrm{sc}}_{0,b} & & \mathcal{U}^{\mathrm{sc}}_{0,b_1}\otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2} \\
\Big\downarrow{\scriptstyle \iota_{b,m_2,m_1}} & & \Big\downarrow{\scriptstyle \iota_{b_1,0,m_1}\otimes\iota_{b_2,m_2,0}} \\
\mathcal{U}^{\mathrm{sc}}_{0,b+m_1+m_2} & \xrightarrow[\Delta=\Delta_{b_1+m_1,b_2+m_2}]{} & \mathcal{U}^{\mathrm{sc}}_{0,b_1+m_1}\otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2+m_2}
\end{array}$$

where the bottom horizontal arrow $\Delta = \Delta_{b_1+m_1,b_2+m_2}$ is defined in Theorem 10.5. Since the homomorphisms ι_{b,m_2,m_1} and $\iota_{b_1,0,m_1}\otimes\iota_{b_2,m_2,0}$ are injective, the homomorphism $\Delta_{b_1+m_1,b_2+m_2}$ gives rise to a uniquely determined homomorphism Δ_{b_1,b_2} making the above diagram commutative as far as we can prove

$$\Delta(\iota_{b,m_2,m_1}(\mathcal{U}^{\mathrm{sc}}_{0,b})) \subset (\iota_{b_1,0,m_1}\otimes\iota_{b_2,m_2,0})(\mathcal{U}^{\mathrm{sc}}_{0,b_1}\otimes\mathcal{U}^{\mathrm{sc}}_{0,b_2}). \qquad (\diamondsuit)$$

As before, we use $\mathcal{U}^{\mathrm{sc},>}_{0,b'}, \mathcal{U}^{\mathrm{sc},\geq}_{0,b'}, \mathcal{U}^{\mathrm{sc},<}_{0,b'}, \mathcal{U}^{\mathrm{sc},\leq}_{0,b'}$ to denote the $\mathbb{C}(\boldsymbol{v})$-subalgebras of $\mathcal{U}^{\mathrm{sc}}_{0,b'}$ generated by $\{e_r\}, \{e_r, \psi^{\pm}_{\pm s^{\pm}}\}, \{f_r\}, \{f_r, \psi^{\pm}_{\pm s^{\pm}}\}$, respectively. For $r\in\mathbb{Z}$, we claim that

$$\Delta(e_r) \in 1\otimes e_r + \mathcal{U}^{\mathrm{sc},>}_{0,b_1+m_1}\otimes\mathcal{U}^{\mathrm{sc},\leq}_{0,b_2+m_2},\ \ \Delta(f_r)\in f_r\otimes 1 + \mathcal{U}^{\mathrm{sc},\geq}_{0,b_1+m_1}\otimes\mathcal{U}^{\mathrm{sc},<}_{0,b_2+m_2}. \qquad (\diamond_1)$$

This follows by combining iteratively the formulas for $\Delta(e_{-1}), \Delta(f_0), \Delta(h_{\pm 1})$ with the relations $[h_{\pm 1}, e_r] = [2]_{\boldsymbol{v}}\cdot e_{r\pm 1}$, $[h_{\pm 1}, f_r] = -[2]_{\boldsymbol{v}}\cdot f_{r\pm 1}$. We also note that

$$\mathcal{U}^{\mathrm{sc},\geq}_{0,b_1}\otimes\mathcal{U}^{\mathrm{sc},\leq}_{0,b_2} \subset (\iota_{b_1,0,m_1}\otimes\iota_{b_2,m_2,0})(\mathcal{U}^{\mathrm{sc}}_{0,b_1}\otimes\mathcal{U}^{\mathrm{sc}}_{0,b_2}). \qquad (\diamond_2)$$

According to ($\diamond_1$), we get

$$\Delta(\iota_{b,m_2,m_1}(e_r)) \in 1\otimes\sum_{s=0}^{-m_2}(-1)^s\binom{-m_2}{s}e_{r-s} + \mathcal{U}^{\mathrm{sc},>}_{0,b_1+m_1}\otimes\mathcal{U}^{\mathrm{sc},\leq}_{0,b_2+m_2}.$$

The right-hand side is an element of $(\iota_{b_1,0,m_1}\otimes\iota_{b_2,m_2,0})(\mathcal{U}^{\mathrm{sc}}_{0,b_1}\otimes\mathcal{U}^{\mathrm{sc}}_{0,b_2})$, due to ($\diamond_2$) and the equality $1\otimes\sum_{s=0}^{-m_2}(-1)^s\binom{-m_2}{s}e_{r-s} = (\iota_{b_1,0,m_1}\otimes\iota_{b_2,m_2,0})(1\otimes e_r)$. Likewise,

$$\Delta(\iota_{b,m_2,m_1}(f_r)) \in \sum_{s=0}^{-m_1}(-1)^s\binom{-m_1}{s}f_{r-s}\otimes 1 + \mathcal{U}^{\mathrm{sc},\geq}_{0,b_1+m_1}\otimes\mathcal{U}^{\mathrm{sc},<}_{0,b_2+m_2}.$$

The right-hand side is an element of $(\iota_{b_1,0,m_1}\otimes\iota_{b_2,m_2,0})(\mathcal{U}^{\mathrm{sc}}_{0,b_1}\otimes\mathcal{U}^{\mathrm{sc}}_{0,b_2})$, due to ($\diamond_2$) and the equality $\sum_{s=0}^{-m_1}(-1)^s\binom{-m_1}{s}f_{r-s}\otimes 1 = (\iota_{b_1,0,m_1}\otimes\iota_{b_2,m_2,0})(f_r\otimes 1)$. We also have

$$\Delta(\iota_{b,m_2,m_1}((\psi_0^+)^{\pm 1})) = (\iota_{b_1,0,m_1} \otimes \iota_{b_2,m_2,0})((\psi_0)^{\pm 1} \otimes (\psi_0)^{\pm 1}),$$
$$\Delta(\iota_{b,m_2,m_1}((\psi_b^-)^{\pm 1})) = (\iota_{b_1,0,m_1} \otimes \iota_{b_2,m_2,0})((\psi_{b_1}^-)^{\pm 1} \otimes (\psi_{b_2}^-)^{\pm 1}).$$

Finally, combining the relations $\psi_r^+ = (\boldsymbol{v} - \boldsymbol{v}^{-1})[e_r, f_0], \psi_{b-r}^- = (\boldsymbol{v}^{-1} - \boldsymbol{v})[e_{b-r}, f_0]$ $(r \in \mathbb{Z}_{>0})$ in $\mathcal{U}^{\mathrm{sc}}_{0,b+m_1+m_2}$ with ($\diamond_1$) and ($\diamond_2$), we get

$$\Delta(\psi_r^+), \Delta(\psi_{b-r}^-) \in \mathcal{U}^{\mathrm{sc},\geq}_{0,b_1+m_1} \otimes \mathcal{U}^{\mathrm{sc},\leq}_{0,b_2+m_2} \subset (\iota_{b_1,0,m_1} \otimes \iota_{b_2,m_2,0})(\mathcal{U}^{\mathrm{sc}}_{0,b_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2}).$$

This completes our proof of ($\diamond$).

Therefore, we obtain the homomorphism Δ_{b_1,b_2} for the particular choice of m_1, m_2.

F(ii) Independence of the Choice of m_1, m_2

Let us now prove that the homomorphism Δ_{b_1,b_2} constructed above does not depend on the choice of m_1, m_2. To this end, fix another pair $m_1', m_2' \in \mathbb{Z}_{\leq 0}$ such that $b_1 + m_1', b_2 + m_2' \in \mathbb{Z}_{\leq 0}$, and set $m = m_1 + m_2, m' = m_1' + m_2'$.

Consider the following diagram:

$$\begin{array}{ccc}
\mathcal{U}^{\mathrm{sc}}_{0,b} & & \mathcal{U}^{\mathrm{sc}}_{0,b_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2} \\
\downarrow{\scriptstyle \iota_{b,m_2,m_1}} & & \downarrow{\scriptstyle \iota_{b_1,0,m_1} \otimes \iota_{b_2,m_2,0}} \\
\mathcal{U}^{\mathrm{sc}}_{0,b+m} & \xrightarrow{\Delta_{b_1+m_1,b_2+m_2}} & \mathcal{U}^{\mathrm{sc}}_{0,b_1+m_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2+m_2} \\
\downarrow{\scriptstyle \iota_{b+m,m_2',m_1'}} & & \downarrow{\scriptstyle \iota_{b_1+m_1,0,m_1'} \otimes \iota_{b_2+m_2,m_2',0}} \\
\mathcal{U}^{\mathrm{sc}}_{0,b+m+m'} & \xrightarrow{\Delta_{b_1+m_1+m_1',b_2+m_2+m_2'}} & \mathcal{U}^{\mathrm{sc}}_{0,b_1+m_1+m_1'} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2+m_2+m_2'}
\end{array}$$

According to Lemma 10.9(a): $\iota_{b+m,m_2',m_1'} \circ \iota_{b,m_2,m_1} = \iota_{b,m_2+m_2',m_1+m_1'}$ and $(\iota_{b_1+m_1,0,m_1'} \otimes \iota_{b_2+m_2,m_2',0}) \circ (\iota_{b_1,0,m_1} \otimes \iota_{b_2,m_2,0}) = (\iota_{b_1,0,m_1+m_1'} \otimes \iota_{b_2,m_2+m_2',0})$. On the other hand, tracing back the explicit formulas for $\Delta_{b_1+m_1,b_2+m_2}$ and $\Delta_{b_1+m_1+m_1',b_2+m_2+m_2'}$ of Theorem 10.5, it is easy to check that the lower square is commutative.

The above two observations imply that the maps Δ_{b_1,b_2} are the same for both (m_1, m_2) and $(m_1 + m_1', m_2 + m_2')$. Due to the symmetry, we also see that the maps Δ_{b_1,b_2} are the same for both (m_1', m_2') and $(m_1 + m_1', m_2 + m_2')$. Therefore, the maps Δ_{b_1,b_2} are the same for both (m_1, m_2) and (m_1', m_2'). This completes our verification.

F(iii) Commutativity of the Diagram for Any $m_1, m_2 \in \mathbb{Z}_{\leq 0}$

It remains to prove the commutativity of the diagram of Theorem 10.10. To this end, choose $m_1', m_2' \in \mathbb{Z}_{\leq 0}$ such that $b_1 + m_1 + m_1', b_2 + m_2 + m_2' \in \mathbb{Z}_{\leq 0}$. Consider a diagram analogous to the previous one:

$$
\begin{array}{ccc}
\mathcal{U}^{\mathrm{sc}}_{0,b} & \xrightarrow{\Delta_{b_1,b_2}} & \mathcal{U}^{\mathrm{sc}}_{0,b_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2} \\
\downarrow{\scriptstyle \iota_{b,m_2,m_1}} & & \downarrow{\scriptstyle \iota_{b_1,0,m_1} \otimes \iota_{b_2,m_2,0}} \\
\mathcal{U}^{\mathrm{sc}}_{0,b+m} & \xrightarrow{\Delta_{b_1+m_1,b_2+m_2}} & \mathcal{U}^{\mathrm{sc}}_{0,b_1+m_1} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2+m_2} \\
\downarrow{\scriptstyle \iota_{b+m,m_2',m_1'}} & & \downarrow{\scriptstyle \iota_{b_1+m_1,0,m_1'} \otimes \iota_{b_2+m_2,m_2',0}} \\
\mathcal{U}^{\mathrm{sc}}_{0,b+m+m'} & \xrightarrow{\Delta_{b_1+m_1+m_1',b_2+m_2+m_2'}} & \mathcal{U}^{\mathrm{sc}}_{0,b_1+m_1+m_1'} \otimes \mathcal{U}^{\mathrm{sc}}_{0,b_2+m_2+m_2'}
\end{array}
$$

By our construction, the lower square is commutative. Applying Lemma 10.9(a) as in Sect. F(ii), we also see that the outer square is commutative. Hence, the commutativity of the top square follows from the injectivity of the homomorphism $\iota_{b_1+m_1,0,m_1'} \otimes \iota_{b_2+m_2,m_2',0}$, due to Lemma 10.9(b).

Appendix G Proof of Theorem 10.13

The proof of Theorem 10.13 proceeds in several steps. First, we recall the RTT presentation of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, and derive the equalities of the right-hand sides of (10.6). Then, we compute the RTT coproduct of certain elements $\tilde{g}_i^{(\pm 1)}$ from the RTT presentation, see Theorems G.10, G.13 (this is the most technical part). This allows us to derive formulas (10.2) and (10.3). Based on these, we deduce (10.4) and (10.5).

G(i) RTT Presentation of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$

Let $R_{\mathrm{trig}}(z/w) \in \mathrm{End}(\mathbb{C}^n \otimes \mathbb{C}^n)$ be the standard trigonometric R-matrix of $\mathfrak{sl}_n$-type:

$$
\begin{aligned}
R_{\mathrm{trig}}(z/w) := \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + \sum_{1 \leq i \neq j \leq n} \frac{z-w}{\boldsymbol{v}z - \boldsymbol{v}^{-1}w} E_{ii} \otimes E_{jj} + \\
\sum_{1 \leq j < i \leq n} \left(\frac{(\boldsymbol{v}-\boldsymbol{v}^{-1})z}{\boldsymbol{v}z - \boldsymbol{v}^{-1}w} E_{ji} \otimes E_{ij} + \frac{(\boldsymbol{v}-\boldsymbol{v}^{-1})w}{\boldsymbol{v}z - \boldsymbol{v}^{-1}w} E_{ij} \otimes E_{ji} \right)
\end{aligned}
\qquad \text{(G.1)}
$$

(for $n = 2$, this definition coincides with formula (11.3)).

Define the RTT algebra of $\mathfrak{sl}_n$, denoted by $U^{\mathrm{rtt}}(\mathfrak{sl}_n)$, to be the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by $\{t^{\pm}_{ij}[\pm r]\}^{r\in\mathbb{N}}_{1\le i,j\le n}$ subject to the following defining relations:

$$t^{\pm}_{ii}[0]t^{\mp}_{ii}[0]=1 \ \text{ for } 1\le i\le n,\ \ t^{+}_{ij}[0]=t^{-}_{ji}[0]=0 \ \text{ for } j<i, \tag{G.2}$$

$$R_{\mathrm{trig}}(z/w)(T^{\epsilon}(z)\otimes 1)(1\otimes T^{\epsilon'}(w))=(1\otimes T^{\epsilon'}(w))(T^{\epsilon}(z)\otimes 1)R_{\mathrm{trig}}(z/w), \tag{G.3}$$

$$\mathrm{qdet}\, T^{\pm}(z)=1, \tag{G.4}$$

for all $\epsilon,\epsilon'\in\{\pm\}$, where the matrices $T^{\pm}(z)\in \mathrm{Mat}_{n\times n}(U^{\mathrm{rtt}}(\mathfrak{sl}_n))$ are given by

$$T^{\pm}(z):=\sum_{i,j=1}^{n} T^{\pm}_{ij}(z)\cdot E_{ij} \ \text{ with } \ T^{\pm}_{ij}(z):=\sum_{r\ge 0} t^{\pm}_{ij}[\pm r]z^{\mp r},$$

and the quantum determinant qdet is defined in a standard way as

$$\mathrm{qdet}\, T^{\pm}(z):=\sum_{\tau\in\mathfrak{S}_n}(-\boldsymbol{v})^{-l(\tau)}T^{\pm}_{1,\tau(1)}(z)T^{\pm}_{2,\tau(2)}(\boldsymbol{v}^{-2}z)\cdots T^{\pm}_{n,\tau(n)}(\boldsymbol{v}^{2-2n}z)$$

(cf. Sect. 11.4 and a footnote there).

Remark G.1 Let us point out right away that the RTT presentation of $U_q(\widehat{\mathfrak{gl}}_n)$ (with a nontrivial central charge), given in [17, Definition 3.2], involves only three out of four relations (G.3), namely for $(\epsilon,\epsilon')=(+,+),(-,-),(-,+)$. However, as pointed out in [32, 2.3], if the central charge is trivial, then the fourth relation for $(\epsilon,\epsilon')=(+,-)$ is equivalent to the one for $(\epsilon,\epsilon')=(-,+)$. Indeed, in our notations, this follows from the equalities $R_{\mathrm{trig}}(z/w)^{-1}=R'_{\mathrm{trig}}(z/w)$, $PR'_{\mathrm{trig}}(w/z)P^{-1}=R_{\mathrm{trig}}(z/w)$, where $R'_{\mathrm{trig}}(z/w)$ is obtained from $R_{\mathrm{trig}}(z/w)$ by replacing $\boldsymbol{v}$ with $\boldsymbol{v}^{-1}$ and $P\in\mathrm{End}(\mathbb{C}^n\otimes\mathbb{C}^n)$ denotes the permutation operator.

Note that $T^{\pm}(z)$ admits the following unique Gauss decomposition:

$$T^{\pm}(z)=\widetilde{F}^{\pm}(z)\cdot\widetilde{G}^{\pm}(z)\cdot\widetilde{E}^{\pm}(z)$$

with $\widetilde{F}^{\pm}(z),\widetilde{G}^{\pm}(z),\widetilde{E}^{\pm}(z)\in\mathrm{Mat}_{n\times n}(U^{\mathrm{rtt}}(\mathfrak{sl}_n))$ of the form

$$\widetilde{F}^{\pm}(z)=\sum_i E_{ii}+\sum_{j<i}\tilde{f}^{\pm}_{ij}(z)\cdot E_{ij},\ \widetilde{G}^{\pm}(z)=\sum_i \tilde{g}^{\pm}_i(z)\cdot E_{ii},\ \widetilde{E}^{\pm}(z)=\sum_i E_{ii}+\sum_{j<i}\tilde{e}^{\pm}_{ji}(z)\cdot E_{ji}.$$

We endow $U^{\mathrm{rtt}}(\mathfrak{sl}_n)$ with the coproduct structure (also known as the RTT coproduct) via

$$\Delta^{\mathrm{rtt}}\colon U^{\mathrm{rtt}}(\mathfrak{sl}_n)\longrightarrow U^{\mathrm{rtt}}(\mathfrak{sl}_n)\otimes U^{\mathrm{rtt}}(\mathfrak{sl}_n) \ \text{ given by } \ \Delta^{\mathrm{rtt}}(T^{\pm}(z)):=T^{\pm}(z)\otimes T^{\pm}(z).$$

Theorem G.2 ([17]) *There exists a unique $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism*

$$\Upsilon\colon U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{sl}_n) \xrightarrow{\sim} U^{\mathrm{rtt}}(\mathfrak{sl}_n),$$

such that

$$e^{\pm}_j(z) \mapsto \frac{\tilde{e}^{\pm}_{j,j+1}(\boldsymbol{v}^j z)}{\boldsymbol{v}-\boldsymbol{v}^{-1}},\ f^{\pm}_j(z) \mapsto \frac{\tilde{f}^{\pm}_{j+1,j}(\boldsymbol{v}^j z)}{\boldsymbol{v}-\boldsymbol{v}^{-1}},$$
$$\psi^{\pm}_j(z) \mapsto \tilde{g}^{\pm}_{j+1}(\boldsymbol{v}^j z)(\tilde{g}^{\pm}_j(\boldsymbol{v}^j z))^{-1},\ \phi^{\pm}_j \mapsto t^{\mp}_{11}[0]t^{\mp}_{22}[0]\cdots t^{\mp}_{jj}[0] \text{ for } 1 \le j < n.$$

Moreover, this isomorphism intertwines the Drinfeld-Jimbo coproduct Δ^{ad} on $U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ with the RTT coproduct Δ^{rtt} on $U^{\mathrm{rtt}}(\mathfrak{sl}_n)$.

Remark G.3 Restricting Υ to $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, viewed as a Hopf subalgebra of $U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, we get an embedding $U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \hookrightarrow U^{\mathrm{rtt}}(\mathfrak{sl}_n)$. We will deliberately refer to $U^{\mathrm{rtt}}(\mathfrak{sl}_n)$ as an RTT presentation of both algebras $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

Let us express the matrix coefficients of $\widetilde{F}^{\pm}(z), \widetilde{G}^{\pm}(z), \widetilde{E}^{\pm}(z)$ as Taylor series in $z^{\mp 1}$: $\tilde{e}^{+}_{ji}(z) = \sum_{r\ge 0} \tilde{e}^{(r)}_{ji} z^{-r}$, $\tilde{e}^{-}_{ji}(z) = \sum_{r<0} \tilde{e}^{(r)}_{ji} z^{-r}$, $\tilde{f}^{+}_{ij}(z) = \sum_{r>0} \tilde{f}^{(r)}_{ij} z^{-r}$, $\tilde{f}^{-}_{ij}(z) = \sum_{r\le 0} \tilde{f}^{(r)}_{ij} z^{-r}$, $\tilde{g}^{\pm}_i(z) = \tilde{g}^{\pm}_i + \sum_{r>0} \tilde{g}^{(\pm r)}_i z^{\mp r}$. According to Theorem G.2, we have

$$\begin{aligned}
&\Upsilon^{-1}(\tilde{e}^{(0)}_{j,j+1}) = (\boldsymbol{v}-\boldsymbol{v}^{-1})e_{j,0},\ \Upsilon^{-1}(\tilde{f}^{(0)}_{j+1,j}) = -(\boldsymbol{v}-\boldsymbol{v}^{-1})f_{j,0},\\
&\Upsilon^{-1}(\tilde{e}^{(-1)}_{j,j+1}) = -\boldsymbol{v}^{-j}(\boldsymbol{v}-\boldsymbol{v}^{-1})e_{j,-1},\ \Upsilon^{-1}(\tilde{f}^{(1)}_{j+1,j}) = \boldsymbol{v}^{j}(\boldsymbol{v}-\boldsymbol{v}^{-1})f_{j,1}.
\end{aligned} \tag{G.5}$$

The following is the key technical result of this subsection.

Proposition G.4 *For any $1 \le j < k < i \le n$, we have:*

(a) $\tilde{e}^{(0)}_{ji} = \frac{1}{\boldsymbol{v}-\boldsymbol{v}^{-1}}[\tilde{e}^{(0)}_{ki}, \tilde{e}^{(0)}_{jk}]_{\boldsymbol{v}^{-1}}$.
(b) $\tilde{f}^{(0)}_{ij} = \frac{-1}{\boldsymbol{v}-\boldsymbol{v}^{-1}}[\tilde{f}^{(0)}_{kj}, \tilde{f}^{(0)}_{ik}]_{\boldsymbol{v}}$.
(c) $\tilde{e}^{(-1)}_{ji} = \frac{1}{\boldsymbol{v}-\boldsymbol{v}^{-1}}[\tilde{e}^{(0)}_{ki}, \tilde{e}^{(-1)}_{jk}]_{\boldsymbol{v}^{-1}}$.
(d) $\tilde{f}^{(1)}_{ij} = \frac{-1}{\boldsymbol{v}-\boldsymbol{v}^{-1}}[\tilde{f}^{(1)}_{kj}, \tilde{f}^{(0)}_{ik}]_{\boldsymbol{v}}$.

Proof

(a) Comparing the matrix coefficients $\langle v_j \otimes v_k|\cdots|v_k \otimes v_i\rangle$ of both sides of the equality $R_{\mathrm{trig}}(z/w)(T^{+}(z)\otimes 1)(1\otimes T^{+}(w)) = (1\otimes T^{+}(w))(T^{+}(z)\otimes 1)R_{\mathrm{trig}}(z/w)$, we get

$$(z-w)T^{+}_{jk}(z)T^{+}_{ki}(w)+(\boldsymbol{v}-\boldsymbol{v}^{-1})zT^{+}_{kk}(z)T^{+}_{ji}(w) = (z-w)T^{+}_{ki}(w)T^{+}_{jk}(z)+(\boldsymbol{v}-\boldsymbol{v}^{-1})wT^{+}_{kk}(w)T^{+}_{ji}(z).$$

Evaluating the coefficients of z^1w^0 in both sides of this equality, we find

$$\tilde{g}_j^+\tilde{e}_{jk}^{(0)}\tilde{g}_k^+\tilde{e}_{ki}^{(0)} + (\boldsymbol{v}-\boldsymbol{v}^{-1})\tilde{g}_k^+\tilde{g}_j^+\tilde{e}_{ji}^{(0)} = \tilde{g}_k^+\tilde{e}_{ki}^{(0)}\tilde{g}_j^+\tilde{e}_{jk}^{(0)}.$$

Combining this with Lemma G.5 below, we obtain

$$(\boldsymbol{v}-\boldsymbol{v}^{-1})\tilde{g}_k^+\tilde{g}_j^+\tilde{e}_{ji}^{(0)} = \tilde{g}_k^+\tilde{g}_j^+[\tilde{e}_{ki}^{(0)},\tilde{e}_{jk}^{(0)}]_{\boldsymbol{v}^{-1}} \Longrightarrow \tilde{e}_{ji}^{(0)} = [\tilde{e}_{ki}^{(0)},\tilde{e}_{jk}^{(0)}]_{\boldsymbol{v}^{-1}}/(\boldsymbol{v}-\boldsymbol{v}^{-1}).$$

(b) Comparing the matrix coefficients $\langle v_i\otimes v_k|\cdots|v_k\otimes v_j\rangle$ of both sides of the equality $R_{\mathrm{trig}}(z/w)(T^-(z)\otimes 1)(1\otimes T^-(w)) = (1\otimes T^-(w))(T^-(z)\otimes 1) R_{\mathrm{trig}}(z/w)$, we get

$$(z-w)T_{ik}^-(z)T_{kj}^-(w)+(\boldsymbol{v}-\boldsymbol{v}^{-1})wT_{kk}^-(z)T_{ij}^-(w) = (z-w)T_{kj}^-(w)T_{ik}^-(z)+(\boldsymbol{v}-\boldsymbol{v}^{-1})zT_{kk}^-(w)T_{ij}^-(z).$$

Evaluating the coefficients of z^0w^1 in both sides of this equality, we find

$$-\tilde{f}_{ik}^{(0)}\tilde{g}_k^-\tilde{f}_{kj}^{(0)}\tilde{g}_j^- + (\boldsymbol{v}-\boldsymbol{v}^{-1})\tilde{g}_k^-\tilde{f}_{ij}^{(0)}\tilde{g}_j^- = -\tilde{f}_{kj}^{(0)}\tilde{g}_j^-\tilde{f}_{ik}^{(0)}\tilde{g}_k^-.$$

Combining this with Lemma G.5 below, we obtain

$$-(\boldsymbol{v}-\boldsymbol{v}^{-1})\tilde{f}_{ij}^{(0)}\tilde{g}_k^-\tilde{g}_j^- = [\tilde{f}_{kj}^{(0)},\tilde{f}_{ik}^{(0)}]_{\boldsymbol{v}}\cdot\tilde{g}_k^-\tilde{g}_j^- \Longrightarrow \tilde{f}_{ij}^{(0)} = -[\tilde{f}_{kj}^{(0)},\tilde{f}_{ik}^{(0)}]_{\boldsymbol{v}}/(\boldsymbol{v}-\boldsymbol{v}^{-1}).$$

(c) Comparing the matrix coefficients $\langle v_k\otimes v_j|\cdots|v_i\otimes v_k\rangle$ of both sides of the equality $R_{\mathrm{trig}}(z/w)(T^+(z)\otimes 1)(1\otimes T^-(w)) = (1\otimes T^-(w))(T^+(z)\otimes 1) R_{\mathrm{trig}}(z/w)$, we get

$$(z-w)T_{ki}^+z)T_{jk}^-(w)+(\boldsymbol{v}-\boldsymbol{v}^{-1})wT_{ji}^+(z)T_{kk}^-(w) = (z-w)T_{jk}^-(w)T_{ki}^+(z)+(\boldsymbol{v}-\boldsymbol{v}^{-1})zT_{ji}^-(w)T_{kk}^+(z).$$

Evaluating the coefficients of z^1w^1 in both sides of this equality, we find

$$\tilde{g}_k^+\tilde{e}_{ki}^{(0)}\left(\tilde{g}_j^-\tilde{e}_{jk}^{(-1)}+\sum_{j'<j}\tilde{f}_{jj'}^{(0)}\tilde{g}_{j'}^-\tilde{e}_{j'k}^{(-1)}\right)=$$
$$\left(\tilde{g}_j^-\tilde{e}_{jk}^{(-1)}+\sum_{j'<j}\tilde{f}_{jj'}^{(0)}\tilde{g}_{j'}^-\tilde{e}_{j'k}^{(-1)}\right)\tilde{g}_k^+\tilde{e}_{ki}^{(0)}+(\boldsymbol{v}-\boldsymbol{v}^{-1})\left(\tilde{g}_j^-\tilde{e}_{ji}^{(-1)}+\sum_{j'<j}\tilde{f}_{jj'}^{(0)}\tilde{g}_{j'}^-\tilde{e}_{j'i}^{(-1)}\right)\tilde{g}_k^+. \tag{G.6}$$

This equation actually implies $\tilde{g}_k^+\tilde{e}_{ki}^{(0)}\tilde{g}_j^-\tilde{e}_{jk}^{(-1)} = \tilde{g}_j^-\tilde{e}_{jk}^{(-1)}\tilde{g}_k^+\tilde{e}_{ki}^{(0)} + (\boldsymbol{v}-\boldsymbol{v}^{-1})\tilde{g}_j^-\tilde{e}_{ji}^{(-1)}\tilde{g}_k^+$. We prove this by induction in j. For $j=1$, this is just (G.6). In general, note that for $j'<j<k<i$, the element $\tilde{f}_{jj'}^{(0)}$ commutes with $\tilde{e}_{ki}^{(0)}$ and $\tilde{g}_k^+$. The latter follows from Lemma G.5, while the equality $[\tilde{f}_{jj'}^{(0)},\tilde{e}_{ki}^{(0)}]=0$ follows by combining parts (a,b) from

above with $[e_{a,0}, f_{b,0}] = 0$ for $a \neq b$. Hence, (G.6) implies $A(j,k,i) + \sum_{j'<j} \tilde{f}^{(0)}_{jj'} A(j',k,i) = 0$, where we set

$$A(j,k,i) := \tilde{g}^+_k \tilde{e}^{(0)}_{ki} \tilde{g}^-_j \tilde{e}^{(-1)}_{jk} - \tilde{g}^-_j \tilde{e}^{(-1)}_{jk} \tilde{g}^+_k \tilde{e}^{(0)}_{ki} - (\boldsymbol{v} - \boldsymbol{v}^{-1}) \tilde{g}^-_j \tilde{e}^{(-1)}_{ji} \tilde{g}^+_k.$$

By the induction assumption $A(j',k,i) = 0$ for $j' < j$, hence, $A(j,k,i) = 0$.

Combining this with Lemma G.5 below, we obtain

$$(\boldsymbol{v} - \boldsymbol{v}^{-1}) \tilde{g}^-_j \tilde{g}^+_k \tilde{e}^{(-1)}_{ji} = \tilde{g}^-_j \tilde{g}^+_k [\tilde{e}^{(0)}_{ki}, \tilde{e}^{(-1)}_{jk}]_{\boldsymbol{v}^{-1}} \Longrightarrow \tilde{e}^{(-1)}_{ji} = [\tilde{e}^{(0)}_{ki}, \tilde{e}^{(-1)}_{jk}]_{\boldsymbol{v}^{-1}} / (\boldsymbol{v} - \boldsymbol{v}^{-1}).$$

(d) Comparing the matrix coefficients $\langle v_k \otimes v_i | \cdots | v_j \otimes v_k \rangle$ of both sides of the equality $R_{\mathrm{trig}}(z/w)(T^+(z) \otimes 1)(1 \otimes T^-(w)) = (1 \otimes T^-(w))(T^+(z) \otimes 1) R_{\mathrm{trig}}(z/w)$, we get

$$(z-w)T^+_{kj}(z)T^-_{ik}(w) + (\boldsymbol{v} - \boldsymbol{v}^{-1}) z T^+_{ij}(z) T^-_{kk}(w) = (z-w) T^-_{ik}(w) T^+_{kj}(z) + (\boldsymbol{v} - \boldsymbol{v}^{-1}) w T^-_{ij}(w) T^+_{kk}(z).$$

Evaluating the coefficients of $z^0 w^0$ in both sides of this equality, we find

$$\left(\tilde{f}^{(1)}_{kj} \tilde{g}^+_j + \sum_{j'<j} \tilde{f}^{(1)}_{kj'} \tilde{g}^+_{j'} \tilde{e}^{(0)}_{j'j} \right) \tilde{f}^{(0)}_{ik} \tilde{g}^-_k + (\boldsymbol{v} - \boldsymbol{v}^{-1}) \left(\tilde{f}^{(1)}_{ij} \tilde{g}^+_j + \sum_{j'<j} \tilde{f}^{(1)}_{ij'} \tilde{g}^+_{j'} \tilde{e}^{(0)}_{j'j} \right) \tilde{g}^-_k = \tilde{f}^{(0)}_{ik} \tilde{g}^-_k \left(\tilde{f}^{(1)}_{kj} \tilde{g}^+_j + \sum_{j'<j} \tilde{f}^{(1)}_{kj'} \tilde{g}^+_{j'} \tilde{e}^{(0)}_{j'j} \right). \tag{G.7}$$

This equation actually implies $\tilde{f}^{(1)}_{kj} \tilde{g}^+_j \tilde{f}^{(0)}_{ik} \tilde{g}^-_k + (\boldsymbol{v} - \boldsymbol{v}^{-1}) \tilde{f}^{(1)}_{ij} \tilde{g}^+_j \tilde{g}^-_k = \tilde{f}^{(0)}_{ik} \tilde{g}^-_k \tilde{f}^{(1)}_{kj} \tilde{g}^+_j$. We prove this by induction in j. For $j = 1$, this is just (G.7). Analogously to part (c) above, we note that the element $\tilde{e}^{(0)}_{j'j}$ commutes with $\tilde{f}^{(0)}_{ik}$ and $\tilde{g}^-_k$ for $j' < j < k < i$. Hence, (G.7) implies $B(j,k,i) + \sum_{j'<j} B(j',k,i) \tilde{e}^{(0)}_{j'j} = 0$, where we set

$$B(j,k,i) := \tilde{f}^{(1)}_{kj} \tilde{g}^+_j \tilde{f}^{(0)}_{ik} \tilde{g}^-_k + (\boldsymbol{v} - \boldsymbol{v}^{-1}) \tilde{f}^{(1)}_{ij} \tilde{g}^+_j \tilde{g}^-_k - \tilde{f}^{(0)}_{ik} \tilde{g}^-_k \tilde{f}^{(1)}_{kj} \tilde{g}^+_j.$$

By the induction assumption $B(j',k,i) = 0$ for $j' < j$, hence, $B(j,k,i) = 0$.

Combining this with Lemma G.5 below, we obtain

$$-(\boldsymbol{v} - \boldsymbol{v}^{-1}) \tilde{f}^{(1)}_{ij} \tilde{g}^-_k \tilde{g}^+_j = [\tilde{f}^{(1)}_{kj}, \tilde{f}^{(0)}_{ik}]_{\boldsymbol{v}} \cdot \tilde{g}^-_k \tilde{g}^+_j$$

$$\Longrightarrow \tilde{f}^{(1)}_{ij} = -[\tilde{f}^{(1)}_{kj}, \tilde{f}^{(0)}_{ik}]_{\boldsymbol{v}} / (\boldsymbol{v} - \boldsymbol{v}^{-1}).$$

□

Lemma G.5 *For any* $1 \le j < i \le n$ *and* $1 \le a, b \le n$*, we have:*

(a) $\tilde{g}^{\epsilon}_a \tilde{g}^{\epsilon'}_b = \tilde{g}^{\epsilon'}_b \tilde{g}^{\epsilon}_a$ *for any* $\epsilon, \epsilon' \in \{\pm\}$.
(b) $\tilde{g}^{\pm}_a \tilde{e}^{(0)}_{ji} = \boldsymbol{v}^{\pm\delta_{ai}\mp\delta_{aj}} \tilde{e}^{(0)}_{ji} \tilde{g}^{\pm}_a$.
(c) $\tilde{g}^{\pm}_a \tilde{f}^{(0)}_{ij} = \boldsymbol{v}^{\mp\delta_{ai}\pm\delta_{aj}} \tilde{f}^{(0)}_{ij} \tilde{g}^{\pm}_a$.
(d) $\tilde{g}^{\pm}_a \tilde{e}^{(-1)}_{ji} = \boldsymbol{v}^{\pm\delta_{ai}\mp\delta_{aj}} \tilde{e}^{(-1)}_{ji} \tilde{g}^{\pm}_a$.
(e) $\tilde{g}^{\pm}_a \tilde{f}^{(1)}_{ij} = \boldsymbol{v}^{\mp\delta_{ai}\pm\delta_{aj}} \tilde{f}^{(1)}_{ij} \tilde{g}^{\pm}_a$.

Proof First, we note that $t^{\pm}_{ii}[0] = \tilde{g}^{\pm}_i$. Hence, we have $\tilde{g}^{\pm}_i \tilde{g}^{\mp}_i = 1$, due to relation (G.2).

(a) Due to the above observation, it suffices to prove $\tilde{g}^{+}_a \tilde{g}^{+}_b = \tilde{g}^{+}_b \tilde{g}^{+}_a$ for $a < b$. This follows by evaluating the coefficients of $z^0 w^1$ in the equality of the matrix coefficients $\langle v_a \otimes v_b | \cdots | v_a \otimes v_b \rangle$ of both sides of the equality $((\boldsymbol{v}z - \boldsymbol{v}^{-1}w)R_{\mathrm{trig}}(z/w))(T^+(z) \otimes 1)(1 \otimes T^+(w)) = (1 \otimes T^+(w))(T^+(z) \otimes 1)((\boldsymbol{v}z - \boldsymbol{v}^{-1}w)R_{\mathrm{trig}}(z/w))$.

(b) Due to the above observation, it suffices to prove $\tilde{g}^{+}_a \tilde{e}^{(0)}_{ji} = \boldsymbol{v}^{\delta_{ai}-\delta_{aj}} \tilde{e}^{(0)}_{ji} \tilde{g}^{+}_a$. This follows by evaluating the coefficients of $z^0 w^1$ in the equality of the matrix coefficients $\langle v_a \otimes v_j | \cdots | v_a \otimes v_i \rangle$ of both sides of the equality $((\boldsymbol{v}z - \boldsymbol{v}^{-1}w)R_{\mathrm{trig}}(z/w))(T^+(z) \otimes 1)(1 \otimes T^+(w)) = (1 \otimes T^+(w))(T^+(z) \otimes 1)((\boldsymbol{v}z - \boldsymbol{v}^{-1}w)R_{\mathrm{trig}}(z/w))$. Note that the cases $a < j, a = j, j < a < i, a = i, a > i$ have to be treated separately.

(c) Due to the above observation, it suffices to prove $\tilde{g}^{-}_a \tilde{f}^{(0)}_{ij} = \boldsymbol{v}^{\delta_{ai}-\delta_{aj}} \tilde{f}^{(0)}_{ij} \tilde{g}^{-}_a$. This follows by evaluating the coefficients of $z^0 w^1$ in the equality of the matrix coefficients $\langle v_i \otimes v_a | \cdots | v_j \otimes v_a \rangle$ of both sides of the equality $((\boldsymbol{v}z - \boldsymbol{v}^{-1}w)R_{\mathrm{trig}}(z/w))(T^-(z) \otimes 1)(1 \otimes T^-(w)) = (1 \otimes T^-(w))(T^-(z) \otimes 1)((\boldsymbol{v}z - \boldsymbol{v}^{-1}w)R_{\mathrm{trig}}(z/w))$. Note that the cases $a < j, a = j, j < a < i, a = i, a > i$ have to be treated separately.

(d) Due to the above observation, it suffices to prove $\tilde{g}^{+}_a \tilde{e}^{(-1)}_{ji} = \boldsymbol{v}^{\delta_{ai}-\delta_{aj}} \tilde{e}^{(-1)}_{ji} \tilde{g}^{+}_a$. This follows by evaluating the coefficients of $z^1 w^1$ in the equality of the matrix coefficients $\langle v_a \otimes v_j | \cdots | v_a \otimes v_i \rangle$ of both sides of the equality $((\boldsymbol{v}z - \boldsymbol{v}^{-1}w)R_{\mathrm{trig}}(z/w))(T^+(z) \otimes 1)(1 \otimes T^-(w)) = (1 \otimes T^-(w))(T^+(z) \otimes 1)((\boldsymbol{v}z - \boldsymbol{v}^{-1}w)R_{\mathrm{trig}}(z/w))$. Note that the cases $a < j, a = j, j < a < i, a = i, a > i$ have to be treated separately.

Let us emphasize that this case is less trivial than part (b), due to the fact that

$$[w^1]T^-_{ji}(w) = \tilde{g}^-_j \tilde{e}^{(-1)}_{ji} + \sum_{j'<j} \tilde{f}^{(0)}_{jj'} \tilde{g}^-_{j'} \tilde{e}^{(-1)}_{j'i}.$$

Hence, the proof proceeds by induction in j, while we also use part (c) from above.

(e) Due to the above observation, it suffices to prove $\tilde{g}^{-}_a \tilde{f}^{(1)}_{ij} = \boldsymbol{v}^{\delta_{ai}-\delta_{aj}} \tilde{f}^{(1)}_{ij} \tilde{g}^{-}_a$. This follows by evaluating the coefficients of $z^0 w^0$ in the equality of the matrix coefficients $\langle v_i \otimes v_a | \cdots | v_j \otimes v_a \rangle$ of both sides of the equality $((\boldsymbol{v}z - \boldsymbol{v}^{-1}w)R_{\mathrm{trig}}(z/w))(T^+(z) \otimes 1)(1 \otimes T^-(w)) = (1 \otimes T^-(w))(T^+(z) \otimes 1)((\boldsymbol{v}z - \boldsymbol{v}^{-1}w)R_{\mathrm{trig}}(z/w))$. Note that the cases $a < j, a = j, j < a < i, a = i, a > i$ have to be treated separately.

Analogously to part (d), this case is less trivial than part (c), due to the fact that

$$[z^{-1}]T^+_{ij}(z) = \tilde{f}^{(1)}_{ij}\tilde{g}^+_j + \sum_{j'<j} \tilde{f}^{(1)}_{ij'}\tilde{g}^+_{j'}\tilde{e}^{(0)}_{j'j}.$$

Hence, the proof proceeds by induction in j, while we also use part (b) from above. □

The following explicit formulas follow immediately from Proposition G.4.

Corollary G.6 *For any $1 \leq j < i \leq n$, we have:*

$$\tilde{e}^{(0)}_{ji} = (\boldsymbol{v} - \boldsymbol{v}^{-1})^{j-i+1}[\tilde{e}^{(0)}_{i-1,i}, [\tilde{e}^{(0)}_{i-2,i-1}, \cdots, [\tilde{e}^{(0)}_{j+1,j+2}, \tilde{e}^{(0)}_{j,j+1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}^{-1}} = (\boldsymbol{v} - \boldsymbol{v}^{-1})^{j-i+1}[[\cdots[\tilde{e}^{(0)}_{i-1,i}, \tilde{e}^{(0)}_{i-2,i-1}]_{\boldsymbol{v}^{-1}}, \cdots, \tilde{e}^{(0)}_{j+1,j+2}]_{\boldsymbol{v}^{-1}}, \tilde{e}^{(0)}_{j,j+1}]_{\boldsymbol{v}^{-1}}, \tag{G.8}$$

$$\tilde{f}^{(0)}_{ij} = (\boldsymbol{v}^{-1} - \boldsymbol{v})^{j-i+1}[\tilde{f}^{(0)}_{j+1,j}, [\tilde{f}^{(0)}_{j+2,j+1}, \cdots, [\tilde{f}^{(0)}_{i-1,i-2}, \tilde{f}^{(0)}_{i,i-1}]_{\boldsymbol{v}} \cdots]_{\boldsymbol{v}}]_{\boldsymbol{v}} = (\boldsymbol{v}^{-1} - \boldsymbol{v})^{j-i+1}[[\cdots[\tilde{f}^{(0)}_{j+1,j}, \tilde{f}^{(0)}_{j+2,j+1}]_{\boldsymbol{v}}, \cdots, \tilde{f}^{(0)}_{i-1,i-2}]_{\boldsymbol{v}}, \tilde{f}^{(0)}_{i,i-1}]_{\boldsymbol{v}}, \tag{G.9}$$

$$\tilde{e}^{(-1)}_{ji} = (\boldsymbol{v} - \boldsymbol{v}^{-1})^{j-i+1}[\tilde{e}^{(0)}_{i-1,i}, [\tilde{e}^{(0)}_{i-2,i-1}, \cdots, [\tilde{e}^{(0)}_{j+1,j+2}, \tilde{e}^{(-1)}_{j,j+1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}^{-1}} = (\boldsymbol{v} - \boldsymbol{v}^{-1})^{j-i+1}[[\cdots[\tilde{e}^{(0)}_{i-1,i}, \tilde{e}^{(0)}_{i-2,i-1}]_{\boldsymbol{v}^{-1}}, \cdots, \tilde{e}^{(0)}_{j+1,j+2}]_{\boldsymbol{v}^{-1}}, \tilde{e}^{(-1)}_{j,j+1}]_{\boldsymbol{v}^{-1}}, \tag{G.10}$$

$$\tilde{f}^{(1)}_{ij} = (\boldsymbol{v}^{-1} - \boldsymbol{v})^{j-i+1}[\tilde{f}^{(1)}_{j+1,j}, [\tilde{f}^{(0)}_{j+2,j+1}, \cdots, [\tilde{f}^{(0)}_{i-1,i-2}, \tilde{f}^{(0)}_{i,i-1}]_{\boldsymbol{v}} \cdots]_{\boldsymbol{v}}]_{\boldsymbol{v}} = (\boldsymbol{v}^{-1} - \boldsymbol{v})^{j-i+1}[[\cdots[\tilde{f}^{(1)}_{j+1,j}, \tilde{f}^{(0)}_{j+2,j+1}]_{\boldsymbol{v}}, \cdots, \tilde{f}^{(0)}_{i-1,i-2}]_{\boldsymbol{v}}, \tilde{f}^{(0)}_{i,i-1}]_{\boldsymbol{v}}. \tag{G.11}$$

Recall elements $E^{(0)}_{ji}, F^{(0)}_{ij}, E^{(-1)}_{ji}, F^{(1)}_{ij} \in U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ of (10.6). Combining Corollary G.6 with (G.5), we get the following result.

Corollary G.7

(a) We have

$$\Upsilon^{-1}(\tilde{e}^{(0)}_{ji}) = (\boldsymbol{v} - \boldsymbol{v}^{-1})E^{(0)}_{ji},\ \Upsilon^{-1}(\tilde{f}^{(0)}_{ij}) = -(\boldsymbol{v} - \boldsymbol{v}^{-1})F^{(0)}_{ij},$$
$$\Upsilon^{-1}(\tilde{e}^{(-1)}_{ji}) = -\boldsymbol{v}^{-j}(\boldsymbol{v} - \boldsymbol{v}^{-1})E^{(-1)}_{ji},\ \Upsilon^{-1}(\tilde{f}^{(1)}_{ij}) = \boldsymbol{v}^{j}(\boldsymbol{v} - \boldsymbol{v}^{-1})F^{(1)}_{ij}. \tag{G.12}$$

(b) The right equalities in each of the first four lines of (10.6) hold.

To derive the right equalities of the last two lines of (10.6), we introduce

$$A^+_{ji} := \sum_{s\geq 1} \sum_{j=j_1<\ldots<j_{s+1}=i} (-1)^{s-1} \tilde{e}^{(0)}_{j_1 j_2} \cdots \tilde{e}^{(0)}_{j_s j_{s+1}},$$
$$A^-_{ij} := \sum_{s\geq 1} \sum_{j=j_1<\ldots<j_{s+1}=i} (-1)^{s-1} \tilde{f}^{(0)}_{j_{s+1} j_s} \cdots \tilde{f}^{(0)}_{j_2 j_1} \tag{G.13}$$

for $1 \leq j < i \leq n$. These elements will play an important role in Sect. G(ii) below.

Lemma G.8 *For any $1 \leq j < i \leq n$, we have*

$$A^+_{ji} = (\boldsymbol{v} - \boldsymbol{v}^{-1})^{j-i+1} [\tilde{e}^{(0)}_{i-1,i}, [\tilde{e}^{(0)}_{i-2,i-1}, \cdots, [\tilde{e}^{(0)}_{j+1,j+2}, \tilde{e}^{(0)}_{j,j+1}]_{\boldsymbol{v}} \cdots]_{\boldsymbol{v}}]_{\boldsymbol{v}} = (\boldsymbol{v} - \boldsymbol{v}^{-1})^{j-i+1} [[\cdots[\tilde{e}^{(0)}_{i-1,i}, \tilde{e}^{(0)}_{i-2,i-1}]_{\boldsymbol{v}}, \cdots, \tilde{e}^{(0)}_{j+1,j+2}]_{\boldsymbol{v}}, \tilde{e}^{(0)}_{j,j+1}]_{\boldsymbol{v}}, \tag{G.14}$$

$$A^-_{ij} = (\boldsymbol{v}^{-1} - \boldsymbol{v})^{j-i+1} [\tilde{f}^{(0)}_{j+1,j}, [\tilde{f}^{(0)}_{j+2,j+1}, \cdots, [\tilde{f}^{(0)}_{i-1,i-2}, \tilde{f}^{(0)}_{i,i-1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}^{-1}} = (\boldsymbol{v}^{-1} - \boldsymbol{v})^{j-i+1} [[\cdots[\tilde{f}^{(0)}_{j+1,j}, \tilde{f}^{(0)}_{j+2,j+1}]_{\boldsymbol{v}^{-1}}, \cdots, \tilde{f}^{(0)}_{i-1,i-2}]_{\boldsymbol{v}^{-1}}, \tilde{f}^{(0)}_{i,i-1}]_{\boldsymbol{v}^{-1}}. \tag{G.15}$$

Proof We prove (G.14) by induction in $i - j$. The result is obvious for $i - j = 1$. To perform the induction step, note that $A^+_{ji} = \tilde{e}^{(0)}_{ji} - \sum_{j<k<i} \tilde{e}^{(0)}_{jk} \cdot A^+_{ki}$. Applying the first equality of (G.8) together with the induction assumption, we get

$$(\boldsymbol{v} - \boldsymbol{v}^{-1})^{i-j-1} A^+_{ji} = [\tilde{e}^{(0)}_{i-1,i}, [\tilde{e}^{(0)}_{i-2,i-1}, \cdots, [\tilde{e}^{(0)}_{j+1,j+2}, \tilde{e}^{(0)}_{j,j+1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}^{-1}} - (\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{j<k<i} [\tilde{e}^{(0)}_{k-1,k}, \cdots, [\tilde{e}^{(0)}_{j+1,j+2}, \tilde{e}^{(0)}_{j,j+1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}} \cdot [\tilde{e}^{(0)}_{i-1,i}, \cdots, [\tilde{e}^{(0)}_{k+1,k+2}, \tilde{e}^{(0)}_{k,k+1}]_{\boldsymbol{v}} \cdots]_{\boldsymbol{v}}.$$

Rewriting $[\tilde{e}^{(0)}_{i-1,i}, X]_{\boldsymbol{v}^{\pm 1}}$ as $\tilde{e}^{(0)}_{i-1,i} \cdot X - \boldsymbol{v}^{\pm 1} X \cdot \tilde{e}^{(0)}_{i-1,i}$ and using the equality $[\tilde{e}^{(0)}_{i-1,i}, \tilde{e}^{(0)}_{l,l+1}] = 0$ for any $l < i - 2$ (due to the quadratic Serre relations in $U^{\mathrm{ad}}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$), we immediately find

$$(\boldsymbol{v} - \boldsymbol{v}^{-1})^{i-j-1} A^+_{ji} = \Bigg[\tilde{e}^{(0)}_{i-1,i}, [\tilde{e}^{(0)}_{i-2,i-1}, \cdots, [\tilde{e}^{(0)}_{j+1,j+2}, \tilde{e}^{(0)}_{j,j+1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}} - (\boldsymbol{v} - \boldsymbol{v}^{-1}) \cdot \sum_{j<k<i-1} [\tilde{e}^{(0)}_{k-1,k}, \cdots, [\tilde{e}^{(0)}_{j+1,j+2}, \tilde{e}^{(0)}_{j,j+1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}} \cdot [\tilde{e}^{(0)}_{i-2,i-1}, \cdots, [\tilde{e}^{(0)}_{k+1,k+2}, \tilde{e}^{(0)}_{k,k+1}]_{\boldsymbol{v}} \cdots]_{\boldsymbol{v}} \Bigg]_{\boldsymbol{v}} =$$
$$[\tilde{e}^{(0)}_{i-1,i}, (\boldsymbol{v} - \boldsymbol{v}^{-1})^{i-j-2} A^+_{j,i-1}]_{\boldsymbol{v}} = [\tilde{e}^{(0)}_{i-1,i}, [\tilde{e}^{(0)}_{i-2,i-1}, \cdots, [\tilde{e}^{(0)}_{j+1,j+2}, \tilde{e}^{(0)}_{j,j+1}]_{\boldsymbol{v}} \cdots]_{\boldsymbol{v}}]_{\boldsymbol{v}}.$$

Note that the last equality follows from the induction assumption applied to $A^+_{j,i-1}$.

To prove that A^+_{ji} also equals the rightmost commutator of (G.14), we apply similar arguments to the equality $A^+_{ji} = \tilde{e}^{(0)}_{ji} - \sum_{j<k<i} A^+_{jk} \cdot \tilde{e}^{(0)}_{ki}$. We evaluate the

right-hand side by applying the rightmost expression of (G.8) to the terms $\tilde{e}^{(0)}_{ji}, \tilde{e}^{(0)}_{ki}$ and the induction assumption to A^{+}_{jk}. Rewriting $[X, \tilde{e}^{(0)}_{j,j+1}]_{\boldsymbol{v}^{\pm1}}$ as $X \cdot \tilde{e}^{(0)}_{j,j+1} - \boldsymbol{v}^{\pm1}\tilde{e}^{(0)}_{j,j+1} \cdot X$ and taking $\tilde{e}^{(0)}_{j,j+1}$ to the leftmost or the rightmost sides, we get the result.

The proof of (G.15) is completely analogous and is left to the interested reader. □

The following result follows by combining Lemma G.8 with formula (G.5).

Corollary G.9

(a) We have

$$\Upsilon^{-1}(A^{+}_{ji}) = (\boldsymbol{v} - \boldsymbol{v}^{-1})\widetilde{E}^{(0)}_{ji}, \quad \Upsilon^{-1}(A^{-}_{ij}) = -(\boldsymbol{v} - \boldsymbol{v}^{-1})\widetilde{F}^{(0)}_{ij}. \tag{G.16}$$

(b) The right equalities in the last two lines of (10.6) hold.

G(ii) Computation of $\Delta^{\mathrm{rtt}}(\tilde{g}_i^{(\pm1)})$

Given a Laurent series $F(z)$, we use $[z^r]F(z)$ to denote the coefficient of z^r in $F(z)$. In this subsection, we compute explicitly $\Delta^{\mathrm{rtt}}(\tilde{g}_i^{(\pm1)})$, see Theorems G.10 and G.13.

Theorem G.10 *For $1 \leq i \leq n$, we have*

$$\begin{aligned}
&\Delta^{\mathrm{rtt}}(\tilde{g}_i^{(1)}) = \tilde{g}_i^{(1)} \otimes \tilde{g}_i^{+} + \tilde{g}_i^{+} \otimes \tilde{g}_i^{(1)} + \sum_{l>i} \tilde{g}_i^{+}\tilde{e}^{(0)}_{il} \otimes \tilde{f}^{(1)}_{li}\tilde{g}_i^{+} + \\
&\sum_{s\geq 1}\sum_{j_1<\ldots<j_{s+1}=i} (-1)^s \tilde{g}_i^{+}\tilde{e}^{(0)}_{j_1j_2}\cdots\tilde{e}^{(0)}_{j_sj_{s+1}} \otimes \tilde{f}^{(1)}_{ij_1}\tilde{g}_i^{+} + \\
&\sum_{l>i}\sum_{s\geq 1}\sum_{j_1<\ldots<j_{s+1}=i} (-1)^s \tilde{g}_i^{+}\tilde{e}^{(0)}_{il}\tilde{e}^{(0)}_{j_1j_2}\cdots\tilde{e}^{(0)}_{j_sj_{s+1}} \otimes \tilde{f}^{(1)}_{lj_1}\tilde{g}_i^{+}.
\end{aligned} \tag{G.17}$$

Proof Our starting point is the equality

$$[z^{-1}]T^{+}_{ii}(z) = \tilde{g}_i^{(1)} + \sum_{j<i} \tilde{f}^{(1)}_{ij}\tilde{g}_j^{+}\tilde{e}^{(0)}_{ji}. \tag{G.18}$$

We also note that $[z^{-1}]T^{+}_{ij}(z) = \tilde{f}^{(1)}_{ij}\tilde{g}_j^{+} + \sum_{j'<j} \tilde{f}^{(1)}_{ij'}\tilde{g}_{j'}^{+}\tilde{e}^{(0)}_{j'i}$ for any $i > j$. Rewriting this as $\tilde{f}^{(1)}_{ij}\tilde{g}_j^{+} = [z^{-1}]T^{+}_{ij}(z) - \sum_{j'<j} \tilde{f}^{(1)}_{ij'}\tilde{g}_{j'}^{+}\tilde{e}^{(0)}_{j'i}$ and applying this formula iteratively, we finally get

$$\tilde{f}^{(1)}_{ij}\tilde{g}_j^{+} = \sum_{s\geq 1}\sum_{j_1<\ldots<j_s=j} (-1)^{s-1}\left([z^{-1}]T^{+}_{ij_1}(z)\right)\tilde{e}^{(0)}_{j_1j_2}\cdots\tilde{e}^{(0)}_{j_{s-1}j_s}. \tag{G.19}$$

Combining formulas (G.18) and (G.19), we get

$$\tilde{g}_i^{(1)} = [z^{-1}]T_{ii}^+(z) - \sum_{j<i} \Big([z^{-1}]T_{ij}^+(z)\Big) \cdot A_{ji}^+, \tag{G.20}$$

where A_{ji}^+ was defined in (G.13).

Thus, it remains to compute explicitly $\Delta^{\mathrm{rtt}}([z^{-1}]T_{ii}^+(z))$, $\Delta^{\mathrm{rtt}}([z^{-1}]T_{ij}^+(z))$, $\Delta^{\mathrm{rtt}}(A_{ji}^+)$ for $i > j$. Evaluating the coefficients of z^{-1} in $\Delta^{\mathrm{rtt}}(T_{ii}^+(z)) = \sum_{a=1}^n T_{ia}^+(z) \otimes T_{ai}^+(z)$, we find

$$\begin{gathered}\Delta^{\mathrm{rtt}}([z^{-1}]T_{ii}^+(z)) = \sum_{j<i} \tilde{f}_{ij}^{(1)}\tilde{g}_j^+ \otimes \tilde{g}_j^+\tilde{e}_{ji}^{(0)} + \sum_{j'<j<i} \tilde{f}_{ij'}^{(1)}\tilde{g}_{j'}^+\tilde{e}_{j'j}^{(0)} \otimes \tilde{g}_j^+\tilde{e}_{ji}^{(0)} + \\ \tilde{g}_i^{(1)} \otimes \tilde{g}_i^+ + \tilde{g}_i^+ \otimes \tilde{g}_i^{(1)} + \sum_{j<i} \tilde{f}_{ij}^{(1)}\tilde{g}_j^+\tilde{e}_{ji}^{(0)} \otimes \tilde{g}_i^+ + \sum_{j<i} \tilde{g}_i^+ \otimes \tilde{f}_{ij}^{(1)}\tilde{g}_j^+\tilde{e}_{ji}^{(0)} + \\ \sum_{l>i} \tilde{g}_i^+\tilde{e}_{il}^{(0)} \otimes \tilde{f}_{li}^{(1)}\tilde{g}_i^+ + \sum_{l>i}^{j<i} \tilde{g}_i^+\tilde{e}_{il}^{(0)} \otimes \tilde{f}_{lj}^{(1)}\tilde{g}_j^+\tilde{e}_{ji}^{(0)},\end{gathered} \tag{G.21}$$

where the first, second, and third lines in the right-hand side correspond to the contributions arising from the cases $a < i$, $a = i$, and $a > i$, respectively.

Evaluating the coefficients of z^{-1} in $\Delta^{\mathrm{rtt}}(T_{ij}^+(z)) = \sum_{a=1}^n T_{ia}^+(z) \otimes T_{aj}^+(z)$, we find

$$\begin{gathered}\Delta^{\mathrm{rtt}}([z^{-1}]T_{ij}^+(z)) = \sum_{j'<j} \tilde{f}_{ij'}^{(1)}\tilde{g}_{j'}^+ \otimes \tilde{g}_{j'}^+\tilde{e}_{j'j}^{(0)} + \sum_{j''<j'<j} \tilde{f}_{ij''}^{(1)}\tilde{g}_{j''}^+\tilde{e}_{j''j'}^{(0)} \otimes \tilde{g}_{j'}^+\tilde{e}_{j'j}^{(0)} + \\ \tilde{f}_{ij}^{(1)}\tilde{g}_j^+ \otimes \tilde{g}_j^+ + \sum_{j'<j} \tilde{f}_{ij'}^{(1)}\tilde{g}_{j'}^+\tilde{e}_{j'j}^{(0)} \otimes \tilde{g}_j^+ + \tilde{g}_i^+ \otimes \tilde{f}_{ij}^{(1)}\tilde{g}_j^+ + \sum_{j'<j} \tilde{g}_i^+ \otimes \tilde{f}_{ij'}^{(1)}\tilde{g}_{j'}^+\tilde{e}_{j'j}^{(0)} + \\ \sum_{l>i} \tilde{g}_i^+\tilde{e}_{il}^{(0)} \otimes \tilde{f}_{lj}^{(1)}\tilde{g}_j^+ + \sum_{l>i}^{j'<j} \tilde{g}_i^+\tilde{e}_{il}^{(0)} \otimes \tilde{f}_{lj'}^{(1)}\tilde{g}_{j'}^+\tilde{e}_{j'j}^{(0)},\end{gathered} \tag{G.22}$$

where the first, second, and third lines in the right-hand side correspond to the contributions arising from $a < j$, $a = j$ or i, and $a > i$, respectively. Note that for $j < a < i$ both $T_{ia}^+(z)$, $T_{aj}^+(z)$ contain only negative powers of z and hence do not contribute above.

Finally, let us compute the coproduct of A_{ji}^+.

Lemma G.11 *We have*

$$\Delta^{\mathrm{rtt}}(A_{ji}^+) = \sum_{s\geq 1} \sum_{j=j_1<\ldots<j_{s+1}=i} \sum_{r=1}^{s+1} (-1)^{s-1}\tilde{e}_{j_r j_{r+1}}^{(0)} \cdots \tilde{e}_{j_s j_{s+1}}^{(0)} \otimes \tilde{e}_{j_1 j_2}^{(0)} \cdots \tilde{e}_{j_{r-1} j_r}^{(0)} (\tilde{g}_{j_r}^+)^{-1}\tilde{g}_i^+.$$

Proof We prove this by induction in $i - j$. The base of induction $i = j + 1$ follows from the equality $A^+_{j,j+1} = \tilde{e}^{(0)}_{j,j+1}$ and Lemma G.12 below. To perform the induction step, note that

$$A^+_{ji} = \tilde{e}^{(0)}_{ji} - \sum_{j<j'<i} \tilde{e}^{(0)}_{jj'} A^+_{j'i}. \tag{G.23}$$

Next, we compute the coproduct of $\tilde{e}^{(0)}_{ji}$.

Lemma G.12 *We have*

$$\Delta^{\mathrm{rtt}}(\tilde{e}^{(0)}_{ji}) = 1 \otimes \tilde{e}^{(0)}_{ji} + \tilde{e}^{(0)}_{ji} \otimes (\tilde{g}^+_j)^{-1}\tilde{g}^+_i + \sum_{j<a<i} \tilde{e}^{(0)}_{ja} \otimes (\tilde{g}^+_j)^{-1}\tilde{g}^+_a \tilde{e}^{(0)}_{ai}.$$

Proof First, let us note that $\tilde{g}^+_j = [z^0]T^+_{jj}(z)$. Thus,

$$\Delta^{\mathrm{rtt}}(\tilde{g}^+_j) = [z^0]\left(\sum_{a=1}^{n} T^+_{ja}(z) \otimes T^+_{aj}(z)\right) = [z^0](T^+_{jj}(z) \otimes T^+_{jj}(z)) = \tilde{g}^+_j \otimes \tilde{g}^+_j.$$

We also note that $[z^0]T^+_{ji}(z) = \tilde{g}^+_j \tilde{e}^{(0)}_{ji}$. Hence, we have

$$\Delta^{\mathrm{rtt}}(\tilde{g}^+_j \tilde{e}^{(0)}_{ji}) = [z^0]\left(T^+_{jj}(z) \otimes T^+_{ji}(z) + T^+_{ji}(z) \otimes T^+_{ii}(z) + \sum_{j<a<i} T^+_{ja}(z) \otimes T^+_{ai}(z)\right) =$$
$$\tilde{g}^+_j \otimes \tilde{g}^+_j \tilde{e}^{(0)}_{ji} + \tilde{g}^+_j \tilde{e}^{(0)}_{ji} \otimes \tilde{g}^+_i + \sum_{j<a<i} \tilde{g}^+_j \tilde{e}^{(0)}_{ja} \otimes \tilde{g}^+_a \tilde{e}^{(0)}_{ai}.$$

Note that in the first equality we used $[z^0](T^+_{ja}(z) \otimes T^+_{ai}(z)) = 0$ for $a < j$ or $a > i$.

Evaluating $\Delta^{\mathrm{rtt}}(\tilde{e}^{(0)}_{ji}) = \Delta^{\mathrm{rtt}}(\tilde{g}^+_j)^{-1} \Delta^{\mathrm{rtt}}(\tilde{g}^+_j \tilde{e}^{(0)}_{ji})$ via these formulas completes our proof. □

Combining (G.23) with Lemma G.12 and applying the induction assumption to $\Delta^{\mathrm{rtt}}(A^+_{j'i})$, we immediately get the formula for $\Delta^{\mathrm{rtt}}(A^+_{ji})$ of Lemma G.11. □

Combining (G.20–G.22) with Lemma G.11, we get (G.17) after tedious computations. □

Theorem G.13 *For* $1 \le i \le n$*, we have*

$$\begin{aligned}
\Delta^{\mathrm{rtt}}(\tilde{g}^{(-1)}_i) = \tilde{g}^{(-1)}_i \otimes \tilde{g}^-_i + \tilde{g}^-_i \otimes \tilde{g}^{(-1)}_i + \sum_{l>i} \tilde{g}^-_i \tilde{e}^{(-1)}_{il} \otimes \tilde{f}^{(0)}_{li} \tilde{g}^-_i + \\
\sum_{s\ge 1} \sum_{j_1<\ldots<j_{s+1}=i} (-1)^s \tilde{g}^-_i \tilde{e}^{(-1)}_{ji} \otimes \tilde{f}^{(0)}_{j_{s+1}j_s} \cdots \tilde{f}^{(0)}_{j_2 j_1} \tilde{g}^-_i + \\
\sum_{l>i} \sum_{s\ge 1} \sum_{j_1<\ldots<j_{s+1}=i} (-1)^s \tilde{g}^-_i \tilde{e}^{(-1)}_{jl} \otimes \tilde{f}^{(0)}_{j_{s+1}j_s} \cdots \tilde{f}^{(0)}_{j_2 j_1} \tilde{f}^{(0)}_{li} \tilde{g}^-_i.
\end{aligned} \tag{G.24}$$

Proof Our starting point is the equality

$$[z]T^-_{ii}(z) = \tilde{g}_i^{(-1)} + \sum_{j<i} \tilde{f}^{(0)}_{ij}\tilde{g}^-_j\tilde{e}^{(-1)}_{ji}. \tag{G.25}$$

We also note that $[z]T^-_{ji}(z) = \tilde{g}^-_j\tilde{e}^{(-1)}_{ji} + \sum_{j'<j}\tilde{f}^{(0)}_{jj'}\tilde{g}^-_{j'}\tilde{e}^{(-1)}_{j'i}$ for any $i > j$. Rewriting this as $\tilde{g}^-_j\tilde{e}^{(-1)}_{ji} = [z]T^-_{ji}(z) - \sum_{j'<j}\tilde{f}^{(0)}_{jj'}\tilde{g}^-_{j'}\tilde{e}^{(-1)}_{j'i}$ and applying this formula iteratively, we finally get

$$\tilde{g}^-_j\tilde{e}^{(-1)}_{ji} = \sum_{s\geq 1}\sum_{j_1<\ldots<j_s=j}(-1)^{s-1}\tilde{f}^{(0)}_{j_sj_{s-1}}\cdots\tilde{f}^{(0)}_{j_2j_1}\cdot\Big([z]T^-_{j_1i}(z)\Big). \tag{G.26}$$

Combining formulas (G.25) and (G.26), we get

$$\tilde{g}_i^{(-1)} = [z]T^-_{ii}(z) - \sum_{j<i}A^-_{ij}\cdot\Big([z]T^-_{ji}(z)\Big), \tag{G.27}$$

where A^-_{ij} was defined in (G.13).

Thus, it remains to compute explicitly $\Delta^{\mathrm{rtt}}([z]T^-_{ii}(z))$, $\Delta^{\mathrm{rtt}}([z]T^-_{ji}(z))$, $\Delta^{\mathrm{rtt}}(A^-_{ij})$ for $i > j$. Evaluating the coefficients of z^1 in $\Delta^{\mathrm{rtt}}(T^-_{ii}(z)) = \sum_{a=1}^n T^-_{ia}(z)\otimes T^-_{ai}(z)$, we find

$$\begin{aligned}
\Delta^{\mathrm{rtt}}([z]T^-_{ii}(z)) &= \sum_{j<i}\tilde{f}^{(0)}_{ij}\tilde{g}^-_j\otimes\tilde{g}^-_j\tilde{e}^{(-1)}_{ji} + \sum_{j'<j<i}\tilde{f}^{(0)}_{ij}\tilde{g}^-_j\otimes\tilde{f}^{(0)}_{jj'}\tilde{g}^-_{j'}\tilde{e}^{(-1)}_{j'i} + \\
&\tilde{g}^-_i\otimes\tilde{g}_i^{(-1)} + \tilde{g}_i^{(-1)}\otimes\tilde{g}^-_i + \sum_{j<i}\tilde{g}^-_i\otimes\tilde{f}^{(0)}_{ij}\tilde{g}^-_j\tilde{e}^{(-1)}_{ji} + \sum_{j<i}\tilde{f}^{(0)}_{ij}\tilde{g}^-_j\tilde{e}^{(-1)}_{ji}\otimes\tilde{g}^-_i + \\
&\sum_{l>i}\tilde{g}^-_i\tilde{e}^{(-1)}_{il}\otimes\tilde{f}^{(0)}_{li}\tilde{g}^-_i + \sum_{l>i}^{j<i}\tilde{f}^{(0)}_{ij}\tilde{g}^-_j\tilde{e}^{(-1)}_{jl}\otimes\tilde{f}^{(0)}_{li}\tilde{g}^-_i,
\end{aligned} \tag{G.28}$$

where the first, second, and third lines in the right-hand side correspond to the contributions arising from the cases $a < i$, $a = i$, and $a > i$, respectively.

Evaluating the coefficients of z^1 in $\Delta^{\mathrm{rtt}}(T^-_{ji}(z)) = \sum_{a=1}^n T^-_{ja}(z)\otimes T^-_{ai}(z)$, we find

$$\begin{aligned}
\Delta^{\mathrm{rtt}}([z]T^-_{ji}(z)) &= \sum_{j'<j}\tilde{f}^{(0)}_{jj'}\tilde{g}^-_{j'}\otimes\tilde{g}^-_{j'}\tilde{e}^{(-1)}_{j'i} + \sum_{j''<j'<j}\tilde{f}^{(0)}_{jj'}\tilde{g}^-_{j'}\otimes\tilde{f}^{(0)}_{j'j''}\tilde{g}^-_{j''}\tilde{e}^{(-1)}_{j''i} + \\
&\tilde{g}^-_j\otimes\tilde{g}^-_j\tilde{e}^{(-1)}_{ji} + \sum_{j'<j}\tilde{g}^-_j\otimes\tilde{f}^{(0)}_{jj'}\tilde{g}^-_{j'}\tilde{e}^{(-1)}_{j'i} + \tilde{g}^-_j\tilde{e}^{(-1)}_{ji}\otimes\tilde{g}^-_i + \sum_{j'<j}\tilde{f}^{(0)}_{jj'}\tilde{g}^-_{j'}\tilde{e}^{(-1)}_{j'i}\otimes\tilde{g}^-_i + \\
&\sum_{l>i}\tilde{g}^-_j\tilde{e}^{(-1)}_{jl}\otimes\tilde{f}^{(0)}_{li}\tilde{g}^-_i + \sum_{l>i}^{j'<j}\tilde{f}^{(0)}_{jj'}\tilde{g}^-_{j'}\tilde{e}^{(-1)}_{j'l}\otimes\tilde{f}^{(0)}_{li}\tilde{g}^-_i,
\end{aligned} \tag{G.29}$$

where the first, second, and third lines in the right-hand side correspond to the contributions arising from $a < j$, $a = j$ or i, and $a > i$, respectively. Note that for $j < a < i$ both $T^-_{ja}(z)$, $T^-_{ai}(z)$ contain only positive powers of z and hence do not contribute above.

Finally, let us compute the coproduct of A^-_{ij}.

Lemma G.14 *We have*

$$\Delta^{\mathrm{rtt}}(A^-_{ij}) = \sum_{s\geq 1} \sum_{j=j_1<\ldots<j_{s+1}=i} \sum_{r=1}^{s+1} (-1)^{s-1} \tilde{g}^-_i (\tilde{g}^-_{j_r})^{-1} \tilde{f}^{(0)}_{j_r j_{r-1}} \cdots \tilde{f}^{(0)}_{j_2 j_1} \otimes \tilde{f}^{(0)}_{j_{s+1} j_s} \cdots \tilde{f}^{(0)}_{j_{r+1} j_r}.$$

Proof We prove this by induction in $i - j$. The base of induction $i = j + 1$ follows from the equality $A^-_{j+1,j} = \tilde{f}^{(0)}_{j+1,j}$ and Lemma G.15 below. To perform the induction step, note that

$$A^-_{ij} = \tilde{f}^{(0)}_{ij} - \sum_{j<j'<i} A^-_{ij'} \tilde{f}^{(0)}_{j'j}. \tag{G.30}$$

Next, we compute the coproduct of $\tilde{f}^{(0)}_{ij}$.

Lemma G.15 *We have*

$$\Delta^{\mathrm{rtt}}(\tilde{f}^{(0)}_{ij}) = \tilde{f}^{(0)}_{ij} \otimes 1 + \tilde{g}^-_i (\tilde{g}^-_j)^{-1} \otimes \tilde{f}^{(0)}_{ij} + \sum_{j<a<i} \tilde{f}^{(0)}_{ia} \tilde{g}^-_a (\tilde{g}^-_j)^{-1} \otimes \tilde{f}^{(0)}_{aj}.$$

Proof First, let us note that $\tilde{g}^-_j = [z^0] T^-_{jj}(z)$. Thus,

$$\Delta^{\mathrm{rtt}}(\tilde{g}^-_j) = [z^0] \left(\sum_{a=1}^{n} T^-_{ja}(z) \otimes T^-_{aj}(z) \right) = [z^0](T^-_{jj}(z) \otimes T^-_{jj}(z)) = \tilde{g}^-_j \otimes \tilde{g}^-_j.$$

We also note that $[z^0] T^-_{ij}(z) = \tilde{f}^{(0)}_{ij} \tilde{g}^-_j$. Hence, we have

$$\Delta^{\mathrm{rtt}}(\tilde{f}^{(0)}_{ij} \tilde{g}^-_j) = [z^0] \left(T^-_{ij}(z) \otimes T^-_{jj}(z) + T^-_{ii}(z) \otimes T^-_{ij}(z) + \sum_{j<a<i} T^-_{ia}(z) \otimes T^-_{aj}(z) \right) =$$
$$\tilde{f}^{(0)}_{ij} \tilde{g}^-_j \otimes \tilde{g}^-_j + \tilde{g}^-_i \otimes \tilde{f}^{(0)}_{ij} \tilde{g}^-_j + \sum_{j<a<i} \tilde{f}^{(0)}_{ia} \tilde{g}^-_a \otimes \tilde{f}^{(0)}_{aj} \tilde{g}^-_j.$$

Note that in the first equality we used $[z^0](T^-_{ia}(z) \otimes T^-_{aj}(z)) = 0$ for $a < j$ or $a > i$.

Evaluating $\Delta^{\mathrm{rtt}}(\tilde{f}^{(0)}_{ij}) = \Delta^{\mathrm{rtt}}(\tilde{f}^{(0)}_{ij} \tilde{g}^-_j) \Delta^{\mathrm{rtt}}(\tilde{g}^-_j)^{-1}$ via these formulas completes our proof. □

Combining (G.30) with Lemma G.15 and applying the induction assumption to $\Delta^{\mathrm{rtt}}(A^-_{ij'})$, we immediately get the formula for $\Delta^{\mathrm{rtt}}(A^-_{ij})$ of Lemma G.14. □

Combining (G.27–G.29) with Lemma G.14, we get (G.24) after tedious computations. □

For $1 \le i \le n$, define $H_{i,\pm 1} \in U^{\mathrm{rtt}}(\mathfrak{sl}_n)$ via $H_{i,\pm 1} := (\tilde{g}_i^{\pm})^{-1}\tilde{g}_i^{(\pm 1)}$. Recall the elements A^+_{ji} and A^-_{ij} of (G.13). Combining Theorems G.10, G.13 with Lemma G.5 and the formula $\Delta^{\mathrm{rtt}}(\tilde{g}_i^{\pm}) = \tilde{g}_i^{\pm} \otimes \tilde{g}_i^{\pm}$, we get the following expressions for $\Delta^{\mathrm{rtt}}(H_{i,\pm 1})$.

Corollary G.16 *We have*

$$\Delta^{\mathrm{rtt}}(H_{i,1}) = H_{i,1}\otimes 1+1\otimes H_{i,1}+\boldsymbol{v}^{-1}\sum_{l>i}\tilde{e}_{il}^{(0)}\otimes\tilde{f}_{li}^{(1)}-\boldsymbol{v}\sum_{j<i}A^+_{ji}\otimes\tilde{f}_{ij}^{(1)}-\sum_{l>i}^{j<i}\tilde{e}_{il}^{(0)}A^+_{ji}\otimes\tilde{f}_{lj}^{(1)}, \tag{G.31}$$

$$\Delta^{\mathrm{rtt}}(H_{i,-1}) = H_{i,-1}\otimes 1+1\otimes H_{i,-1}+\boldsymbol{v}\sum_{l>i}\tilde{e}_{il}^{(-1)}\otimes\tilde{f}_{li}^{(0)}-\boldsymbol{v}^{-1}\sum_{j<i}\tilde{e}_{ji}^{(-1)}\otimes A^-_{ij}-\sum_{l>i}^{j<i}\tilde{e}_{jl}^{(-1)}\otimes A^-_{ij}\tilde{f}_{li}^{(0)}. \tag{G.32}$$

G(iii) Proof of Formula (10.2)

Recall the Hopf algebra embedding $\Upsilon\colon U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \hookrightarrow U^{\mathrm{rtt}}(\mathfrak{sl}_n)$ of Theorem G.2 (see also Remark G.3). It is easy to see that

$$\Upsilon(h_{i,1}) = \frac{H_{i+1,1} - H_{i,1}}{\boldsymbol{v}^i(\boldsymbol{v}-\boldsymbol{v}^{-1})}.$$

Combining Corollaries G.7, G.9 with formula (G.31) and the fact that Υ intertwines Δ and Δ^{rtt}, we immediately get

$$\begin{gathered}\Delta(h_{i,1}) - h_{i,1}\otimes 1 - 1\otimes h_{i,1} = \boldsymbol{v}^{-i}(\boldsymbol{v}-\boldsymbol{v}^{-1})^{-1}\times \\ \Bigg(\boldsymbol{v}^i(\boldsymbol{v}-\boldsymbol{v}^{-1})^2\sum_{l>i+1}E^{(0)}_{i+1,l}\otimes F^{(1)}_{l,i+1} - (\boldsymbol{v}-\boldsymbol{v}^{-1})^2\sum_{k<i+1}\boldsymbol{v}^{k+1}\widetilde{E}^{(0)}_{k,i+1}\otimes F^{(1)}_{i+1,k} - \\ (\boldsymbol{v}-\boldsymbol{v}^{-1})^3\sum_{k<i+1<l}\boldsymbol{v}^k E^{(0)}_{i+1,l}\widetilde{E}^{(0)}_{k,i+1}\otimes F^{(1)}_{lk} - \boldsymbol{v}^{i-1}(\boldsymbol{v}-\boldsymbol{v}^{-1})^2\sum_{l>i}E^{(0)}_{il}\otimes F^{(1)}_{li} + \\ (\boldsymbol{v}-\boldsymbol{v}^{-1})^2\sum_{k<i}\boldsymbol{v}^{k+1}\widetilde{E}^{(0)}_{ki}\otimes F^{(1)}_{ik} + (\boldsymbol{v}-\boldsymbol{v}^{-1})^3\sum_{k<i<l}\boldsymbol{v}^k E^{(0)}_{il}\widetilde{E}^{(0)}_{ki}\otimes F^{(1)}_{lk}\Bigg).\end{gathered} \tag{G.33}$$

This formula implies (10.2) after the following simplifications:

$$\sum_{k<i<l} \boldsymbol{v}^k E_{il}^{(0)} \widetilde{E}_{ki}^{(0)} \otimes F_{lk}^{(1)} - \sum_{k<i+1<l} \boldsymbol{v}^k E_{i+1,l}^{(0)} \widetilde{E}_{k,i+1}^{(0)} \otimes F_{lk}^{(1)} =$$

$$\sum_{l>i+1}^{k<i} \boldsymbol{v}^k (E_{il}^{(0)} \widetilde{E}_{ki}^{(0)} - E_{i+1,l}^{(0)} \widetilde{E}_{k,i+1}^{(0)}) \otimes F_{lk}^{(1)} + \sum_{k<i} \boldsymbol{v}^k E_{i,i+1}^{(0)} \widetilde{E}_{ki}^{(0)} \otimes F_{i+1,k}^{(1)} - \boldsymbol{v}^i \sum_{l>i+1} E_{i+1,l}^{(0)} \widetilde{E}_{i,i+1}^{(0)} \otimes F_{li}^{(1)},$$

$$- \boldsymbol{v}^{-1} \sum_{l>i} E_{il}^{(0)} \otimes F_{li}^{(1)} - (\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{l>i+1} E_{i+1,l}^{(0)} E_{i,i+1}^{(0)} \otimes F_{li}^{(1)} =$$

$$- \boldsymbol{v}^{-1} E_{i,i+1}^{(0)} \otimes F_{i+1,i}^{(1)} + \boldsymbol{v}^{-2} \sum_{l>i+1} [E_{i,i+1}^{(0)}, E_{i+1,l}^{(0)}]_{\boldsymbol{v}^3} \otimes F_{li}^{(1)},$$

$$- \sum_{k<i+1} \boldsymbol{v}^{k+1-i} \widetilde{E}_{k,i+1}^{(0)} \otimes F_{i+1,k}^{(1)} + (\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{k-i} E_{i,i+1}^{(0)} \widetilde{E}_{ki}^{(0)} \otimes F_{i+1,k}^{(1)} =$$

$$- \boldsymbol{v} E_{i,i+1}^{(0)} \otimes F_{i+1,i}^{(1)} - \sum_{k<i} \boldsymbol{v}^{k-i-1} [E_{i,i+1}^{(0)}, \widetilde{E}_{ki}^{(0)}]_{\boldsymbol{v}^3} \otimes F_{i+1,k}^{(1)},$$

where in the second and third equalities we used

$$E_{il}^{(0)} = [E_{i+1,l}^{(0)}, E_{i,i+1}^{(0)}]_{\boldsymbol{v}^{-1}}, \ \ \widetilde{E}_{k,i+1}^{(0)} = [E_{i,i+1}^{(0)}, \widetilde{E}_{ki}^{(0)}]_{\boldsymbol{v}}.$$

G(iv) Proof of Formula (10.3)

The proof of (10.3) is completely analogous and is based on the formula

$$\Upsilon(h_{i,-1}) = \frac{H_{i,-1} - H_{i+1,-1}}{\boldsymbol{v}^{-i}(\boldsymbol{v} - \boldsymbol{v}^{-1})}.$$

Combining this with Corollaries G.7, G.9, formula (G.32) and the fact that Υ intertwines Δ and Δ^{rtt}, one derives (10.3). The computations are similar to the above proof of (10.2) and are left to the interested reader.

G(v) Proof of Formula (10.4)

Recall that $[h_{i,-1}, e_{i,0}] = [2]_{\boldsymbol{v}} \cdot e_{i,-1}$, so that

$$\Delta(e_{i,-1}) = [2]_{\boldsymbol{v}}^{-1} \cdot [\Delta(h_{i,-1}), \Delta(e_{i,0})] = [2]_{\boldsymbol{v}}^{-1} \cdot [\Delta(h_{i,-1}), 1 \otimes e_{i,0} + e_{i,0} \otimes \psi_{i,0}^{+}].$$

Applying formula (10.3) to $\Delta(h_{i,-1})$ and using Lemma G.17 below, we recover (10.4).

Lemma G.17 *For $k < i$ and $l > i+1$, the following equalities hold:*

(a) $[F^{(0)}_{l,i+1}, e_{i,0}] = 0$.
(b) $[\widetilde{F}^{(0)}_{ik}, e_{i,0}] = 0$.
(c) $[F^{(0)}_{li}, e_{i,0}] = -F^{(0)}_{l,i+1}\psi^-_{i,0}$.
(d) $[\widetilde{F}^{(0)}_{i+1,k}, e_{i,0}] = \boldsymbol{v}^{-1}\widetilde{F}^{(0)}_{ik}\psi^-_{i,0}$.
(e) $[[F^{(0)}_{l,i+1}, F^{(0)}_{i+1,i}]_{\boldsymbol{v}^{-3}}, e_{i,0}] = \frac{1-\boldsymbol{v}^{-4}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}F^{(0)}_{l,i+1}\psi^-_{i,0} - \frac{1-\boldsymbol{v}^{-2}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}F^{(0)}_{l,i+1}\psi^+_{i,0}$.
(f) $[[\widetilde{F}^{(0)}_{ik}, F^{(0)}_{i+1,i}]_{\boldsymbol{v}^{-3}}, e_{i,0}] = \frac{1-\boldsymbol{v}^{-4}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\widetilde{F}^{(0)}_{ik}\psi^-_{i,0} - \frac{1-\boldsymbol{v}^{-2}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\widetilde{F}^{(0)}_{ik}\psi^+_{i,0}$.
(g) $[E^{(-1)}_{i+1,l}, e_{i,0}]_{\boldsymbol{v}} = \boldsymbol{v}E^{(-1)}_{il}$.
(h) $[E^{(-1)}_{ki}, e_{i,0}]_{\boldsymbol{v}} = -\boldsymbol{v}E^{(-1)}_{k,i+1}$.
(i) $[E^{(-1)}_{il}, e_{i,0}]_{\boldsymbol{v}^{-1}} = 0$.
(j) $[E^{(-1)}_{k,i+1}, e_{i,0}]_{\boldsymbol{v}^{-1}} = 0$.
(k) $[E^{(-1)}_{kl}, e_{i,0}] = 0$.

Proof Recall that $[f_{j,0}, e_{i,0}] = \frac{\delta_{ji}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}(\psi^-_{i,0} - \psi^+_{i,0})$.

Parts (a, b) are obvious as $e_{i,0}$ commutes with $f_{i+1,0},\ldots,f_{l-1,0}$ and $f_{k,0},\ldots,f_{i-1,0}$. Combining (a, b) with equalities $F^{(0)}_{li} = [f_{i,0}, F^{(0)}_{l,i+1}]_{\boldsymbol{v}}$ and $\widetilde{F}^{(0)}_{i+1,k} = [\widetilde{F}^{(0)}_{ik}, f_{i,0}]_{\boldsymbol{v}^{-1}}$, we get $[F^{(0)}_{li}, e_{i,0}] = [\frac{\psi^-_{i,0}-\psi^+_{i,0}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}, F^{(0)}_{l,i+1}]_{\boldsymbol{v}} = -F^{(0)}_{l,i+1}\psi^-_{i,0}$ and $[\widetilde{F}^{(0)}_{i+1,k}, e_{i,0}] = [\widetilde{F}^{(0)}_{ik}, \frac{\psi^-_{i,0}-\psi^+_{i,0}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}]_{\boldsymbol{v}^{-1}} = \boldsymbol{v}^{-1}\widetilde{F}^{(0)}_{ik}\psi^-_{i,0}$, which proves parts (c, d). Parts (e, f) also follow immediately from (a, b).

(g) Due to the quadratic Serre relations $e_{i,0}$ commutes with $e_{i+2,0},\ldots,e_{l-1,0}$, hence, also with $E^{(0)}_{i+2,l}$. Meanwhile, we have $[e_{i+1,-1}, e_{i,0}]_{\boldsymbol{v}} = \boldsymbol{v}[e_{i+1,0}, e_{i,-1}]_{\boldsymbol{v}^{-1}}$, due to (U2). Thus, $[E^{(-1)}_{i+1,l}, e_{i,0}]_{\boldsymbol{v}} = [[E^{(0)}_{i+2,l}, e_{i+1,-1}]_{\boldsymbol{v}^{-1}}, e_{i,0}]_{\boldsymbol{v}} = [E^{(0)}_{i+2,l}, \boldsymbol{v}[e_{i+1,0}, e_{i,-1}]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}^{-1}} = \boldsymbol{v}E^{(-1)}_{il}$.

(h) We have $[E^{(-1)}_{ki}, e_{i,0}]_{\boldsymbol{v}} = -\boldsymbol{v}[e_{i,0}, E^{(-1)}_{ki}]_{\boldsymbol{v}^{-1}} = -\boldsymbol{v}E^{(-1)}_{k,i+1}$.

(i) Note that $[[e_{i+1,0}, e_{i,-1}]_{\boldsymbol{v}^{-1}}, e_{i,0}]_{\boldsymbol{v}^{-1}} = \boldsymbol{v}^{-1}[[e_{i+1,-1}, e_{i,0}]_{\boldsymbol{v}}, e_{i,0}]_{\boldsymbol{v}^{-1}} = 0$, due to (U2) and (U7). Since also $e_{i,0}$ commutes with $e_{i+2,0},\ldots,e_{l-1,0}$, we get $[E^{(-1)}_{il}, e_{i,0}]_{\boldsymbol{v}^{-1}} = 0$.

(j) As in (i), $[E^{(-1)}_{k,i+1}, e_{i,0}]_{\boldsymbol{v}^{-1}} = 0$ follows from $[[e_{i,0}, e_{i-1,0}]_{\boldsymbol{v}^{-1}}, e_{i,0}]_{\boldsymbol{v}^{-1}} = 0$, due to (U7).

(k) Comparing the matrix coefficients $\langle v_i \otimes v_k|\cdots|v_{i+1}\otimes v_l\rangle$ of both sides of the equality $R_{\mathrm{trig}}(z/w)(T^+(z)\otimes 1)(1\otimes T^-(w)) = (1\otimes T^-(w))(T^+(z)\otimes 1)R_{\mathrm{trig}}(z/w)$, we get

$$(z-w)T^+_{i,i+1}(z)T^-_{kl}(w) + (\boldsymbol{v}-\boldsymbol{v}^{-1})wT^+_{k,i+1}(z)T^-_{il}(w) =$$
$$(z-w)T^-_{kl}(w)T^+_{i,i+1}(z) + (\boldsymbol{v}-\boldsymbol{v}^{-1})wT^-_{k,i+1}(w)T^+_{il}(z).$$

Evaluating the coefficients of $z^1 w^1$ in both sides of this equality, we find

$$[\tilde{g}_i^+ \tilde{e}_{i,i+1}^{(0)}, \tilde{g}_k^- \tilde{e}_{kl}^{(-1)} + \sum_{j<k} \tilde{f}_{kj}^{(0)} \tilde{g}_j^- \tilde{e}_{jl}^{(-1)}] = 0.$$

Hence, by induction in k, we find $[\tilde{e}_{i,i+1}^{(0)}, \tilde{e}_{kl}^{(-1)}] = 0$, which implies $[E_{kl}^{(-1)}, e_{i,0}] = 0$.

□

This completes our proof of (10.4).

G(vi) Proof of Formula (10.5)

Recall that $[h_{i,1}, f_{i,0}] = -[2]_{\boldsymbol{v}} \cdot f_{i,1}$, so that

$$\Delta(f_{i,1}) = -[2]_{\boldsymbol{v}}^{-1} \cdot [\Delta(h_{i,1}), \Delta(f_{i,0})] = -[2]_{\boldsymbol{v}}^{-1} \cdot [\Delta(h_{i,1}), f_{i,0} \otimes 1 + \psi_{i,0}^- \otimes f_{i,0}].$$

Applying formula (10.2) to $\Delta(h_{i,1})$ and using Lemma G.18 below, we recover (10.5).

Lemma G.18 *For $k < i$ and $l > i+1$, the following equalities hold:*

(a) $[E_{i+1,l}^{(0)}, f_{i,0}] = 0.$
(b) $[\widetilde{E}_{ki}^{(0)}, f_{i,0}] = 0.$
(c) $[E_{il}^{(0)}, f_{i,0}] = \boldsymbol{v}^{-1} E_{i+1,l}^{(0)} \psi_{i,0}^+.$
(d) $[\widetilde{E}_{k,i+1}^{(0)}, f_{i,0}] = -\widetilde{E}_{ki}^{(0)} \psi_{i,0}^+.$
(e) $[[E_{i,i+1}^{(0)}, E_{i+1,l}^{(0)}]_{\boldsymbol{v}^3}, f_{i,0}] = \frac{\boldsymbol{v}^{-1}-\boldsymbol{v}^3}{\boldsymbol{v}-\boldsymbol{v}^{-1}} E_{i+1,l}^{(0)} \psi_{i,0}^+ - \frac{\boldsymbol{v}-\boldsymbol{v}^3}{\boldsymbol{v}-\boldsymbol{v}^{-1}} E_{i+1,l}^{(0)} \psi_{i,0}^-.$
(f) $[[E_{i,i+1}^{(0)}, \widetilde{E}_{ki}^{(0)}]_{\boldsymbol{v}^3}, f_{i,0}] = \frac{\boldsymbol{v}^{-1}-\boldsymbol{v}^3}{\boldsymbol{v}-\boldsymbol{v}^{-1}} \widetilde{E}_{ki}^{(0)} \psi_{i,0}^+ - \frac{\boldsymbol{v}-\boldsymbol{v}^3}{\boldsymbol{v}-\boldsymbol{v}^{-1}} \widetilde{E}_{ki}^{(0)} \psi_{i,0}^-.$
(g) $[F_{l,i+1}^{(1)}, f_{i,0}]_{\boldsymbol{v}} = -F_{li}^{(1)}.$
(h) $[F_{ik}^{(1)}, f_{i,0}]_{\boldsymbol{v}} = F_{i+1,k}^{(1)}.$
(i) $[F_{li}^{(1)}, f_{i,0}]_{\boldsymbol{v}^{-1}} = 0.$
(j) $[F_{i+1,k}^{(1)}, f_{i,0}]_{\boldsymbol{v}^{-1}} = 0.$
(k) $[F_{lk}^{(1)}, f_{i,0}] = 0.$

This lemma is proved completely analogously to Lemma G.17. The details are left to the interested reader.

This completes our proof of Theorem 10.13.

Appendix H Proof of Theorem 10.16 and Homomorphisms $\jmath^{\pm}_{\mu_1,\mu_2}$

Our proof of Theorem 10.16 proceeds in three steps. First, we introduce subalgebras $\mathcal{U}^{\mathrm{sc},\pm}_{0,\mu_1,\mu_2}$ of $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1+\mu_2}$ and construct homomorphisms $\jmath^{\pm}_{\mu_1,\mu_2}$ which we referred to in Remark 10.17. Then, we prove Theorem 10.16, reducing some of the verifications to the case of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ via the aforementioned $\jmath^{\pm}_{\mu_1,\mu_2}$. Finally, we verify the commutativity of the diagram from Remark 10.17.

Throughout this section, we assume $\mu_1, \mu_2 \in \Lambda^-$.

H(i) Homomorphisms $\jmath^{\pm}_{\mu_1,\mu_2}$

First, we introduce subalgebras $\mathcal{U}^{\mathrm{sc},\pm}_{0,\mu_1,\mu_2}$ of $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1+\mu_2}$. To this end, recall the explicit identification of the Drinfeld-Jimbo and the new Drinfeld realizations of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ from Theorem 8.10:

$$E_i \mapsto e_{i,0},\ F_i \mapsto f_{i,0},\ K_i^{\pm 1} \mapsto (\psi^+_{i,0})^{\pm 1} = \psi^{\pm}_{i,0} = (\psi^-_{i,0})^{\mp 1} \text{ for } 1 \leq i \leq n-1,$$

$$(K_{i_0})^{\pm 1} \mapsto (\psi^+_{1,0}\cdots\psi^+_{n-1,0})^{\mp 1},$$

$$E_{i_0} \mapsto (-\boldsymbol{v})^{-n}\cdot(\psi^+_{1,0}\cdots\psi^+_{n-1,0})^{-1}\cdot[\cdots[f_{1,1}, f_{2,0}]_{\boldsymbol{v}}, \cdots, f_{n-1,0}]_{\boldsymbol{v}},$$

$$F_{i_0} \mapsto (-\boldsymbol{v})^{n}\cdot[e_{n-1,0}, \cdots, [e_{2,0}, e_{1,-1}]_{\boldsymbol{v}^{-1}}\cdots]_{\boldsymbol{v}^{-1}}\cdot\psi^+_{1,0}\cdots\psi^+_{n-1,0}.$$

Hence, the images $U^+_{\boldsymbol{v}}$ and $U^-_{\boldsymbol{v}}$ of the Drinfeld-Jimbo Borel subalgebras are the subalgebras of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated by $\{e_{i,0}, (\psi^+_{i,0})^{\pm 1}, F^{(1)}_{n1}\}_{i=1}^{n-1}$ and $\{f_{i,0}, (\psi^+_{i,0})^{\pm 1}, E^{(-1)}_{1n}\}_{i=1}^{n-1}$, respectively.

Likewise, let $\mathcal{U}^{\mathrm{sc},+}_{0,\mu_1,\mu_2}$ and $\mathcal{U}^{\mathrm{sc},-}_{0,\mu_1,\mu_2}$ be the $\mathbb{C}(\boldsymbol{v})$-subalgebras of $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1+\mu_2}$ generated by the elements $\{e_{i,0}, (\psi^+_{i,0})^{\pm 1}, F^{(1)}_{n1}\}_{i=1}^{n-1}$ and $\{f_{i,b_{1,i}}, (\psi^-_{i,b_{1,i}+b_{2,i}})^{\pm 1}, \hat{E}^{(-1)}_{1n}\}_{i=1}^{n-1}$, respectively, where as before $b_{1,i} = \alpha_i^\vee(\mu_1), b_{2,i} = \alpha_i^\vee(\mu_2), b_i = b_{1,i} + b_{2,i}$. Here, the elements $\{\hat{E}^{(-1)}_{ji}\}_{j<i}$ are defined via $\hat{E}^{(-1)}_{ji} := [e_{i-1,b_{2,i-1}}, [e_{i-2,b_{2,i-2}}, \cdots, [e_{j+1,b_{2,j+1}}, e_{j,b_{2,j}-1}]_{\boldsymbol{v}^{-1}}\cdots]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}^{-1}}$.

Proposition H.1

(a) *There is a unique $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism $\jmath^+_{\mu_1,\mu_2}\colon U^+_{\boldsymbol{v}} \to U^{\mathrm{sc},+}_{0,\mu_1,\mu_2}$, such that $e_{i,0} \mapsto e_{i,0}$, $(\psi^+_{i,0})^{\pm 1} \mapsto (\psi^+_{i,0})^{\pm 1}$, $F^{(1)}_{n1} \mapsto F^{(1)}_{n1}$.*

(b) *There is a unique $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism $\jmath^-_{\mu_1,\mu_2}\colon U^-_{\boldsymbol{v}} \to U^{\mathrm{sc},-}_{0,\mu_1,\mu_2}$, such that $f_{i,0} \mapsto f_{i,b_{1,i}}$, $(\psi^-_{i,0})^{\pm 1} \mapsto (\psi^-_{i,b_i})^{\pm 1}$, $E^{(-1)}_{1n} \mapsto \hat{E}^{(-1)}_{1n}$.*

Proof

(a) Converting the defining relations of the positive Drinfeld-Jimbo Borel subalgebra into the new Drinfeld realization, we see that $U_{\boldsymbol{v}}^{+}$ is generated by $\{e_{i,0}, (\psi_{i,0}^{+})^{\pm 1}, F_{n1}^{(1)}\}_{i=1}^{n-1}$ with the following defining relations:

$$(\psi_{i,0}^{+})^{\pm 1} \cdot (\psi_{i,0}^{+})^{\mp 1} = 1,\ \ \psi_{i,0}^{+}\psi_{j,0}^{+} = \psi_{j,0}^{+}\psi_{i,0}^{+}, \tag{H.1}$$

$$\psi_{i,0}^{+}e_{j,0} = \boldsymbol{v}^{c_{ij}}e_{j,0}\psi_{i,0}^{+},\ \ \psi_{i,0}^{+}F_{n1}^{(1)} = \boldsymbol{v}^{-\delta_{i1}-\delta_{i,n-1}}F_{n1}^{(1)}\psi_{i,0}^{+}, \tag{H.2}$$

$$[e_{i,0}, [e_{i,0}, e_{i\pm 1,0}]_{\boldsymbol{v}}]_{\boldsymbol{v}^{-1}} = 0,\ \ [e_{i,0}, e_{j,0}] = 0 \text{ if } c_{ij} = 0, \tag{H.3}$$

$$[e_{i,0}, F_{n1}^{(1)}] = 0 \text{ for } 1 < i < n-1, \tag{H.4}$$

$$[e_{1,0}, [e_{1,0}, F_{n1}^{(1)}]]_{\boldsymbol{v}^{-2}} = 0,\ \ [e_{n-1,0}, [e_{n-1,0}, F_{n1}^{(1)}]]_{\boldsymbol{v}^{-2}} = 0, \tag{H.5}$$

$$[F_{n1}^{(1)}, [F_{n1}^{(1)}, e_{1,0}]]_{\boldsymbol{v}^{2}} = 0,\ \ [F_{n1}^{(1)}, [F_{n1}^{(1)}, e_{n-1,0}]]_{\boldsymbol{v}^{2}} = 0. \tag{H.6}$$

Thus, it suffices to check that these relations are preserved under the specified assignment $e_{i,0} \mapsto e_{i,0}, (\psi_{i,0}^{+})^{\pm 1} \mapsto (\psi_{i,0}^{+})^{\pm 1}, F_{n1}^{(1)} \mapsto F_{n1}^{(1)}$. The validity of (H.1–H.4) is obvious.

To verify the first equality of (H.5), we note that $[\psi_{1,1}^{+}, f_{2,0}]_{\boldsymbol{v}} = (\boldsymbol{v}^2 - 1) f_{2,1}\psi_{1,0}^{+}$, due to (U5). Combining this with (U6), we get

$$[e_{1,0}, F_{n1}^{(1)}] = (\boldsymbol{v} - \boldsymbol{v}^{-1})^{-1} \cdot [\cdots[\psi_{1,1}^{+}, f_{2,0}]_{\boldsymbol{v}}, \cdots, f_{n-1,0}]_{\boldsymbol{v}} = \boldsymbol{v}F_{n2}^{(1)}\psi_{1,0}^{+}.$$

Hence, $[e_{1,0}, [e_{1,0}, F_{n1}^{(1)}]]_{\boldsymbol{v}^{-2}} = \boldsymbol{v}[e_{1,0}, F_{n2}^{(1)}\psi_{1,0}^{+}]_{\boldsymbol{v}^{-2}} = \boldsymbol{v}[e_{1,0}, F_{n2}^{(1)}]\psi_{1,0}^{+} = 0$, due to (U6).

The verification of the second equality of (H.5) is similar and is based on

$$[e_{n-1,0}, F_{n1}^{(1)}] = \frac{[[\cdots[f_{1,1}, f_{2,0}]_{\boldsymbol{v}}, \cdots, f_{n-2,0}]_{\boldsymbol{v}}, \psi_{n-1,0}^{+} - \delta_{b_{n-1},0}\psi_{n-1,0}^{-}]_{\boldsymbol{v}}}{\boldsymbol{v} - \boldsymbol{v}^{-1}} = -\boldsymbol{v}F_{n-1,1}^{(1)}\psi_{n-1,0}^{+}.$$

Due to the above equality $[e_{1,0}, F_{n1}^{(1)}] = \boldsymbol{v}F_{n2}^{(1)}\psi_{1,0}^{+}$ and (U4), the verification of the first equality of (H.6) boils down to the proof of $[F_{n1}^{(1)}, F_{n2}^{(1)}]_{\boldsymbol{v}} = 0$. This is an equality in $\mathcal{U}_{0,\mu_1+\mu_2}^{\mathrm{sc},<}$. However, $\mathcal{U}_{0,\mu_1+\mu_2}^{\mathrm{sc},<} \simeq U_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n)$, due to Proposition 5.1(b). Hence, it suffices to check this equality in $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. The latter follows immediately from the validity of (H.6) for $U_{\boldsymbol{v}}^{+}$.

Due to $[e_{n-1,0}, F_{n1}^{(1)}] = -\boldsymbol{v}F_{n-1,1}^{(1)}\psi_{n-1,0}^{+}$ from above and (U4), the verification of the second equality of (H.6) boils down to the proof of $[F_{n1}^{(1)}, F_{n-1,1}^{(1)}]_{\boldsymbol{v}} = 0$. Analogously to the previous verification, the latter follows from the same equality in $U_{\boldsymbol{v}}^{+}$.

(b) The proof of part (b) is completely analogous and is left to the interested reader. □

This completes our construction of the homomorphisms $\jmath^{\pm}_{\mu_1,\mu_2}\colon U^{\pm}_{\boldsymbol{v}} \to \mathcal{U}^{\mathrm{sc},\pm}_{0,\mu_1,\mu_2}$, which we referred to in Remark 10.17. The following results are needed for the next subsection.

Lemma H.2

(a) For any $1 \le j < i \le n$, we have $E^{(0)}_{ji}, \widetilde{E}^{(0)}_{ji}, F^{(1)}_{ij} \in \mathcal{U}^{\mathrm{sc},+}_{0,\mu_1,\mu_2}$.
(b) For any $1 \le j < i \le n$, define

$$\hat{F}^{\pm,(0)}_{ij} := [\cdots[f_{j,b_{1,j}}, f_{j+1,b_{1,j+1}}]_{\boldsymbol{v}^{\pm1}}, \cdots, f_{i-1,b_{1,i-1}}]_{\boldsymbol{v}^{\pm1}}.$$

We have $\hat{F}^{\pm,(0)}_{ij}, \hat{E}^{(-1)}_{ji} \in \mathcal{U}^{\mathrm{sc},-}_{0,\mu_1,\mu_2}$.

Proof

(a) Since $E^{(0)}_{ji}, \widetilde{E}^{(0)}_{ji}$ are expressed via $\boldsymbol{v}^{\pm1}$-commutators of $e_{k,0} \in \mathcal{U}^{\mathrm{sc},+}_{0,\mu_1,\mu_2}$, we obviously get the first two inclusions. The last inclusion is clear for $(i, j) = (n, 1)$. Applying iteratively $[e_{k,0}, F^{(1)}_{k+1,1}] = -\boldsymbol{v}F^{(1)}_{k1}\psi^+_{k,0}$, $[e_{l,0}, F^{(1)}_{il}] = \boldsymbol{v}F^{(1)}_{i,l+1}\psi^+_{l,0}$, we get $F^{(1)}_{ij} \in \mathcal{U}^{\mathrm{sc},+}_{0,\mu_1,\mu_2}$ for any $j < i$.

(b) The inclusions $\hat{F}^{\pm,(0)}_{ij} \in \mathcal{U}^{\mathrm{sc},-}_{0,\mu_1,\mu_2}$ are obvious. It remains to prove $\hat{E}^{(-1)}_{ji} \in \mathcal{U}^{\mathrm{sc},-}_{0,\mu_1,\mu_2}$. This is clear for $(j, i) = (1, n)$. To deduce the general case, it remains to apply the equalities $[f_{i-1,b_{1,i-1}}, \hat{E}^{(-1)}_{1i}] = \hat{E}^{(-1)}_{1,i-1}\psi^-_{i-1,b_{i-1}}$, $[f_{l,b_{1,l}}, \hat{E}^{(-1)}_{li}] = -\hat{E}^{(-1)}_{l+1,i}\psi^-_{l,b_l}$. □

The proof of the following result is straightforward.

Lemma H.3 *For any $1 \le j < i \le n$, we have:*

$$\jmath^{+}_{\mu_1,\mu_2}\colon E^{(0)}_{ji} \mapsto E^{(0)}_{ji}, \widetilde{E}^{(0)}_{ji} \mapsto \widetilde{E}^{(0)}_{ji}, F^{(1)}_{ij} \mapsto F^{(1)}_{ij}, f_{i,1} \mapsto f_{i,1}, h_{i,1} \mapsto h_{i,1},$$

$$\jmath^{-}_{\mu_1,\mu_2}\colon F^{(0)}_{ij} \mapsto \hat{F}^{+,(0)}_{ij}, \widetilde{F}^{(0)}_{ij} \mapsto \hat{F}^{-,(0)}_{ij}, E^{(-1)}_{ji} \mapsto \hat{E}^{(-1)}_{ji}, e_{i,-1} \mapsto e_{i,b_{2,i}-1}, h_{i,-1} \mapsto h_{i,-1}.$$

H(ii) Proof of Theorem 10.16

Due to Theorem 5.5, it suffices to check that the assignment Δ of Theorem 10.16 preserves defining relations (Û1–Û9). To simplify our exposition, we will assume that μ_1, μ_2 are strictly antidominant: $b_{1,i}, b_{2,i} < 0$ for any $1 \le i < n$. This verification is similar to the $n = 2$ case (carried out in Appendix D) and we only indicate the key technical details, see Lemmas H.4–H.15 (their proofs are similar to that of Lemma G.17 and therefore omitted). For $1 \le a \le b < n$, we define $\alpha^\vee_{[a,b]} := \alpha^\vee_a + \alpha^\vee_{a+1} + \ldots + \alpha^\vee_b$.

H(ii).a Compatibility with (Û1)

- The equalities $\Delta((\psi^+_{i,0})^{\pm 1})\Delta((\psi^+_{i,0})^{\mp 1}) = 1$ and $\Delta((\psi^-_{i,b_i})^{\pm 1})\Delta((\psi^-_{i,b_i})^{\mp 1}) = 1$ follow immediately from relation (Û1) for both $\mathcal{U}^{sc}_{0,\mu_1}, \mathcal{U}^{sc}_{0,\mu_2}$.
- The commutativity of $\{\Delta(\psi^+_{i,0}), \Delta(\psi^-_{i,b_i})\}_{i=1}^{n-1}$ between themselves and with $\{\Delta(h_{j,\pm 1})\}_{j=1}^{n-1}$ is due to relations (Û1, Û4, Û5) for both $\mathcal{U}^{sc}_{0,\mu_1}, \mathcal{U}^{sc}_{0,\mu_2}$.
- Finally, we verify $[\Delta(h_{i,r}), \Delta(h_{j,s})] = 0$ for $r, s \in \{\pm 1\}$. To this end, recall the homomorphism $\iota_{0,0,\mu_1} \otimes \iota_{0,\mu_2,0} \colon \mathcal{U}^{sc}_{0,0} \otimes \mathcal{U}^{sc}_{0,0} \to \mathcal{U}^{sc}_{0,\mu_1} \otimes \mathcal{U}^{sc}_{0,\mu_2}$. The key observation is that $\iota_{0,0,\mu_1} \otimes \iota_{0,\mu_2,0}(\Delta(h_{i,r})) = \Delta(h_{i,r}) + \frac{\alpha_i^\vee(\mu_1+\mu_2)}{\boldsymbol{v}^r - \boldsymbol{v}^{-r}}$ for any $i \in I, r \in \{\pm 1\}$ (cf. proof of Corollary 10.11), where by abuse of notation we use $\Delta(h_{i,r})$ to denote elements of both $\mathcal{U}^{sc}_{0,0} \otimes \mathcal{U}^{sc}_{0,0}$ and $\mathcal{U}^{sc}_{0,\mu_1} \otimes \mathcal{U}^{sc}_{0,\mu_2}$. Hence, it suffices to prove $[\Delta(h_{i,r}), \Delta(h_{j,s})] = 0$ in $\mathcal{U}^{sc}_{0,0} \otimes \mathcal{U}^{sc}_{0,0}$. The latter follows immediately from the corresponding result for $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, in which case the assignment Δ of Theorem 10.16 coincides with the Drinfeld-Jimbo coproduct, due to Theorem 10.13.

H(ii).b Compatibility with (Û2)

We need to prove $[\Delta(e_{i,r+1}), \Delta(e_{j,s})]_{\boldsymbol{v}^{c_{ij}}} + [\Delta(e_{j,s+1}), \Delta(e_{i,r})]_{\boldsymbol{v}^{c_{ij}}} = 0$ for $b_{2,i} - 1 \le r \le -1, b_{2,j} - 1 \le s \le -1$.

Case $b_{2,i} - 1 < r \le -1, b_{2,j} - 1 < s \le -1$ In this case, the above sum equals $1 \otimes ([e_{i,r+1}, e_{j,s}]_{\boldsymbol{v}^{c_{ij}}} + [e_{j,s+1}, e_{i,r}]_{\boldsymbol{v}^{c_{ij}}}) = 0$, due to relations (Û2) and (Û4) for $\mathcal{U}^{sc}_{0,\mu_2}$.

Case $r = b_{2,i} - 1, b_{2,j} - 1 < s < -1$ Note that $[e_{j,s+1}, f_{a,0}] = 0$ for any $1 \le a < n$, due to (Û6) for $\mathcal{U}^{sc}_{0,\mu_2}$. As a result, we have $[e_{j,s+1}, F^{(0)}_{ba}] = [e_{j,s+1}, \widetilde{F}^{(0)}_{ba}] = 0$ for any $1 \le a < b \le n$. Combining this with (Û2) and (Û4) for $\mathcal{U}^{sc}_{0,\mu_2}$, we get $[\Delta(e_{i,b_{2,i}}), \Delta(e_{j,s})]_{\boldsymbol{v}^{c_{ij}}} + [\Delta(e_{j,s+1}), \Delta(e_{i,b_{2,i}-1})]_{\boldsymbol{v}^{c_{ij}}} = 1 \otimes ([e_{i,b_{2,i}}, e_{j,s}]_{\boldsymbol{v}^{c_{ij}}} + [e_{j,s+1}, e_{i,b_{2,i}-1}]_{\boldsymbol{v}^{c_{ij}}}) = 0$ as above.

Case $r = b_{2,i} - 1, s = b_{2,j} - 1$ Due to relation (Û4) for both $\mathcal{U}^{sc}_{0,\mu_1}, \mathcal{U}^{sc}_{0,\mu_2}$, we get

$$[\Delta(e_{j,b_{2,j}}), \Delta(e_{i,b_{2,i}-1})]_{\boldsymbol{v}^{c_{ij}}} = 1 \otimes [e_{j,b_{2,j}}, e_{i,b_{2,i}-1}]_{\boldsymbol{v}^{c_{ij}}} -$$

$$(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{l>i+1} E^{(-1)}_{il} \otimes [e_{j,b_{2,j}}, F^{(0)}_{l,i+1}]\psi^-_{i,b_{2,i}} + (\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{i-k-1} E^{(-1)}_{k,i+1} \otimes [e_{j,b_{2,j}}, \widetilde{F}^{(0)}_{ik}]\psi^-_{i,b_{2,i}} -$$

$$(\boldsymbol{v} - \boldsymbol{v}^{-1})^2 \sum_{l>i+1}^{k<i} \boldsymbol{v}^{i-k-1} E^{(-1)}_{kl} \otimes [e_{j,b_{2,j}}, \widetilde{F}^{(0)}_{ik} F^{(0)}_{l,i+1}]\psi^-_{i,b_{2,i}}.$$

Using this formula and Lemma H.4 below, it is straightforward to check that again we obtain $[\Delta(e_{i,b_{2,i}}), \Delta(e_{j,b_{2,j}-1})]_{\boldsymbol{v}^{c_{ij}}} + [\Delta(e_{j,b_{2,j}}), \Delta(e_{i,b_{2,i}-1})]_{\boldsymbol{v}^{c_{ij}}} = 1 \otimes ([e_{i,b_{2,i}}, e_{j,b_{2,j}-1}]_{\boldsymbol{v}^{c_{ij}}} + [e_{j,b_{2,j}}, e_{i,b_{2,i}-1}]_{\boldsymbol{v}^{c_{ij}}}) = 0$.

Lemma H.4 *For any $1 \leq k < i,\ i+1 < l \leq n,\ 1 \leq j < n$, the following holds in $\mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$:*

(a) $[e_{j,b_{2,j}}, F^{(0)}_{l,i+1}] = \delta_{j,i+1} F^{(0)}_{l,i+2} \psi^-_{j,b_{2,j}}$*, where we set* $F^{(0)}_{i+2,i+2} := \frac{-1}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$.
(b) $[e_{j,b_{2,j}}, \widetilde{F}^{(0)}_{ik}] = -\boldsymbol{v}^{-1} \delta_{j,i-1} \widetilde{F}^{(0)}_{i-1,k} \psi^-_{j,b_{2,j}}$*, where we set* $\widetilde{F}^{(0)}_{i-1,i-1} := \frac{\boldsymbol{v}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$.

Case $r = b_{2,i} - 1, s = -1$ Clearly, $[\Delta(e_{i,b_{2,i}}), \Delta(e_{j,-1})]_{\boldsymbol{v}^{c_{ij}}} = 1 \otimes [e_{i,b_{2,i}}, e_{j,-1}]_{\boldsymbol{v}^{c_{ij}}}$ and $[\Delta(e_{j,0}), \Delta(e_{i,b_{2,i}-1})]_{\boldsymbol{v}^{c_{ij}}} = [1 \otimes e_{j,0} + e_{j,0} \otimes \psi^+_{j,0}, \Delta(e_{i,b_{2,i}-1})]_{\boldsymbol{v}^{c_{ij}}}$. We claim that as in the previous cases, one gets $[\Delta(e_{i,b_{2,i}}), \Delta(e_{j,-1})]_{\boldsymbol{v}^{c_{ij}}} + [\Delta(e_{j,0}), \Delta(e_{i,b_{2,i}-1})]_{\boldsymbol{v}^{c_{ij}}} = 1 \otimes ([e_{i,b_{2,i}}, e_{j,-1}]_{\boldsymbol{v}^{c_{ij}}} + [e_{j,0}, e_{i,b_{2,i}-1}]_{\boldsymbol{v}^{c_{ij}}}) = 0$. To this end, we note that the computations of $[1 \otimes e_{j,0}, \Delta(e_{i,b_{2,i}-1})]_{\boldsymbol{v}^{c_{ij}}}$ and $[e_{j,0} \otimes \psi^+_{j,0}, \Delta(e_{i,b_{2,i}-1})]_{\boldsymbol{v}^{c_{ij}}}$ are straightforward and are crucially based on Lemmas H.5 and H.6 below, respectively.

Lemma H.5 *For any $1 \leq k < i,\ i+1 < l \leq n,\ 1 \leq j < n$, the following holds in $\mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$:*

(a) $[e_{j,0}, F^{(0)}_{l,i+1}] = -\boldsymbol{v} \delta_{j,l-1} F^{(0)}_{j,i+1} \psi^+_{j,0}$*, where we set* $F^{(0)}_{i+1,i+1} := \frac{-1}{\boldsymbol{v}(\boldsymbol{v}-\boldsymbol{v}^{-1})}$.
(b) $[e_{j,0}, \widetilde{F}^{(0)}_{ik}] = \delta_{jk} \widetilde{F}^{(0)}_{i,j+1} \psi^+_{j,0}$*, where we set* $\widetilde{F}^{(0)}_{ii} := \frac{1}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$.

Lemma H.6 *For any $1 \leq k < l-1 < n,\ 1 \leq j < n$, the following holds in $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}$:* $[e_{j,0}, E^{(-1)}_{kl}]_{\boldsymbol{v}^{(\alpha^\vee_j, \alpha^\vee_{[k,l-1]})}} = \delta_{jl} E^{(-1)}_{k,l+1} - \delta_{j,k-1} E^{(-1)}_{k-1,l}$.

H(ii).c Compatibility with (Û3)

We need to prove $[\Delta(f_{i,r+1}), \Delta(f_{j,s})]_{\boldsymbol{v}^{-c_{ij}}} + [\Delta(f_{j,s+1}), \Delta(f_{i,r})]_{\boldsymbol{v}^{-c_{ij}}} = 0$ for $b_{1,i} \leq r \leq 0, b_{1,j} \leq s \leq 0$.

Case $b_{1,i} \leq r < 0, b_{1,j} \leq s < 0$ In this case, the above sum equals $([f_{i,r+1}, f_{j,s}]_{\boldsymbol{v}^{-c_{ij}}} + [f_{j,s+1}, f_{i,r}]_{\boldsymbol{v}^{-c_{ij}}}) \otimes 1 = 0$, due to relations (Û3) and (Û5) for $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}$.

Case $r = 0, b_{1,j} < s < 0$ Note that $[f_{j,s}, e_{a,0}] = 0$ for any $1 \leq a < n$, due to (Û6) for $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}$. As a result, we have $[f_{j,s}, E^{(0)}_{ab}] = [f_{j,s}, \widetilde{E}^{(0)}_{ab}] = 0$ for any $1 \leq a < b \leq n$. Combining this with (Û3) and (Û5) for $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}$, we get $[\Delta(f_{i,1}), \Delta(f_{j,s})]_{\boldsymbol{v}^{-c_{ij}}} + [\Delta(f_{j,s+1}), \Delta(f_{i,0})]_{\boldsymbol{v}^{-c_{ij}}} = ([f_{i,1}, f_{j,s}]_{\boldsymbol{v}^{-c_{ij}}} + [f_{j,s+1}, f_{i,0}]_{\boldsymbol{v}^{-c_{ij}}}) \otimes 1 = 0$ as above.

Case $r = 0, s = 0$ Due to relation ($\hat{\mathrm{U}}5$) for both $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}, \mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$, we get

$$[\Delta(f_{i,1}), \Delta(f_{j,0})]_{\boldsymbol{v}^{-c_{ij}}} = [f_{i,1}, f_{j,0}]_{\boldsymbol{v}^{-c_{ij}}} \otimes 1 + (\boldsymbol{v} - \boldsymbol{v}^{-1})\boldsymbol{v}^{-c_{ij}-1} \sum_{l>i+1} [E^{(0)}_{i+1,l}, f_{j,0}]\psi^+_{i,0} \otimes F^{(1)}_{li} -$$

$$(\boldsymbol{v} - \boldsymbol{v}^{-1})\boldsymbol{v}^{-c_{ij}} \sum_{k<i} \boldsymbol{v}^{k-i}[\widetilde{E}^{(0)}_{ki}, f_{j,0}]\psi^+_{i,0} \otimes F^{(1)}_{i+1,k} - (\boldsymbol{v} - \boldsymbol{v}^{-1})^2\boldsymbol{v}^{-c_{ij}} \sum_{l>i+1}^{k<i} \boldsymbol{v}^{k-i-1}[E^{(0)}_{i+1,l}\widetilde{E}^{(0)}_{ki}, f_{j,0}]\psi^+_{i,0} \otimes F^{(1)}_{lk}.$$

Using this formula and Lemma H.7 below, it is straightforward to check that we obtain $[\Delta(f_{i,1}), \Delta(f_{j,0})]_{\boldsymbol{v}^{-c_{ij}}} + [\Delta(f_{j,1}), \Delta(f_{i,0})]_{\boldsymbol{v}^{-c_{ij}}} = ([f_{i,1}, f_{j,0}]_{\boldsymbol{v}^{-c_{ij}}} + [f_{j,1}, f_{i,0}]_{\boldsymbol{v}^{-c_{ij}}}) \otimes 1 = 0$.

Lemma H.7 *For any* $1 \le k < i,\ i+1 < l \le n,\ 1 \le j < n$, *the following holds in* $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}$:

(a) $[E^{(0)}_{i+1,l}, f_{j,0}] = \boldsymbol{v}^{-1}\delta_{j,i+1}E^{(0)}_{i+2,l}\psi^+_{j,0}$, *where we set* $E^{(0)}_{i+2,i+2} := \frac{\boldsymbol{v}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$.
(b) $[\widetilde{E}^{(0)}_{ki}, f_{j,0}] = -\delta_{j,i-1}\widetilde{E}^{(0)}_{k,i-1}\psi^+_{j,0}$, *where we set* $\widetilde{E}^{(0)}_{i-1,i-1} := \frac{-1}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$.

Case $r = 0, s = b_{1,j}$ Clearly, $[\Delta(f_{j,b_{1,j}+1}), \Delta(f_{i,0})]_{\boldsymbol{v}^{-c_{ij}}} = [f_{j,b_{1,j}+1}, f_{i,0}]_{\boldsymbol{v}^{-c_{ij}}} \otimes 1$ and $[\Delta(f_{i,1}), \Delta(f_{j,b_{1,j}})]_{\boldsymbol{v}^{-c_{ij}}} = [\Delta(f_{i,1}), f_{j,b_{1,j}} \otimes 1 + \psi^-_{j,b_{1,j}} \otimes f_{j,0}]_{\boldsymbol{v}^{-c_{ij}}}$. We claim that as in the previous cases, one gets $[\Delta(f_{i,1}), \Delta(f_{j,b_{1,j}})]_{\boldsymbol{v}^{-c_{ij}}} + [\Delta(f_{j,b_{1,j}+1}), \Delta(f_{i,0})]_{\boldsymbol{v}^{-c_{ij}}} = ([f_{i,1}, f_{j,b_{1,j}}]_{\boldsymbol{v}^{-c_{ij}}} + [f_{j,b_{1,j}+1}, f_{i,0}]_{\boldsymbol{v}^{-c_{ij}}}) \otimes 1 = 0$. To this end, we note that the computations of $[\Delta(f_{i,1}), f_{j,b_{1,j}} \otimes 1]_{\boldsymbol{v}^{-c_{ij}}}$ and $[\Delta(f_{i,1}), \psi^-_{j,b_{1,j}} \otimes f_{j,0}]_{\boldsymbol{v}^{-c_{ij}}}$ are straightforward and are crucially based on Lemmas H.8 and H.9 below, respectively.

Lemma H.8 *For any* $1 \le k < i,\ i+1 < l \le n,\ 1 \le j < n$, *the following holds in* $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}$:

(a) $[E^{(0)}_{i+1,l}, f_{j,b_{1,j}}] = -\delta_{j,l-1}E^{(0)}_{i+1,j}\psi^-_{j,b_{1,j}}$, *where we set* $E^{(0)}_{i+1,i+1} := \frac{1}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$.
(b) $[\widetilde{E}^{(0)}_{ki}, f_{j,b_{1,j}}] = \boldsymbol{v}\delta_{jk}\widetilde{E}^{(0)}_{j+1,i}\psi^-_{j,b_{1,j}}$, *where we set* $\widetilde{E}^{(0)}_{ii} := \frac{-1}{\boldsymbol{v}(\boldsymbol{v}-\boldsymbol{v}^{-1})}$.

Lemma H.9 *For any* $1 \le k < l-1 < n,\ 1 \le j < n$, *the following holds in* $\mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$: $[F^{(1)}_{lk}, f_{j,0}]_{\boldsymbol{v}^{-(\alpha^\vee_j, \alpha^\vee_{[k,l-1]})}} = \delta_{jl}F^{(1)}_{l+1,k} - \delta_{j,k-1}F^{(1)}_{l,k-1}$.

H(ii).d Compatibility with ($\hat{\mathrm{U}}4$)

Due to relations ($\hat{\mathrm{U}}1$, $\hat{\mathrm{U}}4$, $\hat{\mathrm{U}}5$) for both $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}, \mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$, we immediately obtain the equalities $\Delta(\psi^+_{i,0})\Delta(e_{j,r}) = \boldsymbol{v}^{c_{ij}}\Delta(e_{j,r})\Delta(\psi^+_{i,0})$, $\Delta(\psi^-_{i,b_i})\Delta(e_{j,r}) = \boldsymbol{v}^{-c_{ij}}\Delta(e_{j,r})\Delta(\psi^-_{i,b_i})$ for $b_{2,j-1} \le r \le 0$.

Let us now verify $[\Delta(h_{i,1}), \Delta(e_{j,r})] = [c_{ij}]_{\boldsymbol{v}} \cdot \Delta(e_{j,r+1})$ for $b_{2,j} - 1 \le r \le -1$.

Case $b_{2,j} \leq r < -1$ The verification in this case follows immediately from relation ($\hat{\text{U}}4$) for $\mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$ combined with Lemma H.10 below.

Lemma H.10 *For any* $1 \leq a < b \leq n,\ b_{2,j} \leq r < -1$*, we have* $[F^{(1)}_{ba}, e_{j,r}] = 0$ *in* $\mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$.

Case $r = -1$ Due to relation ($\hat{\text{U}}4$) for $\mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$, we get

$$[\Delta(h_{i,1}), \Delta(e_{j,-1})] = [c_{ij}]_{\boldsymbol{v}} \cdot 1 \otimes e_{j,0} - (\boldsymbol{v}^2 - \boldsymbol{v}^{-2}) E^{(0)}_{i,i+1} \otimes [F^{(1)}_{i+1,i}, e_{j,-1}] +$$
$$(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{l>i+1} E^{(0)}_{i+1,l} \otimes [F^{(1)}_{l,i+1}, e_{j,-1}] + (\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{k+1-i} \widetilde{E}^{(0)}_{ki} \otimes [F^{(1)}_{ik}, e_{j,-1}] +$$
$$\boldsymbol{v}^{-2}(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{l>i+1} [E^{(0)}_{i,i+1}, E^{(0)}_{i+1,l}]_{\boldsymbol{v}^3} \otimes [F^{(1)}_{li}, e_{j,-1}] -$$
$$(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{k-i-1} [E^{(0)}_{i,i+1}, \widetilde{E}^{(0)}_{ki}]_{\boldsymbol{v}^3} \otimes [F^{(1)}_{i+1,k}, e_{j,-1}] +$$
$$(\boldsymbol{v} - \boldsymbol{v}^{-1})^2 \sum_{l>i+1}^{k<i} \boldsymbol{v}^{k-i} (E^{(0)}_{il} \widetilde{E}^{(0)}_{ki} - E^{(0)}_{i+1,l} \widetilde{E}^{(0)}_{k,i+1}) \otimes [F^{(1)}_{lk}, e_{j,-1}].$$

Using this formula and Lemma H.11 below, it is straightforward to check that we obtain $[\Delta(h_{i,1}), \Delta(e_{j,-1})] = [c_{ij}]_{\boldsymbol{v}} \cdot (1 \otimes e_{j,0} + e_{j,0} \otimes \psi^+_{j,0}) = [c_{ij}]_{\boldsymbol{v}} \cdot \Delta(e_{j,0})$.

Lemma H.11 *For any* $1 \leq a < b \leq n$*, we have* $[F^{(1)}_{ba}, e_{j,-1}] = \frac{-1}{\boldsymbol{v}-\boldsymbol{v}^{-1}} \delta_{ja} \delta_{j,b-1} \psi^+_{j,0}$ *in* $\mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$.

Case $r = b_{2,j} - 1$ According to the next step, we have $\Delta(e_{j,b_{2,j}-1}) = \frac{[\Delta(h_{j,-1}), \Delta(e_{j,b_{2,j}})]}{[2]_{\boldsymbol{v}}}$. Apply the Jacobi identity to get $[2]_{\boldsymbol{v}} \cdot [\Delta(h_{i,1}), \Delta(e_{j,b_{2,j}-1})] = [\Delta(h_{j,-1}), [\Delta(h_{i,1}), \Delta(e_{j,b_{2,j}})]] - [\Delta(e_{j,b_{2,j}}), [\Delta(h_{i,1}), \Delta(h_{j,-1})]]$. The second summand is zero as $[\Delta(h_{i,1}), \Delta(h_{j,-1})] = 0$ by above. Due to the $r = b_{2,j}$ case considered above, we have $[\Delta(h_{i,1}), \Delta(e_{j,b_{2,j}})] = [c_{ij}]_{\boldsymbol{v}} \cdot \Delta(e_{j,b_{2,j}+1})$. It remains to apply $[\Delta(h_{j,-1}), \Delta(e_{j,b_{2,j}+1})] = [2]_{\boldsymbol{v}} \cdot \Delta(e_{j,b_{2,j}})$ as proved below.

Let us now verify the equality $[\Delta(h_{i,-1}), \Delta(e_{j,r})] = [c_{ij}]_{\boldsymbol{v}} \cdot \Delta(e_{j,r-1})$ for $b_{2,j} \leq r \leq 0$.

Case $b_{2,j} < r < 0$ The verification in this case follows immediately from relation ($\hat{\text{U}}4$) for $\mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$ combined with Lemma H.12 below.

Lemma H.12 *For* $1 \leq a < b \leq n, b_{2,j} < r < 0$*, we have* $[F^{(0)}_{ba}, e_{j,r}] = [\widetilde{F}^{(0)}_{ba}, e_{j,r}] = 0$ *in* $\mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$.

Case $r = b_{2,j}$ For $i = j$, the verification of $[\Delta(h_{j,-1}), \Delta(e_{j,b_{2,j}})] = [2]_{\boldsymbol{v}} \cdot \Delta(e_{j,b_{2,j}-1})$ coincides with our proof of formula (10.4) from Appendix G. To prove the claim for $i \neq j$, we can either perform similar long computations or we can rather deduce from the aforementioned case $i = j$. To achieve the latter, we apply the Jacobi identity to get $[2]_{\boldsymbol{v}} \cdot [\Delta(h_{i,-1}), \Delta(e_{j,b_{2,j}})] = [\Delta(h_{j,-1}), [\Delta(h_{i,-1}), \Delta(e_{j,b_{2,j}+1})]] - [\Delta(e_{j,b_{2,j}+1}), [\Delta(h_{i,-1}), \Delta(h_{j,-1})]]$. The second summand is zero as $[\Delta(h_{i,-1}), \Delta(h_{j,-1})] = 0$ by above. Due to the $r = b_{2,j}+1$ case considered above, we have $[\Delta(h_{i,-1}), \Delta(e_{j,b_{2,j}+1})] = [c_{ij}]_{\boldsymbol{v}} \cdot \Delta(e_{j,b_{2,j}})$. It remains to apply the aforementioned equality $[\Delta(h_{j,-1}), \Delta(e_{j,b_{2,j}})] = [2]_{\boldsymbol{v}} \cdot \Delta(e_{j,b_{2,j}-1})$.

Case $r = 0$ The verification of $[\Delta(h_{i,-1}), \Delta(e_{j,0})] = [c_{ij}]_{\boldsymbol{v}} \cdot 1 \otimes e_{j,-1}$ is similar to our proof of formula (10.4) from Appendix G. To this end, we note that the computations of $[\Delta(h_{i,-1}), 1 \otimes e_{j,0}]$ and $[\Delta(h_{i,-1}), e_{j,0} \otimes \psi^+_{j,0}]$ are straightforward and are crucially based on the above Lemmas H.5 and H.6.

H(ii).e Compatibility with (Û5)

Due to relations (Û1, Û4, Û5) for both $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}, \mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$, we immediately obtain the equalities $\Delta(\psi^+_{i,0})\Delta(f_{j,r}) = \boldsymbol{v}^{-c_{ij}}\Delta(f_{j,r})\Delta(\psi^+_{i,0})$, $\Delta(\psi^-_{i,b_i})\Delta(f_{j,r}) = \boldsymbol{v}^{c_{ij}}\Delta(f_{j,r})\Delta(\psi^-_{i,b_i})$ for $b_{1,j} \le r \le 1$.

Let us now verify $[\Delta(h_{i,-1}), \Delta(f_{j,r})] = -[c_{ij}]_{\boldsymbol{v}} \cdot \Delta(f_{j,r-1})$ for $b_{1,j}+1 \le r \le 1$.

Case $b_{1,j}+1 < r < 1$ The verification in this case follows immediately from relation (Û5) for $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}$ combined with Lemma H.13 below.

Lemma H.13 *For any* $1 \le a < b \le n$, $b_{1,j}+1 < r < 1$, *we have* $[E^{(-1)}_{ab}, f_{j,r}] = 0$ *in* $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}$.

Case $r = b_{1,j}+1$ Due to relation (Û5) for $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}$, we have

$$
\begin{aligned}
&[\Delta(h_{i,-1}), \Delta(f_{j,b_{1,j}+1})] = -[c_{ij}]_{\boldsymbol{v}} \cdot f_{j,b_{1,j}} \otimes 1 + (\boldsymbol{v}^2 - \boldsymbol{v}^{-2})[E^{(-1)}_{i,i+1}, f_{j,b_{1,j}+1}] \otimes F^{(0)}_{i+1,i} - \\
&(\boldsymbol{v}-\boldsymbol{v}^{-1}) \sum_{l>i+1} [E^{(-1)}_{i+1,l}, f_{j,b_{1,j}+1}] \otimes F^{(0)}_{l,i+1} - (\boldsymbol{v}-\boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{i-k-1}[E^{(-1)}_{ki}, f_{j,b_{1,j}+1}] \otimes \widetilde{F}^{(0)}_{ik} - \\
&\boldsymbol{v}^2(\boldsymbol{v}-\boldsymbol{v}^{-1}) \sum_{l>i+1} [E^{(-1)}_{il}, f_{j,b_{1,j}+1}] \otimes [F^{(0)}_{l,i+1}, F^{(0)}_{i+1,i}]_{\boldsymbol{v}^{-3}} + \\
&(\boldsymbol{v}-\boldsymbol{v}^{-1}) \sum_{k<i} \boldsymbol{v}^{i+1-k}[E^{(-1)}_{k,i+1}, f_{j,b_{1,j}+1}] \otimes [\widetilde{F}^{(0)}_{ik}, F^{(0)}_{i+1,i}]_{\boldsymbol{v}^{-3}} - \\
&(\boldsymbol{v}-\boldsymbol{v}^{-1})^2 \sum_{l>i+1}^{k<i} \boldsymbol{v}^{i-k}[E^{(-1)}_{kl}, f_{j,b_{1,j}+1}] \otimes (\widetilde{F}^{(0)}_{i+1,k} F^{(0)}_{l,i+1} - \widetilde{F}^{(0)}_{ik} F^{(0)}_{li}).
\end{aligned}
$$

Using this formula and Lemma H.14 below, it is straightforward to check that we obtain $[\Delta(h_{i,-1}), \Delta(f_{j,b_{1,j}+1})] = -[c_{ij}]_{\boldsymbol{v}} \cdot (f_{j,b_{1,j}} \otimes 1 + \psi^-_{j,b_{1,j}} \otimes f_{j,0}) = -[c_{ij}]_{\boldsymbol{v}} \cdot \Delta(f_{j,b_{1,j}})$.

Lemma H.14 *For any* $1 \leq a < b \leq n$*, we have* $[E_{ab}^{(-1)}, f_{j,b_{1,j}+1}] = \frac{-\delta_{ja}\delta_{j,b-1}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\psi^-_{j,b_{1,j}}$ *in* $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}$.

Case $r = 1$ According to the next step, we have $\Delta(f_{j,1}) = -[2]_{\boldsymbol{v}}^{-1} \cdot [\Delta(h_{j,1}), \Delta(f_{j,0})]$. Apply the Jacobi identity to get $[2]_{\boldsymbol{v}} \cdot [\Delta(h_{i,-1}), \Delta(f_{j,1})] = [\Delta(h_{j,1}), [\Delta(h_{i,-1}), \Delta(f_{j,0})]] - [\Delta(f_{j,0}), [\Delta(h_{i,-1}), \Delta(h_{j,1})]]$. The second summand is zero as $[\Delta(h_{i,-1}), \Delta(h_{j,1})] = 0$ by above. Due to the $r = 0$ case considered above, we have $[\Delta(h_{i,-1}), \Delta(f_{j,0})] = -[c_{ij}]_{\boldsymbol{v}} \cdot \Delta(f_{j,-1})$. It remains to apply $[\Delta(h_{j,1}), \Delta(f_{j,-1})] = -[2]_{\boldsymbol{v}} \cdot \Delta(f_{j,0})$ as proved below.

Let us now verify $[\Delta(h_{i,1}), \Delta(f_{j,r})] = -[c_{ij}]_{\boldsymbol{v}} \cdot \Delta(f_{j,r+1})$ for $b_{1,j} \leq r \leq 0$.

Case $b_{1,j} < r < 0$ The verification in this case follows immediately from relation ($\hat{\mathrm{U}}5$) for $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}$ combined with Lemma H.15 below.

Lemma H.15 *For* $1 \leq a < b \leq n, b_{1,j} < r < 0$*, we have* $[E_{ab}^{(0)}, f_{j,r}] = [\widetilde{E}_{ab}^{(0)}, f_{j,r}] = 0$ *in* $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}$.

Case $r = 0$ For $i = j$, the verification of $[\Delta(h_{j,1}), \Delta(f_{j,0})] = -[2]_{\boldsymbol{v}} \cdot \Delta(f_{j,1})$ coincides with our proof of formula (10.5), sketched in Appendix G. To prove the claim for $i \neq j$, we can either perform similar long computations or we can rather deduce from the aforementioned case $i = j$. To achieve the latter, we apply the Jacobi identity to get $-[2]_{\boldsymbol{v}} \cdot [\Delta(h_{i,1}), \Delta(f_{j,0})] = [\Delta(h_{j,1}), [\Delta(h_{i,1}), \Delta(f_{j,-1})]] - [\Delta(f_{j,-1}), [\Delta(h_{i,1}), \Delta(h_{j,1})]]$. The second summand is zero as $[\Delta(h_{i,1}), \Delta(h_{j,1})] = 0$ by above. Due to the $r = -1$ case considered above, we have $[\Delta(h_{i,1}), \Delta(f_{j,-1})] = -[c_{ij}]_{\boldsymbol{v}} \cdot \Delta(f_{j,0})$. It remains to apply the aforementioned equality $[\Delta(h_{j,1}), \Delta(f_{j,0})] = -[2]_{\boldsymbol{v}} \cdot \Delta(f_{j,1})$.

Case $r = b_{1,j}$ The verification of $[\Delta(h_{i,1}), \Delta(f_{j,b_{1,j}})] = -[c_{ij}]_{\boldsymbol{v}} \cdot f_{j,b_{1,j}+1} \otimes 1$ is similar to our proof of formula (10.5), sketched in Appendix G. To this end, we note that the computations of $[\Delta(h_{i,1}), f_{j,b_{1,j}} \otimes 1]$ and $[\Delta(h_{i,1}), \psi^-_{j,b_{1,j}} \otimes f_{j,0}]$ are straightforward and are crucially based on the above Lemmas H.8 and H.9.

H(ii).f Compatibility with ($\hat{\mathrm{U}}6$)

We need to verify

$$[\Delta(e_{i,r}), \Delta(f_{j,s})] = \delta_{ij} \cdot \begin{cases} \Delta(\psi^+_{i,0})\Delta(h_{i,1}) & \text{if } r+s=1, \\ \Delta(\psi^-_{i,b_i})\Delta(h_{i,-1}) & \text{if } r+s=b_i-1, \\ \frac{\Delta(\psi^+_{i,0})}{\boldsymbol{v}-\boldsymbol{v}^{-1}} & \text{if } r+s=0, \\ \frac{-\Delta(\psi^-_{i,b_i})}{\boldsymbol{v}-\boldsymbol{v}^{-1}} & \text{if } r+s=b_i, \\ 0 & \text{otherwise,} \end{cases}$$

for $b_{2,i}-1 \leq r \leq 0,\ b_{1,j} \leq s \leq 1$, where we set $b_i := b_{1,i}+b_{2,i}$ as before.

Cases $b_{2,i}-1 < r \leq 0, b_{1,j} \leq s < 1$ Obviously follows from ($\hat{\text{U}}4$, $\hat{\text{U}}5$, $\hat{\text{U}}6$) for both $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}, \mathcal{U}^{\mathrm{sc}}_{0,\mu_2}$.

Case $b_{2,i} \leq r < -1, s=1$ In this case, we get $[\Delta(e_{i,r}), \Delta(f_{j,1})]=0$, due to Lemma H.10.

Case $r=-1, s=1$ Applying Lemma H.11 from above, it is straightforward to see that we get $[\Delta(e_{i,-1}), \Delta(f_{j,1})]=\frac{\delta_{ij}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\psi^+_{i,0}\otimes\psi^+_{i,0}=\frac{\delta_{ij}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\Delta(\psi^+_{i,0})$.

Case $r=b_{2,i}-1, s=1$ According to relation ($\hat{\text{U}}4$) verified above, we have $\Delta(e_{i,b_{2,i}-1})=[2]_{\boldsymbol{v}}^{-1}\cdot[\Delta(h_{i,-1}), \Delta(e_{i,b_{2,i}})]$. Applying the Jacobi identity, we get $[2]_{\boldsymbol{v}}\cdot[\Delta(e_{i,b_{2,i}-1}), \Delta(f_{j,1})]=[\Delta(h_{i,-1}), [\Delta(e_{i,b_{2,i}}), \Delta(f_{j,1})]]-[\Delta(e_{i,b_{2,i}}), [\Delta(h_{i,-1}), \Delta(f_{j,1})]]$. However, both summands in the right-hand side are zero, due to the above cases and relation ($\hat{\text{U}}5$) established above.

Case $r=b_{2,i}-1, b_{1,j}+1<s<1$ In this case, we get $[\Delta(e_{i,b_{2,i}-1}), \Delta(f_{j,s})]=0$, due to Lemma H.13.

Case $r=b_{2,i}-1, s=b_{1,j}+1$ Applying Lemma H.14 from above, it is straightforward to see that we get $[\Delta(e_{i,b_{2,i}-1}), \Delta(f_{j,b_{1,j}+1})]=-\frac{\delta_{ij}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\psi^-_{i,b_{1,i}}\otimes\psi^-_{i,b_{2,i}}=-\frac{\delta_{ij}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}\Delta(\psi^-_{i,b_i})$.

Case $r=0, s=1$ Consider the homomorphism $\jmath^+_{\mu_1,0}\otimes\jmath^+_{0,\mu_2}\colon U^+_{\boldsymbol{v}}\otimes U^+_{\boldsymbol{v}}\to\mathcal{U}^{\mathrm{sc},+}_{0,\mu_1,0}\otimes\mathcal{U}^{\mathrm{sc},+}_{0,0,\mu_2}$. Comparing the formulas of Theorems 10.13, 10.16 and applying Lemma H.3, we get

$$[\Delta_{\mu_1,\mu_2}(e_{i,0}), \Delta_{\mu_1,\mu_2}(f_{j,1})]=[\jmath^+_{\mu_1,0}\otimes\jmath^+_{0,\mu_2}(\Delta(e_{i,0})), \jmath^+_{\mu_1,0}\otimes\jmath^+_{0,\mu_2}(\Delta(f_{j,1}))]=$$

$$\jmath^+_{\mu_1,0}\otimes\jmath^+_{0,\mu_2}\left([\Delta(e_{i,0}), \Delta(f_{j,1})]\right)=\jmath^+_{\mu_1,0}\otimes\jmath^+_{0,\mu_2}(\delta_{ij}\Delta(\psi^+_{i,0})\Delta(h_{i,1}))=\delta_{ij}\Delta_{\mu_1,\mu_2}(\psi^+_{i,0})\Delta_{\mu_1,\mu_2}(h_{i,1}),$$

where the subscripts in Δ_{μ_1,μ_2} are used this time to distinguish it from the Drinfeld-Jimbo coproduct Δ on $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

Case $r=b_{2,i}-1, s=b_{1,j}$ Consider the homomorphism $\jmath^-_{\mu_1,0}\otimes\jmath^-_{0,\mu_2}\colon U^-_{\boldsymbol{v}}\otimes U^-_{\boldsymbol{v}}\to\mathcal{U}^{\mathrm{sc},-}_{0,\mu_1,0}\otimes\mathcal{U}^{\mathrm{sc},-}_{0,0,\mu_2}$. Comparing the formulas of Theorems 10.13, 10.16 and applying Lemma H.3, we get

$$[\Delta_{\mu_1,\mu_2}(e_{i,b_{2,i}-1}), \Delta_{\mu_1,\mu_2}(f_{j,b_{1,j}})]=[\jmath^-_{\mu_1,0}\otimes\jmath^-_{0,\mu_2}(\Delta(e_{i,-1})), \jmath^-_{\mu_1,0}\otimes\jmath^-_{0,\mu_2}(\Delta(f_{j,0}))]=$$

$$\jmath^-_{\mu_1,0}\otimes\jmath^-_{0,\mu_2}\left([\Delta(e_{i,-1}), \Delta(f_{j,0})]\right)=\jmath^-_{\mu_1,0}\otimes\jmath^-_{0,\mu_2}(\delta_{ij}\Delta(\psi^-_{i,0})\Delta(h_{i,-1}))=$$

$$\delta_{ij}\Delta_{\mu_1,\mu_2}(\psi^-_{i,b_i})\Delta_{\mu_1,\mu_2}(h_{i,-1}).$$

H(ii).g Compatibility with ($\hat{\mathrm{U}}7$)

Utilizing the homomorphism $\jmath^+_{\mu_1,0}\otimes\jmath^+_{0,\mu_2}\colon U^+_{\boldsymbol{v}}\otimes U^+_{\boldsymbol{v}}\to\mathcal{U}^{\mathrm{sc},+}_{0,\mu_1,0}\otimes\mathcal{U}^{\mathrm{sc},+}_{0,0,\mu_2}$ as above, we get

$$[\Delta_{\mu_1,\mu_2}(e_{i,0}),[\Delta_{\mu_1,\mu_2}(e_{i,0}),\cdots,[\Delta_{\mu_1,\mu_2}(e_{i,0}),\Delta_{\mu_1,\mu_2}(e_{j,0})]_{\boldsymbol{v}^{c_{ij}}}\cdots]_{\boldsymbol{v}^{-c_{ij}-2}}]_{\boldsymbol{v}^{-c_{ij}}}=$$
$$\jmath^+_{\mu_1,0}\otimes\jmath^+_{0,\mu_2}([\Delta(e_{i,0}),[\Delta(e_{i,0}),\cdots,[\Delta(e_{i,0}),\Delta(e_{j,0})]_{\boldsymbol{v}^{c_{ij}}}\cdots]_{\boldsymbol{v}^{-c_{ij}-2}}]_{\boldsymbol{v}^{-c_{ij}}})=$$
$$\jmath^+_{\mu_1,0}\otimes\jmath^+_{0,\mu_2}(\Delta([e_{i,0},[e_{i,0},\cdots,[e_{i,0},e_{j,0}]_{\boldsymbol{v}^{c_{ij}}}\cdots]_{\boldsymbol{v}^{-c_{ij}-2}}]_{\boldsymbol{v}^{-c_{ij}}}))=0,$$

where the last equality is due to the Serre relation in $U^+_{\boldsymbol{v}}$ (cf. Remark 5.4).

H(ii).h Compatibility with ($\hat{\mathrm{U}}8$)

Due to relation ($\hat{\mathrm{U}}8$) for $\mathcal{U}^{\mathrm{sc}}_{0,\mu_1}$, we have

$$[\Delta_{\mu_1,\mu_2}(f_{i,0}),[\Delta_{\mu_1,\mu_2}(f_{i,0}),\cdots,[\Delta_{\mu_1,\mu_2}(f_{i,0}),\Delta_{\mu_1,\mu_2}(f_{j,0})]_{\boldsymbol{v}^{c_{ij}}}\cdots]_{\boldsymbol{v}^{-c_{ij}-2}}]_{\boldsymbol{v}^{-c_{ij}}}=$$
$$[f_{i,0},[f_{i,0},\cdots,[f_{i,0},f_{j,0}]_{\boldsymbol{v}^{c_{ij}}}\cdots]_{\boldsymbol{v}^{-c_{ij}-2}}]_{\boldsymbol{v}^{-c_{ij}}}\otimes 1=0.$$

H(ii).i Compatibility with ($\hat{\mathrm{U}}9$)

Applying the homomorphisms $\jmath^\pm_{\mu_1,0}\otimes\jmath^\pm_{0,\mu_2}$, we see that it suffices to prove the equalities:

$$[h_{i,1},[f_{i,1},[h_{i,1},e_{i,0}]]]=0\ \text{ in }\ U^+_{\boldsymbol{v}},\ \ [h_{i,-1},[e_{i,-1},[h_{i,-1},f_{i,0}]]]=0\ \text{ in }\ U^-_{\boldsymbol{v}}.$$

These follow from $[h_{i,\pm1},\psi^\pm_{i,\pm2}]=0$ in $U^\pm_{\boldsymbol{v}}$.

This completes our proof of Theorem 10.16.

H(iii) Relation Between Δ *and* Δ_{μ_1,μ_2}

The following result completes our discussion of Remark 10.17.

Proposition H.16 *The following diagram is commutative:*

$$\begin{CD} U^\pm_{\boldsymbol{v}} @>{\Delta}>> U^\pm_{\boldsymbol{v}}\otimes U^\pm_{\boldsymbol{v}} \\ @V{\jmath^\pm_{\mu_1,\mu_2}}VV @VV{\jmath^\pm_{\mu_1,0}\otimes\jmath^\pm_{0,\mu_2}}V \\ \mathcal{U}^{\mathrm{sc},\pm}_{0,\mu_1,\mu_2} @>>{\Delta_{\mu_1,\mu_2}}> \mathcal{U}^{\mathrm{sc},\pm}_{0,\mu_1,0}\otimes\mathcal{U}^{\mathrm{sc},\pm}_{0,0,\mu_2} \end{CD}$$

Proof To simplify our computations, we will assume that μ_1, μ_2 are strictly antidominant.

(a) To prove the commutativity of the above diagram in the '+' case, it suffices to verify that $J^+_{\mu_1,0} \otimes J^+_{0,\mu_2}(\Delta(X)) = \Delta_{\mu_1,\mu_2}(J^+_{\mu_1,\mu_2}(X))$ for $X \in \{e_{i,0}, (\psi^+_{i,0})^{\pm 1}, F^{(1)}_{n1}\}_{i=1}^{n-1}$. The only non-obvious verification is the one for $X = F^{(1)}_{n1}$.

The computation of $\Delta(F^{(1)}_{n1})$ is based on the computation of $\Delta^{\mathrm{rtt}}(\tilde{f}^{(1)}_{n1})$. Comparing the coefficients of z^{-1} in the equality

$$\Delta^{\mathrm{rtt}}(T^+_{n1}(z)) = T^+_{n1}(z) \otimes T^+_{11}(z) + T^+_{nn}(z) \otimes T^+_{n1}(z) + \sum_{1<i<n} T^+_{ni}(z) \otimes T^+_{i1}(z),$$

we get $\Delta^{\mathrm{rtt}}(\tilde{f}^{(1)}_{n1}\tilde{g}^+_1) = \tilde{f}^{(1)}_{n1}\tilde{g}^+_1 \otimes \tilde{g}^+_1 + \tilde{g}^+_n \otimes \tilde{f}^{(1)}_{n1}\tilde{g}^+_1$, so that $\Delta^{\mathrm{rtt}}(\tilde{f}^{(1)}_{n1}) = \tilde{f}^{(1)}_{n1} \otimes 1 + \tilde{g}^+_n(\tilde{g}^+_1)^{-1} \otimes \tilde{f}^{(1)}_{n1}$. Applying Υ^{-1} of Theorem G.2 and formula (G.12), we finally find

$$\Delta(F^{(1)}_{n1}) = F^{(1)}_{n1} \otimes 1 + \psi^+_{1,0} \cdots \psi^+_{n-1,0} \otimes F^{(1)}_{n1}.$$

Therefore, $J^+_{\mu_1,0} \otimes J^+_{0,\mu_2}(\Delta(F^{(1)}_{n1})) = F^{(1)}_{n1} \otimes 1 + \psi^+_{1,0} \cdots \psi^+_{n-1,0} \otimes F^{(1)}_{n1}$.

On the other hand, we have $\Delta_{\mu_1,\mu_2}(J^+_{\mu_1,\mu_2}(F^{(1)}_{n1})) = \Delta_{\mu_1,\mu_2}(F^{(1)}_{n1})$ and

$$\Delta_{\mu_1,\mu_2}(F^{(1)}_{n1}) = [\cdots[\Delta_{\mu_1,\mu_2}(f_{1,1}), \Delta_{\mu_1,\mu_2}(f_{2,0})]_{\boldsymbol{v}}, \cdots, \Delta_{\mu_1,\mu_2}(f_{n-1,0})]_{\boldsymbol{v}}.$$

Let us first note that $[E^{(0)}_{2l}, f_{2,0}] = \boldsymbol{v}^{-1}E^{(0)}_{3l}\psi^+_{2,0}$, where we set $E^{(0)}_{33} := \frac{\boldsymbol{v}}{\boldsymbol{v}-\boldsymbol{v}^{-1}}$. Combining this with relation (U5) and the formula

$$\Delta_{\mu_1,\mu_2}(f_{1,1}) = f_{1,1} \otimes 1 + \psi^+_{1,0} \otimes f_{1,1} + \boldsymbol{v}^{-1}(\boldsymbol{v} - \boldsymbol{v}^{-1}) \sum_{l>2} E^{(0)}_{2l}\psi^+_{1,0} \otimes F^{(1)}_{l1},$$

we find

$$[\Delta_{\mu_1,\mu_2}(f_{1,1}), \Delta_{\mu_1,\mu_2}(f_{2,0})]_{\boldsymbol{v}} = [f_{1,1}, f_{2,0}]_{\boldsymbol{v}} \otimes 1 + \boldsymbol{v}^{-1}(\boldsymbol{v}-\boldsymbol{v}^{-1}) \sum_{l>2} E^{(0)}_{3l}\psi^+_{1,0}\psi^+_{2,0} \otimes F^{(1)}_{l1}.$$

Further $\boldsymbol{v}$-commuting this with $\Delta_{\mu_1,\mu_2}(f_{3,0}), \ldots, \Delta_{\mu_1,\mu_2}(f_{n-1,0})$, we finally obtain

$$\Delta_{\mu_1,\mu_2}(F^{(1)}_{n1}) = F^{(1)}_{n1} \otimes 1 + \psi^+_{1,0} \cdots \psi^+_{n-1,0} \otimes F^{(1)}_{n1}.$$

This completes our verification of $J^+_{\mu_1,0} \otimes J^+_{0,\mu_2}(\Delta(F^{(1)}_{n1})) = \Delta_{\mu_1,\mu_2}(J^+_{\mu_1,\mu_2}(F^{(1)}_{n1}))$.

(b) The proof of the commutativity in the '−' case is completely analogous. □

Appendix I Proof of Theorem 10.19

Our proof of Theorem 10.19 proceeds in three steps. First, we reduce the problem to its unshifted counterpart, see Theorem I.1. To prove this theorem, we recall the shuffle realization of $U^{>}_{\boldsymbol{v}}$, see Theorem I.3. In the last and final step, we apply a simple result Proposition I.4.

I(i) Reduction to an Unshifted Case

Given $\mu \in \Lambda$ and $\nu_1, \nu_2 \in \Lambda^-$, recall the shift homomorphisms $\iota_{\mu,\nu_1,\nu_2}\colon \mathcal{U}^{\mathrm{sc}}_{0,\mu} \to \mathcal{U}^{\mathrm{sc}}_{0,\mu+\nu_1+\nu_2}$ introduced in Lemma 10.18. Note that ι_{μ,ν_1,ν_2} gives rise to the homomorphisms (restrictions)

$$\iota^{>}_{\mu,\nu_1,\nu_2}\colon \mathcal{U}^{\mathrm{sc},>}_{0,\mu} \to \mathcal{U}^{\mathrm{sc},>}_{0,\mu+\nu_1+\nu_2},\ \iota^{<}_{\mu,\nu_1,\nu_2}\colon \mathcal{U}^{\mathrm{sc},<}_{0,\mu} \to \mathcal{U}^{\mathrm{sc},<}_{0,\mu+\nu_1+\nu_2},\ \iota^{0}_{\mu,\nu_1,\nu_2}\colon \mathcal{U}^{\mathrm{sc},0}_{0,\mu} \to \mathcal{U}^{\mathrm{sc},0}_{0,\mu+\nu_1+\nu_2}.$$

Moreover, evoking the triangular decomposition of Proposition 5.1(a) for both algebras $\mathcal{U}^{\mathrm{sc}}_{0,\mu}$ and $\mathcal{U}^{\mathrm{sc}}_{0,\mu+\nu_1+\nu_2}$, we see that ι_{μ,ν_1,ν_2} is "glued" from the aforementioned three homomorphisms $\iota^{>}_{\mu,\nu_1,\nu_2}, \iota^{<}_{\mu,\nu_1,\nu_2}, \iota^{0}_{\mu,\nu_1,\nu_2}$. Hence, Theorem 10.19 is equivalent to the injectivity of these restrictions $\iota^{>}_{\mu,\nu_1,\nu_2}, \iota^{<}_{\mu,\nu_1,\nu_2}, \iota^{0}_{\mu,\nu_1,\nu_2}$. The injectivity of $\iota^{0}_{\mu,\nu_1,\nu_2}$ is clear. On the other hand, according to Proposition 5.1(b), we have $\mathcal{U}^{\mathrm{sc},>}_{0,\mu} \simeq U^{>}_{\boldsymbol{v}} \simeq \mathcal{U}^{\mathrm{sc},>}_{0,\mu+\nu_1+\nu_2}, \mathcal{U}^{\mathrm{sc},<}_{0,\mu} \simeq U^{<}_{\boldsymbol{v}} \simeq \mathcal{U}^{\mathrm{sc},<}_{0,\mu+\nu_1+\nu_2}$, where $U^{>}_{\boldsymbol{v}}, U^{<}_{\boldsymbol{v}}$ denote the corresponding subalgebras of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. As such, the injectivity of $\iota^{>}_{\mu,\nu_1,\nu_2}$ (resp. $\iota^{<}_{\mu,\nu_1,\nu_2}$) is equivalent to the injectivity of $\iota^{>}_{\nu_1}\colon U^{>}_{\boldsymbol{v}} \to U^{>}_{\boldsymbol{v}}$ (resp. $\iota^{<}_{\nu_2}\colon U^{<}_{\boldsymbol{v}} \to U^{<}_{\boldsymbol{v}}$) given by $e_i(z) \mapsto (1-z^{-1})^{-\alpha_i^\vee(\nu_1)} e_i(z)$ (resp. $f_i(z) \mapsto (1-z^{-1})^{-\alpha_i^\vee(\nu_2)} f_i(z)$) for $i \in I$.

Thus, we have reduced Theorem 10.19 to its unshifted counterpart:

Theorem I.1

(a) The homomorphism $\iota^{>}_{\nu}\colon U^{>}_{\boldsymbol{v}} \to U^{>}_{\boldsymbol{v}}$ is injective for any $\nu \in \Lambda^-$.
(b) The homomorphism $\iota^{<}_{\nu}\colon U^{<}_{\boldsymbol{v}} \to U^{<}_{\boldsymbol{v}}$ is injective for any $\nu \in \Lambda^-$.

Our proof of part (a) is crucially based on the *shuffle realization* of $U^{>}_{\boldsymbol{v}}$, which we recall next (the proof of part (b) is completely analogous).

I(ii) Shuffle Algebra (of Type A_{n-1})

Consider an $\mathbb{N}^I$-graded $\mathbb{C}(\boldsymbol{v})$-vector space $\mathbb{S} = \bigoplus_{\underline{k}=(k_1,\ldots,k_{n-1})\in\mathbb{N}^I} \mathbb{S}_{\underline{k}}$, where $\mathbb{S}_{(k_1,\ldots,k_{n-1})}$ consists of $\prod \mathfrak{S}_{k_i}$-symmetric rational functions in the variables $\{x_{i,r}\}_{i\in I}^{1\le r\le k_i}$. We also fix an $I \times I$ matrix of rational functions $(\zeta_{i,j}(z))_{i,j\in I} \in \mathrm{Mat}_{I\times I}(\mathbb{C}(z))$ by setting $\zeta_{i,j}(z) = \frac{z-\boldsymbol{v}^{-c_{ij}}}{z-1}$, where $(c_{ij})_{i,j=1}^{n-1}$ is the Cartan matrix of $\mathfrak{sl}_n$ as before. Let us now introduce the bilinear $\star$ product on $\mathbb{S}$: given $F \in \mathbb{S}_{\underline{k}}$ and $G \in \mathbb{S}_{\underline{l}}$, define $F \star G \in \mathbb{S}_{\underline{k}+\underline{l}}$ by

$$(F \star G)(x_{1,1}, \ldots, x_{1,k_1+l_1}; \ldots; x_{n-1,1}, \ldots, x_{n-1,k_{n-1}+l_{n-1}}) := \prod_{i=1}^{n-1} k_i! \cdot l_i! \times$$
$$\mathrm{Sym}_{\prod \mathfrak{S}_{k_i+l_i}} \left(F(\{x_{i,r}\}_{i\in I}^{1\le r\le k_i}) G(\{x_{i',r'}\}_{i'\in I}^{k_{i'}<r'\le k_{i'}+l_{i'}}) \cdot \prod_{i\in I}^{i'\in I} \prod_{r\le k_i}^{r'>k_{i'}} \zeta_{i,i'}(x_{i,r}/x_{i',r'}) \right).$$

Here and afterwards, given a function $f \in \mathbb{C}(\{x_{i,1}, \ldots, x_{i,m_i}\}_{i\in I})$, we define

$$\mathrm{Sym}_{\prod \mathfrak{S}_{m_i}}(f) := \prod_{i\in I} \frac{1}{m_i!} \cdot \sum_{(\sigma_1,\ldots,\sigma_{n-1})\in \mathfrak{S}_{m_1}\times\ldots\times\mathfrak{S}_{m_{n-1}}} f(\{x_{i,\sigma_i(1)}, \ldots, x_{i,\sigma_i(m_i)}\}_{i\in I}).$$

This endows $\mathbb{S}$ with a structure of an associative unital algebra with the unit $\mathbf{1} \in \mathbb{S}_{(0,\ldots,0)}$. We will be interested only in a certain subspace of $\mathbb{S}$, defined by the *pole* and *wheel conditions*:

- We say that $F \in \mathbb{S}_{\underline{k}}$ satisfies the *pole conditions* if and only if

$$F = \frac{f(x_{1,1}, \ldots, x_{n-1,k_{n-1}})}{\prod_{i=1}^{n-2} \prod_{r\le k_i}^{r'\le k_{i+1}} (x_{i,r} - x_{i+1,r'})}, \text{ where } f \in (\mathbb{C}(\boldsymbol{v})[x_{i,r}^{\pm 1}]_{i\in I}^{1\le r\le k_i})^{\prod \mathfrak{S}_{k_i}}.$$

- We say that $F \in \mathbb{S}_{\underline{k}}$ satisfies the *wheel conditions* if and only if

$$F(\{x_{i,r}\}) = 0 \text{ once } x_{i,r_1} = \boldsymbol{v} x_{i+\epsilon,l} = \boldsymbol{v}^2 x_{i,r_2} \text{ for some } \epsilon, i, r_1, r_2, l,$$

where $\epsilon \in \{\pm 1\},\ i, i+\epsilon \in I,\ 1 \le r_1, r_2 \le k_i,\ 1 \le l \le k_{i+\epsilon}$.

Let $S_{\underline{k}} \subset \mathbb{S}_{\underline{k}}$ be the subspace of all elements F satisfying these two conditions and set $S := \bigoplus_{\underline{k}\in\mathbb{N}^I} S_{\underline{k}}$. It is straightforward to check that the subspace $S \subset \mathbb{S}$ is $\star$-closed.

Definition I.2 The algebra $(S, \star)$ is called the *shuffle algebra* (of A_{n-1}-type).

The following key result, identifying this algebra with $U_{\boldsymbol{v}}^{>}$, is due to [53][12] (see also [63]).

Theorem I.3 *There is a unique $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $\Psi : U_{\boldsymbol{v}}^{>} \xrightarrow{\sim} S$ such that $e_{i,r} \mapsto x_{i,1}^r$ for any $i \in I, r \in \mathbb{Z}$.*

[12] To be more precise, [53, Theorem 1.1] establishes such a shuffle realization for the half of the quantum toroidal algebra of $\mathfrak{sl}_n$. Since the latter naturally contains $U_{\boldsymbol{v}}^{>}$ as a subalgebra, we get the claimed result.

I(iii) Proof of Theorem I.1(a)

The following result is straightforward:

Proposition I.4

(a) *For any $\nu \in \Lambda^-$, there is a unique algebra homomorphism $\iota'_\nu \colon S \to S$ such that $f(\{x_{i,r}\}_{i\in I}^{1\le r\le k_i}) \mapsto \prod_{i\in I}^{1\le r\le k_i}(1-x_{i,r}^{-1})^{-\alpha_i^\vee(\nu)} \cdot f(\{x_{i,r}\}_{i\in I}^{1\le r\le k_i})$ for any $f \in S_{(k_1,\ldots,k_{n-1})}$.*

(b) *The homomorphisms $\iota_\nu^>$ and ι'_ν are compatible: $\iota'_\nu(\Psi(X)) = \Psi(\iota_\nu^>(X))$ for any $X \in U_{\boldsymbol{v}}^>$.*

(c) *ι'_ν is injective.*

Combining Theorem I.3 and Proposition I.4 immediately yields Theorem I.1(a).

This completes our proof of Theorem 10.19.

Appendix J Proof of Proposition 11.18

Consider the $n = 0$ case of Sect. 11.4. Let $\tilde{e}^\pm(z), \tilde{f}^\pm(z), \tilde{g}_1^\pm(z), \tilde{g}_2^\pm(z)$ be the currents entering the Gauss decomposition of $T^\pm(z)$, and set $\tilde{\psi}^\pm(z) := \tilde{g}_2^\pm(z)(\tilde{g}_1^\pm(z))^{-1}$. According to [17] (see also Theorem G.2) there is a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism

$$\Upsilon \colon U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_2) \xrightarrow{\sim} \mathfrak{U}_{0,0}^{\mathrm{rtt}}/(t_{11}^\pm[0]t_{11}^\mp[0]-1),$$

defined by

$$e^\pm(z) \mapsto \frac{\tilde{e}^\pm(\boldsymbol{v}z)}{\boldsymbol{v}-\boldsymbol{v}^{-1}},\ f^\pm(z) \mapsto \frac{\tilde{f}^\pm(\boldsymbol{v}z)}{\boldsymbol{v}-\boldsymbol{v}^{-1}},\ \psi^\pm(z) \mapsto \tilde{\psi}^\pm(\boldsymbol{v}z),\ \phi^\pm \mapsto t_{11}^\mp[0] \tag{J.1}$$

(a slight modification of $\Upsilon_{0,0}$). The isomorphism Υ intertwines coproducts $\Delta^{\mathrm{rtt}} := \Delta^{\mathrm{rtt}}_{0,0}$ and Δ^{ad}. In particular, the restriction of the pull-back of Δ^{rtt} to the subalgebra $U_{\boldsymbol{v}}(L\mathfrak{sl}_2)$ of $U_{\boldsymbol{v}}^{\mathrm{ad}}(L\mathfrak{sl}_2)$ recovers the Drinfeld-Jimbo coproduct Δ on $U_{\boldsymbol{v}}(L\mathfrak{sl}_2)$.

J(i) Computation of $\Delta(e^\pm(z))$ and $\Delta(f^\pm(z))$

The verification of formulas (11.10) and (11.11) is based on the following result.

Lemma J.1 *We have $T_{11}^\pm(z)^{-1}T_{21}^\pm(z) = \boldsymbol{v}\tilde{f}^\pm(\boldsymbol{v}^2z)$, $T_{12}^\pm(z)T_{11}^\pm(z)^{-1} = \boldsymbol{v}^{-1}\tilde{e}^\pm(\boldsymbol{v}^2z)$.*

Proof Comparing the matrix coefficients $\langle v_1 \otimes v_2|\cdots|v_1 \otimes v_1\rangle$ of both sides of the equality $R_{\mathrm{trig}}(z/w)(T^{\pm}(z)\otimes 1)(1\otimes T^{\pm}(w)) = (1\otimes T^{\pm}(w))(T^{\pm}(z)\otimes 1)R_{\mathrm{trig}}(z/w)$, we get

$$(z-w)T^{\pm}_{11}(z)T^{\pm}_{21}(w) + (\boldsymbol{v}-\boldsymbol{v}^{-1})zT^{\pm}_{21}(z)T^{\pm}_{11}(w) = (\boldsymbol{v}z-\boldsymbol{v}^{-1}w)T^{\pm}_{21}(w)T^{\pm}_{11}(z).$$

Plugging $w = \boldsymbol{v}^2 z$ into this identity, we obtain the first equality:

$$T^{\pm}_{11}(z)^{-1}T^{\pm}_{21}(z) = \boldsymbol{v}T^{\pm}_{21}(\boldsymbol{v}^2 z)T^{\pm}_{11}(\boldsymbol{v}^2 z)^{-1} = \boldsymbol{v}\tilde{f}^{\pm}(\boldsymbol{v}^2 z).$$

Likewise, comparing the matrix coefficients $\langle v_1\otimes v_1|\cdots|v_1\otimes v_2\rangle$, we get the second equality. □

- We have $\tilde{e}^{\pm}(z) = (T^{\pm}_{11}(z))^{-1}T^{\pm}_{12}(z)$. Hence,

$$\Delta^{\mathrm{rtt}}(\tilde{e}^{\pm}(z)) = \left(T^{\pm}_{11}(z)\otimes T^{\pm}_{11}(z) + T^{\pm}_{12}(z)\otimes T^{\pm}_{21}(z)\right)^{-1}\left(T^{\pm}_{11}(z)\otimes T^{\pm}_{12}(z) + T^{\pm}_{12}(z)\otimes T^{\pm}_{22}(z)\right) =$$
$$\left(1 + T^{\pm}_{11}(z)^{-1}T^{\pm}_{12}(z)\otimes T^{\pm}_{11}(z)^{-1}T^{\pm}_{21}(z)\right)^{-1}\left(1\otimes\tilde{e}^{\pm}(z) + \tilde{e}^{\pm}(z)\otimes T^{\pm}_{11}(z)^{-1}T^{\pm}_{22}(z)\right) =$$
$$\left(\sum_{r=0}^{\infty}(-\boldsymbol{v})^r\tilde{e}^{\pm}(z)^r\otimes\tilde{f}^{\pm}(\boldsymbol{v}^2 z)^r\right)\left(1\otimes\tilde{e}^{\pm}(z) + \tilde{e}^{\pm}(z)\otimes(\boldsymbol{v}\tilde{f}^{\pm}(\boldsymbol{v}^2 z)\tilde{e}^{\pm}(z) + \tilde{g}^{\pm}_1(z)^{-1}\tilde{g}^{\pm}_2(z))\right) =$$
$$1\otimes\tilde{e}^{\pm}(z) + \sum_{r=0}^{\infty}(-\boldsymbol{v})^r\cdot\tilde{e}^{\pm}(z)^{r+1}\otimes\tilde{f}^{\pm}(\boldsymbol{v}^2 z)^r\tilde{\psi}^{\pm}(z),$$

  where we used Lemma J.1 twice in the third equality. Applying Υ^{-1}, we recover (11.10).
- We have $\tilde{f}^{\pm}(z) = T^{\pm}_{21}(z)(T^{\pm}_{11}(z))^{-1}$. Hence,

$$\Delta^{\mathrm{rtt}}(\tilde{f}^{\pm}(z)) = \left(T^{\pm}_{21}(z)\otimes T^{\pm}_{11}(z) + T^{\pm}_{22}(z)\otimes T^{\pm}_{21}(z)\right)\left(T^{\pm}_{11}(z)\otimes T^{\pm}_{11}(z) + T^{\pm}_{12}(z)\otimes T^{\pm}_{21}(z)\right)^{-1} =$$
$$\left(\tilde{f}^{\pm}(z)\otimes 1 + T^{\pm}_{22}(z)T^{\pm}_{11}(z)^{-1}\otimes\tilde{f}^{\pm}(z)\right)\left(1 + T^{\pm}_{12}(z)T^{\pm}_{11}(z)^{-1}\otimes\tilde{f}^{\pm}(z)\right)^{-1} =$$
$$\left(\tilde{f}^{\pm}(z)\otimes 1 + (\boldsymbol{v}^{-1}\tilde{f}^{\pm}(z)\tilde{e}^{\pm}(\boldsymbol{v}^2 z) + \tilde{g}^{\pm}_2(z)\tilde{g}^{\pm}_1(z)^{-1})\otimes\tilde{f}^{\pm}(z)\right)\times$$
$$\left(\sum_{r=0}^{\infty}(-\boldsymbol{v})^{-r}\tilde{e}^{\pm}(\boldsymbol{v}^2 z)^r\otimes\tilde{f}^{\pm}(z)^r\right) = \tilde{f}^{\pm}(z)\otimes 1 + \sum_{r=0}^{\infty}(-\boldsymbol{v})^{-r}\cdot\tilde{\psi}^{\pm}(z)\tilde{e}^{\pm}(\boldsymbol{v}^2 z)^r\otimes\tilde{f}^{\pm}(z)^{r+1},$$

  where we used Lemma J.1 twice in the third equality. Applying Υ^{-1}, we recover (11.11).

J(ii) Computation of $\Delta(\psi^{\pm}(z))$

We have $\tilde{\psi}^{\pm}(z) = \tilde{g}^{\pm}(z)^{-1}\tilde{g}_2^{\pm}(z) = T_{11}^{\pm}(z)^{-1}T_{22}^{\pm}(z) - \boldsymbol{v}\tilde{f}^{\pm}(\boldsymbol{v}^2z)\tilde{e}^{\pm}(z)$, due to Lemma J.1. Evaluating $\Delta^{\mathrm{rtt}}(T_{11}^{\pm}(z)^{-1}T_{22}^{\pm}(z))$ as before, we get the following formula:

$$\Delta^{\mathrm{rtt}}(\tilde{\psi}^{\pm}(z)) = \sum_{r=0}^{\infty}(-1)^{r+1}\boldsymbol{v}^{r+2}\tilde{e}^{\pm}(z)^r[\tilde{e}^{\pm}(z), \tilde{f}^{\pm}(\boldsymbol{v}^2z)] \otimes \tilde{f}^{\pm}(\boldsymbol{v}^2z)^{r+1}\tilde{e}^{\pm}(z)+$$
$$\sum_{r=0}^{\infty}(-1)^r(\boldsymbol{v}^{r+1}\tilde{e}^{\pm}(z)^r\tilde{\psi}^{\pm}(z) - \boldsymbol{v}^{1-r}\tilde{\psi}^{\pm}(\boldsymbol{v}^2z)\tilde{e}^{\pm}(\boldsymbol{v}^4z)^r) \otimes \tilde{f}^{\pm}(\boldsymbol{v}^2z)^{r+1}\tilde{e}^{\pm}(z)+$$
$$\sum_{r=0}^{\infty}(-1)^r\boldsymbol{v}^{r+1}[\tilde{e}^{\pm}(z)^r, \tilde{f}^{\pm}(\boldsymbol{v}^2z)]\tilde{e}^{\pm}(z) \otimes \tilde{f}^{\pm}(\boldsymbol{v}^2z)^r\tilde{\psi}^{\pm}(z)+$$
$$\sum_{r=0}^{\infty}(-1)^r\boldsymbol{v}^r\tilde{e}^{\pm}(z)^r\tilde{\psi}^{\pm}(z) \otimes \tilde{f}^{\pm}(\boldsymbol{v}^2z)^r\tilde{\psi}^{\pm}(z)+$$
$$\sum_{r,s=0}^{\infty}(-1)^{r+s+1}\boldsymbol{v}^{-r+s+1}\tilde{\psi}^{\pm}(\boldsymbol{v}^2z)\tilde{e}^{\pm}(\boldsymbol{v}^4z)^r\tilde{e}^{\pm}(z)^{s+1} \otimes \tilde{f}^{\pm}(\boldsymbol{v}^2z)^{r+s+1}\tilde{\psi}^{\pm}(z). \tag{J.2}$$

To simplify the right-hand side of this equality, we need the following result.

Lemma J.2 *We have:*

(a) $[\tilde{e}^{\pm}(z), \tilde{f}^{\pm}(w)] = \frac{(\boldsymbol{v}-\boldsymbol{v}^{-1})z}{w-z}\cdot(\tilde{\psi}^{\pm}(z) - \tilde{\psi}^{\pm}(w))$.
(b) $[\tilde{e}^{\pm}(z), \tilde{f}^{\pm}(\boldsymbol{v}^2z)] = \frac{\tilde{\psi}^{\pm}(z)-\tilde{\psi}^{\pm}(\boldsymbol{v}^2z)}{\boldsymbol{v}}$.
(c) $(z-\boldsymbol{v}^2w)\tilde{\psi}^{\pm}(z)\tilde{e}^{\pm}(w) = (\boldsymbol{v}^2z-w)\tilde{e}^{\pm}(w)\tilde{\psi}^{\pm}(z) \pm w\cdot[\tilde{e}_0, \tilde{\psi}^{\pm}(z)]_{\boldsymbol{v}^2}$.
(d) $\tilde{\psi}^{\pm}(z)\tilde{e}^{\pm}(\boldsymbol{v}^2z) = \boldsymbol{v}^2\tilde{e}^{\pm}(\boldsymbol{v}^{-2}z)\tilde{\psi}^{\pm}(z) = \frac{\tilde{e}^{\pm}(z)\tilde{\psi}^{\pm}(z)+\tilde{\psi}^{\pm}(z)\tilde{e}^{\pm}(z)}{1+\boldsymbol{v}^{-2}}$.
(e) $(z-\boldsymbol{v}^2w)\tilde{e}^{\pm}(z)\tilde{e}^{\pm}(w) - z\cdot[\tilde{e}_0, \tilde{e}^{\pm}(w)]_{\boldsymbol{v}^2} = (\boldsymbol{v}^2z-w)\tilde{e}^{\pm}(w)\tilde{e}^{\pm}(z) + w\cdot[\tilde{e}_0, \tilde{e}^{\pm}(z)]_{\boldsymbol{v}^2}$.
(f) $\tilde{e}^{\pm}(\boldsymbol{v}^2z)^2 - (1+\boldsymbol{v}^2)\tilde{e}^{\pm}(z)\tilde{e}^{\pm}(\boldsymbol{v}^2z) + \boldsymbol{v}^2\tilde{e}^{\pm}(z)^2 = 0$.

Proof Parts (a, c, e) follow from the corresponding relations for $e^{\pm}(z), f^{\pm}(z), \psi^{\pm}(z)$, established in Lemma B.1(c, f1, d1), respectively.

Part (b) is obtained by specializing $w = \boldsymbol{v}^2z$ in (a). Part (d) is obtained by comparing the specializations of (c) at $w = \boldsymbol{v}^2z, w = \boldsymbol{v}^{-2}z$, and $w = z$. Part (f) is obtained by comparing the specializations of (e) at $w = \boldsymbol{v}^2z$ and $w = z$. □

The first two sums of (J.2) add up to zero, due to Lemma J.2(b, d). Applying Lemma J.2(b) to the third sum of (J.2) and Lemma J.2(d) to the last sum of (J.2), we get

$$\Delta^{\mathrm{rtt}}(\tilde{\psi}^{\pm}(z)) = \sum_{r=0}^{\infty}(-\boldsymbol{v})^r A_r(z) \otimes \tilde{f}^{\pm}(\boldsymbol{v}^2 z)^r \tilde{\psi}^{\pm}(z) \tag{J.3}$$

with

$$A_r(z) = \tilde{e}^{\pm}(z)^r \tilde{\psi}^{\pm}(z) + \tilde{e}^{\pm}(z)^{r-1}\tilde{\psi}^{\pm}(z)\tilde{e}^{\pm}(z) + \ldots + \tilde{e}^{\pm}(z)\tilde{\psi}^{\pm}(z)\tilde{e}^{\pm}(z)^{r-1} + \tilde{\psi}^{\pm}(z)\tilde{e}^{\pm}(z)^r.$$

Finally, a simple induction argument based on Lemma J.2(d, f) yields the equality

$$A_r(z) = \tilde{\psi}^{\pm}(z)\tilde{e}^{\pm}(\boldsymbol{v}^2 z)^r(1+\boldsymbol{v}^{-2}+\boldsymbol{v}^{-4}+\ldots+\boldsymbol{v}^{-2r}) = \boldsymbol{v}^{-r}[r+1]_{\boldsymbol{v}}\cdot\tilde{\psi}^{\pm}(z)\tilde{e}^{\pm}(\boldsymbol{v}^2 z)^r.$$

Plugging this into (J.3) and applying Υ^{-1}, we recover (11.12).

This completes our proof of Proposition 11.18.

Acknowledgements We are deeply grateful to M. Bershtein, R. Bezrukavnikov, A. Braverman, A. Brochier, S. Cautis, S. Cherkis, T. Dimofte, P. Etingof, B. Feigin, S. Gautam, M. Gekhtman, V. Ginzburg, D. Hernandez, D. Jordan, J. Kamnitzer, S. Khoroshkin, A. Marshakov, A. Molev, H. Nakajima, V. Pestun, L. Rybnikov, A. Shapiro, A. Weekes, M. Yakimov, whose generous help and advice was crucial in the process of our work. We also thank the anonymous referee.

M.F. was partially funded within the framework of the HSE University Basic Research Program and the Russian Academic Excellence Project '5-100'.

A.T. gratefully acknowledges support from the Simons Center for Geometry and Physics, Stony Brook University and is extremely grateful to IHES for invitation and wonderful working conditions in the summer 2017, where most of the research for this paper was performed. A.T. also thanks Yale University and Max Planck Institute for Mathematics in Bonn for the hospitality and support in the summer 2018, where the final version of this paper was completed. A.T. was partially supported by the NSF Grants DMS–1502497, DMS–1821185.

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