Letters in Mathematical Physics (2020) 110:2083–2111
https://doi.org/10.1007/s11005-020-01287-9

Shuffle algebra realizations of type *A* super Yangians and quantum affine superalgebras for all Cartan data

Alexander Tsymbaliuk[1]

Received: 21 October 2019 / Revised: 24 February 2020 / Accepted: 3 March 2020 /
Published online: 2 April 2020

Abstract
We introduce super Yangians of $\mathfrak{gl}(V)$, $\mathfrak{sl}(V)$ (in the new Drinfeld realization) associated with all Dynkin diagrams. We show that all of them are isomorphic to the super Yangians introduced by Nazarov (Lett Math Phys 21(2), 123–131, 1991), by identifying them with the corresponding RTT super Yangians. However, their "positive halves" are not pairwise isomorphic, and we obtain the shuffle algebra realizations for all of those in spirit of Tsymbaliuk (PBWD bases and shuffle algebra realizations for $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$, $U_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ and their integral forms, preprint, arXiv:1808.09536). We adapt the latter to the trigonometric setup by obtaining the shuffle algebra realizations of the "positive halves" of type A quantum loop superalgebras associated with arbitrary Dynkin diagrams.

Keywords Super Yangian · Shuffle algebra · quantum affine superalgebra

Mathematics Subject Classification 17B37 · 81R10

1 Introduction

1.1 Summary

Recall that a novel feature of Lie superalgebras (in contrast to Lie algebras) is that they admit several non-isomorphic Dynkin diagrams. The isomorphism of the Lie superalgebras corresponding to different Dynkin diagrams of finite/affine type has been obtained by Serganova in the Appendix to [16]. Likewise, one may define various quantizations of universal enveloping superalgebras starting from different Dynkin diagrams, and establishing their isomorphism is a non-trivial question. In the case of quantum finite/affine superalgebras, this question has been addressed 20 years ago by

✉ Alexander Tsymbaliuk
sashikts@gmail.com

1 Department of Mathematics, Yale University, New Haven, CT 06511, USA

Yamane [26]. However, an answer to a similar question for super Yangians seems to be missing in the literature.

In this short note, we study type A super Yangians and their shuffle realizations. We define those in the Drinfeld realization, generalizing the construction of [24] for a distinguished Dynkin diagram. Following [1,13,22], we obtain their RTT realization and thus identify all of them with the super Yangian of Nazarov [19]. We also describe their centers, following [13].

However, the "positive halves" of these algebras, denoted by $Y_\hbar^+$, do essentially depend on the choice of Dynkin diagrams. In the second part of this note, we obtain the shuffle algebra realizations of all such $Y_\hbar^+$ and their Drinfeld-Gavarini dual $\mathbf{Y}_\hbar^+$. We also establish the trigonometric (aka q-deformed) counterparts of these results.

This note is a companion to [25] (the shuffle realizations were announced in [25, §8.2]).

1.2 Outline of the paper

- In Sect. 2.2, we introduce the Drinfeld super Yangians $Y(\mathfrak{gl}(V))$ associated with arbitrary Dynkin diagrams of $\mathfrak{gl}(V)$, where $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a finite-dimensional superspace. For the distinguished Dynkin diagram, this recovers the super Yangians $Y_{m|n}(1)$ of [19], due to [13], while for a general Dynkin diagram, this recovers the construction of [22]. The key result of this section, Theorem 2.18, establishes that $Y(\mathfrak{gl}(V))$ is independent (up to isomorphisms) of the choice of Dynkin diagrams. The latter may be viewed as a rational counterpart of a similar statement for quantum affine superalgebras, due to [26], see Remark 2.19.
  Our proof of Theorem 2.18 is crucially based on the identification of $Y(\mathfrak{gl}(V))$ with the RTT super Yangians $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$ introduced in Sect. 2.3, see Theorem 2.32 and Lemma 2.24.
  In Sect. 2.4, we introduce the RTT super Yangians $Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$ following the classical approach of [18]. For $\dim(V_{\bar{0}}) \neq \dim(V_{\bar{1}})$, we obtain a decomposition $Y^{\mathrm{rtt}}(\mathfrak{gl}(V)) \simeq Y^{\mathrm{rtt}}(\mathfrak{sl}(V)) \otimes ZY^{\mathrm{rtt}}(\mathfrak{gl}(V))$, Theorem 2.48(a), similar to [18]. Here, $ZY^{\mathrm{rtt}}(\mathfrak{gl}(V))$ denotes the center of $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$, which is a polynomial algebra in the coefficients of the quantum Berezinian $b(z)$ defined in (2.41), Theorem 2.43 (for the distinguished Dynkin diagram, $b(z)$ coincides with the quantum Berezinian of [19], due to [12, Theorem 1]). In contrast, $ZY^{\mathrm{rtt}}(\mathfrak{gl}(V)) \subset Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$ if $\dim(V_{\bar{0}}) = \dim(V_{\bar{1}})$, Theorem 2.48(b), and we introduce the RTT super Yangian $Y^{\mathrm{rtt}}(A(V))$ as the corresponding central reduction of $Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$.
  In Sect. 2.5, we introduce the Drinfeld super Yangians $Y(\mathfrak{sl}(V))$ associated with arbitrary Dynkin diagrams of $\mathfrak{gl}(V)$ and construct superalgebra embeddings $Y(\mathfrak{sl}(V)) \hookrightarrow Y(\mathfrak{gl}(V))$ and isomorphisms $Y(\mathfrak{sl}(V)) \xrightarrow{\sim} Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$, Theorem 2.67. The latter implies that super Yangians $Y(\mathfrak{sl}(V))$ associated with various Dynkin diagrams are pairwise isomorphic, Theorem 2.69.
  In Sect. 2.6, we recall the PBW theorem and the triangular decomposition for $Y(\mathfrak{sl}(V))$, Theorem 2.72 and Proposition 2.73.
  In Sect. 2.7, we introduce a $\mathbb{C}[\hbar]$-version $Y_\hbar(\mathfrak{sl}(V))$ and its Drinfeld-Gavarini dual subalgebra $\mathbf{Y}_\hbar(\mathfrak{sl}(V))$. They can be viewed as Rees algebras (2.83) of $Y(\mathfrak{sl}(V))$

with respect to two standard filtrations on it defined via (2.84), Remark 2.82. The PBW Theorems for $Y_\hbar(\mathfrak{sl}(V))$ and $\mathbf{Y}_\hbar(\mathfrak{sl}(V))$, Theorems 2.79 and 2.81, follow from [11, Theorem B.3, Theorem A.7].

- In Sect. 3.1, we introduce the rational shuffle (super)algebra $\bar{W}^V$, Definition 3.6, which may be viewed as a rational super counterpart of the elliptic shuffle algebras of Feigin–Odesskii, [7–9]. It is related to the "positive half" $Y_\hbar^+(\mathfrak{sl}(V))$ of the super Yangian $Y_\hbar(\mathfrak{sl}(V))$ via an explicit homomorphism $\Psi\colon Y_\hbar^+(\mathfrak{sl}(V)) \to \bar{W}^V$ of Proposition 3.7. The injectivity of Ψ is established in Corollary 3.26. The key results of Sect. 3 describe the images of $Y_\hbar^+(\mathfrak{sl}(V))$ and its Drinfeld-Gavarini dual subalgebra $\mathbf{Y}_\hbar^+(\mathfrak{sl}(V))$ under Ψ, Theorems 3.30, 3.9. The latter is used to obtain a new proof of the PBW property for $\mathbf{Y}_\hbar^+(\mathfrak{sl}(V))$, Theorem 3.10.
  In Sect. 3.2, we establish the key result in the simplest case $\dim(V) = 2$, Theorem 3.11.
  In Sect. 3.3, we introduce our key technical tool in the study of the shuffle algebras, the *specialization maps* $\phi_{\underline{d}}$ (3.15). Their two main properties are established in Lemmas 3.17, 3.18, which immediately imply the injectivity of Ψ, Corollary 3.26.
  In Sect. 3.4, we finally describe the images of $Y_\hbar^+(\mathfrak{sl}(V))$ and its subalgebra $\mathbf{Y}_\hbar^+(\mathfrak{sl}(V))$, Theorems 3.30, 3.9. The former consists of all *good* shuffle elements, Definition 3.27, while the latter consists of all *integral* shuffle elements, Definition 3.8. We also prove Theorem 3.10.
- In Sect. 4, we recall the definition of $U_{\boldsymbol{v}}^>(L\mathfrak{gl}(V))$, the "positive half" of the quantum loop superalgebra of $\mathfrak{gl}(V)$, and obtain its shuffle algebra realization, Theorem 4.14. This provides the trigonometric counterpart of Theorem 3.30 and generalizes [25, Theorem 5.17], where this result was established for the distinguished Dynkin diagram of $\mathfrak{gl}(V)$.

2 Type *A* super Yangians

2.1 Setup and notations

Consider a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with a $\mathbb{C}$-basis $\mathsf{v}_1, \ldots, \mathsf{v}_n$ such that each v_i is either *even* ($\mathsf{v}_i \in V_{\bar{0}}$) or *odd* ($\mathsf{v}_i \in V_{\bar{1}}$). We set $n_+ := \dim(V_{\bar{0}})$, $n_- := \dim(V_{\bar{1}})$, and $n := n_+ + n_- = \dim(V)$. For $1 \le i \le n$, define $\bar{i} \in \mathbb{Z}_2$ via $\bar{i} = \begin{cases} \bar{0}, & \text{if } \mathsf{v}_i \in V_{\bar{0}} \\ \bar{1}, & \text{if } \mathsf{v}_i \in V_{\bar{1}} \end{cases}$. Consider a free $\mathbb{Z}$-module $P := \oplus_{i=1}^n \mathbb{Z}\epsilon_i$ with the bilinear form determined by $(\epsilon_i, \epsilon_j) = \delta_{ij}(-1)^{\bar{i}}$ (we set $(-1)^{\bar{0}} := 1$ and, $(-1)^{\bar{1}} := -1$). For $1 \le i < n$, let $\alpha_i := \epsilon_i - \epsilon_{i+1} \in P$ be the simple roots of $\mathfrak{gl}(V)$, and $\Delta^+ := \{\epsilon_j - \epsilon_i\}_{1 \le j < i \le n} \subset P$ be the set of positive roots of $\mathfrak{gl}(V)$. Let $I = \{1, 2, \ldots, n-1\}$ and set $|\alpha_i| := \bar{i} + \overline{i+1} \in \mathbb{Z}_2$ for $i \in I$. Finally, let $(c_{ij})_{i,j \in I}$ be the associated Cartan matrix, that is, $c_{ij} := (\alpha_i, \alpha_j)$.

For a superalgebra A and its two homogeneous elements x, x', we define

$$[x, x'] := xx' - (-1)^{|x|\cdot|x'|}x'x \quad \text{and} \quad \{x, x'\} := xx' + (-1)^{|x|\cdot|x'|}x'x, \tag{2.1}$$

where $|x|$ denotes the $\mathbb{Z}_2$-grading of x (that is, $x \in A_{|x|}$).

Given two superspaces $A = A_{\bar{0}} \oplus A_{\bar{1}}$ and $B = B_{\bar{0}} \oplus B_{\bar{1}}$, their tensor product $A \otimes B$ is also a superspace with $(A \otimes B)_{\bar{0}} = A_{\bar{0}} \otimes B_{\bar{0}} \oplus A_{\bar{1}} \otimes B_{\bar{1}}$ and $(A \otimes B)_{\bar{1}} = A_{\bar{0}} \otimes B_{\bar{1}} \oplus A_{\bar{1}} \otimes B_{\bar{0}}$. Furthermore, if A and B are superalgebras, then $A \otimes B$ is made into a superalgebra, the *graded tensor product* of the superalgebras A and B, via the following multiplication:

$$\begin{gathered}(x \otimes y)(x' \otimes y') = (-1)^{|y|\cdot|x'|}(xx') \otimes (yy') \\ \text{for any } x \in A_{|x|},\, x' \in A_{|x'|},\, y \in B_{|y|},\, y' \in B_{|y'|}.\end{gathered} \tag{2.2}$$

We will use only graded tensor products of superalgebras throughout this paper.

2.2 The Drinfeld super Yangian of $\mathfrak{gl}(V)$

Following [1,3,13,22], define the *Drinfeld super Yangian of* $\mathfrak{gl}(V)$, denoted by $Y(\mathfrak{gl}(V))$, to be the associative $\mathbb{C}$-superalgebra generated by $\{d_j^{(s)}, \widetilde{d}_j^{(s)}, e_i^{(r)}, f_i^{(r)}\}_{1\le i<n, 1\le j\le n}^{r\ge 1, s\ge 0}$ with the $\mathbb{Z}_2$-grading $|d_j^{(r)}| = |\widetilde{d}_j^{(r)}| = \bar{0}$, $|e_i^{(r)}| = |f_i^{(r)}| = |\alpha_i|$, and subject to the following defining relations:

$$d_j^{(0)} = 1,\ \widetilde{d}_j^{(0)} = 1,\ \sum_{t=0}^{r} \widetilde{d}_i^{(t)} d_i^{(r-t)} = \delta_{r,0}, \tag{2.3}$$

$$[d_i^{(r)}, d_j^{(s)}] = 0, \tag{2.4}$$

$$[d_i^{(r)}, e_j^{(s)}] = (-1)^{\bar{i}}(\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} d_i^{(t)} e_j^{(r+s-t-1)}, \tag{2.5}$$

$$[d_i^{(r)}, f_j^{(s)}] = (-1)^{\bar{i}}(-\delta_{i,j} + \delta_{i,j+1}) \sum_{t=0}^{r-1} f_j^{(r+s-t-1)} d_i^{(t)}, \tag{2.6}$$

$$[e_i^{(r)}, f_j^{(s)}] = -(-1)^{\overline{i+1}} \delta_{i,j} \sum_{t=0}^{r+s-1} \widetilde{d}_i^{(t)} d_{i+1}^{(r+s-t-1)}, \tag{2.7}$$

$$\begin{aligned}
&[e_i^{(r)}, e_j^{(s)}] = 0 \text{ if } c_{ij} = 0,\\
&[e_i^{(r)}, e_{i+1}^{(s+1)}] - [e_i^{(r+1)}, e_{i+1}^{(s)}] = -(-1)^{\overline{i+1}} e_i^{(r)} e_{i+1}^{(s)},\\
&[e_i^{(r)}, e_i^{(s)}] = (-1)^{\bar{i}} \sum_{t=1}^{s-1} e_i^{(t)} e_i^{(r+s-t-1)} - (-1)^{\bar{i}} \sum_{t=1}^{r-1} e_i^{(t)} e_i^{(r+s-t-1)} \text{ if } |\alpha_i| = \bar{0},
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
&[f_i^{(r)}, f_j^{(s)}] = 0 \text{ if } c_{ij} = 0,\\
&[f_{i+1}^{(s+1)}, f_i^{(r)}] - [f_{i+1}^{(s)}, f_i^{(r+1)}] = -(-1)^{\overline{i+1}} f_{i+1}^{(s)} f_i^{(r)},\\
&[f_i^{(r)}, f_i^{(s)}] = (-1)^{\bar{i}} \sum_{t=1}^{r-1} f_i^{(r+s-t-1)} f_i^{(t)} - (-1)^{\bar{i}} \sum_{t=1}^{s-1} f_i^{(r+s-t-1)} f_i^{(t)} \text{ if } |\alpha_i| = \bar{0},
\end{aligned} \tag{2.9}$$

as well as cubic Serre relations

$$[e_i^{(r)},[e_i^{(s)},e_j^{(t)}]]+[e_i^{(s)},[e_i^{(r)},e_j^{(t)}]]=0 \text{ if } j=i\pm 1 \text{ and } |\alpha_i|=\bar{0}, \tag{2.10}$$

$$[f_i^{(r)},[f_i^{(s)},f_j^{(t)}]]+[f_i^{(s)},[f_i^{(r)},f_j^{(t)}]]=0 \text{ if } j=i\pm 1 \text{ and } |\alpha_i|=\bar{0}, \tag{2.11}$$

and quartic Serre relations

$$[[e_{j-1}^{(r)},e_j^{(1)}],[e_j^{(1)},e_{j+1}^{(s)}]]=0 \text{ if } |\alpha_j|=\bar{1} \text{ and } |\alpha_{j-1}|=|\alpha_{j+1}|=\bar{0}, \tag{2.12}$$

$$[[f_{j-1}^{(r)},f_j^{(1)}],[f_j^{(1)},f_{j+1}^{(s)}]]=0 \text{ if } |\alpha_j|=\bar{1} \text{ and } |\alpha_{j-1}|=|\alpha_{j+1}|=\bar{0}. \tag{2.13}$$

Remark 2.14 (a) The cubic Serre relations (2.10, 2.11) are also valid for $|\alpha_i|=\bar{1}$, but in that case, they already follow from $[e_i^{(r)},e_i^{(s)}]=0=[f_i^{(r)},f_i^{(s)}]$, due to quadratic relations (2.8, 2.9).

(b) The quartic Serre relations (2.12, 2.13) are also valid for any other parities of $\alpha_{j-1},\alpha_j,\alpha_{j+1}$, but in those cases, they already follow from the quadratic and cubic relations (2.8–2.11).

(c) Generalizing the quartic Serre relations (2.12, 2.13), the following relations also hold:

$$[[e_{j-1}^{(r)},e_j^{(k)}],[e_j^{(l)},e_{j+1}^{(s)}]]+[[e_{j-1}^{(r)},e_j^{(l)}],[e_j^{(k)},e_{j+1}^{(s)}]]=0, \tag{2.15}$$

$$[[f_{j-1}^{(r)},f_j^{(k)}],[f_j^{(l)},f_{j+1}^{(s)}]]+[[f_{j-1}^{(r)},f_j^{(l)}],[f_j^{(k)},f_{j+1}^{(s)}]]=0, \tag{2.16}$$

cf. Remark 2.61(b) and the explanations therein. We note that these relations (2.15, 2.16) play a crucial role in the recent paper [23].

As pointed out to us by Peng, the above definition of $Y(\mathfrak{gl}(V))$ is actually equivalent to the one from [21]. In the particular case (associated with the distinguished Dynkin diagram)

$$\mathsf{v}_1,\ldots,\mathsf{v}_{n_+}\in V_{\bar{0}} \text{ and } \mathsf{v}_{n_++1},\ldots,\mathsf{v}_n\in V_{\bar{1}} \tag{2.17}$$

(so that $|\alpha_{n_+}|=\bar{1}$ and $|\alpha_{i\neq n_+}|=\bar{0}$), the defining relations (2.3–2.13) first appeared in [13, Theorem 3],[1] where it was shown that the corresponding super Yangian is isomorphic to the super Yangian $Y^{\mathrm{rtt}}(\mathfrak{gl}_{n_+|n_-})$ first introduced in [19] (thus generalizing [1, Theorem 5.2]). The same arguments can be used to establish the following result (mentioned first in [21]):

Theorem 2.18 *The superalgebra $Y(\mathfrak{gl}(V))$ depends only on (n_+,n_-), up to an isomorphism.*

This is a direct consequence of Theorem 2.32 and Lemma 2.24 (see Remark 2.35).

[1] We note the following typos in [13]: $j\leq m+1$ should be replaced by $j\geq m+1$ in the third line of (39), the sign $(-1)^{\bar{j}}$ should be replaced by $(-1)^{\overline{j+1}}$ in the right-hand sides of (44, 45).

Remark 2.19 The quantum affine superalgebras corresponding to different Dynkin diagrams of the same affine Lie superalgebra are known to be pairwise isomorphic, due to [26]. A similar statement for super Yangians seems to be missing in the literature. Thus, Theorem 2.18 and its $\mathfrak{sl}(V)$-counterpart, Theorem 2.69, fill this gap at least in type A.

2.3 The RTT super Yangian of $\mathfrak{gl}(V)$

Let $P\colon V\otimes V\to V\otimes V$ be the permutation operator given by $P:=\sum_{i,j}(-1)^{\overline{j}}E_{ij}\otimes E_{ji}$, so that $P(\mathsf{v}_j\otimes\mathsf{v}_i)=(-1)^{\overline{i}\cdot\overline{j}}\mathsf{v}_i\otimes\mathsf{v}_j$. Consider the *rational* R-matrix $R_{\mathrm{rat}}(z)=1-\frac{1}{z}P\in(\mathrm{End}\ V)^{\otimes 2}$.

Following [10,17,19,21], define the *RTT super Yangian of* $\mathfrak{gl}(V)$, denoted by $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$, to be the associative $\mathbb{C}$-superalgebra generated by $\{t^{(r)}_{ij}\}^{r\geq 1}_{1\leq i,j\leq n}$ with the $\mathbb{Z}_2$-grading $|t^{(r)}_{ij}|=\overline{i}+\overline{j}$ and subject to the following defining relation:

$$R_{\mathrm{rat}}(z-w)T_1(z)T_2(w)=T_2(w)T_1(z)R_{\mathrm{rat}}(z-w). \tag{2.20}$$

Here, $T(z)$ is the series in z^{-1} with coefficients in the algebra $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))\otimes\mathrm{End}\ V$, defined by

$$T(z)=\sum_{i,j}(-1)^{\overline{j}(\overline{i}+1)}t_{ij}(z)\otimes E_{ij}\ \text{with}\ t_{ij}(z):=\delta_{i,j}+\sum_{r\geq 1}t^{(r)}_{ij}z^{-r}. \tag{2.21}$$

Remark 2.22 Here, we identify the operator $\sum_{i,j=1}^{n}(-1)^{\overline{j}(\overline{i}+1)}t_{ij}(z)\otimes E_{ij}$ with the matrix $(t_{ij}(z))_{i,j=1}^{n}$. Evoking the multiplication (2.2) on the graded tensor products, we see that the extra sign $(-1)^{\overline{j}(\overline{i}+1)}$ ensures that the product of two matrices is calculated in the usual way.

Multiplying both sides of (2.20) by $z-w$, we obtain an equality of series in z,w with coefficients in $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))\otimes(\mathrm{End}\ V)^{\otimes 2}$. Thus, relation (2.20) is equivalent to the following relations:

$$(z-w)[t_{ij}(z),t_{kl}(w)]=(-1)^{\overline{i}\cdot\overline{j}+\overline{i}\cdot\overline{k}+\overline{j}\cdot\overline{k}}\left(t_{kj}(z)t_{il}(w)-t_{kj}(w)t_{il}(z)\right) \tag{2.23}$$

for all $1\leq i,j,k,l\leq n$.

In the particular case (2.17), we recover the super Yangian $Y^{\mathrm{rtt}}(\mathfrak{gl}_{n_+|n_-})$ of [19] (denoted by $Y_{n_+|n_-}(1)$ in [19]), while for a general case, we actually get isomorphic algebras, due to the following simple result:

Lemma 2.24 *The superalgebra $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$ depends only on (n_+,n_-), up to an isomorphism. In particular, $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$ is isomorphic to the super Yangian $Y^{\mathrm{rtt}}(\mathfrak{gl}_{n_+|n_-})$ of* [19].

Proof Let V' be another superspace with a $\mathbb{C}$-basis $\mathsf{v}'_1,\ldots,\mathsf{v}'_n$ such that each v'_i is either *even* or *odd* and $n'_+=n_+, n'_-=n_-$. Pick a permutation $\sigma\in\Sigma_n$, such that $\mathsf{v}_i\in V$

and $\mathsf{v}'_{\sigma(i)} \in V'$ have the same parity for all i. Then, the assignment $t^{(r)}_{ij} \mapsto t^{(r)}_{\sigma(i),\sigma(j)}$ is compatible with the defining relations (2.23), thus giving rise to an isomorphism $Y^{\mathrm{rtt}}(\mathfrak{gl}(V)) \xrightarrow{\sim} Y^{\mathrm{rtt}}(\mathfrak{gl}(V'))$. □

We also have two standard relations between $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$ and $U(\mathfrak{gl}(V))$ (cf. [13,22]):

Lemma 2.25 (a) *The assignment* $E_{ij} \mapsto (-1)^{\bar{i}} t^{(1)}_{ij}$ *gives rise to a superalgebra embedding*

$$\iota\colon U(\mathfrak{gl}(V)) \hookrightarrow Y^{\mathrm{rtt}}(\mathfrak{gl}(V)).$$

(b) *The assignment* $t^{(r)}_{ij} \mapsto (-1)^{\bar{i}} \delta_{r,1} E_{ij}$ *gives rise to a superalgebra epimorphism*

$$\mathrm{ev}\colon Y^{\mathrm{rtt}}(\mathfrak{gl}(V)) \twoheadrightarrow U(\mathfrak{gl}(V)).$$

Proof Straightforward. □

The superalgebra $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$ is also endowed with two different filtrations, defined via

$$\deg_1(t^{(r)}_{ij}) = r \quad \text{and} \quad \deg_2(t^{(r)}_{ij}) = r - 1. \tag{2.26}$$

Let $\mathrm{gr}_1 Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$, $\mathrm{gr}_2 Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$ denote the corresponding associated graded superalgebras.

Lemma 2.27 (a) *The assignment* $t^{(r)}_{ij} \mapsto \mathsf{t}^{(r)}_{ij}$ *gives rise to a superalgebra isomorphism*

$$\mathrm{gr}_1 Y^{\mathrm{rtt}}(\mathfrak{gl}(V)) \xrightarrow{\sim} \mathbb{C}[\{\mathsf{t}^{(r)}_{ij}\}^{r\geq 1}_{1\leq i,j\leq n}] \tag{2.28}$$

with the polynomial superalgebra in the variables $\mathsf{t}^{(r)}_{ij}$ *with the* $\mathbb{Z}_2$*-grading* $|\mathsf{t}^{(r)}_{ij}| = \bar{i} + \bar{j}$.

(b) *The assignment* $t^{(r)}_{ij} \mapsto (-1)^{\bar{i}} E_{ij} \cdot t^{r-1}$ *gives rise to a superalgebra isomorphism*

$$\mathrm{gr}_2 Y^{\mathrm{rtt}}(\mathfrak{gl}(V)) \xrightarrow{\sim} U(\mathfrak{gl}(V)[t]) \tag{2.29}$$

with the universal enveloping of $\mathfrak{gl}(V)[t] = \mathfrak{gl}(V) \otimes \mathbb{C}[t]$.

Proof Analogous to [13, Theorem 1, Corollary 1], cf. [22, Proposition 2.2, Corollary 2.3]. □

Let us now relate $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$ to $Y(\mathfrak{gl}(V))$. Consider the Gauss decomposition of $T(z)$:

$$T(z) = F(z) \cdot D(z) \cdot E(z).$$

Here, $F(z), D(z), E(z) \in \big(Y^{\mathrm{rtt}}(\mathfrak{gl}(V)) \otimes \mathrm{End}\, V\big)[[z^{-1}]]$ are of the form

$$F(z) = \sum_i E_{ii} + \sum_{j<i} (-1)^{\bar{j}(\bar{i}+1)} F_{ij}(z) \otimes E_{ij},\ \ D(z) = \sum_i D_i(z) \otimes E_{ii},$$
$$E(z) = \sum_i E_{ii} + \sum_{j<i} (-1)^{\bar{i}(\bar{j}+1)} E_{ji}(z) \otimes E_{ji},$$

cf. Remark 2.22. Define the elements $\{D_k^{(s)}, \widetilde{D}_k^{(s)}, E_{ji}^{(r)}, F_{ij}^{(r)}\}_{1\leq j<i\leq n, 1\leq k\leq n}^{r\geq 1, s\geq 0}$ of $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$ via

$$E_{ji}(z) = \sum_{r\geq 1} E_{ji}^{(r)} z^{-r},\ \ F_{ij}(z) = \sum_{r\geq 1} F_{ij}^{(r)} z^{-r},$$
$$D_k(z) = \sum_{s\geq 0} D_k^{(s)} z^{-s},\ \ D_k(z)^{-1} = \sum_{s\geq 0} \widetilde{D}_k^{(s)} z^{-s}.$$

For $1 \leq i < n$ and $r \geq 1$, set $E_i^{(r)} := E_{i,i+1}^{(r)}$ and $F_i^{(r)} := F_{i+1,i}^{(r)}$. Due to [22, Lemma 3.3] (generalizing [1, (5.5)] in the classical setup as well as [13, (10)] for the distinguished Dynkin diagram), we have:

Lemma 2.30 *For any $1 \leq j < i-1 < n$, the following equalities hold in $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$:*

$$E_{ji}^{(r)} = (-1)^{\overline{i-1}} [E_{j,i-1}^{(r)}, E_{i-1}^{(1)}],\ \ F_{ij}^{(r)} = (-1)^{\overline{i-1}} [F_{i-1}^{(1)}, F_{i-1,j}^{(r)}].$$

Corollary 2.31 *$Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$ is generated by $\{D_j^{(s)}, \widetilde{D}_j^{(s)}, E_i^{(r)}, F_i^{(r)}\}_{1\leq i<n, 1\leq j\leq n}^{r\geq 1, s\geq 0}$.*

Similar to [1,4,13,22], we have the following result:

Theorem 2.32 *There is a unique superalgebra isomorphism*

$$\Upsilon\colon Y(\mathfrak{gl}(V)) \xrightarrow{\sim} Y^{\mathrm{rtt}}(\mathfrak{gl}(V)) \tag{2.33}$$

defined by $e_i^{(r)} \mapsto E_i^{(r)}$, $f_i^{(r)} \mapsto F_i^{(r)}$, $d_j^{(s)} \mapsto D_j^{(s)}$, $\widetilde{d}_j^{(s)} \mapsto \widetilde{D}_j^{(s)}$.

Proof The proof is completely analogous to that in the classical case (when $n_- = 0$) presented in [1, §5]; see [13, Theorem 3] for the particular case of (2.17). □

Remark 2.34 The presence of the quartic Serre relations (2.12, 2.13) is solely due to the fact that they also appear among the defining relations of the Lie superalgebra $\mathfrak{gl}(V)$ via the Chevalley generators [16] (see the argument right after [13, (59)]).

Remark 2.35 Theorem 2.32 together with Lemma 2.24 implies Theorem 2.18.

2.4 The RTT super Yangians of $\mathfrak{sl}(V)$, $A(V)$

For any formal power series $f(z) \in 1 + z^{-1}\mathbb{C}[[z^{-1}]]$, the assignment

$$\mu_f : T(z) \mapsto f(z)T(z) \tag{2.36}$$

defines a superalgebra automorphism μ_f of $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$. Following [18], define the *RTT super Yangian of* $\mathfrak{sl}(V)$, denoted by $Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$, as the $\mathbb{C}$-subalgebra of $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$ via

$$Y^{\mathrm{rtt}}(\mathfrak{sl}(V)) := \{y \in Y^{\mathrm{rtt}}(\mathfrak{gl}(V)) | \mu_f(y) = y \text{ for all } f\}. \tag{2.37}$$

In the particular case (2.17), this recovers the super Yangian $Y^{\mathrm{rtt}}(\mathfrak{sl}_{n_+|n_-})$ of [13, §8].

In view of Lemma 2.24, we immediately obtain:

Corollary 2.38 *The superalgebra $Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$ depends only on (n_+, n_-), up to an isomorphism. In particular, $Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$ is isomorphic to the super Yangian $Y^{\mathrm{rtt}}(\mathfrak{sl}_{n_+|n_-})$ of* [13].

Explicitly, this subalgebra can be described as follows:

Lemma 2.39 *$Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$ is generated by coefficients of $D_i(z)^{-1}D_{i+1}(z)$, $E_{i,i+1}(z)$, $F_{i+1,i}(z)$.*

Proof Completely analogous to the proof of [13, Lemma 7] for the particular case of (2.17). □

Definition 2.40 Define the *charge* $c(V) \in \mathbb{Z}$ of V via

$$c(V) := n_+ - n_- = \dim(V_{\bar{0}}) - \dim(V_{\bar{1}}).$$

If V has a nonzero charge, then $Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$ also may be realized as a quotient of $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$. For the latter construction, let us first obtain an explicit description of the center $ZY^{\mathrm{rtt}}(\mathfrak{gl}(V))$ of $Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$. Following [12], define the *quantum Berezinian* $b(z) \in Y^{\mathrm{rtt}}(\mathfrak{gl}(V))[[z^{-1}]]$ via

$$b(z) := 1 + \sum_{r\geq 1} b_r z^{-r} = D'_1(z_1)D'_2(z_2)\cdots D'_n(z_n), \tag{2.41}$$

where $D'_i(z) := \begin{cases} D_i(z), & \text{if } \bar{i} = \bar{0} \\ D_i(z)^{-1}, & \text{if } \bar{i} = \bar{1} \end{cases}$, while $z_1 = z$ and $z_{i+1} = \begin{cases} z_i + c_{i,i+1}, & \text{if } |\alpha_i| = \bar{0} \\ z_i, & \text{if } |\alpha_i| = \bar{1} \end{cases}$.

Remark 2.42 For the distinguished Dynkin diagram, that is for (2.17), this definition recovers the original *quantum Berezinian* of [19, §2], due to the main result (Theorem 1) of [12].

Theorem 2.43 (a) *The elements* $\{b_r\}_{r\geq 1}$ *are central.*

(b) *The elements* $\{b_r\}_{r\geq 1}$ *are algebraically independent and generate the center* $ZY^{\mathrm{rtt}}(\mathfrak{gl}(V))$. *In other words, we have an algebra isomorphism* $ZY^{\mathrm{rtt}}(\mathfrak{gl}(V)) \simeq \mathbb{C}[b_1, b_2, \ldots]$.

Proof (a) To prove that all b_r are central, it suffices to verify that $[b(z), E_i(w)] = 0 = [b(z), F_i(w)]$ for any $1 \leq i < n$. We shall check only the first equality (the second is analogous).

Case 1: $|\alpha_i| = \bar{0}$.

Due to the isomorphism of Theorem 2.32 and the relation (2.5), we have

$$(u-v)E_i(v)D_i(u) = (u-v-(-1)^{\bar{i}})D_i(u)E_i(v) + (-1)^{\bar{i}}D_i(u)E_i(u), \tag{2.44}$$

$$(w-v)E_i(v)D_{i+1}(w) = (w-v+(-1)^{\overline{i+1}})D_{i+1}(w)E_i(v) - (-1)^{\overline{i+1}}D_{i+1}(w)E_i(w). \tag{2.45}$$

Plugging $v=u$, $w=u-(-1)^{\bar{i}}$ into (2.45) and using $\bar{i} = \overline{i+1}$ (as $|\alpha_i| = \bar{0}$), we get

$$E_i(u)D_{i+1}(u-(-1)^{\bar{i}}) = D_{i+1}(u-(-1)^{\bar{i}})E_i(u-(-1)^{\bar{i}}). \tag{2.46}$$

Due to (2.44–2.46):

$$(u-v)E_i(v)D_i(u)D_{i+1}(u-(-1)^{\bar{i}}) = (u-v)D_i(u)D_{i+1}(u-(-1)^{\bar{i}})E_i(v).$$

Hence, $[b(z), E_i(w)] = 0$ as $c_{i,i+1} = -(-1)^{\overline{i+1}} = -(-1)^{\bar{i}}$ and $[E_i(v), D_j(u)] = 0$ for $j \neq i, i+1$.

Case 2: $|\alpha_i| = \bar{1}$.

In this case, $(-1)^{\overline{i+1}} = -(-1)^{\bar{i}}$ and the equality (2.45) is equivalent to

$$(w-v)D_{i+1}(w)^{-1}E_i(v) = (w-v-(-1)^{\bar{i}})E_i(v)D_{i+1}(w)^{-1} + (-1)^{\bar{i}}E_i(w)D_{i+1}(w)^{-1}. \tag{2.47}$$

Combining the equalities (2.44, 2.47), we immediately obtain

$$(u-v)E_i(v)D_i(u)D_{i+1}(u)^{-1} = (u-v)D_i(u)D_{i+1}(u)^{-1}E_i(v).$$

Hence, $[b(z), E_i(w)] = 0$ as $[E_i(v), D_j(u)] = 0$ for $j \neq i, i+1$.
This completes the proof of part (a).

(b) The proof of part (b) is analogous to that of [18, Theorem 2.13] and [1, Theorem 7.2] in the classical case (when $n_- = 0$), and of [13, Theorem 4] for the particular case (2.17). □

Similar to the classical case (when $n_- = 0$) treated in [18] as well as the particular case (2.17) treated in [13], we have:

Theorem 2.48 (a) *If $c(V) \neq 0$, then we have a superalgebra isomorphism*

$$Y^{\mathrm{rtt}}(\mathfrak{gl}(V)) \simeq Y^{\mathrm{rtt}}(\mathfrak{sl}(V)) \otimes ZY^{\mathrm{rtt}}(\mathfrak{gl}(V)). \tag{2.49}$$

(b) *If $c(V) = 0$, then $ZY^{\mathrm{rtt}}(\mathfrak{gl}(V)) \subset Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$.*

Proof (a) Analogous to the proof of [13, Proposition 3] for the particular case of (2.17).
(b) If $n_+ = n_-$, then z_i defined after (2.41) satisfy $\{z_i | \bar{i} = \bar{0}\} = \{z_i | \bar{i} = \bar{1}\}$. Hence, $\mu_f(b(z)) = b(z)$ for all automorphisms (2.36). Thus, $ZY^{\mathrm{rtt}}(\mathfrak{gl}(V)) \subset Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$ by Theorem 2.43. □

Corollary 2.50 *If $c(V) \neq 0$, then the isomorphism (2.49) gives rise to a natural epimorphism $\pi : Y^{\mathrm{rtt}}(\mathfrak{gl}(V)) \twoheadrightarrow Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$ with $\mathrm{Ker}(\pi) = (b_1, b_2, \ldots)$.*

Recall that the classical Lie superalgebra $A(n_+ - 1, n_- - 1)$ coincides with $\mathfrak{sl}(n_+|n_-)$ for $n_+ \neq n_-$, and with the quotient $\mathfrak{sl}(n_+|n_-)/(I)$ for $n_+ = n_-$, where $I = \sum_{i=1}^{n} E_{ii}$ is the central element. Motivated by this and Theorem 2.48(b), if $c(V) = 0$, define the *RTT super Yangian of* $A(V)$, denoted by $Y^{\mathrm{rtt}}(A(V))$, via $Y^{\mathrm{rtt}}(A(V)) := Y^{\mathrm{rtt}}(\mathfrak{sl}(V))/(b_1, b_2, \ldots)$, cf. [13, (67)].

Corollary 2.51 *$Y^{\mathrm{rtt}}(A(V))$ depends only on $n_+ = n_-$, up to an isomorphism.*

Proof Similar to [13, Corollary 2], the center $ZY^{\mathrm{rtt}}(\mathfrak{sl}(V))$ of $Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$ is a polynomial algebra in $\{b_r\}_{r=1}^{\infty}$. Combining this with Corollary 2.38 implies the result. □

2.5 The Drinfeld super Yangian of $\mathfrak{sl}(V)$

Following [3] (cf. [24][2] and [13]), define the *Drinfeld super Yangian of* $\mathfrak{sl}(V)$, denoted by $Y(\mathfrak{sl}(V))$, to be the associative $\mathbb{C}$-superalgebra generated by $\{h_{i,r}, \mathsf{x}^{\pm}_{i,r}\}_{1\leq i<n}^{r\geq 0}$ with the $\mathbb{Z}_2$-grading $|h_{i,r}| = \bar{0}$, $|\mathsf{x}^{\pm}_{i,r}| = |\alpha_i|$, and subject to the following defining relations:

$$[h_{i,r}, h_{j,s}] = 0, \tag{2.52}$$

$$[h_{i,0}, \mathsf{x}^{\pm}_{j,s}] = \pm c_{ij}\mathsf{x}^{\pm}_{j,s}, \tag{2.53}$$

$$[h_{i,r+1}, \mathsf{x}^{\pm}_{j,s}] - [h_{i,r}, \mathsf{x}^{\pm}_{j,s+1}] = \pm\frac{c_{ij}}{2}\{h_{i,r}, \mathsf{x}^{\pm}_{j,s}\} \text{ unless } i = j \text{ and } |\alpha_i| = \bar{1}, \tag{2.54}$$

$$[h_{i,r}, \mathsf{x}^{\pm}_{i,s}] = 0 \text{ if } |\alpha_i| = \bar{1}, \tag{2.55}$$

$$[\mathsf{x}^{+}_{i,r}, \mathsf{x}^{-}_{j,s}] = \delta_{i,j} h_{i,r+s}, \tag{2.56}$$

$$[\mathsf{x}^{\pm}_{i,r+1}, \mathsf{x}^{\pm}_{j,s}] - [\mathsf{x}^{\pm}_{i,r}, \mathsf{x}^{\pm}_{j,s+1}] = \pm\frac{c_{ij}}{2}\{\mathsf{x}^{\pm}_{i,r}, \mathsf{x}^{\pm}_{j,s}\} \text{ unless } i = j \text{ and } |\alpha_i| = \bar{1}, \tag{2.57}$$

$$[\mathsf{x}^{\pm}_{i,r}, \mathsf{x}^{\pm}_{j,s}] = 0 \text{ if } c_{ij} = 0, \tag{2.58}$$

[2] As noticed in [13], the relation (2.62) should replace the wrong quartic Serre relations of [24, Definition 2].

as well as cubic Serre relations

$$[\mathsf{x}^{\pm}_{i,r}, [\mathsf{x}^{\pm}_{i,s}, \mathsf{x}^{\pm}_{j,t}]] + [\mathsf{x}^{\pm}_{i,s}, [\mathsf{x}^{\pm}_{i,r}, \mathsf{x}^{\pm}_{j,t}]] = 0 \text{ if } j = i \pm 1 \text{ and } |\alpha_i| = \bar{0}, \quad (2.59)$$

and quartic Serre relations

$$[[\mathsf{x}^{\pm}_{j-1,r}, \mathsf{x}^{\pm}_{j,0}], [\mathsf{x}^{\pm}_{j,0}, \mathsf{x}^{\pm}_{j+1,s}]] = 0 \text{ if } |\alpha_j| = \bar{1} \text{ and } |\alpha_{j-1}| = |\alpha_{j+1}| = \bar{0}. \quad (2.60)$$

Remark 2.61 (a) Similar to Remark 2.14, Serre relations (2.59) and (2.60) also hold for all other parities, but in those cases, they already follow from (2.57, 2.58) and (2.57, 2.58, 2.59).

(b) Generalizing the quartic Serre relations (2.60), the following relations also hold:

$$[[\mathsf{x}^{\pm}_{j-1,r}, \mathsf{x}^{\pm}_{j,k}], [\mathsf{x}^{\pm}_{j,l}, \mathsf{x}^{\pm}_{j+1,s}]] + [[\mathsf{x}^{\pm}_{j-1,r}, \mathsf{x}^{\pm}_{j,l}], [\mathsf{x}^{\pm}_{j,k}, \mathsf{x}^{\pm}_{j+1,s}]] = 0. \quad (2.62)$$

One way to prove this is to use the classical argument of deducing all Serre relations from the basic ones by commuting the latter with certain Cartan elements. Let $Q^{\pm}_j(r; k, l; s)$ denote the left-hand side of (2.62). Our goal is to prove $Q^{\pm}_j(r; k, l; s) = 0$ for any $r, k, l, s \geq 0$, while we know it only for $k = l = 0$ and $r, s \geq 0$, due to (2.60) and Remark 2.61(a). Define the elements $\{t_{i,r}\}^{r\geq 0}_{1\leq i<n}$ of $Y(\mathfrak{sl}(V))$ via

$$\sum_{r\geq 0} t_{i,r} u^{-r-1} = \log\left(1 + \sum_{r\geq 0} h_{i,r} u^{-r-1}\right), \quad (2.63)$$

cf. (2.52). The relations (2.53, 2.54) imply the following commutation relations:

$$[t_{i,r}, \mathsf{x}^{\pm}_{j,s}] = \pm c_{ij} \sum_{l=0}^{[r/2]} \binom{r}{2l} \frac{(c_{ij}/2)^{2l}}{2l+1} \mathsf{x}^{\pm}_{j,r+s-2l}, \quad (2.64)$$

cf. [14, Remark of §2.9]. Commuting both sides of the equality $Q^{\pm}_j(r; 0, 0; s) = 0$ with $t_{j-1,k}$ and using (2.64), one obtains $Q^{\pm}_j(r; k, 0; s) = 0$ by an induction in k. Commuting the latter equality with $t_{j-1,l}$, one derives the desired equality $Q^{\pm}_j(r; k, l; s) = 0$ by an induction in l.

Another way to prove (2.62) is to verify the corresponding equality on the shuffle side, see Sect. 3 (though we set $\hbar = 1$ for the current purpose). According to Corollary 3.26, it suffices to prove $\Psi(Q^{+}_j(r; k, l; s)) = 0$. The latter follows from the obvious equality

$$\Psi(Q^{+}_j(r; k, l; s)) = (x^k_{j,1} x^l_{j,2} + x^l_{j,1} x^k_{j,2}) \Psi(Q^{+}_j(r; 0, 0; s)) \quad (2.65)$$

in the notations of *loc.cit.*, combined with $Q^{+}_j(r; 0, 0; s) = 0$.

Let us now relate $Y(\mathfrak{sl}(V))$ to $Y(\mathfrak{gl}(V))$ of Sect. 2.2. Define $u_1, \ldots, u_{n-1}$ via

$$u_1 := u \text{ and } u_{i+1} = u_i + \frac{c_{i,i+1}}{2} = u_i - \frac{(-1)^{\overline{i+1}}}{2}. \tag{2.66}$$

Consider the generating series $e_i(u), f_i(u), d_j(u)$ with coefficients in $Y(\mathfrak{gl}(V))$, defined via

$$e_i(u) := \sum_{r\geq 1} e_i^{(r)} u^{-r},\ \ f_i(u) := \sum_{r\geq 1} f_i^{(r)} u^{-r},\ \ d_j(u) := 1 + \sum_{r\geq 1} d_j^{(r)} u^{-r}.$$

We also introduce the elements $\{X^{\pm}_{i,r}, H_{i,r}\}_{1\leq i<n}^{r\geq 0}$ of $Y(\mathfrak{gl}(V))$ via

$$\sum_{r\geq 0} X^{+}_{i,r} u^{-r-1} = f_i(u_i),\ \sum_{r\geq 0} X^{-}_{i,r} u^{-r-1} = (-1)^{\bar{i}} e_i(u_i),$$
$$1 + \sum_{r\geq 0} H_{i,r} u^{-r-1} = d_i(u_i)^{-1} d_{i+1}(u_i).$$

Theorem 2.67 *The assignment $\mathsf{x}^{\pm}_{i,r} \mapsto X^{\pm}_{i,r}$, $h_{i,r} \mapsto H_{i,r}$ gives rise to a superalgebra embedding*

$$\jmath \colon Y(\mathfrak{sl}(V)) \hookrightarrow Y(\mathfrak{gl}(V)). \tag{2.68}$$

Moreover, the superalgebra isomorphism $\Upsilon \colon Y(\mathfrak{gl}(V)) \xrightarrow{\sim} Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$ of Theorem 2.32 gives rise to a superalgebra isomorphism $\Upsilon \colon Y(\mathfrak{sl}(V)) \xrightarrow{\sim} Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$.

Proof The compatibility of the assignment $\mathsf{x}^{\pm}_{i,r} \mapsto X^{\pm}_{i,r}$, $h_{i,r} \mapsto H_{i,r}$ with the defining relations (2.52–2.60) is straightforward (cf. [1, Remark 5.12]). Hence, we obtain a superalgebra homomorphism $\jmath \colon Y(\mathfrak{sl}(V)) \to Y(\mathfrak{gl}(V))$. Its image coincides with the pre-image of $Y^{\mathrm{rtt}}(\mathfrak{sl}(V))$ under Υ, due to Lemma 2.39. Finally, the injectivity of $\jmath$ is established in the same way as it was proved in [13, Proposition 5] for the particular case of $Y(\mathfrak{gl}_{n_+|n_-})$ (the proof crucially uses the construction of PBW bases for the Yangians of [15,24], recalled in Theorem 2.72). ◻

As an immediate corollary of Theorem 2.67 and Corollary 2.38, we obtain:

Theorem 2.69 *The superalgebra $Y(\mathfrak{sl}(V))$ depends only on (n_+, n_-), up to an isomorphism.*

2.6 The PBW theorem and the triangular decomposition for $Y(\mathfrak{sl}(V))$

Let $Y^{\pm}(\mathfrak{sl}(V))$ and $Y^{0}(\mathfrak{sl}(V))$ be the subalgebras of $Y(\mathfrak{sl}(V))$ generated by $\{\mathsf{x}^{\pm}_{i,r}\}$ and $\{h_{i,r}\}$, respectively. Likewise, let $\widetilde{Y}^{\pm}(\mathfrak{sl}(V))$ and $\widetilde{Y}^{0}(\mathfrak{sl}(V))$ be the associative $\mathbb{C}$-superalgebras generated by $\{\mathsf{x}^{\pm}_{i,r}\}_{1\leq i<n}^{r\geq 0}$ and $\{h_{i,r}\}_{1\leq i<n}^{r\geq 0}$, respectively, with the $\mathbb{Z}_2$-grading $|h_{i,r}| = \bar{0}$, $|\mathsf{x}^{\pm}_{i,r}| = |\alpha_i|$, and subject to the defining relations (2.57–2.60)

and (2.52), respectively. The assignments $\mathsf{x}^{\pm}_{i,r} \mapsto \mathsf{x}^{\pm}_{i,r}$ and $h_{i,r} \mapsto h_{i,r}$ clearly give rise to epimorphisms $\widetilde{Y}^{\pm}(\mathfrak{sl}(V)) \twoheadrightarrow Y^{\pm}(\mathfrak{sl}(V))$ and $\widetilde{Y}^{0}(\mathfrak{sl}(V)) \twoheadrightarrow Y^{0}(\mathfrak{sl}(V))$ (which are actually isomorphisms, due to Proposition 2.73(a)).

Pick any total ordering $\preceq$ on $\Delta^+ \times \mathbb{N}$. For every $(\beta, r) \in \Delta^+ \times \mathbb{N}$, we choose:

(1) a decomposition $\beta = \alpha_{i_1} + \ldots + \alpha_{i_p}$ such that $[\cdots[e_{\alpha_{i_1}}, e_{\alpha_{i_2}}], \cdots, e_{\alpha_{i_p}}]$ is a nonzero root vector e_β of $\mathfrak{sl}(V)$ (here, e_{α_i} denotes the standard Chevalley generator of $\mathfrak{sl}(V)$);
(2) a decomposition $r = r_1 + \ldots + r_p$ with $r_i \in \mathbb{N}$.

Define the *PBW basis elements* $\mathsf{x}^{\pm}_{\beta,r}$ of $Y^{\pm}(\mathfrak{sl}(V))$ or $\widetilde{Y}^{\pm}(\mathfrak{sl}(V))$ via

$$\mathsf{x}^{\pm}_{\beta,r} := [\cdots[[\mathsf{x}^{\pm}_{i_1,r_1}, \mathsf{x}^{\pm}_{i_2,r_2}], \mathsf{x}^{\pm}_{i_3,r_3}], \cdots, \mathsf{x}^{\pm}_{i_p,r_p}]. \tag{2.70}$$

Let H denote the set of all functions $h\colon \Delta^+ \times \mathbb{N} \to \mathbb{N}$ with finite support and such that $h(\beta, r) \leq 1$ if $|\beta| = \bar{1}$ (we set $|\pm(\alpha_j + \ldots + \alpha_i)| := |\alpha_j| + \ldots + |\alpha_i| \in \mathbb{Z}_2$). The monomials

$$\mathsf{x}^{\pm}_h := \prod^{\rightarrow}_{(\beta,r)\in\Delta^+\times\mathbb{N}} {\mathsf{x}^{\pm}_{\beta,r}}^{h(\beta,r)} \quad \text{with} \quad h \in H \tag{2.71}$$

will be called the *ordered PBW monomials* of $Y^{\pm}(\mathfrak{sl}(V))$ or $\widetilde{Y}^{\pm}(\mathfrak{sl}(V))$.

The following PBW result for Yangians is originally due to [15][3] (cf. [1, Theorem 5.11] and [13, proof of Proposition 5]):

Theorem 2.72 (a) *The ordered PBW monomials* $\{\mathsf{x}^{\pm}_h\}_{h\in H}$ *form a* $\mathbb{C}$*-basis of* $\widetilde{Y}^{\pm}(\mathfrak{sl}(V))$.
(b) *The ordered (in any way) monomials in* $\{h_{i,r}\}^{r\geq 0}_{1\leq i<n}$ *form a* $\mathbb{C}$*-basis of* $\widetilde{Y}^{0}(\mathfrak{sl}(V))$.
(c) *The products of ordered PBW monomials* $\{\mathsf{x}^{-}_h\}_{h\in H}$, $\{\mathsf{x}^{+}_{h'}\}_{h'\in H}$, *and the ordered monomials in* $\{h_{i,r}\}^{r\geq 0}_{1\leq i<n}$ *form a* $\mathbb{C}$*-basis of* $Y(\mathfrak{sl}(V))$.

As an important corollary, we obtain the *triangular decomposition* for $Y(\mathfrak{sl}(V))$:

Proposition 2.73 (a) *The assignments* $\mathsf{x}^{\pm}_{i,r} \mapsto \mathsf{x}^{\pm}_{i,r}$ *and* $h_{i,r} \mapsto h_{i,r}$ *give rise to isomorphisms*

$$\widetilde{Y}^{\pm}(\mathfrak{sl}(V)) \xrightarrow{\sim} Y^{\pm}(\mathfrak{sl}(V)) \text{ and } \widetilde{Y}^{0}(\mathfrak{sl}(V)) \xrightarrow{\sim} Y^{0}(\mathfrak{sl}(V)).$$

(b) *The multiplication map*

$$m\colon Y^{-}(\mathfrak{sl}(V)) \otimes Y^{0}(\mathfrak{sl}(V)) \otimes Y^{+}(\mathfrak{sl}(V)) \longrightarrow Y(\mathfrak{sl}(V))$$

is an isomorphism of $\mathbb{C}$*-vector superspaces.*

[3] The original proof of [15] contains a substantial gap, see [11, Appendix B] for an alternative proof.

Remark 2.74 In Sect. 3, we will use a particular total ordering $\preceq$ on $\Delta^+ \times \mathbb{N}$:

$$(\beta, r) \preceq (\beta', r') \text{ iff } \beta \prec \beta' \text{ or } \beta = \beta', r \leq r', \tag{2.75}$$

where the total ordering $\preceq$ on Δ^+ is as follows:

$$\alpha_j + \alpha_{j+1} + \cdots + \alpha_i \preceq \alpha_{j'} + \alpha_{j'+1} + \cdots + \alpha_{i'} \text{ iff } j < j' \text{ or } j = j', i \leq i'. \tag{2.76}$$

2.7 The super Yangians $Y_\hbar(\mathfrak{sl}(V))$ and $\mathbf{Y}_\hbar(\mathfrak{sl}(V))$

For the sake of the next section, let us introduce a $\mathbb{C}[\hbar]$-version of $Y(\mathfrak{sl}(V))$ by homogenizing the defining relations of the latter. More precisely, let $Y_\hbar(\mathfrak{sl}(V))$ be the associative $\mathbb{C}[\hbar]$-superalgebra generated by $\{h_{i,r}, \mathsf{x}^\pm_{i,r}\}^{r\geq 0}_{1\leq i<n}$ with the $\mathbb{Z}_2$-grading $|h_{i,r}| = \bar{0}$, $|\mathsf{x}^\pm_{i,r}| = |\alpha_i|$, and subject to (2.52, 2.53, 2.55, 2.56, 2.58, 2.59, 2.60) and the following modifications of (2.54, 2.57):

$$[h_{i,r+1}, \mathsf{x}^\pm_{j,s}] - [h_{i,r}, \mathsf{x}^\pm_{j,s+1}] = \pm\frac{c_{ij}\hbar}{2}\{h_{i,r}, \mathsf{x}^\pm_{j,s}\} \text{ unless } i = j \text{ and } |\alpha_i| = \bar{1}, \tag{2.77}$$

$$[\mathsf{x}^\pm_{i,r+1}, \mathsf{x}^\pm_{j,s}] - [\mathsf{x}^\pm_{i,r}, \mathsf{x}^\pm_{j,s+1}] = \pm\frac{c_{ij}\hbar}{2}\{\mathsf{x}^\pm_{i,r}, \mathsf{x}^\pm_{j,s}\} \text{ unless } i = j \text{ and } |\alpha_i| = \bar{1}. \tag{2.78}$$

The algebra $Y_\hbar(\mathfrak{sl}(V))$ is $\mathbb{N}$-graded via $\deg(h_{i,r}) = \deg(\mathsf{x}^\pm_{i,r}) = r$, $\deg(\hbar) = 1$.

Following Sect. 2.6, let $Y^\pm_\hbar(\mathfrak{sl}(V))$ and $Y^0_\hbar(\mathfrak{sl}(V))$ be the $\mathbb{C}[\hbar]$-subalgebras of $Y_\hbar(\mathfrak{sl}(V))$ generated by $\{\mathsf{x}^\pm_{i,r}\}$ and $\{h_{i,r}\}$, respectively. We also define the *PBW basis elements* $\{\mathsf{x}^\pm_{\beta,r}\}^{r\in\mathbb{N}}_{\beta\in\Delta^+}$ and the *ordered PBW monomials* $\{\mathsf{x}^\pm_h\}_{h\in H}$ of $Y_\hbar(\mathfrak{sl}(V))$ via (2.70) and (2.71), respectively. We have the following counterparts of Theorem 2.72(c) and Proposition 2.73 (cf. [11]):

Theorem 2.79 (a) *The products of ordered PBW monomials* $\{\mathsf{x}^-_h\}_{h\in H}$, $\{\mathsf{x}^+_{h'}\}_{h'\in H}$, *and the ordered monomials in* $\{h_{i,r}\}^{r\geq 0}_{1\leq i<n}$ *form a basis of a free* $\mathbb{C}[\hbar]$*-module* $Y_\hbar(\mathfrak{sl}(V))$.

(b) *The multiplication map* $m\colon Y^-_\hbar(\mathfrak{sl}(V)) \otimes_{\mathbb{C}[\hbar]} Y^0_\hbar(\mathfrak{sl}(V)) \otimes_{\mathbb{C}[\hbar]} Y^+_\hbar(\mathfrak{sl}(V)) \to Y_\hbar(\mathfrak{sl}(V))$ *is an isomorphism of* $\mathbb{C}[\hbar]$*-modules.*

(c) $Y^\pm_\hbar(\mathfrak{sl}(V))$ *are isomorphic to the associative* $\mathbb{C}[\hbar]$*-superalgebras generated by* $\{\mathsf{x}^\pm_{i,r}\}^{r\geq 0}_{1\leq i<n}$ *with the* $\mathbb{Z}_2$*-grading* $|\mathsf{x}^\pm_{i,r}| = |\alpha_i|$ *and subject to the defining relations* (2.58–2.60, 2.78).

The Drinfeld-Gavarini dual $\mathbf{Y}_\hbar(\mathfrak{sl}(V))$ is the $\mathbb{C}[\hbar]$-subalgebra of $Y_\hbar(\mathfrak{sl}(V))$ generated by

$$\mathsf{H}_{i,r} := \hbar \cdot h_{i,r} \text{ and } \mathsf{X}^\pm_{\beta,r} := \hbar\mathsf{x}^\pm_{\beta,r} \text{ for } i \in I, \beta \in \Delta^+, r \in \mathbb{N}. \tag{2.80}$$

For $h \in H$ (Sect. 2.6), set $\mathsf{X}^\pm_h := \overrightarrow{\prod}_{(\beta,r)\in\Delta^+\times\mathbb{N}} \mathsf{X}^{\pm\ h(\beta,r)}_{\beta,r}$. The following is [11, Theorem A.7]:

Theorem 2.81 (a) *The subalgebra* $\mathbf{Y}_\hbar(\mathfrak{sl}(V))$ *is independent of all our choices in* (2.70).

(b) *The products of ordered PBW monomials* $\{\mathsf{X}^-_h\}_{h\in H}$, $\{\mathsf{X}^+_{h'}\}_{h'\in H}$, *and the ordered monomials in* $\{\mathsf{H}_{i,r}\}_{1\le i<n}^{r\ge 0}$ *form a basis of a free* $\mathbb{C}[\hbar]$*-module* $\mathbf{Y}_\hbar(\mathfrak{sl}(V))$.

Let $\mathbf{Y}^+_\hbar(\mathfrak{sl}(V))$ be the $\mathbb{C}[\hbar]$-subalgebra of $Y^+_\hbar(\mathfrak{sl}(V))$ generated by $\{\mathsf{X}^+_{\beta,r}\}_{\beta\in\Delta^+}^{r\in\mathbb{N}}$. A new proof of Theorem 2.81 but with $\mathbf{Y}^+_\hbar(\mathfrak{sl}(V))$ in place of $\mathbf{Y}_\hbar(\mathfrak{sl}(V))$ is provided in the next section.

Remark 2.82 (a) In view of Theorems 2.79 and 2.81, the algebras $Y_\hbar(\mathfrak{sl}(V))$ and $\mathbf{Y}_\hbar(\mathfrak{sl}(V))$ may be defined as the Rees algebras:

$$Y_\hbar(\mathfrak{sl}(V)) = \mathrm{Rees}^{F_*}Y(\mathfrak{sl}(V)) \text{ and } \mathbf{Y}_\hbar(\mathfrak{sl}(V)) = \mathrm{Rees}^{F'_*}Y(\mathfrak{sl}(V)). \quad (2.83)$$

Here, $F'_*Y(\mathfrak{sl}(V))$ and $F_*Y(\mathfrak{sl}(V))$ are the two algebra filtrations on $Y(\mathfrak{sl}(V))$, defined by specifying the degrees of PBW basis elements $\{\mathsf{x}^\pm_{\beta,r}, h_{i,r}\}_{\beta\in\Delta^+,1\le i<n}^{r\ge 0}$ as follows:

$$\deg_1(\mathsf{x}^\pm_{\beta,r}) = \deg_1(h_{i,r}) = r+1 \ \text{ and } \ \deg_2(\mathsf{x}^\pm_{\beta,r}) = \deg_2(h_{i,r}) = r. \quad (2.84)$$

They are pre-images of the filtrations (2.26) under the embedding $\Upsilon\colon Y(\mathfrak{sl}(V)) \hookrightarrow Y^{\mathrm{rtt}}(\mathfrak{gl}(V))$.

(b) For $a \in \mathbb{C}^\times$:

$$Y_\hbar(\mathfrak{sl}(V))/(\hbar-a)Y_\hbar(\mathfrak{sl}(V)) \simeq \mathbf{Y}_\hbar(\mathfrak{sl}(V))/(\hbar-a)\mathbf{Y}_\hbar(\mathfrak{sl}(V)) \simeq Y(\mathfrak{sl}(V)),$$

but $Y_\hbar(\mathfrak{sl}(V))/\hbar Y_\hbar(\mathfrak{sl}(V)) \simeq U(\mathfrak{sl}(V)\otimes\mathbb{C}[t])$, while $\mathbf{Y}_\hbar(\mathfrak{sl}(V))/\hbar\mathbf{Y}_\hbar(\mathfrak{sl}(V))$ is supercommutative.

3 Shuffle algebra realizations of $Y^+_\hbar(\mathfrak{sl}(V))$ and $\mathbf{Y}^+_\hbar(\mathfrak{sl}(V))$

In this section, we obtain shuffle algebra realizations[4] of the superalgebras $Y^+_\hbar(\mathfrak{sl}(V))$ and $\mathbf{Y}^+_\hbar(\mathfrak{sl}(V))$ of Sect. 2.7, generalizing [25, Theorems 7.15, 7.16] for the particular case of (2.17).

3.1 The rational shuffle algebra W^V and its integral form $\mathfrak{W}^V$

We follow the notations of [25, §7.2]. Let Σ_k denote the symmetric group in k elements, and set $\Sigma_{(k_1,\ldots,k_{n-1})} := \Sigma_{k_1}\times\cdots\times\Sigma_{k_{n-1}}$ for $k_1,\ldots,k_{n-1}\in\mathbb{N}$. Consider an $\mathbb{N}^I$-graded $\mathbb{C}[\hbar]$-module $\bar{\mathbb{W}}^V = \bigoplus_{\underline{k}=(k_1,\ldots,k_{n-1})\in\mathbb{N}^I}\bar{\mathbb{W}}^V_{\underline{k}}$, where $\bar{\mathbb{W}}^V_{(k_1,\ldots,k_{n-1})}$ consists of rational functions from $\mathbb{C}[\hbar](\{x_{i,r}\}_{i\in I}^{1\le r\le k_i})$ which are *supersymmetric* in $\{x_{i,r}\}_{r=1}^{k_i}$ for any $i\in I$, that is, symmetric if $|\alpha_i| = \bar{0}$ and skew-symmetric if $|\alpha_i| = \bar{1}$.

[4] These are rational super counterparts of the elliptic shuffle algebras of Feigin–Odesskii [7–9].

We fix an $I \times I$ matrix of rational functions $(\zeta_{i,j}(z))_{i,j\in I} \in \mathrm{Mat}_{I\times I}(\mathbb{C}[\hbar](z))$ via

$$\zeta_{i,j}(z) = (-1)^{\delta_{i>j}\delta_{|\alpha_i|,\bar{1}}\delta_{|\alpha_j|,\bar{1}}} \left(1 + \frac{c_{ij}\hbar}{2z}\right). \tag{3.1}$$

Let us now introduce the bilinear *shuffle product* $\star$ on $\bar{\mathbb{W}}^V$: given $F \in \bar{\mathbb{W}}^V_{\underline{k}}$ and $G \in \bar{\mathbb{W}}^V_{\underline{l}}$, define $F\star G \in \bar{\mathbb{W}}^V_{\underline{k}+\underline{l}}$ via

$$(F\star G)(x_{1,1},\ldots,x_{1,k_1+l_1};\ldots;x_{n-1,1},\ldots,x_{n-1,k_{n-1}+l_{n-1}}) := \underline{k}!\cdot\underline{l}!\times \\ \mathrm{SSym}_{\Sigma_{\underline{k}+\underline{l}}}\left(F\left(\{x_{i,r}\}_{i\in I}^{1\le r\le k_i}\right) G\left(\{x_{i',r'}\}_{i'\in I}^{k_{i'}<r'\le k_{i'}+l_{i'}}\right)\cdot \prod_{i\in I}^{i'\in I}\prod_{r\le k_i}^{r'>k_{i'}} \zeta_{i,i'}(x_{i,r}-x_{i',r'})\right). \tag{3.2}$$

Here, $\underline{k}! = \prod_{i\in I} k_i!$, and for $f \in \mathbb{C}(\{x_{i,1},\ldots,x_{i,m_i}\}_{i\in I})$ we define its *supersymmetrization* via

$$\mathrm{SSym}_{\Sigma_{\underline{m}}}(f)(\{x_{i,1},\ldots,x_{i,m_i}\}_{i\in I}) := \\ \sum_{(\sigma_1,\ldots,\sigma_{n-1})\in\Sigma_{\underline{m}}} \frac{(-1)^{\sum_{i\in I}\ell(\sigma_i)|\alpha_i|} f(\{x_{i,\sigma_i(1)},\ldots,x_{i,\sigma_i(m_i)}\})}{\underline{m}!}.$$

This endows $\bar{\mathbb{W}}^V$ with a structure of an associative unital algebra with the unit $\mathbf{1} \in \bar{\mathbb{W}}^V_{(0,\ldots,0)}$.

We will be interested only in the submodule of $\bar{\mathbb{W}}^V$ defined by the *pole* and *wheel conditions*:

- We say that $F \in \bar{\mathbb{W}}^V_{\underline{k}}$ satisfies the *pole conditions* if

$$F = \frac{f(x_{1,1},\ldots,x_{n-1,k_{n-1}})}{\prod_{i=1}^{n-2}\prod_{r\le k_i}^{r'\le k_{i+1}}(x_{i,r}-x_{i+1,r'})},\ \ f \in \mathbb{C}[\hbar][\{x_{i,r}\}_{i\in I}^{1\le r\le k_i}], \tag{3.3}$$

  where the polynomial f is supersymmetric in $\{x_{i,r}\}_{r=1}^{k_i}$ for all $i \in I$.
- We say that $F \in \bar{\mathbb{W}}^V_{\underline{k}}$ satisfies the *first kind wheel conditions* if

$$F(\{x_{i,r}\}) = 0 \text{ once } x_{i,r_1} = x_{i+\epsilon,s} + \hbar/2 = x_{i,r_2} + \hbar \text{ for some } \epsilon, i, r_1, r_2, s, \tag{3.4}$$

  where $\epsilon \in \{\pm 1\}$, $i, i+\epsilon \in I$, $1 \le r_1, r_2 \le k_i$, $1 \le s \le k_{i+\epsilon}$ and $|\alpha_i| = \bar{0}$.
- We say that $F \in \bar{\mathbb{W}}^V_{\underline{k}}$ satisfies the *second kind wheel conditions* if

$$F(\{x_{i,r}\}) = 0 \text{ once } x_{i-1,s} = x_{i,r_1} + \hbar/2 = x_{i+1,s'} = x_{i,r_2} - \hbar/2 \\ \text{for some } i, r_1, r_2, s, s', \tag{3.5}$$

  where $i, i-1, i+1 \in I$, $1 \le r_1, r_2 \le k_i$, $1 \le s \le k_{i-1}$, $1 \le s' \le k_{i+1}$ and $|\alpha_i| = \bar{1}$.

Let $\bar{W}^V_{\underline{k}} \subset \bar{\mathbb{W}}^V_{\underline{k}}$ denote the $\mathbb{C}[\hbar]$-submodule of all elements F satisfying these three conditions and set $\bar{W}^V := \bigoplus_{\underline{k}\in\mathbb{N}^I} \bar{W}^V_{\underline{k}}$. It is straightforward to check that $\bar{W}^V \subset \bar{\mathbb{W}}^V$ is $\star$-closed.

Definition 3.6 The algebra $\left(\bar{W}^V, \star\right)$ shall be called the *rational shuffle (super)algebra*.

This algebra is related to $Y^+_\hbar(\mathfrak{sl}(V))$ of Sect. 2.7 via the following construction:

Proposition 3.7 *The assignment* $\mathsf{x}^+_{i,r} \mapsto x^r_{i,1}$ $(i \in I, r \in \mathbb{N})$ *gives rise to a* $\mathbb{C}[\hbar]$-*algebra homomorphism* $\Psi\colon Y^+_\hbar(\mathfrak{sl}(V)) \to \bar{W}^V$.

Proof The assignment $\mathsf{x}^+_{i,r} \mapsto x^r_{i,1}$ $(i \in I, r \in \mathbb{N})$ is compatible with the defining relations (2.58–2.60, 2.78) of $Y^+_\hbar(\mathfrak{sl}(V))$, due to Theorem 2.79(c). Hence, it gives rise to a $\mathbb{C}[\hbar]$-algebra homomorphism $\Psi\colon Y^+_\hbar(\mathfrak{sl}(V)) \to \bar{W}^V$. □

The injectivity of Ψ will be proved in Corollary 3.26, while its image will be identified with the submodule W^V of *good* elements, see Definition 3.27 and Theorem 3.30 (in particular, the cokernel of Ψ is an $\hbar$-torsion module), resulting in the algebra isomorphism $Y^+_\hbar(\mathfrak{sl}(V)) \xrightarrow{\sim} W^V$. This constitutes the first main result of this section.

Recall the $\mathbb{C}[\hbar]$-subalgebra $\mathbf{Y}^+_\hbar(\mathfrak{sl}(V))$ of $Y^+_\hbar(\mathfrak{sl}(V))$, generated by $\{\mathsf{X}^+_{\beta,r}\}^{r\in\mathbb{N}}_{\beta\in\Delta^+}$ of (2.80). Our second key result of this section provides an explicit description of the image $\Psi(\mathbf{Y}^+_\hbar(\mathfrak{sl}(V)))$.

Definition 3.8 $F \in \bar{W}^V_{\underline{k}}$ is *integral* if F is divisible by $\hbar^{k_1+\ldots+k_{n-1}}$.

Set $\mathfrak{W}^V := \bigoplus_{\underline{k}\in\mathbb{N}^I} \mathfrak{W}^V_{\underline{k}}$, where $\mathfrak{W}^V_{\underline{k}} \subset \bar{W}^V_{\underline{k}}$ denotes the $\mathbb{C}[\hbar]$-submodule of all *integral* elements. The following is the second main result of this section:

Theorem 3.9 *The* $\mathbb{C}[\hbar]$-*algebra homomorphism* $\Psi\colon Y^+_\hbar(\mathfrak{sl}(V)) \to \bar{W}^V$ *gives rise to a* $\mathbb{C}[\hbar]$-*algebra isomorphism* $\Psi\colon \mathbf{Y}^+_\hbar(\mathfrak{sl}(V)) \xrightarrow{\sim} \mathfrak{W}^V$.

As a corollary, we will also obtain a new proof of the following result (cf. Theorem 2.81):

Theorem 3.10 (a) *The subalgebra* $\mathbf{Y}^+_\hbar(\mathfrak{sl}(V))$ *is independent of all our choices in* (2.70).

(b) *The ordered PBW monomials* $\{X^+_h\}_{h\in H}$ *form a basis of a free* $\mathbb{C}[\hbar]$-*module* $\mathbf{Y}^+_\hbar(\mathfrak{sl}(V))$.

3.2 The image of $Y^+_\hbar(\mathfrak{sl}(V))$ for $\dim(V) = 2$

In the simplest case $\dim(V) = 2$, all shuffle elements are *good* (Definition 3.27), that is $W^V = \bar{W}^V$ (see Remark 3.28(a)). Therefore, Theorem 3.30 is equivalent to

Theorem 3.11 *If* $\dim(V) = 2$, *then* $\Psi\colon Y^+_\hbar(\mathfrak{sl}(V)) \to \bar{W}^V$ *is an algebra isomorphism.*

There are two cases to consider: $\bar{1} \neq \bar{2}$ (so that $|\mathsf{x}^+_{1,r}| = \bar{1}$) and $\bar{1} = \bar{2}$ (so that $|\mathsf{x}^+_{1,r}| = \bar{0}$). First, assume $\bar{1} \neq \bar{2}$. Due to Theorem 2.79, the following result implies Theorem 3.11:

Lemma 3.12 *The ordered products* $\{x^{r_1} \star x^{r_2} \star \cdots \star x^{r_k}\}_{k\in\mathbb{N}}^{0\le r_1<\cdots<r_k}$ *form a* $\mathbb{C}[\hbar]$*-basis of* $\bar{W}^V$.

Proof This follows from the $\mathbb{C}[\hbar]$-algebra isomorphism $\bar{W}^V \simeq \bigoplus_k \Lambda_k$, where Λ_k denotes the $\mathbb{C}[\hbar]$-module of skew-symmetric $\mathbb{C}[\hbar]$-polynomials in k variables, while the algebra structure on the direct sum arises via the standard skew-symmetrization maps $\Lambda_k \otimes \Lambda_l \to \Lambda_{k+l}$. □

Next, assume $\bar{1} = \bar{2}$. Due to Theorem 2.79, the following result implies Theorem 3.11:

Lemma 3.13 *The ordered products* $\{x^{r_1} \star x^{r_2} \star \cdots \star x^{r_k}\}_{k\in\mathbb{N}}^{0\le r_1\le\cdots\le r_k}$ *form a* $\mathbb{C}[\hbar]$*-basis of* $\bar{W}^V$.

Proof Recall from [25, Lemma 6.22] that the k-th power of $x^r \in \bar{W}^V_1$ $(k \ge 1, r \ge 0)$ equals $x^r \star \cdots \star x^r = k \cdot (x_1 \cdots x_k)^r$. Therefore, for any ordered collection

$$0 \le r_1 = \cdots = r_{k_1} < r_{k_1+1} = \cdots = r_{k_1+k_2} < \cdots < r_{k_1+\ldots+k_{l-1}+1} = \cdots = r_{k=k_1+\ldots+k_l},$$

it is clear that $x^{r_1} \star \cdots \star x^{r_k}$ is a symmetric polynomial of the form

$$\nu_{\underline{r}} m_{(r_1,\ldots,r_k)}(x_1,\ldots,x_k) + \sum \nu_{\underline{r}'} m_{\underline{r}'}(x_1,\ldots,x_k).$$

Here, $m_{\underline{r}}(x_1,\ldots,x_k)$ are the monomial symmetric polynomials, the sum is over $\underline{r}' = (r'_1 \le \cdots \le r'_k)$ satisfying $r_1 \le r'_1 \le r'_k \le r_k$, $\nu_{\underline{r}'} \in \mathbb{C}[\hbar]$, and $\nu_{\underline{r}} = \prod_{i=1}^{l} k_i$.

This completes the proof of Lemma 3.13 as $\{m_{(s_1,\ldots,s_k)}(x_1,\ldots,x_k)\}_{0\le s_1\le\cdots\le s_k}$ form a $\mathbb{C}[\hbar]$-basis of $\mathbb{C}[\hbar][\{x_r\}_{r=1}^k]^{\Sigma_k} \simeq \bar{W}^V_k$. □

Combining Lemmas 3.12 and 3.13, we obtain the proof of Theorem 3.11.

3.3 The specialization maps and the injectivity of Ψ

For an ordered PBW monomial x^+_h $(h \in H)$, define its *degree* $\deg(\mathsf{x}^+_h) = \deg(h) \in \mathbb{N}^{\frac{n(n-1)}{2}}$ as a collection of $d_\beta := \sum_{r\in\mathbb{N}} h(\beta, r)$ $(\beta \in \Delta^+)$ ordered with respect to the total ordering (2.76) on Δ^+. We consider the *lexicographical ordering* on the collections $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+}$ of $\mathbb{N}^{\frac{n(n-1)}{2}}$:

$$\{d_\beta\}_{\beta\in\Delta^+} < \{d'_\beta\}_{\beta\in\Delta^+} \text{ iff there is } \gamma \in \Delta^+ \text{such that } d_\gamma > d'_\gamma \text{ and } d_\beta = d'_\beta \text{ for all } \beta < \gamma.$$

In what follows, we shall need an explicit formula for $\Psi(\mathsf{x}^+_{\beta,r})$:

Lemma 3.14 *For* $1 \leq j < i < n$ *and* $r \in \mathbb{N}$, *we have*

$$\Psi(\mathsf{x}^{+}_{\alpha_j+\alpha_{j+1}+\ldots+\alpha_i,r}) = \hbar^{i-j}\frac{p(x_{j,1},\ldots,x_{i,1})}{(x_{j,1}-x_{j+1,1})\cdots(x_{i-1,1}-x_{i,1})},$$

where $p(x_{j,1},\ldots,x_{i,1})$ *is a degree r monomial, up to a sign.*

Proof Straightforward computation. □

For $\beta = \alpha_j + \alpha_{j+1} + \ldots + \alpha_i$, define $j(\beta) := j, i(\beta) := i$, and let $[\beta]$ denote the integer interval $[j(\beta); i(\beta)]$. Consider a collection of the intervals $\{[\beta]\}_{\beta\in\Delta^+}$ each taken with a multiplicity $d_\beta \in \mathbb{N}$ and ordered with respect to (2.76) (the order inside each group is irrelevant), denoted by $\cup_{\beta\in\Delta^+}[\beta]^{d_\beta}$. Define $\underline{l} \in \mathbb{N}^I$ via $\underline{l} := \sum_{\beta\in\Delta^+} d_\beta[\beta]$. Let us now define the *specialization map*

$$\phi_{\underline{d}}\colon \bar{W}^V_{\underline{l}} \longrightarrow \mathbb{C}[\hbar][\{y_{\beta,s}\}^{1\leq s\leq d_\beta}_{\beta\in\Delta^+}]. \tag{3.15}$$

Split the variables $\{x_{i,r}\}^{1\leq r\leq l_i}_{i\in I}$ into $\sum_{\beta\in\Delta^+} d_\beta$ groups corresponding to the above intervals, and specialize the variable $x_{k,*}$ in the s-th copy of $[\beta]$ to $y_{\beta,s} + \frac{c_{12}+\ldots+c_{k-1,k}}{2}\hbar$ (so that the $x_{*,*}$-variables in the s-th copy of the interval $[\beta]$ are specialized to various $\hbar$-shifts of the same new variable $y_{\beta,s}$). For $F = \frac{f(x_{1,1},\ldots,x_{n-1,l_{n-1}})}{\prod_{i=1}^{n-2}\prod_{1\leq r\leq k_i}^{1\leq r'\leq k_{i+1}}(x_{i,r}-x_{i+1,r'})} \in \bar{W}^V_{\underline{l}}$, we finally define $\phi_{\underline{d}}(F)$ as the corresponding specialization of its numerator f.

Remark 3.16 Note that $\phi_{\underline{d}}(F)$ is independent of our splitting of the variables $\{x_{i,r}\}^{1\leq r\leq l_i}_{i\in I}$ into groups and is supersymmetric in $\{y_{\beta,s}\}^{d_\beta}_{s=1}$ for each $\beta \in \Delta^+$ (recall $|\beta| = |\alpha_{j(\beta)}| + \ldots + |\alpha_{i(\beta)}|$).

The key properties of the *specialization maps* $\phi_{\underline{d}}$ are summarized in the next two lemmas.

Lemma 3.17 *If* $\deg(h) < \underline{d}$, *then* $\phi_{\underline{d}}(\Psi(\mathsf{x}^+_h)) = 0$.

Proof The above condition guarantees that $\phi_{\underline{d}}$-specialization of any summand of the supersymmetrization appearing in $\Psi(\mathsf{x}^+_h)$ contains among all the ζ-factors at least one factor of the form $\zeta_{i,i+1}(-\frac{c_{i,i+1}}{2}\hbar) = 0$; hence, it is zero. The result follows. □

Lemma 3.18 *The specializations* $\{\phi_{\underline{d}}(\Psi(\mathsf{x}^+_h))\}^{\deg(h)=\underline{d}}_{h\in H}$ *are linearly independent over* $\mathbb{C}[\hbar]$.

Proof Consider the image of $\mathsf{x}^+_h = \overrightarrow{\prod}_{(\beta,r)\in\Delta^+\times\mathbb{N}} {\mathsf{x}^+_{\beta,r}}^{h(\beta,r)}$ under Ψ. It is a sum of $(\sum_{\beta\in\Delta^+} d_\beta)!$ terms, and as in the proof of Lemma 3.17, most of them specialize to zero under $\phi_{\underline{d}}$, $\underline{d} := \deg(h)$. The summands which do not specialize to zero are parametrized by $\Sigma_{\underline{d}} := \prod_{\beta\in\Delta^+}\Sigma_{d_\beta}$. More precisely, given $(\sigma_\beta)_{\beta\in\Delta^+} \in \Sigma_{\underline{d}}$, the associated summand corresponds to the case when for all $\beta \in \Delta^+$ and $1 \leq s \leq d_\beta$, the $(\sum_{\beta'<\beta} d_{\beta'} + s)$-th factor of the corresponding term of $\Psi(\mathsf{x}^+_h)$ is evaluated at

$\{y_{\beta,\sigma_\beta(s)} + \frac{c_{12}+\ldots+c_{k-1,k}}{2}\hbar\}_{j(\beta)\le k\le i(\beta)}$. Similar to [25, Lemma 3.15], the image of this summand under $\phi_{\underline{d}}$ may be written in the form $\prod_{\beta,\beta'\in\Delta^+}^{\beta<\beta'} G_{\beta,\beta'} \cdot \prod_{\beta\in\Delta^+} G_\beta \cdot \prod_{\beta\in\Delta^+} G_\beta^{(\sigma_\beta)}$ (up to a sign) with the factors $G_{\beta,\beta'}, G_\beta, G_\beta^{(\sigma_\beta)}$ to be specified below.

The factor $G_{\beta,\beta'}$ $(\beta<\beta')$ arises as a product of the specializations of the ζ-factors (note that we ignore the denominator z in $\zeta_{k,k\pm1}(z)$, but not in $\zeta_{k,k}(z)$) among two variables, which are getting specialized to $\hbar$-shifts of $y_{\beta,*}$ and $y_{\beta',*}$. Explicitly, we have

$$G_{\beta,\beta'} = \prod_{1\le s\le d_\beta}^{1\le s'\le d_{\beta'}} \prod_{k=j(\beta)}^{i(\beta)} \Big\{ (y_{\beta,s}-y_{\beta',s'})^{\delta_{k,j(\beta')-1}-\delta_{k,i(\beta')}} \times (y_{\beta,s}-y_{\beta',s'}-(-1)^{\overline{k}}\hbar)^{\delta_{k-1\in[\beta']}} (y_{\beta,s}-y_{\beta',s'}+\delta_{|\alpha_k|,\bar{0}}(-1)^{\overline{k}}\hbar)^{\delta_{k\in[\beta']}} \Big\}. \tag{3.19}$$

In particular, the total power of $(y_{\beta,s}-y_{\beta',s'})$ in $G_{\beta,\beta'}$ is nonnegative and equals

$$\#\{k|[\beta]\ni k\in[\beta'], |\alpha_k|=\bar{1}\} + \delta_{j(\beta)<j(\beta')}\delta_{i(\beta)+1\in[\beta']}. \tag{3.20}$$

Likewise, the total factor $G_\beta \cdot G_\beta^{(\sigma_\beta)}$ arises as a product of:

1) the specializations of $\Psi(\mathsf{x}^+_{\beta,*})$,
2) the specializations of the ζ-factors (note that we ignore the denominator z in $\zeta_{k,k\pm1}(z)$, but not in $\zeta_{k,k}(z)$) among two variables, which are getting specialized to $\hbar$-shifts of $y_{\beta,*}$.

Due to Lemma 3.14, the total contribution of the specializations in 1) equals

$$\hbar^{d_\beta(i(\beta)-j(\beta))} \cdot \prod_{s=1}^{d_\beta} p_{\beta,r_\beta(h,s)}(y_{\beta,\sigma_\beta(s)}), \tag{3.21}$$

where the collection $\{r_\beta(h,1),\ldots,r_\beta(h,d_\beta)\}$ is obtained by listing every $r\in\mathbb{N}$ with multiplicity $h(\beta,r)>0$ in the non-decreasing order and $p_{\beta,r}(y)$ are degree r monic polynomials (obtained by evaluating the monomials p of Lemma 3.14 at $\hbar$-shifts of y). On the other hand, the total contribution of the specializations in 2) equals

$$\prod_{1\le s<s'\le d_\beta} \prod_{j(\beta)<j\le i(\beta)} \Big((y_{\beta,\sigma_\beta(s)}-y_{\beta,\sigma_\beta(s')}-\hbar)(y_{\beta,\sigma_\beta(s)}-y_{\beta,\sigma_\beta(s')}+\hbar) \Big)^{\delta_{|\alpha_j|,\bar{0}}} \times \prod_{1\le s<s'\le d_\beta} \prod_{j(\beta)<j\le i(\beta)} \Big((y_{\beta,\sigma_\beta(s)}-y_{\beta,\sigma_\beta(s')}-(-1)^{\bar{j}}\hbar)(y_{\beta,\sigma_\beta(s)}-y_{\beta,\sigma_\beta(s')}) \Big)^{\delta_{|\alpha_j|,\bar{1}}} \times \prod_{1\le s<s'\le d_\beta} \left(\frac{y_{\beta,\sigma_\beta(s)}-y_{\beta,\sigma_\beta(s')}+(-1)^{\overline{j(\beta)}}\hbar}{y_{\beta,\sigma_\beta(s)}-y_{\beta,\sigma_\beta(s')}} \right)^{\delta_{|\alpha_{j(\beta)}|,\bar{0}}}. \tag{3.22}$$

While the product of the factors G_β and $G_\beta^{(\sigma_\beta)}$ equals the product of expressions (3.21, 3.22), we define each of them separately as follows:

$$G_\beta = \hbar^{d_\beta(i(\beta)-j(\beta))} \cdot \prod_{1\le s\ne s'\le d_\beta} (y_{\beta,s}-y_{\beta,s'})^{\lfloor \frac{\mathrm{odd}(\beta)}{2}\rfloor}(y_{\beta,s}-y_{\beta,s'}+\hbar)^{\mathrm{even}(\beta)+\lfloor\frac{\mathrm{odd}(\beta)-1}{2}\rfloor}, \tag{3.23}$$

$$G_\beta^{(\sigma_\beta)} = \prod_{s=1}^{d_\beta} p_{\beta,r_\beta(h,s)}(y_{\beta,\sigma_\beta(s)}) \cdot \begin{cases} \prod_{s<s'} \frac{y_{\beta,\sigma_\beta(s)}-y_{\beta,\sigma_\beta(s')}+(-1)^{\overline{j(\beta)}}\hbar}{y_{\beta,\sigma_\beta(s)}-y_{\beta,\sigma_\beta(s')}}, & \text{if } |\beta|=\bar{0} \\ (-1)^{\sigma_\beta}, & \text{if } |\beta|=\bar{1} \end{cases}, \tag{3.24}$$

where

$$\mathrm{even}(\beta) := \#\{k\in[\beta]\,|\,|\alpha_k|=\bar{0}\} \text{ and } \mathrm{odd}(\beta) := \#\{k\in[\beta]\,|\,|\alpha_k|=\bar{1}\}.$$

It is straightforward to verify that the products of (3.23, 3.24) and (3.21, 3.22) indeed coincide.

Note that the factors $\{G_{\beta,\beta'}\}_{\beta<\beta'} \cup \{G_\beta\}_\beta$ of (3.19, 3.23) are independent of $(\sigma_\beta)_{\beta\in\Delta^+} \in \Sigma_{\underline{d}}$. Therefore, the specialization $\phi_{\underline{d}}(\Psi(\mathsf{x}_h^+))$ has the following form:

$$\phi_{\underline{d}}(\Psi(\mathsf{x}_h^+)) = \pm \prod_{\beta,\beta'\in\Delta^+}^{\beta<\beta'} G_{\beta,\beta'} \cdot \prod_{\beta\in\Delta^+} G_\beta \cdot \prod_{\beta\in\Delta^+}\left(\sum_{\sigma_\beta\in\Sigma_{d_\beta}} G_\beta^{(\sigma_\beta)}\right). \tag{3.25}$$

For $\beta\in\Delta^+$, consider a two-dimensional superspace V'_β with basis vectors v'_1 and v'_2 having the parity $\overline{j(\beta)}$ and $\overline{i(\beta)}$, respectively. Then, the sum $\sum_{\sigma_\beta\in\Sigma_{d_\beta}} G_\beta^{(\sigma_\beta)}$ coincides with the value of the shuffle element $p_{\beta,r_\beta(h,1)}(x)\star\cdots\star p_{\beta,r_\beta(h,d_\beta)}(x) \in \bar{W}_{d_\beta}^{V'_\beta}$ evaluated at $\{y_{\beta,s}\}_{s=1}^{d_\beta}$. The latter elements are linearly independent (they form a basis of $\bar{W}_{d_\beta}^{V'_\beta}$), due to Lemmas 3.12 and 3.13.

Thus, (3.25) together with the above observation completes our proof of Lemma 3.18. □

Corollary 3.26 *The homomorphism* $\Psi\colon Y_\hbar^+(\mathfrak{sl}(V)) \to \bar{W}^V$ *is injective.*

Proof Assume the contrary, that there is a nonzero $x\in Y_\hbar^+(\mathfrak{sl}(V))$ such that $\Psi(x)=0$. Due to Theorem 2.79, x may be written in the form $x=\sum_{h\in H} c_h \mathsf{x}_h^+$, where all but finitely many of $c_h\in\mathbb{C}[\hbar]$ are zero. Define $\underline{d} := \max\{\deg(h)|c_h\ne 0\}$. Applying the *specialization map* $\phi_{\underline{d}}$ to $\Psi(x)=0$, we get $\sum_{h\in H}^{\deg(h)=\underline{d}} c_h\phi_{\underline{d}}(\Psi(\mathsf{x}_h^+))=0$ by Lemma 3.17. Furthermore, we get $c_h=0$ for all $h\in H$ with $\deg(h)=\underline{d}$, due to Lemma 3.18. This contradicts our choice of $\underline{d}$.

This completes our proof of the injectivity of Ψ. □

3.4 Proofs of the main results

In this section, we describe the images of $Y_\hbar^+(\mathfrak{sl}(V))$ and $\mathbf{Y}_\hbar^+(\mathfrak{sl}(V))$ under Ψ. To present the former, we make the following definition:

Definition 3.27 $F \in \bar{W}^V_{\underline{k}}$ is *good* if $\phi_{\underline{d}}(F)$ is divisible by $\hbar^{\sum_{\beta\in\Delta^+} d_\beta(i(\beta)-j(\beta))}$ for any degree vector $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+}$ such that $\underline{k} = \sum_{\beta\in\Delta^+} d_\beta[\beta]$.

Set $W^V := \bigoplus_{\underline{k}\in\mathbb{N}^I} W^V_{\underline{k}}$, where $W^V_{\underline{k}} \subset \bar{W}^V_{\underline{k}}$ denotes the $\mathbb{C}[\hbar]$-submodule of all *good* elements.

Remark 3.28 (a) For $\dim(V) = 2$, we have $W^V = \bar{W}^V$.
(b) $\mathfrak{W}^V \subseteq W^V$ as any *integral* shuffle element is obviously *good*.

Lemma 3.29 $\Psi(Y^+_\hbar(\mathfrak{sl}(V))) \subseteq W^V$.

Proof The proof is completely analogous to that of [25, Lemma 6.19] for the particular case of (2.17). For any $\beta \in \Delta^+$, $1 \le s \le d_\beta$ and $j(\beta) \le k < i(\beta)$, ζ-factors between the variables $x_{k,*}$ and $x_{k+1,*}$ that are specialized to $\hbar$-shifts of $y_{\beta,s}$ always specialize under $\phi_{\underline{d}}$ to a multiple of $\hbar$. It remains to note that the total number of such pairs is exactly $\sum_{\beta\in\Delta^+} d_\beta(i(\beta) - j(\beta))$. □

The following is the first key result of this section:

Theorem 3.30 *The $\mathbb{C}[\hbar]$-algebra embedding $\Psi\colon Y^+_\hbar(\mathfrak{sl}(V)) \hookrightarrow \bar{W}^V$ gives rise to a $\mathbb{C}[\hbar]$-algebra isomorphism $\Psi\colon Y^+_\hbar(\mathfrak{sl}(V)) \xrightarrow{\sim} W^V$.*

Proof We need to show that any *good* element $F \in W^V_{\underline{k}}$ belongs to the submodule $M \cap W^V_{\underline{k}}$, where $M \subset W^V$ denotes the $\mathbb{C}[\hbar]$-submodule spanned by $\{\Psi(\mathrm{x}^+_h)\}_{h\in H}$. Let $T_{\underline{k}}$ denote a finite set consisting of all degree vectors $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+} \in \mathbb{N}^{\frac{n(n-1)}{2}}$ such that $\sum_{\beta\in\Delta^+} d_\beta[\beta] = \underline{k}$. We order $T_{\underline{k}}$ with respect to the lexicographical ordering on $\mathbb{N}^{\frac{n(n-1)}{2}}$. In particular, the minimal element $\underline{d}_{\min} = \{d_\beta\}_{\beta\in\Delta^+} \in T_{\underline{k}}$ is characterized by $d_\beta = 0$ for all non-simple roots $\beta \in \Delta^+$.

The proof is crucially based on the following result:

Lemma 3.31 *If $\phi_{\underline{d}'}(F) = 0$ for all $\underline{d}' \in T_{\underline{k}}$ such that $\underline{d}' > \underline{d}$, then there exists an element $F_{\underline{d}} \in M$ such that $\phi_{\underline{d}}(F) = \phi_{\underline{d}}(F_{\underline{d}})$ and $\phi_{\underline{d}'}(F_{\underline{d}}) = 0$ for all $\underline{d}' > \underline{d}$.*

Proof of Lemma 3.31 Consider the following total ordering on the set $\{(\beta, s)\}^{1\le s\le d_\beta}_{\beta\in\Delta^+}$:

$$(\beta, s) \le (\beta', s') \text{ iff } \beta < \beta' \text{ or } \beta = \beta', s \le s'. \tag{3.32}$$

First, we note that the wheel conditions (3.4, 3.5) for F guarantee that $\phi_{\underline{d}}(F)$ (which is a polynomial in $\{y_{\beta,s}\}$) vanishes up to appropriate orders under the following specializations:

(i) $y_{\beta,s} = y_{\beta',s'} + \hbar$ for $(\beta, s) < (\beta', s')$,
(ii) $y_{\beta,s} = y_{\beta',s'} - \hbar$ for $(\beta, s) < (\beta', s')$.

The orders of vanishing are computed similarly to [25, Remark 5.24], cf. [5,20]. Let us view the specialization appearing in the definition of $\phi_{\underline{d}}$ as a step-by-step specialization in each interval $[\beta]$. As we specialize the variables in the new interval, we count only those wheel conditions that arise from the non-specialized yet variables. Varying different orderings of the intervals, we pick the maximal order of vanishing for each of the linear terms $y_{\beta,s} - y_{\beta',s'} \pm \hbar$. We claim that the resulting orders of vanishing under the specializations (i) and (ii) exactly equal the powers of $y_{\beta,s} - y_{\beta',s'} \mp \hbar$ in $G_{\beta,\beta'}$ (if $\beta < \beta'$) or in G_β (if $\beta = \beta'$). More precisely:

- if $i(\beta) < i(\beta')$, then specializing first in the s-th copy of $[\beta]$ and then in the s'-th copy of $[\beta']$, the orders of vanishing under (i, ii) equal the powers of $y_{\beta,s} - y_{\beta',s'} \mp \hbar$ in $G_{\beta,\beta'}$;
- if $i(\beta) > i(\beta')$, then specializing first in the s'-th copy of $[\beta']$ and then in the s-th copy of $[\beta]$, the orders of vanishing under (i, ii) equal the powers of $y_{\beta,s} - y_{\beta',s'} \mp \hbar$ in $G_{\beta,\beta'}$;
- if $i(\beta) = i(\beta')$ (or $\beta' = \beta$), specializing first in the s-th copy of $[\beta]$ and then in the s'-th copy of $[\beta']$, the orders of vanishing under (i, ii) differ from the powers of $y_{\beta,s} - y_{\beta',s'} \mp \hbar$ in $G_{\beta,\beta'}$ or G_β (if $\beta' = \beta$) by an absence of a single factor $y_{\beta,s} - y_{\beta',s'} + (-1)^{\overline{i(\beta)+1}}\hbar$. Reversing the order of specializations, we end up missing only one factor $y_{\beta',s'} - y_{\beta,s} + (-1)^{\overline{i(\beta)+1}}\hbar$. Hence, picking the maximal order of vanishing for each of $y_{\beta,s} - y_{\beta',s'} \mp \hbar$ achieves the result.

Second, we claim that $\phi_{\underline{d}}(F)$ vanishes under the following specializations:

(iii) $y_{\beta,s} = y_{\beta',s'}$ for $(\beta, s) < (\beta', s')$ such that $j(\beta) < j(\beta')$ and $i(\beta) + 1 \in [\beta']$.

Indeed, if $j(\beta) < j(\beta')$ and $i(\beta) + 1 \in [\beta']$, there are positive roots $\gamma, \gamma' \in \Delta^+$ such that $j(\gamma) = j(\beta), i(\gamma) = i(\beta'), j(\gamma') = j(\beta'), i(\gamma') = i(\beta)$. Consider the degree vector $\underline{d}' \in T_{\underline{k}}$ given by $d'_\alpha = d_\alpha + \delta_{\alpha,\gamma} + \delta_{\alpha,\gamma'} - \delta_{\alpha,\beta} - \delta_{\alpha,\beta'}$. Then, $\underline{d}' > \underline{d}$ and thus $\phi_{\underline{d}'}(F) = 0$. The result follows.

Finally, we also note that the skew-symmetry of the elements of W^V with respect to the variables $\{x_{k,*}\}$ with $|\alpha_k| = \bar{1}$ implies that $\phi_{\underline{d}}(F)$ vanishes under the following specializations:

(iv) $y_{\beta,s} = y_{\beta',s'}$ for $(\beta, s) < (\beta', s')$
and vanishing order is $\#\{k | [\beta] \ni k \in [\beta'], |\alpha_k| = \bar{1}\}$.

For $\beta < \beta'$ and any s, s', combining (iii) and (iv), we see that the order of vanishing of $\phi_{\underline{d}}(F)$ at $y_{\beta,s} = y_{\beta',s'}$ exactly equals the power of $y_{\beta,s} - y_{\beta',s'}$ in $G_{\beta,\beta'}$ as computed in (3.20). Similar, for $\beta' = \beta$ and $1 \leq s < s' \leq d_\beta$, combining (iii) and (iv), we see that the order of vanishing of $\phi_{\underline{d}}(F)$ at $y_{\beta,s} = y_{\beta,s'}$ equals the power of $y_{\beta,s} - y_{\beta,s'}$ in G_β of (3.23) plus one if $|\beta| = \bar{1}$.

Combining the above vanishing conditions for $\phi_{\underline{d}}(F)$ with F being *good*, we see that $\phi_{\underline{d}}(F)$ is divisible exactly by the product $\prod_{\beta<\beta'} G_{\beta,\beta'} \cdot \prod_\beta G_\beta$ of (3.19, 3.23). Therefore, we have

$$\phi_{\underline{d}}(F) = \prod_{\beta,\beta' \in \Delta^+}^{\beta<\beta'} G_{\beta,\beta'} \cdot \prod_{\beta \in \Delta^+} G_\beta \cdot G \tag{3.33}$$

for some supersymmetric polynomial

$$G \in \mathbb{C}[\hbar][\{y_{\beta,s}\}_{\beta\in\Delta^+}^{1\le s\le d_\beta}]^{\Sigma_{\underline{d}}} \simeq \bigotimes_{\beta\in\Delta^+} \mathbb{C}[\hbar][\{y_{\beta,s}\}_{s=1}^{d_\beta}]^{\Sigma_{d_\beta}}, \tag{3.34}$$

where $\mathbb{C}[\hbar][\{y_{\beta,s}\}_{s=1}^{d_\beta}]^{\Sigma_{d_\beta}}$ denotes the submodule of symmetric (resp. skew-symmetric) polynomials in $\{y_{\beta,s}\}_{s=1}^{d_\beta}$ if $|\beta| = \bar{0}$ (resp. $|\beta| = \bar{1}$).

Combining this observation with formula (3.25) and the discussion after it, we see that there is a linear combination $F_{\underline{d}} = \sum_{h\in H}^{\deg(h)=\underline{d}} c_h \mathsf{x}_h^+$ such that $\phi_{\underline{d}}(F) = \phi_{\underline{d}}(F_{\underline{d}})$, due to our proof of Lemma 3.18. The equality $\phi_{\underline{d}'}(F_{\underline{d}}) = 0$ for $\underline{d}' > \underline{d}$ is due to Lemma 3.17.

This completes our proof of Lemma 3.31. □

Let $\underline{d}_{\max}$ and $\underline{d}_{\min}$ denote the maximal and the minimal elements of $T_{\underline{k}}$, respectively. The condition of Lemma 3.31 is vacuous for $\underline{d} = \underline{d}_{\max}$. Therefore, Lemma 3.31 applies. Applying it iteratively, we will eventually find an element $\widetilde{F} \in M$ such that $\phi_{\underline{d}}(F) = \phi_{\underline{d}}(\widetilde{F})$ for all $\underline{d} \in T_{\underline{k}}$. In the particular case of $\underline{d} = \underline{d}_{\min}$, this yields $F = \widetilde{F}$ (as the *specialization map* $\phi_{\underline{d}_{\min}}$ essentially does not change the function). Hence, $F \in M$.

This completes our proof of Theorem 3.30. □

Using the same arguments, let us now prove Theorem 3.9.

Proof of Theorem 3.9 The proof proceeds in two steps: first, we establish the inclusion $\Psi(\mathbf{Y}_\hbar^+(\mathfrak{sl}(V))) \subseteq \mathfrak{W}^V$, and then, the opposite inclusion $\Psi(\mathbf{Y}_\hbar^+(\mathfrak{sl}(V))) \supseteq \mathfrak{W}^V$.

Lemma 3.35 $\Psi(\mathbf{Y}_\hbar^+(\mathfrak{sl}(V))) \subseteq \mathfrak{W}^V$.

Proof According to Lemma 3.14, $\Psi(\mathsf{X}_{\beta,r}^+)$ is divisible by $\hbar^{i(\beta)-j(\beta)+1}$. It remains to note that

$$\sum_{\beta\in\Delta^+} d_\beta(i(\beta) - j(\beta) + 1) = \sum_{i\in I} l_i, \tag{3.36}$$

where $\underline{l} = (l_1, \ldots, l_{n-1}) \in \mathbb{N}^I$ is defined via $(l_1, \ldots, l_{n-1}) := \sum_{\beta\in\Delta^+} d_\beta[\beta]$. □

The proof of the opposite inclusion $\Psi(\mathbf{Y}_\hbar^+(\mathfrak{sl}(V))) \supseteq \mathfrak{W}^V$ is completely analogous to our proof of Theorem 3.30 and Lemma 3.31. Indeed, it suffices to note that the factor $\hbar^{d_\beta(i(\beta)-j(\beta))}$ in the definition of G_β (3.23) shall be replaced by $\hbar^{d_\beta(i(\beta)-j(\beta)+1)}$, which does not affect (3.33), due to (3.36) (as we replaced "F being *good*" by "F being *integral*"). □

We conclude this section with a new proof of Theorem 3.10.

Proof of Theorem 3.10 (a) Since $\Psi\colon Y_\hbar^+(\mathfrak{sl}(V)) \to W^V$ is injective and the image of $\mathbf{Y}_\hbar^+(\mathfrak{sl}(V))$, the submodule $\mathfrak{W}^V$, is independent of any choices of $\mathsf{x}_{\beta,r}^+$, Theorem 3.10(a) follows.

(b) Following the proofs of Theorems 3.9 and 3.30, we have already established that $\mathfrak{W}^V$ is $\mathbb{C}[\hbar]$-spanned by the images of the ordered PBW monomials $\{\Psi(\mathsf{X}_h^+)\}_{h\in H}$. Combining this with the injectivity of Ψ and Theorem 2.79, completes the proof of Theorem 3.10(b). □

4 Generalization to type *A* quantum affine superalgebras

In this section, we briefly discuss the trigonometric counterparts of the previous results. The quantum affine superalgebras were first studied 20 years ago by Yamane [26]. In the *loc.cit.*, both the Drinfeld–Jimbo and the new Drinfeld realizations were proposed and the isomorphism between those was obtained. Also, the isomorphisms between the Drinfeld–Jimbo quantum (affine) superalgebras corresponding to different Dynkin diagrams were constructed.

Remark 4.1 Such isomorphisms in the type A toroidal setup, which does not admit the Drinfeld–Jimbo realization, have been recently constructed in [2].

In this section, we obtain the shuffle algebra realizations of the "positive halves" of the quantum affine superalgebras of $\mathfrak{gl}(V)$ corresponding to different Dynkin diagrams of $\mathfrak{gl}(V)$.

Let $\boldsymbol{v}$ be a formal variable. Define the "positive half" of the quantum loop superalgebra of $\mathfrak{gl}(V)$, denoted by $U^>_{\boldsymbol{v}}(L\mathfrak{gl}(V))$, to be the associative $\mathbb{C}(\boldsymbol{v})$-superalgebra generated by $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ with the $\mathbb{Z}_2$-grading $|e_{i,r}|=|\alpha_i|$, and subject to the following defining relations:

$$(z-\boldsymbol{v}^{c_{ij}}w)e_i(z)e_j(w)=(-1)^{|\alpha_i|\cdot|\alpha_j|}(\boldsymbol{v}^{c_{ij}}z-w)e_j(w)e_i(z), \tag{4.2}$$

$$[e_i(z),e_j(w)]=0 \text{ if } c_{ij}=0, \tag{4.3}$$

as well as cubic $\boldsymbol{v}$-Serre relations

$$[e_i(z_1),[e_i(z_2),e_j(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}}+[e_i(z_2),[e_i(z_1),e_j(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}}=0 \text{ if } j=i\pm 1 \text{ and } |\alpha_i|=\bar{0}, \tag{4.4}$$

and quartic $\boldsymbol{v}$-Serre relations

$$[[[e_{i-1}(w),e_i(z_1)]_{\boldsymbol{v}^{-1}},e_{i+1}(u)]_{\boldsymbol{v}},e_i(z_2)]+[[[e_{i-1}(w),e_i(z_2)]_{\boldsymbol{v}^{-1}},e_{i+1}(u)]_{\boldsymbol{v}},e_i(z_1)]=0 \text{ if } |\alpha_i|=\bar{1} \text{ and } |\alpha_{i-1}|=|\alpha_{i+1}|=\bar{0}, \tag{4.5}$$

where $e_i(z):=\sum_{r\in\mathbb{Z}}e_{i,r}z^{-r}$, $[a,b]_x:=ab-(-1)^{|a|\cdot|b|}x\cdot ba$ for homogeneous a,b.

Remark 4.6 The superalgebra $U^>_{\boldsymbol{v}}(L\mathfrak{gl}(V))$ is $\mathbb{N}^I$-graded via $\deg(e_{i,r})=1_i=(0,\ldots,1,\ldots,0)$ with 1 at the i-th spot. Given elements $a,b\in U^>_{\boldsymbol{v}}(L\mathfrak{gl}(V))$ with $\deg(a)=\underline{k}$ and $\deg(b)=\underline{l}$, we set $(a,b):=\sum_{i,j\in I}k_il_jc_{ij}$. Following [26, §6.7], we define $[\![a,b]\!]:=ab-(-1)^{|a|\cdot|b|}\boldsymbol{v}^{(a,b)}\cdot ba$.

(a) The cubic $\boldsymbol{v}$-Serre relations (4.4) can be written in the form

$$[\![e_i(z_1), [\![e_i(z_2), e_j(w)]\!]]\!] + [\![e_i(z_2), [\![e_i(z_1), e_j(w)]\!]]\!] = 0. \tag{4.7}$$

The relation (4.7) is also valid for $|\alpha_i| = \bar{1}$, but in that case, it already follows from (4.3).

(b) The quartic $\boldsymbol{v}$-Serre relations (4.5) can be written in the form

$$[\![[\![[\![e_{i-1}(w), e_i(z_1)]\!], e_{i+1}(u)]\!], e_i(z_2)]\!] + [\![[\![[\![e_{i-1}(w), e_i(z_2)]\!], e_{i+1}(u)]\!], e_i(z_1)]\!] = 0. \tag{4.8}$$

The relation (4.8) is also valid for any other parities of $\alpha_{i-1}, \alpha_i, \alpha_{i+1}$, but in those cases, it already follows from the quadratic and cubic relations (4.2–4.4).

(c) We finally note that (4.8) may be replaced by the following equivalent relations:

$$[\![[\![e_{i-1}(w), e_i(z_1)]\!], [\![e_{i+1}(u), e_i(z_2)]\!]]\!] + [\![[\![e_{i-1}(w), e_i(z_2)]\!], [\![e_{i+1}(u), e_i(z_1)]\!]]\!] = 0. \tag{4.9}$$

This is an affinization of the quartic $\boldsymbol{v}$-Serre relations of [6] for finite quantum superalgebras.

Let us now define the trigonometric shuffle (super)algebra $\left(S^V, \star\right)$ analogously to the rational shuffle (super)algebra $\left(\bar{W}^V, \star\right)$ of Sect. 3.1 with the following modifications:

(1) All rational functions $F \in S^V$ are defined over $\mathbb{C}(\boldsymbol{v})$;

(2) The analogue of (3.1) is the matrix $(\zeta_{i,j}(z))_{i,j\in I} \in \mathrm{Mat}_{I\times I}(\mathbb{C}(\boldsymbol{v})(z))$ defined via

$$\zeta_{i,j}(z) = (-1)^{\delta_{i>j}\delta_{|\alpha_i|,\bar{1}}\delta_{|\alpha_j|,\bar{1}}}(z - \boldsymbol{v}^{-c_{ij}})/(z-1); \tag{4.10}$$

(3) The *shuffle product* is defined via (3.2) with $\zeta_{i,i'}(x_{i,r} - x_{i',r'})$ replaced by $\zeta_{i,i'}(x_{i,r}/x_{i',r'})$;

(4) The *pole conditions* (3.3) for $F \in S^V_{\underline{k}}$ are modified as follows:

$$F = \frac{f(x_{1,1}, \ldots, x_{n-1,k_{n-1}})}{\prod_{i=1}^{n-2}\prod_{r\le k_i}^{r'\le k_{i+1}}(x_{i,r} - x_{i+1,r'})}, \tag{4.11}$$

where $f \in (\mathbb{C}(\boldsymbol{v})[\{x_{i,r}^{\pm 1}\}_{i\in I}^{1\le r\le k_i}])^{\Sigma_{\underline{k}}}$ is a supersymmetric Laurent polynomial;

(5) The *first kind wheel conditions* (3.4) for $F \in S^V$ are modified as follows:

$$F(\{x_{i,r}\}) = 0 \text{ once } x_{i,r_1} = \boldsymbol{v}x_{i+\epsilon,s} = \boldsymbol{v}^2 x_{i,r_2} \text{ for some } \epsilon, i, r_1, r_2, s, \tag{4.12}$$

with $|\alpha_i| = \bar{0}$;

(6) The *second kind wheel conditions* (3.5) for $F \in S^V$ are modified as follows:

$$F(\{x_{i,r}\}) = 0 \text{ once } x_{i-1,s} = \boldsymbol{v} x_{i,r_1} = x_{i+1,s'} = \boldsymbol{v}^{-1} x_{i,r_2} \text{ for some } i, r_1, r_2, s, s', \quad (4.13)$$

with $|\alpha_i| = \bar{1}$.

The following is the main result of this section (announced in [25, §8.2]), generalizing [25, Theorem 5.17] for the particular case of (2.17):

Theorem 4.14 *The assignment* $e_{i,r} \mapsto x_{i,1}^r$ $(i \in I, r \in \mathbb{Z})$ *gives rise to an algebra isomorphism*

$$\Psi \colon U_{\boldsymbol{v}}^{>}(L\mathfrak{gl}(V)) \xrightarrow{\sim} S^V. \quad (4.15)$$

The proof of this theorem is completely analogous to our proof of Theorem 3.30.

Remark 4.16 We note that [25, Theorems 3.34, 8.8] providing the shuffle algebra realizations of the RTT and Lusztig/Grojnowski/Chari-Pressley integral forms of $U_{\boldsymbol{v}}^{>}(L\mathfrak{gl}_n)$ can be straightforwardly generalized to the case of $U_{\boldsymbol{v}}^{>}(L\mathfrak{gl}(V))$. The former has potential applications to the geometric representation theory (cf. [11, Proposition 4.12, Remark 4.16] for $n_- = 0$).

Acknowledgements I am indebted to Boris Feigin, Michael Finkelberg, and Andrei Neguţ for numerous interesting discussions over the years; to Luan Bezerra and Evgeny Mukhin for a useful correspondence on quantum affine superalgebras; to Vasily Pestun for his invitation to IHES; to Yung-Ning Peng for bringing [21,22] (related to our Sects. 2.2, 2.3) to my attention after the first version of this note was posted on arXiv; to anonymous referees for many useful suggestions. This note was prepared during the author's visit to IHES (Bures-sur-Yvette, France) in the summer 2019, sponsored by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation program (QUASIFT grant agreement 677368). I gratefully acknowledge the NSF Grant DMS-1821185.

References

1. Brundan, J., Kleshchev, A.: Parabolic presentations of the Yangian $Y(\mathfrak{gl}_n)$. Commun. Math. Phys. **254**(1), 191–220 (2005)
2. Bezerra, L., Mukhin, E.: Braid actions on quantum toroidal superalgebras, preprint, arXiv:1912.08729
3. Drinfeld, V.: A new realization of Yangians and quantized affine algebras. Sov. Math. Dokl. **36**(2), 212–216 (1988)
4. Ding, J., Frenkel, I.: Isomorphism of two realizations of quantum affine algebra $U_q(\widehat{\mathfrak{gl}(n)})$. Commun. Math. Phys. **156**(2), 277–300 (1993)
5. Feigin, B., Hashizume, K., Hoshino, A., Shiraishi, J., Yanagida, S.: A commutative algebra on degenerate $\mathbb{CP}^1$ and Macdonald polynomials. J. Math. Phys. **50**(9), 095215 (2009)
6. Floreanini, R., Leites, D., Vinet, L.: On the defining relations of quantum superalgebras. Lett. Math. Phys. **23**(2), 127–131 (1991)
7. Feigin, B., Odesskii, A.: Sklyanin's elliptic algebras, (Russian) Funktsional. Anal. i Prilozhen. **23**, no. 3, 45–54 (1989); translation in Funct. Anal. Appl. **23**, no. 3, 207–214 (1989)
8. Feigin, B., Odesskii, A.: Elliptic deformations of current algebras and their representations by difference operators, (Russian) Funktsional. Anal. i Prilozhen. **31**, no. 3, 57–70 (1997); translation in Funct. Anal. Appl. **31**, no. 3, 193–203 (1997)
9. Feigin, B., Odesskii, A.: Quantized moduli spaces of the bundles on the elliptic curve and their applications, Integrable structures of exactly solvable two-dimensional models of quantum field theory (Kiev, 2000), 123–137; NATO Sci. Ser. II Math. Phys. Chem., 35, Kluwer, Dordrecht (2001)

10. Faddeev, L., Reshetikhin, N., Takhtadzhyan, L.: Quantization of Lie groups and Lie algebras, (Russian) Algebra i Analiz **1**, no. 1, 178–206 (1989); translation in Leningrad Math. J. **1**, no. 1, 193–225 (1990)
11. Finkelberg, M., Tsymbaliuk, A.: Shifted quantum affine algebras: integral forms in type A (with appendices by A. Tsymbaliuk, A. Weekes), Arnold Math. J. **5**(2), 197–283 (2019)
12. Gow, L.: On the Yangian $Y(\mathfrak{gl}_{m|n})$ and its quantum Berezinian. Czechoslov. J. Phys. **55**(11), 1415–1420 (2005)
13. Gow, L.: Gauss decomposition of the Yangian $Y(\mathfrak{gl}_{m|n})$. Commun. Math. Phys. **276**(3), 799–825 (2007)
14. Gautam, S., Laredo, V Toledano: Yangians and quantum loop algebras. Selecta Math. (N.S.) **19**(2), 271–336 (2013)
15. Levendorskii, S.: On PBW bases for Yangians. Lett. Math. Phys. **27**(1), 37–42 (1993)
16. Leites, D., Saveliev, M., Serganova, V.: Embeddings of $\mathfrak{osp}(N/2)$ and the associated nonlinear supersymmetric equations, Group theoretical methods in physics, Vol. I (Yurmala, 1985), 255–297, VNU Sci. Press, Utrecht (1986)
17. Molev, A.: Yangians and Classical Lie Algebras, Mathematical Surveys and Monographs, vol. 143. American Mathematical Society, Providence (2007)
18. Molev, A., Nazarov, M., Olshanskii, G.: Yangians and classical Lie algebras, (Russian) Uspekhi Mat. Nauk **51**, no. 2 (308), 27–104 (1996); translation in Russian Math. Surveys **51**, no. 2, 205–282 (1996)
19. Nazarov, M.: Quantum Berezinian and the classical Capelli identity. Lett. Math. Phys. **21**(2), 123–131 (1991)
20. Negut, A.: Quantum toroidal and shuffle algebras, preprint, arXiv:1302.6202
21. Peng, Y.-N.: Finite W-superalgebras and truncated super Yangians. Lett. Math. Phys. **104**(1), 89–102 (2014)
22. Peng, Y.-N.: Parabolic presentations of the super Yangian $Y(\mathfrak{gl}_{M|N})$ associated with arbitrary 01-sequences. Commun. Math. Phys. **346**(1), 313–347 (2016)
23. Peng, Y.-N.: *Finite W-superalgebras via super Yangians*, preprint, arXiv:2001.08718
24. Stukopin, V.: Yangians of Lie superalgebras of type $A(m, n)$, (Russian) Funktsional. Anal. i Prilozhen. **28**, no. 3, 85–88 (1994); translation in Funct. Anal. Appl. **28**, no. 3, 217–219 (1994)
25. Tsymbaliuk, A.: PBWD bases and shuffle algebra realizations for $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$, $U_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ and their integral forms, preprint, arXiv:1808.09536
26. Yamane, H.: On defining relations of affine Lie superalgebras and affine quantized universal enveloping superalgebras, Publ. Res. Inst. Math. Sci. **35**, no. 3, 321–390 (1999); Errata to: *On defining relations of affine Lie superalgebras and affine quantized universal enveloping superalgebras*, Publ. Res. Inst. Math. Sci. **37**, no. 4, 615–619 (2001)