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PBWD bases and shuffle algebra realizations for $U_v(L\mathfrak{sl}_n)$, $U_{v_1,v_2}(L\mathfrak{sl}_n)$, $U_v(L\mathfrak{sl}(m|n))$ and their integral forms

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Abstract

We construct a family of PBWD (Poincaré–Birkhoff–Witt–Drinfeld) bases for the quantum loop algebras $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$, $U_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ in the new Drinfeld realizations. In the 2-parameter case, this proves (Hu et al. in Commun Math Phys 278(2):453–486, 2008, Theorem 3.11) (stated in *loc. cit.* without a proof), while in the super case it proves a conjecture of Zhang (Math. Z. 278(3–4):663–703, 2014). The main ingredient in our proofs is the interplay between those quantum loop algebras and the corresponding shuffle algebras, which are trigonometric counterparts of the elliptic shuffle algebras of Feigin and Odesskii (Anal. i Prilozhen 23(3):45–54, 1989; Anal i Prilozhen 31(3):57–70, 1997; Integrable Structures of Exactly Solvable Two-Dimensional Models of Quantum Field Theory (Kiev, 2000). NATO Sci Ser II Math Phys Chem, vol 35, Kluwer Academic Publishers, Dordrecht, pp 123–137, 2001). Our approach is similar to that of Enriquez (J Lie Theory 13(1):21–64, 2003) in the formal setting, but the key novelty is an explicit shuffle algebra realization of the corresponding algebras, which is of independent interest. This also allows us to strengthen the above results by constructing a family of PBWD bases for the RTT forms of those quantum loop algebras as well as for the Lusztig form of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. The rational counterparts provide shuffle algebra realizations of type A (super) Yangians and their Drinfeld–Gavarini dual subalgebras.

Keywords Quantum affine algebra · Yangian · Shuffle algebra · Lusztig form · RTT form

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1 Introduction

1.1 Summary

The quantum loop algebras (aka quantum affine algebras with the trivial central charge) associated to a simple finite dimensional Lie algebra $\mathfrak{g}$ admit two well-known presentations: the original Drinfeld–Jimbo realization $U_v^{\mathrm{DJ}}(L\mathfrak{g})$ and the new Drinfeld (aka loop) realization $U_v(L\mathfrak{g})$. The latter presentation (which is essential to study the representation theory of quantum loop algebras) was introduced by V. Drinfeld in [7], while the explicit isomorphism

$$U_v^{\mathrm{DJ}}(L\mathfrak{g}) \xrightarrow{\sim} U_v(L\mathfrak{g}) \tag{1.1}$$

was stated without a proof in [7, Theorem 3]. Actually, (1.1) was upgraded in *loc. cit.* to the isomorphism of the corresponding quantum affine algebras (with nontrivial central charges)

$$U_v^{\mathrm{DJ}}(\widehat{\mathfrak{g}}) \xrightarrow{\sim} U_v(\widehat{\mathfrak{g}}) \tag{1.2}$$

The proof of the isomorphism (1.2) (hence, also of (1.1)) was properly established in the works of J. Beck [2], I. Damiani [6], and N. Jing [23]. Let us note that [2,6] actually constructed the isomorphism opposite way $U_v(\widehat{\mathfrak{g}}) \xrightarrow{\sim} U_v^{\mathrm{DJ}}(\widehat{\mathfrak{g}})$ by utilizing Lusztig's affine braid group action on $U_v^{\mathrm{DJ}}(\widehat{\mathfrak{g}})$, which is precisely the inverse of (1.1), as shown in [2, Remark of §4].

Since quantum loop algebras are natural quantizations of the universal enveloping of the loop Lie algebras $L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$, one of the first natural tasks is to seek an analogue of PBW bases for the former algebras. This was accomplished more than 25 years ago by J. Beck [1] in the context of $U_v^{\mathrm{DJ}}(\widehat{\mathfrak{g}})$. More precisely, he constructed the bases of each of the subalgebras featuring in the triangular decomposition (viewed as a vector space isomorphism)

$$U_v^{\mathrm{DJ}}(\widehat{\mathfrak{g}}) \simeq U_v^{\mathrm{DJ},>}(\widehat{\mathfrak{g}}) \otimes U_v^{\mathrm{DJ},0}(\widehat{\mathfrak{g}}) \otimes U_v^{\mathrm{DJ},<}(\widehat{\mathfrak{g}}). \tag{1.3}$$

We note that the construction of [1] actually depends on a choice of an element $x \in P^\vee$ of the coweight lattice, which pairs positively with all simple roots of $\mathfrak{g}$, together with a choice of a reduced decomposition of $(1, x) \in W \ltimes P^\vee \simeq \widehat{W}^{ext}$ in the extended affine Weyl group.

The algebra $U_v(\widehat{\mathfrak{g}})$ also admits a triangular decomposition, i.e. a vector space isomorphism

$$U_v(\widehat{\mathfrak{g}}) \simeq U_v^>(\widehat{\mathfrak{g}}) \otimes U_v^0(\widehat{\mathfrak{g}}) \otimes U_v^<(\widehat{\mathfrak{g}}). \tag{1.4}$$

However, the isomorphism (1.2) <u>does not</u> intertwine the triangular decompositions (1.3, 1.4). Therefore, it is desirable to construct PBWD bases (the letter "D" after "PBW" is to indicate the new Drinfeld realization) of $U_v(\widehat{\mathfrak{g}})$, compatible with the triangular decomposition (1.4). As $U_{\boldsymbol{v}}^{>}(\widehat{\mathfrak{g}}) \simeq U_{\boldsymbol{v}}^{>}(L\mathfrak{g})$ is actually isomorphic to $U_{\boldsymbol{v}}^{<}(\widehat{\mathfrak{g}}) \simeq U_{\boldsymbol{v}}^{<}(L\mathfrak{g})$, the above boils down to:

Problem: Construct PBWD bases of the *positive* subalgebras $U_{\boldsymbol{v}}^{>}(L\mathfrak{g})$.

To our surprise, this question seems to remain open. The only case we found addressed in the literature is the type A quantum loop algebras and their two-parameter generalizations $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$, see [22, Theorem 3.11]. However, the proof of that theorem is missing in *loc. cit*. This gap has been also noticed in [37,38], where a weak version of the PBW property has been established for the quantum loop superalgebra $U_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ of [36] by straightforward lengthy arguments.

One objective of this paper is to fill in the above gap by constructing a family of PBWD bases for the aforementioned quantum loop algebras $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$, $U_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$, which we further refine by constructing a family of PBWD bases for their certain integral forms $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, $\mathfrak{U}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$, $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$, defined over $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$ and $\mathbb{C}[\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_1^{-1},\boldsymbol{v}_2^{-1}]$, respectively.[1] This is accomplished by providing the shuffle realization of their positive subalgebras (following the ideas of [12–14] and [10]), which constitutes another main result of our paper. It should be noted that the corresponding shuffle realization of $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ can be implicitly deduced from [31], but we provide an alternative simpler proof which also works for the other two algebras $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}^{>}(L\mathfrak{sl}_n)$, $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}(m|n))$ as well as for their integral forms $\mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$, $\mathfrak{U}_{\boldsymbol{v}_1,\boldsymbol{v}_2}^{>}(L\mathfrak{sl}_n)$, $\mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}(m|n))$.

The aforementioned integral forms are not completely new in the literature. Indeed, the $\mathfrak{gl}_n$-counterpart of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, the integral form $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ of the quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$, appeared recently in [17] where it was used to construct and study integral forms of type A shifted quantum loop algebras. As shown in *loc. cit.*, $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ coincides with the tautological integral form (that is, the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra generated by the same collection of generators) of $U_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n)$, the RTT realization of the quantum loop $\mathfrak{gl}_n$. The RTT approach to quantum groups goes back to the St. Petersburg school of L. Faddeev, see [15], while the isomorphism

$$U_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n) \simeq U_{\boldsymbol{v}}(L\mathfrak{gl}_n) \tag{1.5}$$

is due to [9], where it was actually upgraded to quantum affine algebras: $U_{\boldsymbol{v}}^{\mathrm{rtt}}(\widehat{\mathfrak{gl}}_n) \simeq U_{\boldsymbol{v}}(\widehat{\mathfrak{gl}}_n)$. The isomorphism (1.5) admits natural generalizations to the two-parameter and super-cases:

$$\begin{aligned} U_{\boldsymbol{v}_1,\boldsymbol{v}_2}^{\mathrm{rtt}}(L\mathfrak{gl}_n) &\simeq U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{gl}_n) \\ U_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}(m|n)) &\simeq U_{\boldsymbol{v}}(L\mathfrak{gl}(m|n)) \end{aligned} \tag{1.6}$$

see [24,39] and the references therein. Thus, the tautological integral forms of the algebras in the left-hand side of (1.6) give rise to integral forms of the algebras in the right-hand side, which can be perceived as $\mathfrak{gl}_n$, $\mathfrak{gl}(m|n)$-counterparts of our integral forms $\mathfrak{U}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$, $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$.

[1] It should be noted right away that these forms can be defined over $\mathbb{Z}[\boldsymbol{v},\boldsymbol{v}^{-1}]$ and $\mathbb{Z}[\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_1^{-1},\boldsymbol{v}_2^{-1}]$, respectively, and all our results for the integral forms generalize verbatim to this setting as well.

Let us point out right away both the similarities and the differences between the current work and a much older paper [10] of B. Enriquez. In [10], the author established similar results for the quantum loop algebras in the formal setting, that is, when working over $\mathbb{C}[[\hbar]]$ rather than over $\mathbb{C}(\boldsymbol{v})$. In particular, the PBW theorem of [10, Theorem 1.3] is proved using an embedding of $U^{>}_{\hbar}(L\mathfrak{g})$ into the corresponding type $\mathfrak{g}$ shuffle algebra $S^{(\mathfrak{g})}$ [10, Corollary 1.4] with the image $\bar{S}^{(\mathfrak{g})} \subset S^{(\mathfrak{g})}$ being the subalgebra generated by degree 1 components. In type A, this coincides with our Proposition 3.4. However, the heart of our shuffle algebra realization is the proof of the equality $\bar{S}^{(\mathfrak{g})} = S^{(\mathfrak{g})}$, at least, for $\mathfrak{g} = \mathfrak{sl}_n$ (and similarly for $\mathfrak{g} = \mathfrak{sl}(m|n)$). This implies the (corrected) description of $\bar{S}^{(\mathfrak{g})}$ conjectured in [10, Remark 3.16].

We expect that similar arguments shall provide PBWD bases for $U_{\boldsymbol{v}}(L\mathfrak{g})$ as well as establish their shuffle realizations, at least for simply-laced simple $\mathfrak{g}$, which will be discussed elsewhere. In contrast, the PBWD theorem for the Yangian $Y(\mathfrak{g})$ of any semisimple Lie algebra $\mathfrak{g}$ has been proved long time ago in [28].

A particular PBWD basis of the integral form $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ was used in [17] to define an integral form of type A shifted quantum loop algebras of [16], see Remark 2.13 and Theorem 2.23. Furthermore, an important family of elements of the latter form, which were crucially used in [17, Proof of Theorem 4.15], appear manifestly via their shuffle realizations (3.45).

Another particular PBWD basis of $\mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is very similar to the one arising from the results of [31] by viewing $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ as a "vertical" subalgebra of the quantum toroidal algebra $U_{\boldsymbol{v},\bar{\boldsymbol{v}}}(\ddot{\mathfrak{gl}}_n)$, see Remark 3.27.

Finally, let us make a few general comments about the PBWD bases constructed in this paper. As was pointed out to us by P. Etingof, the linear independence of the ordered monomials (2.15), which is established in Sect. 3.2.2, can be immediately deduced by using the PBW property of $U(\mathfrak{sl}_n[t, t^{-1}])$ as well as flatness of the deformation, cf. [10, Theorem 1.3]. Nevertheless, we provide technical details as they are needed both for Sect. 3.2.3 and for the generalizations to the two-parameter and super cases. At that point, we should note that while the two-parameter quantum affine algebras have been extensively studied since the original work [22], see [24,26,27] for a partial list of references (see also [3,4,25,35] for the case of two-parameter quantum finite groups), not many results have been established for them. In particular, it is still an open question whether these are flat deformations of the corresponding universal enveloping algebras (the results of the current paper give an affirmative answer to this question in type A). In [27], an isomorphism between the Drinfeld–Jimbo and the new Drinfeld realizations of these algebras was established (generalizing [22, Theorem 3.12] for type A), but it is not known (at the moment) whether the former realization admits the PBW basis analogous the one of [1,2].

1.2 Outline of the paper

- In Sect. 2.1, we recall the new Drinfeld realization of the quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. In Proposition 2.9, we invoke its triangular decomposition as well as explicit descriptions of its positive, negative, and Cartan subalgebras $U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, $U^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, $U^{0}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ established in [21].

In Sect. 2.2, we introduce the *PBWD basis elements* $e_\beta(r),\ f_\beta(r)$ of (2.12), which do depend on certain choices [see (1)–(3) prior to (2.12)]. Having picked a specific order (2.11) on $\Delta^+ \times \mathbb{Z}$, we use those elements to construct the *ordered PBWD monomials* $e_h,\ f_h$ of (2.15). This provides a family of the PBWD bases for $U^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, see Theorems 2.16 and 2.18 for the formulation of these results, while their proofs are deferred to Sect. 3.

In Sect. 2.3, we introduce integral forms $\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n), \mathfrak{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ as the $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebras generated by $\widetilde{e}_\beta(r),\ \widetilde{f}_\beta(r)$ of (2.19). While these elements do depend on the choices (1)–(3) made prior to (2.12), we prove that the forms $\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n), \mathfrak{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ are independent of these choices and posses PBWD bases (over $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$), see Theorem 2.20 (which is proved in Sect. 3). Following Remark 2.27, the integral form $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ of the entire $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ introduced in Definition 2.21 is identified with the RTT integral form $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ of [15], which is used in [17] to establish Theorem 2.23. The latter implies the triangular decomposition for $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ as well as provides a whole family of the PBWD bases for it, see Corollary 2.24, Theorem 2.25.

• In Sect. 3.1, we introduce the shuffle algebra $S^{(n)}$, which may be viewed as a trigonometric degeneration of the elliptic shuffle algebra of Feigin–Odesskii, see [12–14]. An algebra embedding $\Psi\colon U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \hookrightarrow S^{(n)}$ of Proposition 3.4 is a "simple version" of the shuffle realization of $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ (which was first used in [10] in the formal setting). The "hard version" of the shuffle algebra realization, Theorem 3.5, establishes that Ψ is an algebra isomorphism. We conclude this section with a construction of the specialization maps $\phi_{\underline{d}}$ (3.7) which constitute the key tool in our study of the shuffle algebra $S^{(n)}$ and the homomorphism Ψ.

In Sect. 3.2, we prove simultaneously Theorems 2.16 and 3.5 by combining the key properties of the specialization maps $\phi_{\underline{d}}$ established in Lemmas 3.16, 3.17, 3.21 with a direct treatment [crucially based on the formula (3.10)] of the simplest "rank 1" case $n = 2$ in Lemma 3.12.

In Sect. 3.3, we provide an explicit description of the image $\mathfrak{S}^{(n)} = \Psi(\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)) \subset S^{(n)}$, see Theorem 3.40 and Definition 3.37. While this description of $\mathfrak{S}^{(n)}$ is rather cumbersome (in particular, it is not even obvious that it is a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra of $S^{(n)}$), we can still establish important properties for it, see Proposition 3.42, which play the crucial role in [17].

In Sect. 3.4, we prove simultaneously Theorems 2.20 and 3.40 by first treating $n = 2$ case in Lemmas 3.46, 3.48 and then following arguments of Sect. 3.2 to treat the general case.

• In Sect. 4, we generalize the key results of Sects. 2, 3 to the two-parameter quantum loop algebras $U^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ of [22] recalled in Sect. 4.1. We construct a family of the PBWD bases for $U^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ in Theorem 4.3, thus generalizing [22, Theorem 3.11] presented in *loc. cit.* without a proof, see Remark 4.4. We further strengthen this by constructing a family of the PBWD bases for the integral form $\mathfrak{U}^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$, see Theorem 4.7. Finally, we provide the shuffle algebra realization of $U^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ in Theorem 4.10.

• In Sect. 5, we generalize the key results of Sects. 2, 3 to the quantum loop superalgebra $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ of [36] recalled in Sect. 5.1. We construct a family of the PBWD bases for $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}(m|n))$ in Theorem 5.7, thus proving a conjecture of [37], see Remark 5.8. We further strengthen this by constructing a family of the PBWD bases for the integral form $\mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}(m|n))$, see Theorem 5.11.

In Sect. 5.4, we introduce the shuffle algebra $S^{(m|n)}$ featuring new wheel conditions and skew-symmetry in one family of variables. Generalizing Theorem 3.5, we construct an algebra isomorphism $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}(m|n)) \xrightarrow{\sim} S^{(m|n)}$, see Theorem 5.18 and its proof in Sect. 5.5.

• In Sect. 6.1, we recall the *positive subalgebra* of the Yangian $Y_{\hbar}^{>}(\mathfrak{sl}_n)$ and its Drinfeld–Gavarini dual subalgebra $\mathbf{Y}_{\hbar}^{>}(\mathfrak{sl}_n)$, as well as the PBWD bases for those, see Theorems 6.5, 6.8.

In Sect. 6.2, we introduce a (rational) counterpart $\bar{W}^{(n)}$ of the (trigonometric) shuffle algebra $S^{(n)}$, equipped with an embedding $\Psi\colon Y_{\hbar}^{>}(\mathfrak{sl}_n) \hookrightarrow \bar{W}^{(n)}$, see Proposition 6.16. In contrast to Theorem 3.5, Ψ is not an isomorphism, and we provide explicit descriptions of the images $W^{(n)} = \Psi(Y_{\hbar}^{>}(\mathfrak{sl}_n))$, $\mathfrak{W}^{(n)} = \Psi(\mathbf{Y}_{\hbar}^{>}(\mathfrak{sl}_n))$ in Theorems 6.20, 6.27, see Definitions 6.17, 6.25.

• In Sect. 7.1, we recall the positive subalgebra of the super Yangian $Y_{\hbar}^{>}(\mathfrak{sl}(m|n))$, its Drinfeld–Gavarini dual subalgebra $\mathbf{Y}_{\hbar}^{>}(\mathfrak{sl}(m|n))$, and their PBWD bases, see Theorems 7.6, 7.7.

In Sect. 7.2, we introduce a (rational) counterpart $\bar{W}^{(m|n)}$ of the (trigonometric) shuffle algebra $S^{(m|n)}$, equipped with an embedding $\Psi\colon Y_{\hbar}^{>}(\mathfrak{sl}(m|n)) \hookrightarrow \bar{W}^{(m|n)}$, see Proposition 7.11. We provide explicit descriptions of the images $W^{(m|n)} = \Psi(Y_{\hbar}^{>}(\mathfrak{sl}(m|n)))$ and $\mathfrak{W}^{(m|n)} = \Psi(\mathbf{Y}_{\hbar}^{>}(\mathfrak{sl}(m|n)))$ in Theorems 7.14, 7.15, see Definition 7.12.

• In Sect. 8.1, we recall another integral form $\mathsf{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ of $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$, first explicitly considered in [20]. We construct a family of the PBWD bases for $\mathsf{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ in Theorem 8.4 and provide its shuffle algebra realization in Theorem 8.7, see Definition 8.5. The former yields a family of the PBWD bases for the Lusztig form [29] of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, due to Remark 8.8 and Theorem 8.9. In the rest of Sect. 8, we recall the key results of the companion papers [33,34].

2 Quantum loop algebra $U_v(L\mathfrak{sl}_n)$ and its integral form $\mathfrak{U}_v(L\mathfrak{sl}_n)$

2.1 Quantum loop algebra $U_v(L\mathfrak{sl}_n)$

Let $I = \{1, \ldots, n-1\}$, $(c_{ij})_{i,j\in I}$ be the Cartan matrix of $\mathfrak{sl}_n$, and $\boldsymbol{v}$ be a formal variable. Following [7], define the quantum loop algebra of $\mathfrak{sl}_n$ (in the new Drinfeld presentation), denoted by $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, to be the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by $\{e_{i,r}, f_{i,r}, \psi_{i,\pm s}^{\pm}\}_{i\in I}^{r\in\mathbb{Z}, s\in\mathbb{N}}$ with the following defining relations:

$$[\psi_i^{\epsilon}(z), \psi_j^{\epsilon'}(w)] = 0,\ \ \psi_{i,0}^{\pm}\cdot\psi_{i,0}^{\mp} = 1, \tag{2.1}$$

$$(z - \boldsymbol{v}^{c_{ij}}w)e_i(z)e_j(w) = (\boldsymbol{v}^{c_{ij}}z - w)e_j(w)e_i(z), \tag{2.2}$$

$$(\boldsymbol{v}^{c_{ij}}z - w)f_i(z)f_j(w) = (z - \boldsymbol{v}^{c_{ij}}w)f_j(w)f_i(z), \tag{2.3}$$

$$(z - \boldsymbol{v}^{c_{ij}}w)\psi_i^{\epsilon}(z)e_j(w) = (\boldsymbol{v}^{c_{ij}}z - w)e_j(w)\psi_i^{\epsilon}(z), \tag{2.4}$$

$$(\boldsymbol{v}^{c_{ij}} z - w)\psi_i^\epsilon(z) f_j(w) = (z - \boldsymbol{v}^{c_{ij}} w) f_j(w)\psi_i^\epsilon(z), \tag{2.5}$$

$$[e_i(z), f_j(w)] = \frac{\delta_{ij}}{\boldsymbol{v} - \boldsymbol{v}^{-1}} \delta\left(\frac{z}{w}\right)\left(\psi_i^+(z) - \psi_i^-(z)\right), \tag{2.6}$$

$$\begin{aligned} &e_i(z)e_j(w) = e_j(w)e_i(z) \text{ if } c_{ij} = 0, \\ &[e_i(z_1), [e_i(z_2), e_j(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} + [e_i(z_2), [e_i(z_1), e_j(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} = 0 \quad \text{if } c_{ij} = -1, \end{aligned} \tag{2.7}$$

$$\begin{aligned} &f_i(z)f_j(w) = f_j(w)f_i(z) \text{ if } c_{ij} = 0, \\ &[f_i(z_1), [f_i(z_2), f_j(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} + [f_i(z_2), [f_i(z_1), f_j(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} = 0 \quad \text{if } c_{ij} = -1, \end{aligned} \tag{2.8}$$

where $[a, b]_x := ab - x \cdot ba$ and the generating series are defined as follows:

$$e_i(z) := \sum_{r\in\mathbb{Z}} e_{i,r} z^{-r}, \;\; f_i(z) := \sum_{r\in\mathbb{Z}} f_{i,r} z^{-r}, \;\; \psi_i^\pm(z) := \sum_{s\geq 0} \psi_{i,\pm s}^\pm z^{\mp s}, \;\; \delta(z) := \sum_{r\in\mathbb{Z}} z^r.$$

Let $U_{\boldsymbol{v}}^<(L\mathfrak{sl}_n)$, $U_{\boldsymbol{v}}^>(L\mathfrak{sl}_n)$, $U_{\boldsymbol{v}}^0(L\mathfrak{sl}_n)$ be the $\mathbb{C}(\boldsymbol{v})$-subalgebras of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated respectively by $\{f_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$, $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$, $\{\psi_{i,\pm s}^\pm\}_{i\in I}^{s\in\mathbb{N}}$. The following is standard (see e.g. [21, Theorem 2]):

Proposition 2.9 (a) *(Triangular decomposition of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$) The multiplication map*

$$m\colon U_{\boldsymbol{v}}^<(L\mathfrak{sl}_n) \otimes_{\mathbb{C}(\boldsymbol{v})} U_{\boldsymbol{v}}^0(L\mathfrak{sl}_n) \otimes_{\mathbb{C}(\boldsymbol{v})} U_{\boldsymbol{v}}^>(L\mathfrak{sl}_n) \longrightarrow U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$$

is an isomorphism of $\mathbb{C}(\boldsymbol{v})$-vector spaces.

(b) *The algebra $U_{\boldsymbol{v}}^>(L\mathfrak{sl}_n)$ (resp. $U_{\boldsymbol{v}}^<(L\mathfrak{sl}_n)$ and $U_{\boldsymbol{v}}^0(L\mathfrak{sl}_n)$) is isomorphic to the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ (resp. $\{f_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ and $\{\psi_{i,\pm s}^\pm\}_{i\in I}^{s\in\mathbb{N}}$) with the defining relations (2.2, 2.7) (resp. (2.3, 2.8) and (2.1)).*

2.2 PBWD bases of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$

Let $\{\alpha_i\}_{i=1}^{n-1}$ be the standard simple positive roots of $\mathfrak{sl}_n$, and Δ^+ be the set of positive roots: $\Delta^+ = \{\alpha_j + \alpha_{j+1} + \cdots + \alpha_i\}_{1\leq j\leq i<n}$. Consider the following total order "$\leq$" on Δ^+:

$$\alpha_j + \alpha_{j+1} + \cdots + \alpha_i \leq \alpha_{j'} + \alpha_{j'+1} + \cdots + \alpha_{i'} \;\text{ iff }\; j < j' \text{ or } j = j', i \leq i'. \tag{2.10}$$

We also pick a total order $\preceq_\beta$ on $\mathbb{Z}$ for any $\beta \in \Delta^+$. This gives rise to the total order "$\leq$" on $\Delta^+ \times \mathbb{Z}$:

$$(\beta, r) \leq (\beta', r') \;\text{ iff }\; \beta < \beta' \text{ or } \beta = \beta', r \preceq_\beta r'. \tag{2.11}$$

For every pair $(\beta, r) \in \Delta^+ \times \mathbb{Z}$, we choose:

(1) a decomposition $\beta = \alpha_{i_1} + \cdots + \alpha_{i_p}$ such that $[\cdots[e_{\alpha_{i_1}}, e_{\alpha_{i_2}}], \cdots, e_{\alpha_{i_p}}]$ is a non-zero root vector e_β of $\mathfrak{sl}_n$ (here, e_{α_i} denotes the standard Chevalley generator of $\mathfrak{sl}_n$);
(2) a decomposition $r = r_1 + \cdots + r_p$ with $r_k \in \mathbb{Z}$;

(3) a sequence $(\lambda_1, \dots, \lambda_{p-1}) \in \{\boldsymbol{v}, \boldsymbol{v}^{-1}\}^{p-1}$.

Then, we define the *PBWD basis elements* $e_\beta(r) \in U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ and $f_\beta(r) \in U_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n)$ via

$$\begin{aligned} e_\beta(r) &:= [\cdots[[e_{i_1,r_1}, e_{i_2,r_2}]_{\lambda_1}, e_{i_3,r_3}]_{\lambda_2}, \cdots, e_{i_p,r_p}]_{\lambda_{p-1}}, \\ f_\beta(r) &:= [\cdots[[f_{i_1,r_1}, f_{i_2,r_2}]_{\lambda_1}, f_{i_3,r_3}]_{\lambda_2}, \cdots, f_{i_p,r_p}]_{\lambda_{p-1}}. \end{aligned} \tag{2.12}$$

In particular, $e_{\alpha_i}(r) = e_{i,r}$ and $f_{\alpha_i}(r) = f_{i,r}$. We note that $e_\beta(r)$, $f_\beta(r)$ degenerate to the corresponding root generators $e_\beta \otimes t^r$, $f_\beta \otimes t^r$ of $\mathfrak{sl}_n[t, t^{-1}] = \mathfrak{sl}_n \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ as $\boldsymbol{v} \to 1$, hence, the terminology.

Remark 2.13 The following particular choice featuresmanifestly in [17](cf. Remark 4.4):

$$\begin{aligned} e_{\alpha_j+\alpha_{j+1}+\cdots+\alpha_i}(r) &:= [\cdots[[e_{j,r}, e_{j+1,0}]_{\boldsymbol{v}}, e_{j+2,0}]_{\boldsymbol{v}}, \cdots, e_{i,0}]_{\boldsymbol{v}}, \\ f_{\alpha_j+\alpha_{j+1}+\cdots+\alpha_i}(r) &:= [\cdots[[f_{j,r}, f_{j+1,0}]_{\boldsymbol{v}}, f_{j+2,0}]_{\boldsymbol{v}}, \cdots, f_{i,0}]_{\boldsymbol{v}}. \end{aligned} \tag{2.14}$$

Let H denote the set of all functions $h \colon \Delta^+ \times \mathbb{Z} \to \mathbb{N}$ with finite support. The monomials

$$e_h := \prod_{(\beta,r)\in\Delta^+\times\mathbb{Z}}^{\rightarrow} e_\beta(r)^{h(\beta,r)}, \qquad f_h := \prod_{(\beta,r)\in\Delta^+\times\mathbb{Z}}^{\leftarrow} f_\beta(r)^{h(\beta,r)}, \qquad \forall\, h \in H \tag{2.15}$$

will be called the *ordered PBWD monomials* of $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ and $U_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n)$. Here, the arrows $\to$ and $\leftarrow$ over the product signs refer to the total order (2.11) and its opposite, respectively.

Our first main result establishes the PBWD property of $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ and $U_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n)$ (cf. [28]):

Theorem 2.16 (a) *The ordered PBWD monomials* $\{e_h\}_{h\in H}$ *form a* $\mathbb{C}(\boldsymbol{v})$*-basis of* $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$.
(b) *The ordered PBWD monomials* $\{f_h\}_{h\in H}$ *form a* $\mathbb{C}(\boldsymbol{v})$*-basis of* $U_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n)$.

The proof of Theorem 2.16 is presented in Sect. 3.2 and is based on the shuffle approach.

Let us relabel the Cartan generators via $\psi_{i,r} := \begin{cases} \psi^+_{i,r}, & \text{if } r \geq 0 \\ \psi^-_{i,r}, & \text{if } r < 0 \end{cases}$, so that $(\psi_{i,0})^{-1} = \psi^-_{i,0}$. Let H_0 denote the set of all functions $g \colon I \times \mathbb{Z} \to \mathbb{Z}$ with finite support and such that $g(i,r) \geq 0$ for $r \neq 0$. The monomials (note that the order of the products is irrelevant, due to (2.1))

$$\psi_g := \prod_{(i,r)\in I\times\mathbb{Z}} \psi_{i,r}^{g(i,r)}, \qquad \forall\, g \in H_0 \tag{2.17}$$

will be called the *PBWD monomials* of $U_{\boldsymbol{v}}^{0}(L\mathfrak{sl}_n)$.

According to Proposition 2.9(b), the PBWD monomials $\{\psi_g\}_{g\in H_0}$ form a $\mathbb{C}(\boldsymbol{v})$-basis of $U^0_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. Combining this with Theorem 2.16 and Proposition 2.9(a), we finally get:

Theorem 2.18 *The elements*

$$\left\{f_{h_-}\cdot\psi_{h_0}\cdot e_{h_+}\mid h_-,h_+\in H, h_0\in H_0\right\}$$

form a $\mathbb{C}(\boldsymbol{v})$*-basis of the quantum loop algebra* $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$*.*

2.3 Integral form $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and its PBWD bases

Following the above notations, define $\widetilde{e}_\beta(r)\in U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $\widetilde{f}_\beta(r)\in U^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ via

$$\widetilde{e}_\beta(r):=(\boldsymbol{v}-\boldsymbol{v}^{-1})e_\beta(r),\quad \widetilde{f}_\beta(r):=(\boldsymbol{v}-\boldsymbol{v}^{-1})f_\beta(r),\quad \forall\,(\beta,r)\in\Delta^+\times\mathbb{Z}. \tag{2.19}$$

We also define $\widetilde{e}_h, \widetilde{f}_h$ via the formula (2.15) but using $\widetilde{e}_\beta(r), \widetilde{f}_\beta(r)$ instead of $e_\beta(r), f_\beta(r)$. Finally, we define integral forms $\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $\mathfrak{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ as the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebras of $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $U^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated by $\{\widetilde{e}_\beta(r)\}^{r\in\mathbb{Z}}_{\beta\in\Delta^+}$ and $\{\widetilde{f}_\beta(r)\}^{r\in\mathbb{Z}}_{\beta\in\Delta^+}$, respectively.

We note that the above definition of $\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n), \mathfrak{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ depends on all the choices (1)–(3) made when defining $e_\beta(r), f_\beta(r)$ in (2.12). Our next result establishes that they are actually independent of these choices and posses PBWD bases analogous to those of Theorem 2.16.

Theorem 2.20 (a) *The subalgebras* $\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ *and* $\mathfrak{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ *are independent of all our choices.*

(b) *The ordered PBWD monomials* $\{\widetilde{e}_h\}_{h\in H}$ *form a basis of the free* $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$*-module* $\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

(c) *The ordered PBWD monomials* $\{\widetilde{f}_h\}_{h\in H}$ *form a basis of the free* $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$*-module* $\mathfrak{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

The proof of Theorem 2.20 is presented in Sect. 3.4 and is based on the shuffle approach.

We also define an integral form $\mathfrak{U}^0_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ as the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $U^0_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated by $\{\psi^\pm_{i,\pm s}\}^{s\in\mathbb{N}}_{i\in I}$. Due to Proposition 2.9(b), the PBWD monomials $\{\psi_g\}_{g\in H_0}$ of (2.17) form a basis of the free $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-module $\mathfrak{U}^0_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

Definition 2.21 The integral form $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is defined as the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated by $\{\widetilde{e}_\beta(r),\widetilde{f}_\beta(r)\}^{r\in\mathbb{Z}}_{\beta\in\Delta^+}\cup\{\psi^\pm_{i,\pm s}\}^{s\in\mathbb{N}}_{i\in I}$.

Remark 2.22 Due to Theorem 2.20(a), the algebra $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ itself is independent of any choices (made in the definition (2.12, 2.19) of the PBWD basis elements $\widetilde{e}_\beta(r),\widetilde{f}_\beta(r)$).

The following result is proved in [17, Theorem 3.24] (cf. Theorem 2.18):

Theorem 2.23 [17] *For the particular choice* (2.14) *of* $e_\beta(r)$, $f_\beta(r)$ *in* (2.19)*, the elements*

$$\left\{ \widetilde{f}_{h_-} \cdot \psi_{h_0} \cdot \widetilde{e}_{h_+} \mid h_-, h_+ \in H, h_0 \in H_0 \right\}$$

form a basis of the free $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$*-module* $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

In view of Theorem 2.20, this gives rise to the *triangular decomposition* of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$:

Corollary 2.24 *The multiplication map*

$$m\colon \mathfrak{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathfrak{U}^0_{\boldsymbol{v}}(L\mathfrak{sl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \longrightarrow \mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$$

is an isomorphism of the free $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$*-modules.*

Combining this Corollary with Theorem 2.20, we finally obtain:

Theorem 2.25 *For any choices (1)–(3) made prior to* (2.12)*, the elements*

$$\left\{ \widetilde{f}_{h_-} \cdot \psi_{h_0} \cdot \widetilde{e}_{h_+} \mid h_-, h_+ \in H, h_0 \in H_0 \right\}$$

form a basis of the free $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$*-module* $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

Remark 2.26 All results of this Sect. also hold when $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$ is replaced with $\mathbb{Z}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$.

We conclude this section with a remark discussing the $\mathfrak{gl}_n$-counterpart of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$:

Remark 2.27 (a) It is often more convenient to work with the quantum loop algebra of $\mathfrak{gl}_n$, denoted by $U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$, which roughly speaking is obtained from $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ by adding a Cartan current. Its integral form $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ is defined similarly to $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ of Definition 2.21, and admits a triangular decomposition $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \simeq \mathfrak{U}^<_{\boldsymbol{v}}(L\mathfrak{gl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathfrak{U}^0_{\boldsymbol{v}}(L\mathfrak{gl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{gl}_n)$, cf. Corollary 2.24. Here, $\mathfrak{U}^<_{\boldsymbol{v}}(L\mathfrak{gl}_n) \simeq \mathfrak{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, $\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{gl}_n) \simeq \mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, $\mathfrak{U}^0_{\boldsymbol{v}}(L\mathfrak{gl}_n) \supset \mathfrak{U}^0_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

(b) The proof of Theorem 2.23 presented in [17] crucially utilizes the identification of $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ and the RTT integral form $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ of [15] (see [17, §3.2, Proposition 3.11]) under the $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $U_{\boldsymbol{v}}(L\mathfrak{gl}_n) \simeq \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}(\boldsymbol{v})$ of [9].

(c) We note that the integral form $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ provides a quantization of the thick slice ${}^\dagger\mathcal{W}_0$ of [16, §4.8], see [17, Remark 3.26]. More precisely, we have

$$\begin{aligned} &\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)/(\boldsymbol{v}-1) \\ &\quad \simeq \mathbb{C}\left[t^\pm_{ji}[\pm r]\right]^{r\geq 0}_{1\leq j,i\leq n} \Big/ \left(\langle t^+_{ij}[0], t^-_{ji}[0], t^\pm_{kk}[0]t^\mp_{kk}[0]-1\rangle^{1\leq k\leq n}_{1\leq j<i\leq n}\right). \end{aligned} \tag{2.28}$$

3 Shuffle algebra realizations of $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ and $\mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$

In this section, we establish the shuffle algebra realizations of $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ and $\mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ [hence, the independence of the latter from all choices made, Theorem 2.20(a)], and use those to prove Theorems 2.16(a) and 2.20(b). As the assignment $e_{i,r} \mapsto f_{i,r}$ $(i \in I, r \in \mathbb{Z})$ gives rise to a $\mathbb{C}(\boldsymbol{v})$-algebra anti-isomorphism $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n) \to U_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n)$ (resp. a $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra anti-isomorphism $\mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n) \to \mathfrak{U}_{\boldsymbol{v}}^{<}(L\mathfrak{sl}_n)$) that maps the ordered PBWD monomials of the source to the ordered PBWD monomials of the target (up to a sign and an integer power of $\boldsymbol{v}$), both Theorems 2.16(b) and 2.20(c) follow as well.

3.1 Shuffle algebra $S^{(n)}$

We follow the notations of [16, Appendix I(ii)] (cf. [31]).[2] Let Σ_k denote the symmetric group in k elements, and set $\Sigma_{(k_1,\ldots,k_{n-1})}:=\Sigma_{k_1}\times\cdots\times\Sigma_{k_{n-1}}$ for $k_1,\ldots,k_{n-1}\in\mathbb{N}$. Consider an $\mathbb{N}^I$-graded $\mathbb{C}(\boldsymbol{v})$-vector space $\mathbb{S}^{(n)} = \bigoplus_{\underline{k}=(k_1,\ldots,k_{n-1})\in\mathbb{N}^I} \mathbb{S}^{(n)}_{\underline{k}}$, where $\mathbb{S}^{(n)}_{(k_1,\ldots,k_{n-1})}$ consists of $\Sigma_{\underline{k}}$-symmetric rational functions in the variables $\{x_{i,r}\}_{i\in I}^{1\le r\le k_i}$. We fix an $I\times I$ matrix of rational functions $(\zeta_{i,j}(z))_{i,j\in I} \in \mathrm{Mat}_{I\times I}(\mathbb{C}(\boldsymbol{v})(z))$ via $\zeta_{i,j}(z) := \frac{z-\boldsymbol{v}^{-c_{ij}}}{z-1}$. Let us now introduce the bilinear *shuffle product* $\star$ on $\mathbb{S}^{(n)}$: given $F\in\mathbb{S}^{(n)}_{\underline{k}}$ and $G\in\mathbb{S}^{(n)}_{\underline{\ell}}$, define $F\star G\in\mathbb{S}^{(n)}_{\underline{k}+\underline{\ell}}$ via

$$(F\star G)(x_{1,1},\ldots,x_{1,k_1+\ell_1};\ldots;x_{n-1,1},\ldots,x_{n-1,k_{n-1}+\ell_{n-1}}) := \frac{1}{\underline{k}!\cdot\underline{\ell}!} \times \mathrm{Sym}_{\Sigma_{\underline{k}+\underline{\ell}}}\left(F\left(\{x_{i,r}\}_{i\in I}^{1\le r\le k_i}\right) G\left(\{x_{i',r'}\}_{i'\in I}^{k_{i'}<r'\le k_{i'}+\ell_{i'}}\right)\cdot \prod_{i\in I}^{i'\in I}\prod_{r\le k_i}^{r'>k_{i'}} \zeta_{i,i'}(x_{i,r}/x_{i',r'})\right). \tag{3.1}$$

Here, $\underline{k}! = \prod_{i\in I} k_i!$, while the *symmetrization* of $f\in\mathbb{C}(\{x_{i,1},\ldots,x_{i,m_i}\}_{i\in I})$ is defined via

$$\mathrm{Sym}_{\Sigma_{\underline{m}}}(f)(\{x_{i,1},\ldots,x_{i,m_i}\}_{i\in I}) := \sum_{(\sigma_1,\ldots,\sigma_{n-1})\in\Sigma_{\underline{m}}} f\left(\{x_{i,\sigma_i(1)},\ldots,x_{i,\sigma_i(m_i)}\}_{i\in I}\right).$$

This endows $\mathbb{S}^{(n)}$ with a structure of an associative unital algebra with the unit $\mathbf{1}\in\mathbb{S}^{(n)}_{(0,\ldots,0)}$.

We will be interested only in the subspace of $\mathbb{S}^{(n)}$ defined by the *pole* and *wheel conditions*:

[2] These are trigonometric counterparts of the elliptic shuffle algebras of Feigin–Odesskii [12–14].

- We say that $F \in \mathbb{S}^{(n)}_{\underline{k}}$ satisfies the *pole conditions* if

$$F = \frac{f(x_{1,1}, \dots, x_{n-1,k_{n-1}})}{\prod_{i=1}^{n-2} \prod_{r\le k_i}^{r'\le k_{i+1}} (x_{i,r} - x_{i+1,r'})}, \text{ where } f \in \mathbb{C}(\boldsymbol{v})[\{x_{i,r}^{\pm 1}\}_{i\in I}^{1\le r\le k_i}]^{\Sigma_{\underline{k}}}. \quad (3.2)$$

- We say that $F \in \mathbb{S}^{(n)}_{\underline{k}}$ satisfies the *wheel conditions*,[3] if

$$F(\{x_{i,r}\}) = 0 \text{ once } x_{i,r_1} = \boldsymbol{v} x_{i+\epsilon,s} = \boldsymbol{v}^2 x_{i,r_2} \text{ for some } \epsilon, i, r_1, r_2, s, \quad (3.3)$$

  where $\epsilon \in \{\pm 1\}$, $i, i+\epsilon \in I$, $1 \le r_1 \ne r_2 \le k_i$, $1 \le s \le k_{i+\epsilon}$.

Let $S^{(n)}_{\underline{k}} \subset \mathbb{S}^{(n)}_{\underline{k}}$ denote the subspace of all elements F satisfying these two conditions and set

$$S^{(n)} := \bigoplus_{\underline{k}\in\mathbb{N}^I} S^{(n)}_{\underline{k}}.$$

It is straightforward to check that the subspace $S^{(n)} \subset \mathbb{S}^{(n)}$ is $\star$-closed. The **shuffle algebra** $\left(S^{(n)}, \star\right)$ is related to $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ via the following construction:[4]

Proposition 3.4 *The assignment* $e_{i,r} \mapsto x_{i,1}^r$ $(i \in I, r \in \mathbb{Z})$ *gives rise to an injective* $\mathbb{C}(\boldsymbol{v})$*-algebra homomorphism* $\Psi \colon U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \to S^{(n)}$.

Proof The assignment $e_{i,r} \mapsto x_{i,1}^r$ is compatible with relations (2.2, 2.7), hence, it gives rise to a $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism $\Psi \colon U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \to S^{(n)}$, due to Proposition 2.9(b).

The injectivity of Ψ follows from the general arguments based on the existence of a non-degenerate pairing on the source and a pairing on the target compatible with the former one via Ψ. This is explained in details in [31, Lemma 2.20, Proposition 2.30, Proposition 3.8]. □

The next result follows from its much harder counterpart [31, Theorem 1.1], but we will provide its alternative simpler proof[5] below, see Remark 3.25:

Theorem 3.5 $\Psi \colon U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \xrightarrow{\sim} S^{(n)}$ *of Proposition* 3.4 *is a* $\mathbb{C}(\boldsymbol{v})$*-algebra isomorphism.*

Before we proceed to the proofs of Theorem 2.16(a) and Theorem 3.5, let us introduce the key tool in our study of the shuffle algebra, the so-called *specialization maps*. For a positive root $\beta = \alpha_j + \alpha_{j+1} + \cdots + \alpha_i$, define $j(\beta) := j$, $i(\beta) := i$, and let $[\beta]$ denote the integer interval $[j(\beta); i(\beta)]$. Consider a collection of the intervals

[3] Following [12–14] the role of the wheel conditions is exactly to replace complicated Serre relations.

[4] In the formal setup (when working over $\mathbb{C}[[\hbar]]$ rather than over $\mathbb{C}(\boldsymbol{v})$), this goes back to [10, Corollary 1.4].

[5] One of the benefits of our proof is that it will be directly generalized to establish the isomorphisms of Theorems 4.10 and 5.18 below, for which no analogue of [31, Theorem 1.1] is known at the moment.

$\{[\beta]\}_{\beta\in\Delta^+}$ each taken with a multiplicity $d_\beta \in \mathbb{N}$ and ordered with respect to the total order (2.10) on Δ^+ (the order inside each group is irrelevant). Let $\underline{d} \in \mathbb{N}^{\frac{n(n-1)}{2}}$ denote the collection $\{d_\beta\}_{\beta\in\Delta^+}$. Define $\underline{\ell} = (\ell_1, \ldots, \ell_{n-1}) \in \mathbb{N}^I$, the $\mathbb{N}^I$-degree of $\underline{d}$, via

$$\sum_{i\in I} \ell_i\alpha_i = \sum_{\beta\in\Delta^+} d_\beta\beta. \tag{3.6}$$

Let us now define the **specialization map**

$$\phi_{\underline{d}} \colon S^{(n)}_{\underline{\ell}} \longrightarrow \mathbb{C}(\boldsymbol{v})[\{y^{\pm 1}_{\beta,s}\}^{1\le s\le d_\beta}_{\beta\in\Delta^+}]^{\Sigma_{\underline{d}}}. \tag{3.7}$$

We split the variables $\{x_{i,r}\}^{1\le r\le \ell_i}_{i\in I}$ into $\sum_{\beta\in\Delta^+} d_\beta$ groups corresponding to the above intervals, and specialize those in the sth copy of $[\beta]$ to $\boldsymbol{v}^{-j(\beta)}\cdot y_{\beta,s}, \ldots, \boldsymbol{v}^{-i(\beta)}\cdot y_{\beta,s}$ in the natural order (the variable $x_{i,r}$ is specialized to $\boldsymbol{v}^{-i}y_{\beta,s}$). Then, for $F = \frac{f(x_{1,1},\ldots,x_{n-1,\ell_{n-1}})}{\prod_{i=1}^{n-2}\prod^{1\le r'\le \ell_{i+1}}_{1\le r\le \ell_i}(x_{i,r}-x_{i+1,r'})} \in S^{(n)}_{\underline{\ell}}$, we define $\phi_{\underline{d}}(F)$ as the corresponding specialization of f. We note that $\phi_{\underline{d}}(F)$ is independent of our splitting of the variables $\{x_{i,r}\}^{1\le r\le \ell_i}_{i\in I}$ into groups and is symmetric in $\{y_{\beta,s}\}^{d_\beta}_{s=1}$ for any β.

The key properties of the specialization maps $\phi_{\underline{d}}$ and their relevance to $\Psi(e_h)$, the images of the ordered PBWD monomials in the shuffle algebra, are discussed in Lemmas 3.16, 3.17, 3.21 below. We conclude this section with an explicit example of the specialization maps:

Example 3.8 Consider $F = \frac{x^a_{1,1}x^b_{2,1}x^c_{3,1}}{(x_{1,1}-x_{2,1})(x_{2,1}-x_{3,1})} \in S^{(n)}_{\underline{\ell}}$, where $\underline{\ell} = (1, 1, 1, 0, \ldots, 0) \in \mathbb{N}^I$ and $a, b, c \in \mathbb{Z}$. Let us compute the images of F under all possible specialization maps:

(a) If $\underline{d}$ encodes a single positive root $\beta = \alpha_1 + \alpha_2 + \alpha_3$, then $\phi_{\underline{d}}(F)$ is a Laurent polynomial in a single variable $y_{\beta,1}$ and equals

$$(\boldsymbol{v}^{-1}y_{\beta,1})^a(\boldsymbol{v}^{-2}y_{\beta,1})^b(\boldsymbol{v}^{-3}y_{\beta,1})^c.$$

(b) If $\underline{d}$ encodes two positive roots $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2 + \alpha_3$, then $\phi_{\underline{d}}(F)$ is a Laurent polynomial in two variables $y_{\beta_1,1}, y_{\beta_2,1}$ and equals

$$(\boldsymbol{v}^{-1}y_{\beta_1,1})^a(\boldsymbol{v}^{-2}y_{\beta_2,1})^b(\boldsymbol{v}^{-3}y_{\beta_2,1})^c.$$

(c) If $\underline{d}$ encodes two positive roots $\beta_1 = \alpha_1 + \alpha_2$, $\beta_2 = \alpha_3$, then $\phi_{\underline{d}}(F)$ is a Laurent polynomial in two variables $y_{\beta_1,1}, y_{\beta_2,1}$ and equals

$$(\boldsymbol{v}^{-1}y_{\beta_1,1})^a(\boldsymbol{v}^{-2}y_{\beta_1,1})^b(\boldsymbol{v}^{-3}y_{\beta_2,1})^c.$$

(d) If $\underline{d}$ encodes three positive roots $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$, $\beta_3 = \alpha_3$, then $\phi_{\underline{d}}(F)$ is a Laurent polynomials in three variables $y_{\beta_1,1}, y_{\beta_2,1}, y_{\beta_3,1}$ and equals

$$(\boldsymbol{v}^{-1}y_{\beta_1,1})^a(\boldsymbol{v}^{-2}y_{\beta_2,1})^b(\boldsymbol{v}^{-3}y_{\beta_3,1})^c.$$

3.2 Proof of Theorem 2.16(a)

Our proof of Theorem 2.16(a) will proceed in two steps: first, we shall establish the linear independence[6] of the ordered PBWD monomials in Sect. 3.2.2, and then we will verify that they linearly span the entire algebra $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ in Sect. 3.2.3 (we note that the order of these two steps is usually opposite in the proofs of PBW-type theorems).

We start by establishing Theorem 2.16(a) for $n = 2$ in Sect. 3.2.1.

3.2.1 Proof of Theorem 2.16(a) for $n = 2$

For $k \in \mathbb{N}$, set $[k]_{\boldsymbol{v}} := \frac{\boldsymbol{v}^k - \boldsymbol{v}^{-k}}{\boldsymbol{v} - \boldsymbol{v}^{-1}}$ and $[k]_{\boldsymbol{v}}! := [1]_{\boldsymbol{v}} \cdots [k]_{\boldsymbol{v}}$. We start from the following simple computation in $S^{(2)}$ (as $I = \{1\}$, we shall denote the variables $x_{1,r}$ simply by x_r):

Lemma 3.9 *For any $k \geq 1$ and $r \in \mathbb{Z}$, the kth power of $x^r \in S^{(2)}_1$ equals*

$$\underbrace{x^r \star \cdots \star x^r}_{k \text{ times}} = \boldsymbol{v}^{-\frac{k(k-1)}{2}} [k]_{\boldsymbol{v}}! \cdot (x_1 \cdots x_k)^r. \tag{3.10}$$

Proof The proof is by induction on k, the base case $k = 1$ being trivial. Applying the induction assumption to the $(k-1)$-st power of x^r, the proof of (3.10) boils down to the verification of

$$\sum_{p=1}^{k} \prod_{1 \leq s \leq k}^{s \neq p} \frac{x_s - \boldsymbol{v}^{-2} x_p}{x_s - x_p} = \boldsymbol{v}^{1-k} [k]_{\boldsymbol{v}}. \tag{3.11}$$

The left-hand side of (3.11) is a rational function in $\{x_p\}_{p=1}^k$ of degree 0 and is easily checked to have no poles, hence, must be a constant. To evaluate this constant, let $x_k \to \infty$: the last summand (corresponding to $p = k$) tends to $\boldsymbol{v}^{-2(k-1)}$, while the sum of the first $k-1$ summands (corresponding to $1 \leq p \leq k-1$) tends to $1 + \boldsymbol{v}^{-2} + \cdots + \boldsymbol{v}^{-2(k-2)}$ by the induction assumption, thus, resulting in $1 + \boldsymbol{v}^{-2} + \boldsymbol{v}^{-4} + \cdots + \boldsymbol{v}^{-2(k-1)} = \boldsymbol{v}^{1-k}[k]_{\boldsymbol{v}}$ as claimed. □

Theorem 2.16(a) for $n = 2$ is equivalent to the following result:

Lemma 3.12 *For any total order $\preceq$ on $\mathbb{Z}$, the ordered monomials $\{e_{r_1} e_{r_2} \cdots e_{r_k}\}_{k \in \mathbb{N}}^{r_1 \preceq \cdots \preceq r_k}$ form a $\mathbb{C}(\boldsymbol{v})$-basis of $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_2)$.*

Proof For $r_1 = \cdots = r_{k_1} \prec r_{k_1+1} = \cdots = r_{k_1+k_2} \prec \cdots \prec r_{k_1+\cdots+k_{\ell-1}+1} = \cdots = r_{k_1+\cdots+k_\ell}$, set $k := k_1 + \cdots + k_\ell$ and choose $\sigma \in \Sigma_k$ so that $r_{\sigma(1)} \leq \cdots \leq r_{\sigma(k)}$. Then $x^{r_1} \star \cdots \star x^{r_k}$ is a symmetric Laurent polynomial of the form

$$\nu_{\underline{r}} m_{(r_{\sigma(1)}, \ldots, r_{\sigma(k)})}(x_1, \ldots, x_k) + \sum \nu_{\underline{r}'} m_{\underline{r}'}(x_1, \ldots, x_k)$$

[6] As pointed out in the introduction, the linear independence can be deduced from the general arguments based on the flatness of the deformation and the PBW property of $U(\mathfrak{sl}_n[t, t^{-1}])$. However, the *specialization maps* of (3.7) and formulas (3.18, 3.19) will be used below to prove that $\{e_h\}_{h \in H}$ span $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. We will use the same approach for two-parameter quantum loop algebra, for which the general arguments do not apply.

where

- $m_{(s_1,\ldots,s_k)}(x_1,\ldots,x_k)$ (with $s_1 \leq \cdots \leq s_k$) are the monomial symmetric polynomials (that is, the sums of all monomials $x_1^{t_1}\cdots x_k^{t_k}$ as $(t_1,\ldots,t_k)$ ranges over all distinct permutations of $(s_1,\ldots,s_k)$),
- the sum is over $\underline{r}' = (r'_1 \leq \cdots \leq r'_k) \in \mathbb{Z}^k$ distinct from $(r_{\sigma(1)},\ldots,r_{\sigma(k)})$ and satisfying

$$r_{\sigma(1)} \leq r'_1 \leq r'_k \leq r_{\sigma(k)} \ \text{ as well as } \ r'_1+\cdots+r'_k = r_1+\cdots+r_k,$$

- the coefficients $\nu_{\underline{r}'}$ are Laurent polynomials in $\boldsymbol{v}$, that is, $\nu_{\underline{r}'} \in \mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$,
- the coefficient $\nu_{\underline{r}}$ is explicitly given by

$$\nu_{\underline{r}} = \prod_{1\leq p\leq \ell} \left(\boldsymbol{v}^{-\frac{k_p(k_p-1)}{2}}[k_p]_{\boldsymbol{v}}!\right), \tag{3.13}$$

due to Lemma 3.9.

Therefore, the shuffle products $\{x^{r_1}\star x^{r_2}\star\cdots\star x^{r_k}\}_{k\in\mathbb{N}}^{r_1\preceq\cdots\preceq r_k}$ form a $\mathbb{C}(\boldsymbol{v})$-basis of $S^{(2)}$, since $\{m_{(s_1,\ldots,s_k)}(x_1,\ldots,x_k)\}_{s_1\leq\cdots\leq s_k}$ form a $\mathbb{C}(\boldsymbol{v})$-basis of $\mathbb{C}[\{x_p^{\pm 1}\}_{p=1}^k]^{\Sigma_k}$ and $S_k^{(2)} \simeq \mathbb{C}[\{x_p^{\pm 1}\}_{p=1}^k]^{\Sigma_k}$ as vector spaces. This completes our proof of Lemma 3.12, due to the injectivity of Ψ. □

3.2.2 Linear independence of e_h and two properties of the specialization maps

For an ordered PBWD monomial e_h ($h \in H$), define its *degree* $\deg(e_h) = \deg(h) \in \mathbb{N}^{\frac{n(n-1)}{2}}$ as a collection of $d_\beta := \sum_{r\in\mathbb{Z}} h(\beta, r) \in \mathbb{N}$ ($\beta \in \Delta^+$) ordered with respect to the total order (2.10) on Δ^+. We consider the *anti-lexicographical order* on $\mathbb{N}^{\frac{n(n-1)}{2}}$:

$$\{d_\beta\}_{\beta\in\Delta^+} < \{d'_\beta\}_{\beta\in\Delta^+} \ \text{ iff } \ \exists\,\gamma \in \Delta^+ \text{ s.t. } d_\gamma > d'_\gamma \text{ and } d_\beta = d'_\beta \ \forall \beta < \gamma.$$

In what follows, we shall need an explicit formula for $\Psi(e_\beta(r))$:

Lemma 3.14 *For $1 \leq j < i < n$ and $r \in \mathbb{Z}$, we have*

$$\Psi(e_{\alpha_j+\alpha_{j+1}+\cdots+\alpha_i}(r)) = (1-\boldsymbol{v}^2)^{i-j}\frac{p(x_{j,1},\ldots,x_{i,1})}{(x_{j,1}-x_{j+1,1})\cdots(x_{i-1,1}-x_{i,1})},$$

where $p(x_{j,1},\ldots,x_{i,1})$ is a degree $r+i-j$ monomial, up to a sign and an integer power of $\boldsymbol{v}$.

Proof Straightforward computation. □

Example 3.15 $p(x_{j,1},\ldots,x_{i,1}) = x_{j,1}^{r+1}x_{j+1,1}\cdots x_{i-1,1}$ for the particular choice (2.14).

Our proof of the linear independence of $\{e_h\}_{h\in H}$ is crucially based on the following two key properties of the specialization maps introduced in (3.7):

Lemma 3.16 *If* $\deg(h) < \underline{d}$ *and* $\mathbb{N}^I$*-degrees of* $\underline{d}$ *and* $\deg(h)$ *coincide, then* $\phi_{\underline{d}}(\Psi(e_h)) = 0$.

Proof The condition $\deg(h) < \underline{d}$ guarantees that $\phi_{\underline{d}}$-specialization of any summand of the symmetrization appearing in $\Psi(e_h)$ contains among all the ζ-factors at least one factor of the form $\zeta_{i,i+1}(\boldsymbol{v}) = 0$, hence, it is zero. The result follows. □

Lemma 3.17 *The specializations* $\{\phi_{\underline{d}}(\Psi(e_h))\}_{h\in H}^{\deg(h)=\underline{d}}$ *are linearly independent over* $\mathbb{C}(\boldsymbol{v})$.

Proof Consider the image $\Psi(e_h) \in S_{\underline{k}}^{(n)}$ (here, $\underline{k}$ is the $\mathbb{N}^I$-degree of $\underline{d} = \deg(h)$). It is a sum of $\underline{k}!$ terms, but most of them specialize to zero under $\phi_{\underline{d}}$, as in the proof of Lemma 3.16. The summands which do not specialize to zero are parametrized by $\Sigma_{\underline{d}} := \prod_{\beta\in\Delta^+} \Sigma_{d_\beta}$. More precisely, given $(\sigma_\beta)_{\beta\in\Delta^+} \in \Sigma_{\underline{d}}$, the associated summand corresponds to the case when for all $\beta \in \Delta^+$ and $1 \le s \le d_\beta$, the $(\sum_{\beta'<\beta} d_{\beta'} + s)$th factor of the corresponding term of $\Psi(e_h)$ is evaluated at $\boldsymbol{v}^{-j(\beta)} y_{\beta,\sigma_\beta(s)}, \ldots, \boldsymbol{v}^{-i(\beta)} y_{\beta,\sigma_\beta(s)}$. The image of this summand under $\phi_{\underline{d}}$ equals

$$\prod_{\beta<\beta'} G_{\beta,\beta'} \cdot \prod_{\beta} G_\beta \cdot \prod_{\beta} G_\beta^{(\sigma_\beta)}$$

(up to a common sign and an integer power of $\boldsymbol{v}$), where

$$\begin{aligned}
G_{\beta,\beta'} &= \prod_{1\le s\le d_\beta}^{1\le s'\le d_{\beta'}} \left(\prod_{j\in[\beta],\, j'\in[\beta']}^{j=j'} (y_{\beta,s} - \boldsymbol{v}^{-2} y_{\beta',s'}) \cdot \prod_{j\in[\beta],\, j'\in[\beta']}^{j=j'+1} (y_{\beta,s} - \boldsymbol{v}^{2} y_{\beta',s'}) \right) \\
&\quad \times \prod_{1\le s\le d_\beta}^{1\le s'\le d_{\beta'}} (y_{\beta,s} - y_{\beta',s'})^{\delta_{j(\beta')>j(\beta)}\delta_{i(\beta)+1\in[\beta']}}, \\
G_\beta &= (1-\boldsymbol{v}^2)^{d_\beta(i(\beta)-j(\beta))} \cdot \prod_{1\le s\ne s'\le d_\beta} (y_{\beta,s} - \boldsymbol{v}^2 y_{\beta,s'})^{i(\beta)-j(\beta)} \\
&\quad \times \prod_{1\le s\le d_\beta} y_{\beta,s}^{i(\beta)-j(\beta)}, \\
G_\beta^{(\sigma_\beta)} &= \prod_{s=1}^{d_\beta} y_{\beta,\sigma_\beta(s)}^{r_\beta(h,s)} \cdot \prod_{s<s'} \frac{y_{\beta,\sigma_\beta(s)} - \boldsymbol{v}^{-2} y_{\beta,\sigma_\beta(s')}}{y_{\beta,\sigma_\beta(s)} - y_{\beta,\sigma_\beta(s')}}.
\end{aligned} \tag{3.18}$$

Here, the collection $\{r_\beta(h,1), \ldots, r_\beta(h,d_\beta)\}$ is obtained by listing every $r \in \mathbb{Z}$ with multiplicity $h(\beta, r) > 0$ with respect to the total order $\preceq_\beta$ on $\mathbb{Z}$, see Sect. 2.2. We also use the standard *delta function* notation: $\delta_{\text{condition}} = \begin{cases} 1, & \text{if condition holds} \\ 0, & \text{if condition fails} \end{cases}$.

Note that the factors $\{G_{\beta,\beta'}\}_{\beta<\beta'} \cup \{G_\beta\}_\beta$ in (3.18) are independent of $(\sigma_\beta)_{\beta\in\Delta^+} \in \Sigma_{\underline{d}}$. Therefore, the specialization $\phi_{\underline{d}}(\Psi(e_h))$ has the following form:

$$\phi_{\underline{d}}(\Psi(e_h)) = c \cdot \prod_{\beta,\beta'\in\Delta^+}^{\beta<\beta'} G_{\beta,\beta'} \cdot \prod_{\beta\in\Delta^+} G_\beta \cdot \prod_{\beta\in\Delta^+} \left(\sum_{\sigma_\beta\in\Sigma_{d_\beta}} G_\beta^{(\sigma_\beta)} \right) \text{ with } c \in \mathbb{C}^\times \cdot \boldsymbol{v}^{\mathbb{Z}}. \tag{3.19}$$

For any $\beta \in \Delta^+$, we note that the sum $\sum_{\sigma_\beta\in\Sigma_{d_\beta}} G_\beta^{(\sigma_\beta)}$ coincides (up to a factor of $\mathbb{C}^\times$) with the value of the shuffle element $x^{r_\beta(h,1)} \star \cdots \star x^{r_\beta(h,d_\beta)} \in S^{(2)}_{d_\beta}$ (in the shuffle algebra $S^{(2)}$!) evaluated at $\{y_{\beta,s}\}_{s=1}^{d_\beta}$. The latter elements are linearly independent, due to Lemma 3.12.

Thus, (3.19) together with the above observation complete our proof of Lemma 3.17. □

Now we can complete our proof of the linear independence of the ordered PBWD monomials $\{e_h\}_{h\in H}$. Assume the contrary, that a nontrivial linear combination $\sum_{h\in H} c_h e_h$ vanishes in $U_{\boldsymbol{v}}^>(L\mathfrak{sl}_n)$ (here, all but finitely many of c_h are zero, but at least one of them is non-zero). Define $\underline{d} := \max\{\deg(h)|c_h \neq 0\}$. Applying the specialization map $\phi_{\underline{d}}$ to $\sum_{h\in H} c_h \Psi(e_h) = 0$, we get $\sum_{h\in H}^{\deg(h)=\underline{d}} c_h \phi_{\underline{d}}(\Psi(e_h)) = 0$ by Lemma 3.16. Furthermore, we get $c_h = 0$ for all $h \in H$ of degree $\deg(h) = \underline{d}$, due to Lemma 3.17. This contradicts our choice of $\underline{d}$.

Remark 3.20 The machinery of the *specialization maps* $\phi_{\underline{d}}$ that was used in the above proof is of its own interest (cf. [11, (1.4)] and [31, (4.24)]).

3.2.3 Spanning property of e_h and dominance property of the specialization maps

Let $M \subset S^{(n)}$ be the $\mathbb{C}(\boldsymbol{v})$-span of $\{\Psi(e_h)\}_{h\in H}$. For $\underline{k} \in \mathbb{N}^I$, let $T_{\underline{k}}$ be a finite set consisting of all degree vectors $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+} \in \mathbb{N}^{\frac{n(n-1)}{2}}$ such that $\sum_{\beta\in\Delta^+} d_\beta \beta = \sum_{i\in I} k_i \alpha_i$. We order $T_{\underline{k}}$ with respect to the anti-lexicographical order on $\mathbb{N}^{\frac{n(n-1)}{2}}$. In particular, the minimal element $\underline{d}_{\min} = \{d_\beta\}_{\beta\in\Delta^+} \in T_{\underline{k}}$ is characterized by $d_\beta = 0$ for all non-simple roots $\beta \in \Delta^+$.

Lemma 3.21 *Let $F \in S^{(n)}_{\underline{k}}$ and $\underline{d} \in T_{\underline{k}}$. If $\phi_{\underline{d}'}(F) = 0$ for all $\underline{d}' \in T_{\underline{k}}$ such that $\underline{d}' > \underline{d}$, then there exists an element $F_{\underline{d}} \in M$ such that $\phi_{\underline{d}}(F) = \phi_{\underline{d}}(F_{\underline{d}})$ and $\phi_{\underline{d}'}(F_{\underline{d}}) = 0$ for all $\underline{d}' > \underline{d}$.*

Proof Consider the following total order on the set $\{(\beta, s)|\beta \in \Delta^+, 1 \leq s \leq d_\beta\}$:

$$(\beta, s) \leq (\beta', s') \text{ iff } \beta < \beta' \text{ or } \beta = \beta', s \leq s'. \tag{3.22}$$

First, we note that the wheel conditions (3.3) for F guarantee that $\phi_{\underline{d}}(F)$ (which is a Laurent polynomial in $\{y_{\beta,s}\}$) vanishes up to appropriate orders under the following specializations:

(i) $y_{\beta,s} = \boldsymbol{v}^{-2} y_{\beta',s'}$ for $(\beta, s) < (\beta', s')$,
(ii) $y_{\beta,s} = \boldsymbol{v}^{2} y_{\beta',s'}$ for $(\beta, s) < (\beta', s')$.

The orders of vanishing are computed similarly to [11,31]. Explicitly, let us view the specialization appearing in the definition of $\phi_{\underline{d}}$ as a step-by-step specialization in each interval $[\beta]$, ordered first in the non-increasing length order, while the intervals of the same length are ordered in the non-decreasing order of $j(\beta)$. As we specialize the variables in the sth interval ($1 \leq s \leq \sum_{\beta\in\Delta^+} d_\beta$), we count only those wheel conditions that arise from the non-specialized yet variables. A straightforward case-by-case verification[7] shows that the corresponding orders of vanishing under the specializations (i) and (ii) equal $\#\{(j, j') \in [\beta] \times [\beta'] | j = j'\} - \delta_{\beta,\beta'}$ and $\#\{(j, j') \in [\beta] \times [\beta'] | j = j' + 1\}$, respectively.

Second, we claim that $\phi_{\underline{d}}(F)$ vanishes under the following specializations:

(iii) $y_{\beta,s} = y_{\beta',s'}$ for $(\beta, s) < (\beta', s')$ such that $j(\beta) < j(\beta')$ and $i(\beta) + 1 \in [\beta']$.

Indeed, if $j(\beta) < j(\beta')$ and $i(\beta) + 1 \in [\beta']$, there are positive roots $\gamma, \gamma' \in \Delta^+$ such that $j(\gamma) = j(\beta), i(\gamma) = i(\beta'), j(\gamma') = j(\beta'), i(\gamma') = i(\beta)$. Consider the degree vector $\underline{d}' \in T_{\underline{k}}$ given by $d'_\alpha = d_\alpha + \delta_{\alpha,\gamma} + \delta_{\alpha,\gamma'} - \delta_{\alpha,\beta} - \delta_{\alpha,\beta'}$. Then, $\underline{d}' > \underline{d}$ and thus $\phi_{\underline{d}'}(F) = 0$. The result follows.

Combining the above vanishing conditions for $\phi_{\underline{d}}(F)$, we see that it is divisible exactly by the product $\prod_{\beta<\beta'} G_{\beta,\beta'} \cdot \prod_\beta G_\beta$ of (3.18). Therefore, we have

$$\phi_{\underline{d}}(F) = \prod_{\beta,\beta'\in\Delta^+}^{\beta<\beta'} G_{\beta,\beta'} \cdot \prod_{\beta\in\Delta^+} G_\beta \cdot G \tag{3.23}$$

for some symmetric Laurent polynomial

$$G \in \mathbb{C}(\boldsymbol{v})[\{y^{\pm 1}_{\beta,s}\}^{1\leq s\leq d_\beta}_{\beta\in\Delta^+}]^{\Sigma_{\underline{d}}} \simeq \bigotimes_{\beta\in\Delta^+} \mathbb{C}(\boldsymbol{v})[\{y^{\pm 1}_{\beta,s}\}^{d_\beta}_{s=1}]^{\Sigma_{d_\beta}}. \tag{3.24}$$

Combining (3.23, 3.24) with formula (3.19) and the discussion after it, we see that there is a (unique) linear combination $F_{\underline{d}} = \sum_{h\in H}^{\deg(h)=\underline{d}} c_h \Psi(e_h)$ with $c_h \in \mathbb{C}(\boldsymbol{v})$ such that $\phi_{\underline{d}}(F) = \phi_{\underline{d}}(F_{\underline{d}})$, due to Lemma 3.12. The equality $\phi_{\underline{d}'}(F_{\underline{d}}) = 0$ for $\underline{d}' > \underline{d}$ is due to Lemma 3.16.

This completes our proof of Lemma 3.21. □

Using Lemma above, we can now show that any shuffle element $F \in S^{(n)}_{\underline{k}}$ belongs to $M \cap S^{(n)}_{\underline{k}}$. Let $\underline{d}_{\max}$ and $\underline{d}_{\min}$ denote the maximal and the minimal elements of $T_{\underline{k}}$, respectively. The condition of Lemma 3.21 is vacuous for $\underline{d} = \underline{d}_{\max}$. Therefore, Lemma 3.21 applies. Applying it iteratively, we eventually find an element $\widetilde{F} \in M$

[7] This can be checked by treating each of the following cases separately: $j = j' = i = i'$, $j = j' = i < i'$, $j = j' < i = i'$, $j = j' < i < i'$, $j < j' \leq i' < i$, $j < j' < i' = i$, $j < j' < i < i'$, $j = i < j' = i'$, $j = i < j' < i'$, $j < j' = i = i'$, $j < j' = i < i'$, $j < i < j' \leq i'$, where we set $j := j(\beta)$, $j' := j(\beta')$, $i := i(\beta)$, $i' := i(\beta')$.

such that $\phi_{\underline{d}}(F) = \phi_{\underline{d}}(\widetilde{F})$ for all $\underline{d} \in T_{\underline{k}}$. In the particular case of $\underline{d} = \underline{d}_{\min}$, this yields $F = \widetilde{F}$, cf. Example 3.8(d). Hence, $F \in M$.

Invoking the injectivity of Ψ (Proposition 3.4), we thus see that $\{e_h\}_{h\in H}$ span $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. Combining this with the linear independence of $\{e_h\}_{h\in H}$ established in Sect. 3.2.2 completes our proof of Theorem 2.16(a). We note that the result of Theorem 2.16(b) follows as well, as explained in the beginning of Sect. 3.

Remark 3.25 The above argument also implies the surjectivity of Ψ. Combining this with the injectivity of Ψ, established in Proposition 3.4, we obtain a new proof of Theorem 3.5.

Remark 3.26 We note that the above argument actually provides a much bigger class of the PBWD bases for $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, with the PBWD basis elements given rather in the shuffle form.

Remark 3.27 In [31], the shuffle realization of the quantum toroidal algebra $U_{\boldsymbol{v},\bar{\boldsymbol{v}}}(\ddot{\mathfrak{gl}}_n)$ (which is an associative $\mathbb{C}(\boldsymbol{v},\bar{\boldsymbol{v}})$-algebra with $\boldsymbol{v},\bar{\boldsymbol{v}}$ being two independent formal variables) was established by studying the crucial *slope* $\leq \mu$ *subalgebras*. In particular, combining the proofs of Proposition 3.9 and Lemma 3.14 of *loc.cit.*, one obtains the PBWD basis of $U_{\boldsymbol{v},\bar{\boldsymbol{v}}}(\ddot{\mathfrak{gl}}_n)$ with the PBWD basis elements given explicitly in the shuffle realization, see elements $E^{\mu}_{[j;i)}$ of [31, (3.46)]. This gives rise to the PBWD basis of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ by viewing the latter as a "vertical" subalgebra of $U_{\boldsymbol{v},\bar{\boldsymbol{v}}}(\ddot{\mathfrak{gl}}_n)$. The corresponding PBWD basis elements are given by $\Psi^{-1}((1-\boldsymbol{v}^2)^{i-j}\frac{p(x_{j,1},\ldots,x_{i,1})}{(x_{j,1}-x_{j+1,1})\cdots(x_{i-1,1}-x_{i,1})})$, where $p(x_{j,1},\ldots,x_{i,1}) = \prod_{a=j}^{i} x_{a,1}^{\lfloor\mu(a-j+1)\rfloor-\lfloor\mu(a-j)\rfloor}$ with $1\leq j\leq i<n$ and $\mu\in\frac{1}{i-j+1}\mathbb{Z}$. As we fix $1\leq j\leq i<n$ and let μ run over $\frac{1}{i-j+1}\mathbb{Z}$, the degree of p varies over $\mathbb{Z}$ multiplicity-free. Comparing this to Lemma 3.14, we see that the corresponding PBWD basis of $U^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is reminiscent to a particular one of Theorem 2.16(a), but it should be stressed right away that the former utilizes a total order on $\Delta^+\times\mathbb{Z}$ that is different from (2.11) used in (2.15).

3.3 Integral form $\mathfrak{S}^{(n)}$

For $\underline{k}\in\mathbb{N}^I$, set $|\underline{k}| := \sum_{i=1}^{n-1} k_i$. Consider a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-submodule $\widetilde{\mathfrak{S}}^{(n)}_{\underline{k}} \subset S^{(n)}_{\underline{k}}$ consisting of those $F\in S^{(n)}_{\underline{k}}$ for which the numerator f in (3.2) is of the form:

$$f \in (\boldsymbol{v}-\boldsymbol{v}^{-1})^{|\underline{k}|}\cdot\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][\{x_{i,r}^{\pm1}\}_{i\in I}^{1\leq r\leq k_i}]^{\Sigma_{\underline{k}}}. \tag{3.28}$$

Then $\widetilde{\mathfrak{S}}^{(n)} := \bigoplus_{\underline{k}\in\mathbb{N}^I} \widetilde{\mathfrak{S}}^{(n)}_{\underline{k}}$ is clearly a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $S^{(n)}$.

Remark 3.29 If one wishes to work with forms defined over $\mathbb{Z}[\boldsymbol{v},\boldsymbol{v}^{-1}]$, formula (3.28) should be replaced accordingly. To this end, we note that the shuffle product $F\star G$ of (3.1) still makes sense as $\frac{1}{\underline{k}!\cdot\underline{\ell}!}\cdot\mathrm{Sym}(\ldots)$ may be written simply as a sum over so-called $(\underline{k},\underline{\ell})$-shuffles, since F and G are symmetric. We leave details to the interested reader.

Moreover, $\Psi(\mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)) \subset \widetilde{\mathfrak{S}}^{(n)}$, due to Lemma 3.14. The key goal of this section is to explicitly describe the image $\Psi(\mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n))$.

Example 3.30 Consider $F(x_1, x_2) = (\boldsymbol{v} - \boldsymbol{v}^{-1})^2(x_1x_2)^r \in \widetilde{\mathfrak{S}}_2^{(2)}$ with $r \in \mathbb{Z}$. Then, we have $[2]_{\boldsymbol{v}} \cdot F \in \Psi(\mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_2))$, but $F \notin \Psi(\mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_2))$, as follows from Lemmas 3.46, 3.48 below.

Pick $F \in \widetilde{\mathfrak{S}}_{\underline{k}}^{(n)}$. For any degree vector $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+} \in \mathbb{N}^{\frac{n(n-1)}{2}}$ such that $\sum_{\beta\in\Delta^+} d_\beta\beta = \sum_{i\in I} k_i\alpha_i$, consider $\phi_{\underline{d}}(F) \in \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}][\{y_{\beta,s}^{\pm 1}\}_{\beta\in\Delta^+}^{1\leq s\leq d_\beta}]^{\Sigma_{\underline{d}}}$ of (3.7). First, we note that $\phi_{\underline{d}}(F)$ is divisible by

$$A_{\underline{d}} := (\boldsymbol{v} - \boldsymbol{v}^{-1})^{|\underline{k}|}. \tag{3.31}$$

Second, following the first part of the proof of Lemma 3.21, $\phi_{\underline{d}}(F)$ is also divisible by

$$\begin{aligned} B_{\underline{d}} := & \prod_{(\beta,s)<(\beta',s')} (y_{\beta,s} - \boldsymbol{v}^{-2} y_{\beta',s'})^{\#\{(j,j')\in[\beta]\times[\beta']\,|\, j=j'\}-\delta_{\beta,\beta'}} \\ & \times \prod_{(\beta,s)<(\beta',s')} (y_{\beta,s} - \boldsymbol{v}^{2} y_{\beta',s'})^{\#\{(j,j')\in[\beta]\times[\beta']\,|\, j=j'+1\}} \end{aligned} \tag{3.32}$$

due to the wheel conditions (3.3), where we use the total order (3.22) on the set $\{(\beta, s)\}_{\beta\in\Delta^+}^{1\leq s\leq d_\beta}$. Combining these observations, we define the *reduced specialization map*

$$\varphi_{\underline{d}} \colon \widetilde{\mathfrak{S}}_{\underline{k}}^{(n)} \longrightarrow \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}][\{y_{\beta,s}^{\pm 1}\}_{\beta\in\Delta^+}^{1\leq s\leq d_\beta}]^{\Sigma_{\underline{d}}} \quad \text{via} \quad \varphi_{\underline{d}}(F) := \frac{\phi_{\underline{d}}(F)}{A_{\underline{d}} B_{\underline{d}}}. \tag{3.33}$$

Let us introduce another type of specialization maps. Given a collection of positive integers $\underline{t} = \{t_{\beta,r}\}_{\beta\in\Delta^+}^{1\leq r\leq \ell_\beta}$ ($\ell_\beta \in \mathbb{N}$), define a degree vector $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+} \in \mathbb{N}^{\frac{n(n-1)}{2}}$ via $d_\beta := \sum_{r=1}^{\ell_\beta} t_{\beta,r}$. Let us now define the *vertical specialization map*

$$\varpi_{\underline{t}} \colon \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}][\{y_{\beta,s}^{\pm 1}\}_{\beta\in\Delta^+}^{1\leq s\leq d_\beta}]^{\Sigma_{\underline{d}}} \longrightarrow \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}][\{z_{\beta,r}^{\pm 1}\}_{\beta\in\Delta^+}^{1\leq r\leq \ell_\beta}]. \tag{3.34}$$

For each $\beta \in \Delta^+$, we split the variables $\{y_{\beta,s}\}_{s=1}^{d_\beta}$ into ℓ_β groups of size $t_{\beta,r}$ each ($1 \leq r \leq \ell_\beta$) and specialize the variables in the rth group to $\boldsymbol{v}^{-2}z_{\beta,r}, \boldsymbol{v}^{-4}z_{\beta,r}, \boldsymbol{v}^{-6}z_{\beta,r}, \ldots, \boldsymbol{v}^{-2t_{\beta,r}}z_{\beta,r}$. For any $K \in \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}][\{y_{\beta,s}^{\pm 1}\}_{\beta\in\Delta^+}^{1\leq s\leq d_\beta}]^{\Sigma_{\underline{d}}}$, we define $\varpi_{\underline{t}}(K)$ as the corresponding specialization of K. We note that $\varpi_{\underline{t}}(K)$ is independent of our splitting of the variables $\{y_{\beta,s}\}_{s=1}^{d_\beta}$ into groups.

Finally, we shall combine (3.33) and (3.34) to obtain the key tool in the study of the integral form $\Psi(\mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n))$ of $S^{(n)}$. Given $\underline{k} \in \mathbb{N}^I$, $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+} \in \mathbb{N}^{\frac{n(n-1)}{2}}$ and a collection of positive integers $\underline{t} = \{t_{\beta,r}\}_{\beta\in\Delta^+}^{1\leq r\leq \ell_\beta}$ such that

$$\sum_{\beta\in\Delta^+} d_\beta \beta = \sum_{i\in I} k_i\alpha_i \qquad \text{and} \qquad \sum_{r=1}^{\ell_\beta} t_{\beta,r} = d_\beta, \tag{3.35}$$

we define the **cross specialization map**

$$\Upsilon_{\underline{d},\underline{t}} \colon \widetilde{\mathfrak{S}}_{\underline{k}}^{(n)} \longrightarrow \mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][\{z_{\beta,r}^{\pm 1}\}_{\beta\in\Delta^+}^{1\leq r\leq \ell_\beta}] \quad \text{via} \quad \Upsilon_{\underline{d},\underline{t}}(F) := \varpi_{\underline{t}}(\varphi_{\underline{d}}(F)). \tag{3.36}$$

Definition 3.37 $F \in S_{\underline{k}}^{(n)}$ is **integral** if $F \in \widetilde{\mathfrak{S}}_{\underline{k}}^{(n)}$ and $\Upsilon_{\underline{d},\underline{t}}(F)$ is divisible by $\prod_{\beta\in\Delta^+}^{1\leq r\leq \ell_\beta} [t_{\beta,r}]_{\boldsymbol{v}}!$ (the product of $\boldsymbol{v}$-factorials) for all possible $\underline{d}$ and $\underline{t}$ satisfying (3.35).

Example 3.38 In the simplest case $n = 2$, a symmetric Laurent polynomial $F \in S_k^{(2)}$ is integral if and only if it has the form $F = (\boldsymbol{v}-\boldsymbol{v}^{-1})^k \cdot \bar{F}$ with $\bar{F} \in \mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][\{x_p^{\pm 1}\}_{p=1}^k]^{\Sigma_k}$ satisfying the following divisibility condition:

$$\bar{F}(\boldsymbol{v}^{-2}z_1, \boldsymbol{v}^{-4}z_1, \ldots, \boldsymbol{v}^{-2k_1}z_1, \ldots, \boldsymbol{v}^{-2}z_\ell, \ldots, \boldsymbol{v}^{-2k_\ell}z_\ell) \ \text{is divisible by} \ [k_1]_{\boldsymbol{v}}! \cdots [k_\ell]_{\boldsymbol{v}}! \tag{3.39}$$

for any decomposition $k = k_1 + \cdots + k_\ell$ into a sum of positive integers.

Let $\mathfrak{S}_{\underline{k}}^{(n)} \subset \widetilde{\mathfrak{S}}_{\underline{k}}^{(n)}$ denote the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-submodule of all integral elements and set

$$\mathfrak{S}^{(n)} := \bigoplus_{\underline{k}\in\mathbb{N}^I} \mathfrak{S}_{\underline{k}}^{(n)}.$$

The following is the key result of this section:

Theorem 3.40 *The $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $\Psi\colon U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n) \xrightarrow{\sim} S^{(n)}$ of Theorem 3.5 gives rise to a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-algebra isomorphism $\Psi\colon \mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n) \xrightarrow{\sim} \mathfrak{S}^{(n)}$.*

The proof of Theorem 3.40 is presented in Sect. 3.4.

Corollary 3.41 *(a) $\mathfrak{S}^{(n)}$ is a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $S^{(n)}$.*

(b) Theorem 2.20(a) holds, that is, the subalgebra $\mathfrak{U}_{\boldsymbol{v}}^{>}(L\mathfrak{sl}_n)$ is independent of all the choices.

The following two properties of the integral form $\mathfrak{S}^{(n)}$ are crucially used in [17]:

Proposition 3.42 (a) *Fix $1 \leq \ell < n$ and consider the linear map $\iota'_\ell \colon S^{(n)} \to S^{(n)}$ given by*

$$\iota'_\ell(F)\left(\{x_{i,r}\}_{i\in I}^{1\leq r\leq k_i}\right) := \prod_{r=1}^{k_\ell}(1-x_{\ell,r}^{-1}) \cdot F\left(\{x_{i,r}\}_{i\in I}^{1\leq r\leq k_i}\right) \ \text{ for } \ F \in S_{\underline{k}}^{(n)}, \underline{k}\in\mathbb{N}^I. \tag{3.43}$$

Then

$$F \in \mathfrak{S}^{(n)} \Longleftrightarrow \iota'_\ell(F) \in \mathfrak{S}^{(n)}. \tag{3.44}$$

(b) *For any* $\underline{k} \in \mathbb{N}^I$ *and a collection* $g_i(\{x_{i,r}\}_{r=1}^{k_i}) \in \mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][\{x_{i,r}^{\pm 1}\}_{r=1}^{k_i}]^{\Sigma_{k_i}}$ $(\forall\, i \in I)$, *consider*

$$F = (\boldsymbol{v}-\boldsymbol{v}^{-1})^{|\underline{k}|}\cdot \frac{\prod_{i=1}^{n-1}\prod_{1\le r\ne r'\le k_i}(x_{i,r}-\boldsymbol{v}^{-2}x_{i,r'})\cdot \prod_{i=1}^{n-1} g_i(\{x_{i,r}\}_{r=1}^{k_i})}{\prod_{i=1}^{n-2}\prod_{1\le r\le k_i}^{1\le r'\le k_{i+1}}(x_{i,r}-x_{i+1,r'})}. \tag{3.45}$$

Then, F *is integral, i.e.* $F \in \mathfrak{S}^{(n)}_{\underline{k}}$.

Proof (a) Obvious from the above definition of the integral form $\mathfrak{S}^{(n)}$.

(b) The presence of the factor $\prod_{i=1}^{n-1}\prod_{1\le r\ne r'\le k_i}(x_{i,r}-\boldsymbol{v}^{-2}x_{i,r'})$ in F of (3.45) guarantees that for any degree vector $\underline{d}=\{d_\beta\}_{\beta\in\Delta^+}$ satisfying $\sum_{\beta\in\Delta^+} d_\beta\beta=\sum_{i\in I}k_i\alpha_i$, the reduced specialization $\varphi_{\underline{d}}(F)$ of (3.33) is divisible by $\prod_{\beta\in\Delta^+}\prod_{1\le s\ne s'\le d_\beta}(y_{\beta,s}-\boldsymbol{v}^{-2}y_{\beta,s'})$. Thus, the further specialization $\varpi_{\underline{t}}(\varphi_{\underline{d}}(F)) = \Upsilon_{\underline{d},\underline{t}}(F)$ vanishes if at least one of $t_{\beta,r}$'s is greater than 1. Meanwhile, the divisibility condition of Definition 3.37 is vacuous in the remaining case when all $t_{\beta,r}=1$. Therefore, $F\in\mathfrak{S}^{(n)}_{\underline{k}}$ as claimed. □

3.4 Proofs of Theorem 2.20(b) and Theorem 3.40

We note that both Theorem 2.20(b) and Theorem 3.40 follow from the following two results:

(I) For any $k\ge 1$, $\{\beta_p\}_{p=1}^k\subset\Delta^+$, $\{r_p\}_{p=1}^k\subset\mathbb{Z}$, we have $\Psi(\widetilde{e}_{\beta_1}(r_1)\cdots\widetilde{e}_{\beta_k}(r_k))\in\mathfrak{S}^{(n)}$.

(II) Any element $F\in\mathfrak{S}^{(n)}$ may be written as a $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-linear combination of $\{\Psi(\widetilde{e}_h)\}_{h\in H}$.

The proof of (I) is straightforward and will be easily deduced from our definition of $\mathfrak{S}^{(n)}$, while the proof of (II) will follow from Lemma 3.21 and the validity of (II) for $n=2$, see Lemma 3.48.

We start by establishing both (I) and (II) for $n=2$ in Sect. 3.4.1.

3.4.1 $n=2$ case

For $n=2$, the description of the integral form $\mathfrak{S}^{(n)}\subset S^{(n)}$ is the simplest, see Example 3.38. Set $\widetilde{e}_r := (\boldsymbol{v}-\boldsymbol{v}^{-1})e_r\in U^>_{\boldsymbol{v}}(L\mathfrak{sl}_2)$ for $r\in\mathbb{Z}$. The following result establishes (I) for $n=2$:

Lemma 3.46 *For any* $k\ge 1$ *and* $r_1,\ldots,r_k\in\mathbb{Z}$, *we have* $\Psi(\widetilde{e}_{r_1}\cdots\widetilde{e}_{r_k})\in\mathfrak{S}^{(2)}_k$.

Proof Pick any decomposition $k=k_1+\cdots+k_\ell$ with all $k_p\ge 1$. We claim that as we specialize the variables $x_1,\ldots,x_k$ to $\{\boldsymbol{v}^{-2r}z_p\}_{1\le p\le \ell}^{1\le r\le k_p}$, the image of any summand of the symmetrization appearing in $\Psi(e_{r_1}\cdots e_{r_k})\in S_k^{(2)}$ is divisible by the product $\prod_{p=1}^{\ell}[k_p]_{\boldsymbol{v}}!$ of $\boldsymbol{v}$-factorials.

To this end, let us fix $1\leq p\leq \ell$ and consider the relative position of the variables $\boldsymbol{v}^{-2}z_p, \boldsymbol{v}^{-4}z_p,\ldots,\boldsymbol{v}^{-2k_p}z_p$. If there is an index $1\leq r<k_p$ such that $\boldsymbol{v}^{-2(r+1)}z_p$ is placed to the left of $\boldsymbol{v}^{-2r}z_p$, then the above specialization of the corresponding ζ-factor equals $\frac{\boldsymbol{v}^{-2(r+1)}z_p-\boldsymbol{v}^{-2}\cdot\boldsymbol{v}^{-2r}z_p}{\boldsymbol{v}^{-2(r+1)}z_p-\boldsymbol{v}^{-2r}z_p}=0$. However, if $\boldsymbol{v}^{-2r}z_p$ stays to the left of $\boldsymbol{v}^{-2(r+1)}z_p$ for all $1\leq r<k_p$, then the total contribution of the specializations of the corresponding ζ-factors equals

$$\prod_{1\leq r<s\leq k_p}\frac{\boldsymbol{v}^{-2r}z_p-\boldsymbol{v}^{-2}\cdot\boldsymbol{v}^{-2s}z_p}{\boldsymbol{v}^{-2r}z_p-\boldsymbol{v}^{-2s}z_p}=\boldsymbol{v}^{-\frac{k_p(k_p-1)}{2}}[k_p]_{\boldsymbol{v}}! \tag{3.47}$$

Combining this over all $1\leq p\leq\ell$, we see that $\prod_{p=1}^{\ell}[k_p]_{\boldsymbol{v}}!$ indeed divides the above specialization of $\Psi(e_{r_1}\cdots e_{r_k})$. This completes our proof of Lemma 3.46. □

For simplicity of the exposition, we will assume now that the total order $\preceq$ on $\mathbb{Z}$ is the usual one $\leq$. The following result establishes (II) for $n=2$:

Lemma 3.48 *Any symmetric Laurent polynomial* $\bar{F}\in\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][\{x_p^{\pm1}\}_{p=1}^{k}]^{\Sigma_k}$ *satisfying the divisibility condition* (3.39) *may be written as a* $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$*-linear combination of* $\{\Psi(e_h)\}_{h\in H}$.

Proof We may assume that $\bar{F}$ is homogeneous of the total degree N. Let V_N denote the set of all ordered k-tuples of integers $\underline{r}=(r_1,r_2,\ldots,r_k)$, $r_1\leq\cdots\leq r_k$, such that $r_1+\cdots+r_k=N$. This set is totally ordered with respect to the lexicographical order:

$$\underline{r}\prec\underline{r}'\ \text{ iff }\ \exists\, 1\leq p\leq k \text{ s.t. } r_p<r'_p \text{ and } r_s=r'_s\ \forall\, s>p.$$

Let us present $\bar{F}$ as a linear combination of the monomial symmetric polynomials:

$$\bar{F}(x_1,\ldots,x_k)=\sum_{\underline{r}\in V_N}\mu_{\underline{r}}m_{\underline{r}}(x_1,\ldots,x_k)\ \text{ with }\ \mu_{\underline{r}}\in\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}].$$

Pick the maximal element $\underline{r}_{\max}=(r_1,\ldots,r_k)$ of the finite set

$$V_N(\bar{F}):=\{\underline{r}\in V_N|\mu_{\underline{r}}\neq 0\}$$

and consider a decomposition $k=k_1+\cdots+k_\ell$ such that

$$r_1=\cdots=r_{k_1}<r_{k_1+1}=\cdots=r_{k_1+k_2}<\cdots<r_{k_1+\cdots+k_{\ell-1}+1}=\cdots=r_k.$$

Evaluating $\bar{F}$ at the corresponding specialization $\{\boldsymbol{v}^{-2s}z_p\}_{1\leq p\leq\ell}^{1\leq s\leq k_p}$, we see that the coefficient of the lexicographically largest monomial in the variables $\{z_p\}_{p=1}^{\ell}$ equals

$\mu_{\underline{r}_{\max}}$, up to an integer power of $\boldsymbol{v}$. Therefore, the divisibility condition (3.39) implies:

$$\frac{\mu_{\underline{r}_{\max}}}{\prod_{p=1}^{\ell}[k_p]_{\boldsymbol{v}}!} \in \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]. \tag{3.49}$$

Set $\bar{F}^{(0)} := \bar{F}$ and define $\bar{F}^{(1)} \in \mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}][\{x_p^{\pm 1}\}_{p=1}^{k}]^{\Sigma_k}$ via

$$\bar{F}^{(1)} := \bar{F}^{(0)} - \boldsymbol{v}^{\sum_{p=1}^{\ell} k_p(k_p-1)/2} \frac{\mu_{\underline{r}_{\max}}}{\prod_{p=1}^{\ell}[k_p]_{\boldsymbol{v}}!} \Psi(e_{r_1} \cdots e_{r_k}). \tag{3.50}$$

We note that $\bar{F}^{(1)}$ satisfies (3.39), due to (3.49) and Lemma 3.46. Applying the same argument to $\bar{F}^{(1)}$ in place of $F = \bar{F}^{(0)}$, we obtain $\bar{F}^{(2)}$ also satisfying (3.39). Proceeding further, we thus construct a sequence of symmetric Laurent polynomials $\{\bar{F}^{(s)}\}_{s\in\mathbb{N}}$ satisfying (3.39).

According to our proof of Lemma 3.12, especially formula (3.13), the sequence $\underline{r}^{(i)}_{\max} \in V_N$ of the maximal elements of $V_N(\bar{F}^{(i)})$ strictly decreases. Meanwhile, the sequence of the minimal powers of any variable in $\bar{F}^{(s)}$ is a non-decreasing sequence. Hence, $\bar{F}^{(s)} = 0$ for some $s \in \mathbb{N}$.

This completes our proof of Lemma 3.48. □

3.4.2 General case

Let us now generalize the arguments of Sect. 3.4.1 to prove (I) and (II) for any $n > 2$. The proof of the former is quite similar (though is more elaborate) to that of Lemma 3.46:

Lemma 3.51 $\Psi(\widetilde{e}_{\beta_1}(r_1)\cdots\widetilde{e}_{\beta_m}(r_m)) \in \mathfrak{S}^{(n)}$ *for any* $m \geq 1$, $\{\beta_p\}_{p=1}^m \subset \Delta^+$, $\{r_p\}_{p=1}^m \subset \mathbb{Z}$.

Proof Define $\underline{k} \in \mathbb{N}^I$ via $\sum_{i\in I} k_i\alpha_i = \sum_{p=1}^m \beta_p$, so that $F := \Psi(\widetilde{e}_{\beta_1}(r_1)\cdots\widetilde{e}_{\beta_m}(r_m)) \in S_{\underline{k}}^{(n)}$. First, we note that F is divisible by $(\boldsymbol{v} - \boldsymbol{v}^{-1})^{|\underline{k}|}$, due to Lemma 3.14. Therefore, $F \in \widetilde{\mathfrak{S}}_{\underline{k}}^{(n)}$.

It remains to show that $\Upsilon_{\underline{d},\underline{t}}(F)$ satisfies the divisibility condition of Definition 3.37 for any $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+} \in \mathbb{N}^{\frac{n(n-1)}{2}}$ and a collection of positive integers $\underline{t} = \{t_{\beta,s}\}_{\beta\in\Delta^+}^{1\leq s\leq \ell_\beta}$ satisfying (3.35). To this end, recall that the cross specialization $\Upsilon_{\underline{d},\underline{t}}(F)$ is computed in three steps:

- first, we specialize $x_{*,*}$-variables in f of (3.2) to $\boldsymbol{v}^?$-multiples of $y_{*,*}$-variables as in (3.7);
- second, we divide that specialization by the product of appropriate powers of $(\boldsymbol{v} - \boldsymbol{v}^{-1})$ and linear terms in $y_{*,*}$-variables arising via the wheel conditions, see (3.31, 3.32) and (3.33);
- finally, we specialize $y_{*,*}$-variables to $\boldsymbol{v}^?$-multiples of $z_{*,*}$-variables as in (3.34).

Fix $\beta \in \Delta^+$ and $1 \leq s \leq \ell_\beta$, and consider those $x_{*,*}$ that eventually got specialized to $\boldsymbol{v}^? z_{\beta,s}$. Without loss of generality, we may assume those are $\{x_{i,r}\}_{j(\beta)\leq i\leq i(\beta)}^{1\leq r\leq t_{\beta,s}}$. We may also assume that $x_{i,r}$ was specialized to $\boldsymbol{v}^{-i}y_{\beta,r}$ under the first specialization (3.7), while $y_{\beta,r}$ was specialized to $\boldsymbol{v}^{-2r}z_{\beta,s}$ under the second specialization (3.34), for any $j(\beta)\leq i\leq i(\beta),\ 1\leq r\leq t_{\beta,s}$.

For $j(\beta)\leq i<i(\beta)$ and $1\leq r\neq r'\leq t_{\beta,s}$, consider the relative position of the variables $x_{i,r}, x_{i,r'}, x_{i+1,r'}$. As $x_{i,r}, x_{i,r'}$ cannot enter the same function $\Psi(\widetilde{e}_*(*))$, $x_{i,r}$ is placed either to the left of $x_{i,r'}$ or to the right. In the former case, we gain the factor $\zeta_{i,i}(x_{i,r}/x_{i,r'})$, which upon the specialization $\phi_{\underline{d}}$ contributes the factor $(y_{\beta,r}-\boldsymbol{v}^{-2}y_{\beta,r'})$. Likewise, if $x_{i+1,r'}$ is placed to the left of $x_{i,r}$, we gain the factor $\zeta_{i+1,i}(x_{i+1,r'}/x_{i,r})$, which upon the specialization $\phi_{\underline{d}}$ contributes the factor $(y_{\beta,r}-\boldsymbol{v}^{-2}y_{\beta,r'})$ as well. In the remaining case, when $x_{i,r'}$ is to the left of $x_{i,r}$ while $x_{i+1,r'}$ is not, we gain the factor $\zeta_{i,i+1}(x_{i,r'}/x_{i+1,r'})$, which upon the specialization $\phi_{\underline{d}}$ specializes to 0. As i ranges from $j(\beta)$ up to $i(\beta)-1$, we thus gain the $(i(\beta)-j(\beta))$th power of $(y_{\beta,r}-\boldsymbol{v}^{-2}y_{\beta,r'})$. Note that this power exactly coincides with the power of $(y_{\beta,r}-\boldsymbol{v}^{-2}y_{\beta,r'})$ in $B_{\underline{d}}$ of (3.32), by which we divide $\phi_{\underline{d}}(F)$ to define the reduced specialization $\varphi_{\underline{d}}(F)$ of (3.33).

However, we have not used yet the ζ-factors $\zeta_{i(\beta),i(\beta)}(x_{i(\beta),r}/x_{i(\beta),r'})$ for $x_{i(\beta),r}$ placed to the left of $x_{i(\beta),r'}$. If there is $1\leq r<t_{\beta,s}$ such that $x_{i(\beta),r+1}$ is placed to the left of $x_{i(\beta),r}$, then $\zeta_{i(\beta),i(\beta)}(x_{i(\beta),r+1}/x_{i(\beta),r})$ specializes to zero upon (3.34). In the remaining case, when each $x_{i(\beta),r}$ stays to the left of $x_{i(\beta),r+1}$, the total contribution of the specializations of the corresponding ζ-factors equals $\boldsymbol{v}^{-t_{\beta,s}(t_{\beta,s}-1)/2}[t_{\beta,s}]_{\boldsymbol{v}}!$ as in formula (3.47).

This completes our proof of Lemma 3.51. □

Let $\widetilde{M}\subset S^{(n)}$ be the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-span of $\{\Psi(\widetilde{e}_h)\}_{h\in H}$. Generalizing Lemma 3.21, we have:

Lemma 3.52 *Let $F\in\mathfrak{S}^{(n)}_{\underline{k}}$ and $\underline{d}\in T_{\underline{k}}$. If $\phi_{\underline{d}'}(F)=0$ for all $\underline{d}'\in T_{\underline{k}}$ such that $\underline{d}'>\underline{d}$, then there exists an element $F_{\underline{d}}\in\widetilde{M}$ such that $\phi_{\underline{d}}(F)=\phi_{\underline{d}}(F_{\underline{d}})$ and $\phi_{\underline{d}'}(F_{\underline{d}})=0$ for all $\underline{d}'>\underline{d}$.*

Proof The proof of this lemma is completely analogous to the one of Lemma 3.21. More precisely, combining formulas (3.23, 3.24) and the condition $F\in\mathfrak{S}^{(n)}_{\underline{k}}$ together with formula (3.19) and the discussion following it, the result follows from its $n=2$ counterpart. The latter has been already taken care of in Lemma 3.48. □

Combining Lemmas 3.51, 3.52 with the argument following our proof of Lemma 3.21 completes our proof of both Theorems 2.20(b) and 3.40. We note that the result of Theorem 2.20(c) follows as well, as explained in the beginning of Sect. 3.

4 Generalizations to $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$

The two-parameter quantum loop algebra $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ was introduced in [22][8] as a generalization of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ (one recovers the latter from the former by setting $\boldsymbol{v}_1=\boldsymbol{v}_2^{-1}=\boldsymbol{v}$ and identifying some Cartan elements, see [22, Remark 3.3(4)]). The key results of [22] are:

[8] To be more precise, this recovers the algebra of *loc.cit.* with the trivial central charges.

1) the Drinfeld–Jimbo type realization of $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$, see [22, Theorem 3.12];
2) the PBW basis of its subalgebras $U^{>}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n), U^{<}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$, see [22, Theorem 3.11].

However, the latter result ([22, Theorem 3.11]) is stated without any glimpse of a proof.

The primary goal of this section is to generalize Theorem 2.16 to the case of $U^{>}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$, thus proving [22, Theorem 3.11]. Along the way, we also generalize Theorem 3.5 by providing the shuffle realization of $U^{>}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$, which is of independent interest. The latter is used to construct the PBWD bases for the integral form of $U^{>}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$, generalizing Theorem 2.20.

4.1 Two-parameter quantum loop algebra $U^{>}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$

For the purpose of this section, it suffices to work only with the subalgebra $U^{>}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ of $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$. Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be two independent formal variables and set $\mathbb{K} := \mathbb{C}(\boldsymbol{v}_1^{1/2}, \boldsymbol{v}_2^{1/2})$. Following [22, Definition 3.1], define $U^{>}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ to be the associative $\mathbb{K}$-algebra generated by $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ with the following defining relations:

$$(z - (\langle j,i\rangle\langle i,j\rangle)^{1/2}w)e_i(z)e_j(w) = (\langle j,i\rangle z - (\langle j,i\rangle\langle i,j\rangle^{-1})^{1/2}w)e_j(w)e_i(z) \tag{4.1}$$

as well as Serre relations:

$$\begin{aligned} &e_i(z)e_j(w) = e_j(w)e_i(z) \text{ if } c_{ij} = 0,\\ &[e_i(z_1), [e_i(z_2), e_{i+1}(w)]_{\boldsymbol{v}_2}]_{\boldsymbol{v}_1} + [e_i(z_2), [e_i(z_1), e_{i+1}(w)]_{\boldsymbol{v}_2}]_{\boldsymbol{v}_1} = 0,\\ &[e_i(z_1), [e_i(z_2), e_{i-1}(w)]_{\boldsymbol{v}_2^{-1}}]_{\boldsymbol{v}_1^{-1}} + [e_i(z_2), [e_i(z_1), e_{i-1}(w)]_{\boldsymbol{v}_2^{-1}}]_{\boldsymbol{v}_1^{-1}} = 0, \end{aligned} \tag{4.2}$$

where $e_i(z) = \sum_{r\in\mathbb{Z}} e_{i,r}z^{-r}$ and $\langle i,j\rangle \in \mathbb{K}$ is defined via $\langle i,j\rangle := \boldsymbol{v}_1^{\delta_{ij}-\delta_{i+1,j}}\boldsymbol{v}_2^{\delta_{i,j+1}-\delta_{ij}}$.

4.2 PBWD bases of $U^{>}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$

We shall follow the notations of Sect. 2.2, except that now $(\lambda_1,\ldots,\lambda_{p-1}) \in \{\boldsymbol{v}_1, \boldsymbol{v}_2\}^{p-1}$. Similarly to (2.12), we define the *PBWD basis elements* $e_\beta(r) \in U^{>}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ via

$$e_\beta(r) := [\cdots[[e_{i_1,r_1}, e_{i_2,r_2}]_{\lambda_1}, e_{i_3,r_3}]_{\lambda_2}, \cdots, e_{i_p,r_p}]_{\lambda_{p-1}}.$$

The monomials $e_h := \overrightarrow{\prod\limits_{(\beta,r)\in\Delta^+\times\mathbb{Z}}} e_\beta(r)^{h(\beta,r)}$ $(h \in H)$ will be called the *ordered PBWD monomials* of $U^{>}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$. Here, the arrow over the product sign refers to the total order (2.11).

Our first main result establishes the PBWD property of $U^{>}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$:

Theorem 4.3 *The ordered PBWD monomials* $\{e_h\}_{h\in H}$ *form a* $\mathbb{K}$*-basis of* $U^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$.

The proof of Theorem 4.3 is outlined in Sect. 4.5 and is based on the shuffle approach.

Remark 4.4 We note that the PBWD basis elements introduced in [22, (3.14)] are

$$e_{\alpha_j+\alpha_{j+1}+\cdots+\alpha_i}(r) := [\cdots[[e_{j,r}, e_{j+1,0}]_{\boldsymbol{v}_1}, e_{j+2,0}]_{\boldsymbol{v}_1}, \cdots, e_{i,0}]_{\boldsymbol{v}_1}. \tag{4.5}$$

In this particular case, Theorem 4.3 recovers [22, Theorem 3.11] provided without a proof.

Remark 4.6 The entire two-parameter quantum loop algebra $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ admits a triangular decomposition, cf. Proposition 2.9(a). Hence, an analogue of Theorem 2.18 holds for $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ as well, thus providing a family of PBWD $\mathbb{K}$-bases for $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$.

4.3 Integral form $\mathfrak{U}^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ and its PBWD bases

For $(\beta, r) \in \Delta^+ \times \mathbb{Z}$, we define $\widetilde{e}_\beta(r) \in U^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ via

$$\widetilde{e}_\beta(r) := (\boldsymbol{v}_1^{1/2}\boldsymbol{v}_2^{-1/2} - \boldsymbol{v}_1^{-1/2}\boldsymbol{v}_2^{1/2})e_\beta(r).$$

We also define $\widetilde{e}_h \in U^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ via (2.15) but using $\widetilde{e}_\beta(r)$ instead of $e_\beta(r)$. Finally, we define integral form $\mathfrak{U}^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ as the $\mathbb{C}[\boldsymbol{v}_1^{1/2}, \boldsymbol{v}_2^{1/2}, \boldsymbol{v}_1^{-1/2}, \boldsymbol{v}_2^{-1/2}]$-subalgebra of $U^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ generated by $\{\widetilde{e}_\beta(r)\}^{r\in\mathbb{Z}}_{\beta\in\Delta^+}$.

The following counterpart of Theorem 2.20 provides a much stronger version of Theorem 4.3:

Theorem 4.7 (a) *The subalgebra* $\mathfrak{U}^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ *is independent of all our choices.*

(b) *Elements* $\{\widetilde{e}_h\}_{h\in H}$ *form a basis of the free* $\mathbb{C}[\boldsymbol{v}_1^{1/2}, \boldsymbol{v}_2^{1/2}, \boldsymbol{v}_1^{-1/2}, \boldsymbol{v}_2^{-1/2}]$*-module* $\mathfrak{U}^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$.

The proof of Theorem 4.7 follows easily from the one of Theorem 4.3 presented below in the same way as we deduced the proof of Theorem 2.20 in Sect. 3.4 from that of Theorem 2.16.

Remark 4.8 We note that it is often more convenient to work with the two-parameter quantum loop algebra $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{gl}_n)$, cf. Remark 2.27(a). Its integral form $\mathfrak{U}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{gl}_n)$ is defined analogously to $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$. Following the arguments of [17, Proposition 3.11], $\mathfrak{U}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{gl}_n)$ is identified with the RTT integral form $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{gl}_n)$ under the $\mathbb{K}$-algebra isomorphism $U_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{gl}_n) \simeq \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{gl}_n) \otimes_{\mathbb{C}[\boldsymbol{v}_1^{1/2},\boldsymbol{v}_2^{1/2},\boldsymbol{v}_1^{-1/2},\boldsymbol{v}_2^{-1/2}]} \mathbb{K}$ of [24], cf. Remark 2.27(b). Thus, the analogue of [17, Theorem 3.24] provides a family of PBWD bases for $\mathfrak{U}_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{gl}_n)$, cf. Theorems 2.23, 2.25.

4.4 Shuffle algebra $\widetilde{S}^{(n)}$

Define the shuffle algebra $(\widetilde{S}^{(n)}, \star)$ analogously to $(S^{(n)}, \star)$ with the following modifications:

(1) all vector spaces are now defined over $\mathbb{K}$ (rather than over $\mathbb{C}(\boldsymbol{v})$);
(2) $(\zeta_{i,j}(z))_{i,j\in I} \in \mathrm{Mat}_{I\times I}(\mathbb{K}(z))$, used in the shuffle product (3.1), are now chosen as:

$$\zeta_{i,j}(z) = \left(\frac{z - \boldsymbol{v}_1^{1/2}\boldsymbol{v}_2^{-1/2}}{z-1}\right)^{\delta_{j,i-1}} \left(\frac{z - \boldsymbol{v}_1^{-1}\boldsymbol{v}_2}{z-1}\right)^{\delta_{ji}} \left(\boldsymbol{v}_1^{1/2}\boldsymbol{v}_2^{1/2}\cdot\frac{z - \boldsymbol{v}_1^{1/2}\boldsymbol{v}_2^{-1/2}}{z-1}\right)^{\delta_{j,i+1}};$$

(3) the wheel conditions (3.3) for F are replaced with

$$F(\{x_{i,r}\}) = 0 \quad \text{once} \quad x_{i,r_1} = \boldsymbol{v}_1^{1/2}\boldsymbol{v}_2^{-1/2} x_{i+\epsilon,s} = \boldsymbol{v}_1\boldsymbol{v}_2^{-1} x_{i,r_2}$$

for some $\epsilon \in \{\pm 1\},\ i,\ r_1 \neq r_2,\ s.$

The following result is analogous to Proposition 3.4 (recovering the latter by setting $\boldsymbol{v}_1 = \boldsymbol{v}_2^{-1} = \boldsymbol{v}$):

Proposition 4.9 *The assignment $e_{i,r} \mapsto x_{i,1}^r$ $(i \in I, r \in \mathbb{Z})$ gives rise to an injective $\mathbb{K}$-algebra homomorphism $\Psi\colon U^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n) \to \widetilde{S}^{(n)}$.*

Our proof of Theorem 4.3 below implies the counterpart of Theorem 3.5, see Remark 4.11:

Theorem 4.10 *$\Psi\colon U^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n) \xrightarrow{\sim} \widetilde{S}^{(n)}$ of Proposition 4.9 is a $\mathbb{K}$-algebra isomorphism.*

4.5 Proof of Theorem 4.3

The proof of Theorem 4.3 is completely analogous to our proof of Theorem 2.16(a) and is based on the embedding $\Psi\colon U^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n) \hookrightarrow \widetilde{S}^{(n)}$ of Proposition 4.9. Indeed, the linear independence of $\{e_h\}_{h\in H}$ is deduced exactly as in Sect. 3.2.2 with the only modification of the *specialization maps* $\phi_{\underline{d}}\colon \widetilde{S}^{(n)}_{\underline{\ell}} \to \mathbb{K}[\{y_{\beta,s}^{\pm 1}\}_{\beta\in\Delta^+}^{1\le s\le d_\beta}]^{\Sigma_{\underline{d}}}$ of (3.7) by replacing $\boldsymbol{v}^{-i}$ with $(\boldsymbol{v}_1^{1/2}\boldsymbol{v}_2^{-1/2})^{-i}$. Then, the results of Lemmas 3.16 and 3.17 still hold, thus implying the linear independence of $\{e_h\}_{h\in H}$. Meanwhile, the fact that $\{e_h\}_{h\in H}$ span $U^>_{\boldsymbol{v}_1,\boldsymbol{v}_2}(L\mathfrak{sl}_n)$ is deduced using the arguments of Sect. 3.2.3. To be more precise, Lemma 3.21 still holds and its iterative application immediately implies that any shuffle element $F \in \widetilde{S}^{(n)}$ belongs to the $\mathbb{K}$-span of $\{\Psi(e_h)\}_{h\in H}$.

Remark 4.11 Combining the last statement in the above proof of Theorem 4.3 with the injectivity of Ψ (Proposition 4.9), we obtain a proof of Theorem 4.10.

5 Generalizations to $U_v(L\mathfrak{sl}(m|n))$

The quantum loop superalgebra $U_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$[9] was introduced in [36], both in the Drinfeld–Jimbo and the new Drinfeld realizations, see [36, Theorem 8.5.1] for an identification of those. The representation theory of these algebras was partially studied in [37] by crucially utilizing a weak version of the PBW theorem for $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$, [37, Theorem 3.12]. Inspired by [22], the author also conjectured the PBW theorem for $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$, see [37, Remark 3.13(2)].

The primary goal of this section is to generalize Theorem 2.16 to the case of $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$, thus proving the conjecture of [37]. Along the way, we also generalize Theorem 3.5 by providing the shuffle realization of $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$, which is of independent interest. The latter is used to construct the PBWD bases for the integral form of $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$, generalizing Theorem 2.20.

Remark 5.1 (a) We should stress right away that in the exposition below we do choose a distinguished Dynkin diagram with a single simple positive root of odd degree. The generalization of all our results to an arbitrary Dynkin diagram is carried out in [33], cf. Sect. 8.2.

(b) The shuffle algebras associated to quantum loop superalgebras seem to be new in the literature as they involve both symmetric and skew-symmetric functions ("bosons" and "fermions").

5.1 Quantum loop superalgebra $U^>_v(L\mathfrak{sl}(m|n))$

For the purpose of this section, it suffices to work only with the subalgebra $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ of $U_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$. Let $I = \{1, \ldots, m+n-1\}$ from now on. Consider a free $\mathbb{Z}$-module $\oplus_{i=1}^{m+n}\mathbb{Z}\epsilon_i$ with the bilinear form $(\cdot,\cdot)$ determined by $(\epsilon_i, \epsilon_j) = (-1)^{\delta_{i>m}}\delta_{ij}$. Let $\boldsymbol{v}$ be a formal variable and define $\{\boldsymbol{v}_i\}_{i\in I} \subset \{\boldsymbol{v}, \boldsymbol{v}^{-1}\}$ via $\boldsymbol{v}_i := \boldsymbol{v}^{(\epsilon_i,\epsilon_i)}$. For $i, j \in I$, set $\bar{c}_{ij} := (\alpha_i, \alpha_j)$ with $\alpha_i := \epsilon_i - \epsilon_{i+1}$.

Following [36] (cf. [37, Theorem 3.3]), define $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ to be the associative $\mathbb{C}(\boldsymbol{v})$-superalgebra generated by $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ with the $\mathbb{Z}_2$-grading $|e_{m,r}| = \bar{1}$, $|e_{i,r}| = \bar{0}$ $(i \neq m, r \in \mathbb{Z})$, and subject to the following defining relations:

$$(z - \boldsymbol{v}^{\bar{c}_{ij}} w)e_i(z)e_j(w) = (\boldsymbol{v}^{\bar{c}_{ij}} z - w)e_j(w)e_i(z) \text{ if } \bar{c}_{ij} \neq 0, \tag{5.2}$$

$$\begin{aligned} &[e_i(z), e_j(w)] = 0 \text{ if } \bar{c}_{ij} = 0,\\ &[e_i(z_1), [e_i(z_2), e_j(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} + [e_i(z_2), [e_i(z_1), e_j(w)]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}} = 0 \quad \text{if } \bar{c}_{ij} = \pm 1 \text{ and } i \neq m, \end{aligned} \tag{5.3}$$

[9] To be more precise, one actually needs to use the classical Lie superalgebra $A(m-1, n-1)$ in place of $\mathfrak{sl}(m|n)$, which do coincide when $m \neq n$. However, we shall ignore this difference, since we will be working only with the *positive subalgebras* and those are isomorphic: $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(m|n)) \simeq U^>_{\boldsymbol{v}}(LA(m-1, n-1))$.

as well as quartic Serre relations:

$$[[[e_{m-1}(w), e_m(z_1)]_{\boldsymbol{v}^{-1}}, e_{m+1}(u)]_{\boldsymbol{v}}, e_m(z_2)] \\ +[[[e_{m-1}(w), e_m(z_2)]_{\boldsymbol{v}^{-1}}, e_{m+1}(u)]_{\boldsymbol{v}}, e_m(z_1)] = 0, \quad (5.4)$$

where $e_i(z) = \sum_{r\in\mathbb{Z}} e_{i,r}z^{-r}$ and we use the super-bracket notations:

$$[a, b]_x := ab - (-1)^{|a||b|}x \cdot ba, \qquad [a, b] := [a, b]_1$$

for $\mathbb{Z}_2$-homogeneous elements a, b (we set $(-1)^{\bar{0}} := 1$ and $(-1)^{\bar{1}} := -1$).

5.2 PBWD bases of $U^>_v(L\mathfrak{sl}(m|n))$

Let $\Delta^+ = \{\alpha_j + \alpha_{j+1} + \cdots + \alpha_i\}_{1\leq j\leq i<m+n}$. For $\beta \in \Delta^+$, define its parity $|\beta| \in \mathbb{Z}_2$ via

$$|\beta| = \begin{cases} \bar{1}, & \text{if } m \in [\beta] \\ \bar{0}, & \text{if } m \notin [\beta] \end{cases}. \quad (5.5)$$

We shall follow the notations of Sect. 2.2. In particular, we define the *PBWD basis elements* $e_\beta(r) \in U^>_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ via (2.12), but with $[\cdot, \cdot]$ denoting the super-bracket.

Let $\bar{H}$ denote the set of all functions $h\colon \Delta^+ \times \mathbb{Z} \to \mathbb{N}$ with finite support and such that $h(\beta, r) \leq 1$ if $|\beta| = \bar{1}$. The monomials

$$e_h := \prod^{\rightarrow}_{(\beta,r)\in\Delta^+\times\mathbb{Z}} e_\beta(r)^{h(\beta,r)}, \qquad \forall\, h \in \bar{H} \quad (5.6)$$

will be called the *ordered PBWD monomials* of $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$. Here, the arrow over the product sign refers to the total order (2.11), as before.

Our first main result establishes the PBWD property of $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$:

Theorem 5.7 *The ordered PBWD monomials $\{e_h\}_{h\in\bar{H}}$ form a $\mathbb{C}(\boldsymbol{v})$-basis of $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$.*

The proof of Theorem 5.7 is presented in Sect. 5.5 and is based on the shuffle approach.

Remark 5.8 We note that the PBWD basis elements introduced in [37, (3.12)] are

$$e_{\alpha_j+\alpha_{j+1}+\cdots+\alpha_i}(r) := [\cdots[[e_{j,r}, e_{j+1,0}]_{\boldsymbol{v}_{j+1}}, e_{j+2,0}]_{\boldsymbol{v}_{j+2}}, \cdots, e_{i,0}]_{\boldsymbol{v}_i}. \quad (5.9)$$

In this particular case, Theorem 5.7 recovers the conjecture of [37, Remark 3.13(2)].

Remark 5.10 The entire quantum loop superalgebra $U_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ admits a triangular decomposition as in Proposition 2.9, see [37, Theorem 3.3]. Hence, an analogue of Theorem 2.18 holds for $U_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ as well, thus providing a family of PBWD $\mathbb{C}(\boldsymbol{v})$-bases for $U_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$.

5.3 Integral form $\mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ and its PBWD bases

Following (2.19), for any $(\beta, r) \in \Delta^+ \times \mathbb{Z}$, we define $\widetilde{e}_\beta(r) \in U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ via

$$\widetilde{e}_\beta(r) := (\boldsymbol{v} - \boldsymbol{v}^{-1})e_\beta(r).$$

We also define $\widetilde{e}_h$ via (5.6) but using $\widetilde{e}_\beta(r)$ instead of $e_\beta(r)$. Finally, let $\mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ denote the $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-subalgebra of $U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ generated by $\{\widetilde{e}_\beta(r)\}^{r\in\mathbb{Z}}_{\beta\in\Delta^+}$.

The following counterpart of Theorem 2.20 provides a much stronger version of Theorem 5.7:

Theorem 5.11 (a) *The subalgebra $\mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ is independent of all our choices.*
(b) *The elements $\{\widetilde{e}_h\}_{h\in\bar{H}}$ form a basis of the free $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-module $\mathfrak{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$.*

The proof of Theorem 5.11 follows easily from the one of Theorem 5.7 presented below in the same way as we deduced the proof of Theorem 2.20 in Sect. 3.4 from that of Theorem 2.16.

Remark 5.12 We note that it is often more convenient to work with the quantum loop superalgebra $U_{\boldsymbol{v}}(L\mathfrak{gl}(m|n))$, cf. Remark 2.27(a). Its integral form $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}(m|n))$ is defined analogously to $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$. Following the arguments of [17, Proposition 3.11], $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}(m|n))$ is identified with the RTT integral form $\mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}(m|n))$, [39, Definition 3.1], under the $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $U_{\boldsymbol{v}}(L\mathfrak{gl}(m|n)) \simeq \mathfrak{U}^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}(m|n)) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}(\boldsymbol{v})$, cf. Remark 2.27(b). Hence, the analogue of [17, Theorem 3.24] provides a family of PBWD bases for $\mathfrak{U}_{\boldsymbol{v}}(L\mathfrak{gl}(m|n))$, cf. Theorems 2.23, 2.25.

5.4 Shuffle algebra $S^{(m|n)}$

Consider an $\mathbb{N}^I$-graded $\mathbb{C}(\boldsymbol{v})$-vector space $\mathbb{S}^{(m|n)} = \bigoplus_{\underline{k}\in\mathbb{N}^I} \mathbb{S}^{(m|n)}_{\underline{k}}$, where $\mathbb{S}^{(m|n)}_{(k_1,\ldots,k_{m+n-1})}$ consists of rational functions in the variables $\{x_{i,r}\}^{1\leq r\leq k_i}_{i\in I}$ which are **supersymmetric**, that is

(1) symmetric in $\{x_{i,r}\}^{k_i}_{r=1}$ for every $i \neq m$;
(2) skew-symmetric in $\{x_{m,r}\}^{k_m}_{r=1}$.

We also fix an $I \times I$ matrix of rational functions $(\zeta_{i,j}(z))_{i,j\in I} \in \mathrm{Mat}_{I\times I}(\mathbb{C}(\boldsymbol{v})(z))$ via

$$\zeta_{i,j}(z) = \frac{z - \boldsymbol{v}^{-\bar{c}_{ij}}}{z - 1}. \tag{5.13}$$

This allows us to endow $\mathbb{S}^{(m|n)}$ with a structure of an associative unital algebra with the shuffle product defined via (3.1), but a supersymmetrization SSym in place of the symmetrization Sym. Here, the *supersymmetrization* of $f \in \mathbb{C}(\{x_{i,1}, \ldots, x_{i,s_i}\}_{i\in I})$ is defined via

$$\mathrm{SSym}_{\Sigma_{\underline{s}}}(f)(\{x_{i,1},\ldots,x_{i,s_i}\}_{i\in I}) := \sum_{(\sigma_1,\ldots,\sigma_{m+n-1})\in\Sigma_{\underline{s}}} \mathrm{sgn}(\sigma_m) f\left(\{x_{i,\sigma_i(1)},\ldots,x_{i,\sigma_i(s_i)}\}_{i\in I}\right).$$

As before, we will be interested only in the subspace of $\mathbb{S}^{(m|n)}$ defined by the *pole* and *wheel conditions* (but now there are two kinds of the latter one):

- We say that $F \in \mathbb{S}^{(m|n)}_{\underline{k}}$ satisfies the *pole conditions* if

$$F = \frac{f(x_{1,1},\ldots,x_{m+n-1,k_{m+n-1}})}{\prod_{i=1}^{m+n-2}\prod_{r\leq k_i}^{r'\leq k_{i+1}}(x_{i,r}-x_{i+1,r'})}, \tag{5.14}$$

  where $f \in \mathbb{C}(\boldsymbol{v})[\{x_{i,r}^{\pm 1}\}_{i\in I}^{1\leq r\leq k_i}]$ is a supersymmetric Laurent polynomial, that is, symmetric in $\{x_{i,r}\}_{r=1}^{k_i}$ for every $i \neq m$ and skew-symmetric in $\{x_{m,r}\}_{r=1}^{k_m}$.
- We say that $F \in \mathbb{S}^{(m|n)}_{\underline{k}}$ satisfies the *first kind wheel conditions* if

$$F(\{x_{i,r}\}) = 0 \quad \text{once} \quad x_{i,r_1} = \boldsymbol{v}_i x_{i+\epsilon,s} = \boldsymbol{v}_i^2 x_{i,r_2} \tag{5.15}$$

  for some $\epsilon \in \{\pm 1\},\ i \in I\backslash\{m\},\ 1 \leq r_1 \neq r_2 \leq k_i,\ 1 \leq s \leq k_{i+\epsilon}$.
- We say that $F \in \mathbb{S}^{(m|n)}_{\underline{k}}$ satisfies the *second kind wheel conditions* if

$$F(\{x_{i,r}\}) = 0 \quad \text{once} \quad x_{m-1,s} = \boldsymbol{v}x_{m,r_1} = x_{m+1,s'} = \boldsymbol{v}^{-1}x_{m,r_2} \tag{5.16}$$

  for some $1 \leq r_1 \neq r_2 \leq k_m,\ 1 \leq s \leq k_{m-1},\ 1 \leq s' \leq k_{m+1}$.

Let $S^{(m|n)}_{\underline{k}} \subset \mathbb{S}^{(m|n)}_{\underline{k}}$ denote the subspace of all elements F satisfying these three conditions. We define

$$S^{(m|n)} := \bigoplus_{\underline{k}\in\mathbb{N}^I} S^{(m|n)}_{\underline{k}}$$

It is straightforward to check that $S^{(m|n)} \subset \mathbb{S}^{(m|n)}$ is $\star$-closed. Similar to Proposition 3.4, the **shuffle algebra** $\left(S^{(m|n)},\star\right)$ is related to $U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$ via:

Proposition 5.17 *The assignment $e_{i,r} \mapsto x_{i,1}^r$ $(i \in I, r \in \mathbb{Z})$ gives rise to an injective $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism $\Psi\colon U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}(m|n)) \to S^{(m|n)}$.*

Our proof of Theorem 5.7 below implies the counterpart of Theorem 3.5, see Remark 5.26:

Theorem 5.18 $\Psi\colon U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}(m|n)) \xrightarrow{\sim} S^{(m|n)}$ *of Proposition 5.17 is an algebra isomorphism.*

5.5 Proof of Theorem 5.7

The proof of Theorem 5.7 is similar to our proof of Theorem 2.16(a) and is based on the embedding $\Psi\colon U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}(m|n)) \hookrightarrow S^{(m|n)}$ of Proposition 5.17. Thus, we will only outline the proof, highlighting the key changes.

We start by establishing Theorem 5.7 in the simplest case $m = n = 1$:

Lemma 5.19 *For any total order $\preceq$ on $\mathbb{Z}$, the ordered monomials $\{e_{r_1} e_{r_2} \cdots e_{r_k}\}_{k\in\mathbb{N}}^{r_1\prec\cdots\prec r_k}$ form a $\mathbb{C}(\boldsymbol{v})$-basis of $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}(1|1))$.*

Proof This follows from the $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $S^{(1|1)} \simeq \bigoplus_k \Lambda_k$, where Λ_k denotes the vector space of skew-symmetric Laurent polynomials in k variables, while the algebra structure on the direct sum arises via the standard skew-symmetrization maps $\Lambda_k \otimes \Lambda_\ell \to \Lambda_{k+\ell}$. □

Let us now treat the general case of m, n. Given a degree vector $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+} \in \mathbb{N}^{\Delta^+}$, define $\underline{\ell} \in \mathbb{N}^I$ via $\sum_{\beta\in\Delta^+} d_\beta \beta = \sum_{i\in I} \ell_i \alpha_i$. The **specialization map**

$$\phi_{\underline{d}}\colon S_{\underline{\ell}}^{(m|n)} \longrightarrow \mathbb{C}(\boldsymbol{v})[\{y_{\beta,s}^{\pm 1}\}_{\beta\in\Delta^+}^{1\leq s\leq d_\beta}]$$

is defined similar to (3.7), but with the only change that the variable $x_{i,r}$ in the sth copy of the interval $[\beta]$ is specialized to $\boldsymbol{v}^{-i} y_{\beta,s}$ if $i \leq m$ and to $\boldsymbol{v}^{i-2m} y_{\beta,s}$ if $i > m$. We note that $\phi_{\underline{d}}(F)$ is a supersymmetric Laurent polynomial, that is, symmetric in $\{y_{\beta,s}\}_{s=1}^{d_\beta}$ if $|\beta| = \bar{0}$ and skew-symmetric in $\{y_{\beta,s}\}_{s=1}^{d_\beta}$ if $|\beta| = \bar{1}$.

Thus defined specialization maps $\phi_{\underline{d}}$ still satisfy Lemmas 3.16 and 3.17 (with $\bar{H}$ used in place of H in the formulation of the latter), hence, the linear independence of $\{e_h\}_{h\in\bar{H}}$ as follows from the argument presented right after our proof of Lemma 3.17.

Furthermore, for any $h \in \bar{H}$ with $\deg(h) = \underline{d}$, we have the following generalization of the formulas (3.18, 3.19) from our proof of Lemma 3.17:

$$\phi_{\underline{d}}(\Psi(e_h)) = c \cdot \prod_{\beta,\beta'\in\Delta^+}^{\beta<\beta'} \bar{G}_{\beta,\beta'} \cdot \prod_{\beta\in\Delta^+} \bar{G}_\beta \cdot \prod_{\beta\in\Delta^+} \left(\sum_{\sigma_\beta\in\Sigma_{d_\beta}} \bar{G}_\beta^{(\sigma_\beta)} \right) \tag{5.20}$$

with $c \in \mathbb{C}^\times \cdot \boldsymbol{v}^{\mathbb{Z}}$, where

$$\begin{aligned}
\bar{G}_{\beta,\beta'} &= \prod_{1\le s\le d_\beta}^{1\le s'\le d_{\beta'}} (y_{\beta,s}-\boldsymbol{v}^{-2}y_{\beta',s'})^{\nu^-(\beta,\beta')}\cdot(y_{\beta,s}-\boldsymbol{v}^{2}y_{\beta',s'})^{\nu^+(\beta,\beta')} \\
&\quad\times \prod_{1\le s\le d_\beta}^{1\le s'\le d_{\beta'}} (y_{\beta,s}-y_{\beta',s'})^{\delta_{j(\beta')>j(\beta)}\delta_{i(\beta)+1\in[\beta']}+\delta_{m\in[\beta]}\delta_{m\in[\beta']}}, \\
\bar{G}_\beta &= (1-\boldsymbol{v}^2)^{d_\beta(i(\beta)-j(\beta))}\cdot \prod_{1\le s\ne s'\le d_\beta}(y_{\beta,s}-\boldsymbol{v}^2 y_{\beta,s'})^{i(\beta)-j(\beta)} \\
&\quad\times \prod_{1\le s\le d_\beta} y_{\beta,s}^{i(\beta)-j(\beta)}, \\
\bar{G}_\beta^{(\sigma_\beta)} &= \prod_{s=1}^{d_\beta} y_{\beta,\sigma_\beta(s)}^{r_\beta(h,s)}\cdot
\begin{cases}
\prod_{s<s'}\frac{y_{\beta,\sigma_\beta(s)}-\boldsymbol{v}^{-2}y_{\beta,\sigma_\beta(s')}}{y_{\beta,\sigma_\beta(s)}-y_{\beta,\sigma_\beta(s')}}, & \text{if } m>i(\beta)\\
\prod_{s<s'}\frac{y_{\beta,\sigma_\beta(s)}-\boldsymbol{v}^{2}y_{\beta,\sigma_\beta(s')}}{y_{\beta,\sigma_\beta(s)}-y_{\beta,\sigma_\beta(s')}}, & \text{if } m<j(\beta)\\
\mathrm{sgn}(\sigma_\beta), & \text{if } m\in[\beta]
\end{cases}.
\end{aligned} \tag{5.21}$$

Here, the collection $\{r_\beta(h,1),\ldots,r_\beta(h,d_\beta)\}$ is defined exactly as after (3.18) (that is, listing every $r\in\mathbb{Z}$ with multiplicity $h(\beta,r)>0$ with respect to the total order $\preceq_\beta$ on $\mathbb{Z}$), while the powers $\nu^\pm(\beta,\beta')$ are given by the following explicit formulas:

$$\begin{aligned}\nu^-(\beta,\beta') = \#\{(j,j')\in[\beta]\times[\beta']\,|\,j=j'<m\}\,+\\ \#\{(j,j')\in[\beta]\times[\beta']\,|\,j=j'+1>m\}\end{aligned} \tag{5.22}$$

and

$$\begin{aligned}\nu^+(\beta,\beta') = \#\{(j,j')\in[\beta]\times[\beta']\,|\,j=j'>m\}\,+\\ \#\{(j,j')\in[\beta]\times[\beta']\,|\,j=j'+1\le m\}.\end{aligned} \tag{5.23}$$

For any $\beta\in\Delta^+$, we note that the sum $\sum_{\sigma_\beta\in\Sigma_{d_\beta}}\bar{G}_\beta^{(\sigma_\beta)}$ coincides (up to a factor of $\mathbb{C}^\times$) with the value of the shuffle element $x^{r_\beta(h,1)}\star\cdots\star x^{r_\beta(h,d_\beta)}$, viewed as an element of

(1) the shuffle algebra $S^{(2|0)}$ if $m>i(\beta)$,
(2) the shuffle algebra $S^{(0|2)}$ if $m<j(\beta)$,
(3) the shuffle algebra $S^{(1|1)}$ if $m\in[\beta]$,

evaluated at $\{y_{\beta,s}\}_{s=1}^{d_\beta}$. The latter elements are linearly independent, due to Lemmas 3.12, 5.19.

Remark 5.24 The above reduction to the rank 1 cases (that is, $m+n=2$) together with Lemma 5.19 explains why H had to be replaced by $\bar{H}$ in the current setting.

The fact that $\{e_h\}_{h\in\bar{H}}$ span $U_{\boldsymbol{v}}^{>}(L\mathfrak{sl}(m|n))$ follows from the validity of Lemma 3.21 in the current setting. Let us now prove the latter using the same ideas and notations as before.

First, we note that the wheel conditions (5.15, 5.16) for F guarantee that $\phi_{\underline{d}}(F)$ (which is a Laurent polynomial in $\{y_{\beta,s}\}$) vanishes up to appropriate orders under the following specializations:

(i) $y_{\beta,s} = \boldsymbol{v}^{-2} y_{\beta',s'}$ for $(\beta, s) < (\beta', s')$,
(ii) $y_{\beta,s} = \boldsymbol{v}^{2} y_{\beta',s'}$ for $(\beta, s) < (\beta', s')$.

A straightforward case-by-case verification shows that these orders of vanishing exactly equal the corresponding powers of $y_{\beta,s} - \boldsymbol{v}^{-2} y_{\beta',s'}$ and $y_{\beta,s} - \boldsymbol{v}^{2} y_{\beta',s'}$ appearing in $\bar{G}_{\beta,\beta'}$ (if $\beta < \beta'$) or $\bar{G}_{\beta}$ (if $\beta = \beta'$) of (5.21). In the former case, those are explicitly given by (5.22, 5.23).

Remark 5.25 We should point out right away that the computation of the corresponding orders requires an extra argument in the case when $\beta = \beta'$ and $m \in [\beta]$. Recall that the way we counted these orders in the proof of Lemma 3.21 was by realizing the specialization $\phi_{\underline{d}}$ as a step-by-step specialization in each interval in the specified order. A priori, we can choose another order of the intervals or even another way to perform this specialization. Let us now illustrate how our argument should be modified in the particular case $\beta = \beta', m \in [\beta]$. Note that if we first specialize the variables in the interval $[\beta]$ to the corresponding $\boldsymbol{v}$-multiples of $y_{\beta,s}$, then the wheel conditions contribute $i(\beta) - j(\beta)$ to the order of vanishing at $y_{\beta,s} = \boldsymbol{v}^{2} y_{\beta,s'}$ and $i(\beta) - j(\beta) - 1$ to the order of vanishing at $y_{\beta,s} = \boldsymbol{v}^{-2} y_{\beta,s'}$. If instead we first specialize the variables in the interval $[\beta]$ to the corresponding $\boldsymbol{v}$-multiples of $y_{\beta,s'}$, then the wheel conditions contribute $i(\beta) - j(\beta) - 1$ to the order of vanishing at $y_{\beta,s} = \boldsymbol{v}^{2} y_{\beta,s'}$ and $i(\beta) - j(\beta)$ to the order of vanishing at $y_{\beta,s} = \boldsymbol{v}^{-2} y_{\beta,s'}$. Thus, none of these two specializations provides the desired orders of vanishing simultaneously for $y_{\beta,s} = \boldsymbol{v}^{2} y_{\beta,s'}$ and $y_{\beta,s} = \boldsymbol{v}^{-2} y_{\beta,s'}$. However, picking the maximal of the orders separately for $y_{\beta,s} = \boldsymbol{v}^{2} y_{\beta,s'}$ and $y_{\beta,s} = \boldsymbol{v}^{-2} y_{\beta,s'}$, we recover $i(\beta) - j(\beta)$ for both of them, so that they equal the corresponding powers of $y_{\beta,s} - \boldsymbol{v}^{2} y_{\beta,s'}$ and $y_{\beta,s} - \boldsymbol{v}^{-2} y_{\beta,s'}$ appearing in $\bar{G}_{\beta}$ of (5.21).

Second, we claim that $\phi_{\underline{d}}(F)$ vanishes under the following specializations:

(iii) $y_{\beta,s} = y_{\beta',s'}$ for $(\beta, s) < (\beta', s')$ such that $j(\beta) < j(\beta')$ and $i(\beta) + 1 \in [\beta']$.

Indeed, if $j(\beta) < j(\beta')$ and $i(\beta) + 1 \in [\beta']$, there are positive roots $\gamma, \gamma' \in \Delta^+$ such that $j(\gamma) = j(\beta), i(\gamma) = i(\beta'), j(\gamma') = j(\beta'), i(\gamma') = i(\beta)$. Consider the degree vector $\underline{d}' \in T_{\underline{k}}$ given by $d'_\alpha = d_\alpha + \delta_{\alpha,\gamma} + \delta_{\alpha,\gamma'} - \delta_{\alpha,\beta} - \delta_{\alpha,\beta'}$. Then, $\underline{d}' > \underline{d}$ and thus $\phi_{\underline{d}'}(F) = 0$. The result follows.

Finally, we also note that the skew-symmetry of the elements of $S^{(m|n)}$ with respect to the variables $\{x_{m,*}\}$ implies that $\phi_{\underline{d}}(F)$ vanishes under the following specializations:

(iv) $y_{\beta,s} = y_{\beta',s'}$ for all $\beta < \beta'$ (and any s, s') such that $[\beta] \ni m \in [\beta']$.

Combining the above vanishing conditions for $\phi_{\underline{d}}(F)$, we see that it is divisible exactly by the product $\prod_{\beta<\beta'} \bar{G}_{\beta,\beta'} \cdot \prod_{\beta} \bar{G}_{\beta}$ of (5.21). Therefore, we have

$$\phi_{\underline{d}}(F) = \prod_{\beta,\beta'\in\Delta^+}^{\beta<\beta'} \bar{G}_{\beta,\beta'} \cdot \prod_{\beta\in\Delta^+} \bar{G}_{\beta} \cdot \bar{G},$$

where $\bar{G}$ is a supersymmetric Laurent polynomial in $\{y_{\beta,s}\}_{\beta\in\Delta^+}^{1\le s\le d_\beta}$, that is:

(1) $\bar{G}$ is symmetric in $\{y_{\beta,s}\}_{s=1}^{d_\beta}$ if $|\beta| = \bar{0}$,

(2) $\bar{G}$ is skew-symmetric in $\{y_{\beta,s}\}_{s=1}^{d_\beta}$ if $|\beta| = \bar{1}$.

Combining this observation with formulas (5.20, 5.21) and the discussion following them, we obtain a proof of Lemma 3.21 in the current setting, due to Lemmas 3.12, 5.19.

Following the arguments from the end of Sect. 3.2.3, we see that $\{\Psi(e_h)\}_{h\in\bar{H}}$ linearly span $S^{(m|n)}$. Invoking the injectivity of Ψ (Proposition 5.17), this implies that $\{e_h\}_{h\in\bar{H}}$ span $U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}(m|n))$. This completes our proof of Theorem 5.7.

Remark 5.26 The above argument also provides a proof of Theorem 5.18.

6 Generalizations to the Yangian $Y_\hbar(\mathfrak{sl}_n)$

The PBWD bases for the Yangian $Y_\hbar(\mathfrak{g})$ of any semisimple Lie algebra $\mathfrak{g}$ have been constructed 25 years ago in [28][10]. Note that while the Yangian deforms the universal enveloping of the loop algebra, that is $Y_\hbar(\mathfrak{g})/(\hbar) \simeq U(\mathfrak{g}[t])$, there is a canonical construction of the *Drinfeld–Gavarini dual* (Hopf) subalgebra $Y'_\hbar(\mathfrak{g}) \subset Y_\hbar(\mathfrak{g})$ such that $Y'_\hbar(\mathfrak{g})/(\hbar)$ is a commutative $\mathbb{C}$-algebra, see [17, Appendix A] and the original references [8,18]. The PBWD bases for the Drinfeld–Gavarini dual $Y'_\hbar(\mathfrak{g})$ were constructed in [17, Theorem A.7], following [18].

As just mentioned, the PBWD results (cf. Theorems 2.16, 2.18, 2.20, 2.25) are known both for $Y_\hbar(\mathfrak{g})$ and $Y'_\hbar(\mathfrak{g})$ for an arbitrary semisimple $\mathfrak{g}$. Thus, the key objective of this section is to provide the shuffle realizations of $Y_\hbar(\mathfrak{sl}_n)$ and $Y'_\hbar(\mathfrak{sl}_n)$ similar to those of Theorems 3.5, 3.40. For the latter purpose, it suffices to consider only the subalgebras $Y^{>}_\hbar(\mathfrak{sl}_n)$, $Y'^{>}_\hbar(\mathfrak{sl}_n) \simeq \mathbf{Y}^{>}_\hbar(\mathfrak{sl}_n)$.

6.1 Algebras $Y^{>}_\hbar(\mathfrak{sl}_n)$ and $\mathbf{Y}^{>}_\hbar(\mathfrak{sl}_n)$

Let $I = \{1, \ldots, n-1\}$, $(c_{ij})_{i,j\in I}$ be the Cartan matrix of $\mathfrak{sl}_n$, and $\hbar$ be a formal variable. Following [7], define the *positive subalgebra* of the Yangian of $\mathfrak{sl}_n$, denoted by $Y^{>}_\hbar(\mathfrak{sl}_n)$, to be the associative $\mathbb{C}[\hbar]$-algebra generated by $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{N}}$ with the following defining relations:

$$[e_{i,r+1}, e_{j,s}] - [e_{i,r}, e_{j,s+1}] = \frac{c_{ij}\hbar}{2}\left(e_{i,r}e_{j,s} + e_{j,s}e_{i,r}\right) \tag{6.1}$$

[10] See [17, Appendix B] for a correction of a gap in the proof of [28].

as well as Serre relations:

$$\begin{aligned}&[e_{i,r}, e_{j,s}] = 0 \text{ if } c_{ij} = 0,\\&[e_{i,r_1}, [e_{i,r_2}, e_{j,s}]] + [e_{i,r_2}, [e_{i,r_1}, e_{j,s}]] = 0 \text{ if } c_{ij} = -1.\end{aligned} \tag{6.2}$$

Let $\{\alpha_i\}_{i=1}^{n-1}$ and Δ^+ be as in Sect. 2.2. For any $(\beta, r) \in \Delta^+ \times \mathbb{N}$, we choose:

(1) a decomposition $\beta = \alpha_{i_1} + \cdots + \alpha_{i_p}$ such that $[\cdots[e_{\alpha_{i_1}}, e_{\alpha_{i_2}}], \cdots, e_{\alpha_{i_p}}]$ is a non-zero root vector e_β of $\mathfrak{sl}_n$ (here, e_{α_i} denotes the standard Chevalley generator of $\mathfrak{sl}_n$);
(2) a decomposition $r = r_1 + \cdots + r_p$ with $r_k \in \mathbb{N}$.

Then, we define the *PBWD basis elements* $e_\beta(r) \in Y_\hbar^>(\mathfrak{sl}_n)$ via

$$e_\beta(r) := [\cdots[[e_{i_1,r_1}, e_{i_2,r_2}], e_{i_3,r_3}], \cdots, e_{i_p,r_p}]. \tag{6.3}$$

Let H^+ denote the set of all functions $h\colon \Delta^+ \times \mathbb{N} \to \mathbb{N}$ with finite support. The monomials

$$e_h := \prod_{(\beta,r)\in\Delta^+\times\mathbb{N}}^{\rightarrow} e_\beta(r)^{h(\beta,r)}, \quad \forall\, h \in H^+ \tag{6.4}$$

will be called the *ordered PBWD monomials* of $Y_\hbar^>(\mathfrak{sl}_n)$. Here, the arrow over the product sign refers to the total order on $\Delta^+ \times \mathbb{N}$ obtained as the restriction of the order (2.11).

The following is due to [28] (cf. [17, Theorem B.3]):

Theorem 6.5 ([28]) *The elements $\{e_h\}_{h\in H^+}$ form a basis of the free $\mathbb{C}[\hbar]$-module $Y_\hbar^>(\mathfrak{sl}_n)$.*

Remark 6.6 This result actually holds for any total order on $\Delta^+ \times \mathbb{N}$ used in (6.4), see [28].

For any $\beta \in \Delta^+$ and $r \in \mathbb{N}$, define $\widetilde{e}_\beta(r) \in Y_\hbar^>(\mathfrak{sl}_n)$ via

$$\widetilde{e}_\beta(r) := \hbar \cdot e_\beta(r). \tag{6.7}$$

For $h \in H^+$, we also define $\widetilde{e}_h$ via the formula (6.4) but using $\widetilde{e}_\beta(r)$ instead of $e_\beta(r)$. Finally, define an integral form $\mathbf{Y}_\hbar^>(\mathfrak{sl}_n)$ as the $\mathbb{C}[\hbar]$-subalgebra of $Y_\hbar^>(\mathfrak{sl}_n)$ generated by $\{\widetilde{e}_\beta(r)\}_{\beta\in\Delta^+}^{r\in\mathbb{N}}$.

The following result is proved in [17, Theorem A.7]:

Theorem 6.8 ([17]) (a) *The subalgebra $\mathbf{Y}_\hbar^>(\mathfrak{sl}_n)$ is independent of all our choices (1)–(2) made when defining $e_\beta(r)$ in (6.3) and hence $\widetilde{e}_\beta(r)$ in (6.7).*

(b) *The ordered PBWD monomials $\{\widetilde{e}_h\}_{h\in H^+}$ form a basis of the free $\mathbb{C}[\hbar]$-module $\mathbf{Y}_\hbar^>(\mathfrak{sl}_n)$.*

6.2 Rational shuffle algebra $W^{(n)}$ and its integral form $\mathfrak{W}^{(n)}$

Define the shuffle algebra $(\bar{W}^{(n)}, \star)$ analogously to the shuffle algebra $(S^{(n)}, \star)$ of Sect. 3.1 with the following modifications:

(1) all rational functions are defined over $\mathbb{C}[\hbar]$;
(2) the rational functions $(\zeta_{i,j}(z))_{i,j\in I} \in \mathrm{Mat}_{I\times I}(\mathbb{C}[\hbar](z))$ are chosen via

$$\zeta_{i,j}(z) = 1 + \frac{c_{ij}\hbar}{2z};$$

(3) shuffle product $\star$ is defined via (3.1), but $\zeta_{i,i'}(x_{i,r}/x_{i',r'})$ are replaced with $\zeta_{i,i'}(x_{i,r} - x_{i',r'})$;
(4) the *pole conditions* (3.2) for $F \in \bar{W}^{(n)}_{\underline{k}}$ are replaced with

$$F = \frac{f(x_{1,1},\dots,x_{n-1,k_{n-1}})}{\prod_{i=1}^{n-2}\prod_{r\le k_i}^{r'\le k_{i+1}}(x_{i,r} - x_{i+1,r'})}, \text{ where } f \in \mathbb{C}[\hbar][\{x_{i,r}\}_{i\in I}^{1\le r\le k_i}]^{\Sigma_{\underline{k}}}; \tag{6.9}$$

(5) the *wheel conditions* (3.3) for $F \in \bar{W}^{(n)}_{\underline{k}}$ are replaced with

$$F(\{x_{i,r}\}) = 0 \quad \text{once} \quad x_{i,r_1} = x_{i+\epsilon,s} + \frac{\hbar}{2} = x_{i,r_2} + \hbar \tag{6.10}$$

for some $\epsilon \in \{\pm 1\}$, i, $r_1 \neq r_2$, s.

The **rational shuffle algebra** $\left(\bar{W}^{(n)}, \star\right)$ is related to $Y^{>}_{\hbar}(\mathfrak{sl}_n)$ via the following construction:

Proposition 6.11 *The assignment* $e_{i,r} \mapsto x^r_{i,1}$ $(i \in I, r \in \mathbb{N})$ *gives rise to a* $\mathbb{C}[\hbar]$-*algebra homomorphism* $\Psi\colon Y^{>}_{\hbar}(\mathfrak{sl}_n) \to \bar{W}^{(n)}$.

The next result is straightforward, cf. Lemma 3.14:

Lemma 6.12 *For* $1 \le j < i < n$ *and* $r \in \mathbb{N}$, *we have*

$$\Psi(e_{\alpha_j+\alpha_{j+1}+\dots+\alpha_i}(r)) = \hbar^{i-j}\frac{p(x_{j,1},\dots,x_{i,1})}{(x_{j,1}-x_{j+1,1})\cdots(x_{i-1,1}-x_{i,1})},$$

where $p(x_{j,1},\dots,x_{i,1}) \in \mathbb{C}[\hbar][x_{j,1},\dots,x_{i,1}]$ *is a degree* r *monomial, up to a sign.*

Example 6.13 For the particular choice

$$e_{\alpha_j+\alpha_{j+1}+\dots+\alpha_i}(r) = [\cdots[e_{j,r}, e_{j+1,0}],\cdots, e_{i,0}]$$

of the PBWD basis elements (see [17, §2]), we have $p(x_{j,1},\dots,x_{i,1}) = (-1)^{i-j}x^r_{j,1}$.

Let us adapt our key tool, the specialization maps, to this setting. Given a degree vector $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+} \in \mathbb{N}^{\Delta^+}$, define $\underline{\ell} \in \mathbb{N}^I$ via $\sum_{\beta\in\Delta^+} d_\beta \beta = \sum_{i\in I} \ell_i \alpha_i$. The **specialization map**

$$\phi_{\underline{d}} \colon \bar{W}^{(n)}_{\underline{\ell}} \longrightarrow \mathbb{C}[\hbar][\{y_{\beta,s}\}^{1\leq s\leq d_\beta}_{\beta\in\Delta^+}]^{\Sigma_{\underline{d}}} \tag{6.14}$$

is defined similar to (3.7), but with the only change that the variable $x_{i,r}$ in the sth copy of the interval $[\beta]$ is specialized to $y_{\beta,s} - \frac{i\hbar}{2}$.

Then, arguing exactly as in Sect. 3.2.2, we get:

Proposition 6.15 *The elements $\{\Psi(e_h)\}_{h\in H^+}$ are linearly independent.*

Combining this with Theorem 6.5, we obtain:

Proposition 6.16 *$\Psi \colon Y^>_\hbar(\mathfrak{sl}_n) \to \bar{W}^{(n)}$ is an injective $\mathbb{C}[\hbar]$-algebra homomorphism.*

However, in contrast to Theorem 3.5, the embedding $\Psi \colon Y^>_\hbar(\mathfrak{sl}_n) \hookrightarrow \bar{W}^{(n)}$ is not an isomorphism. The description of its image is similar to Theorem 3.40, but is significantly simpler.

Definition 6.17 $F \in \bar{W}^{(n)}_{\underline{k}}$ is **good** if $\phi_{\underline{d}}(F)$ is divisible by $\hbar^{\sum_{\beta\in\Delta^+} d_\beta(i(\beta)-j(\beta))}$ for any degree vector $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+} \in \mathbb{N}^{\Delta^+}$ such that $\sum_{\beta\in\Delta^+} d_\beta\beta = \sum_{i\in I} k_i\alpha_i$.

Example 6.18 In the simplest case $n = 2$, any element $F \in \bar{W}^{(n)}_{\underline{k}}$ ($\underline{k} \in \mathbb{N}^I$) is good.

Let $W^{(n)}_{\underline{k}} \subset \bar{W}^{(n)}_{\underline{k}}$ denote the $\mathbb{C}[\hbar]$-submodule of all good elements and set

$$W^{(n)} := \bigoplus_{\underline{k}\in\mathbb{N}^I} W^{(n)}_{\underline{k}}$$

Lemma 6.19 $\Psi(Y^>_\hbar(\mathfrak{sl}_n)) \subseteq W^{(n)}$.

Proof Let $F = \Psi(e_{i_1,r_1}\cdots e_{i_N,r_N}) \in \bar{W}^{(n)}_{\underline{k}}$ and choose $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+} \in \mathbb{N}^{\Delta^+}$ such that $\sum_{\beta\in\Delta^+} d_\beta\beta = \sum_{i\in I} k_i\alpha_i$. For any $\beta \in \Delta^+$ and $1 \leq s \leq d_\beta$, consider ζ-factors between those pairs of $x_{*,*}$-variables that are specialized to $y_{\beta,s} - \frac{\ell\hbar}{2}$, $y_{\beta,s} - \frac{(\ell+1)\hbar}{2}$ with $j(\beta) \leq \ell < i(\beta)$ in the definition of the specialization map (6.14). Each of them contributes a multiple of $\hbar$ into $\phi_{\underline{d}}(F)$ and there are exactly $\sum_{\beta\in\Delta^+} d_\beta(i(\beta) - j(\beta))$ of such pairs. Hence, $F \in W^{(n)}_{\underline{k}}$. □

The following is the key result of this section:

Theorem 6.20 *The $\mathbb{C}[\hbar]$-algebra embedding $\Psi \colon Y^>_\hbar(\mathfrak{sl}_n) \hookrightarrow \bar{W}^{(n)}$ of Proposition 6.16 gives rise to a $\mathbb{C}[\hbar]$-algebra isomorphism $\Psi \colon Y^>_\hbar(\mathfrak{sl}_n) \xrightarrow{\sim} W^{(n)}$.*

In view of Example 6.18, this theorem for $n = 2$ is equivalent to the following result:

Lemma 6.21 *Any symmetric polynomial* $F \in \mathbb{C}[\hbar][\{x_p\}_{p=1}^k]^{\Sigma_k}$ *may be written as a* $\mathbb{C}[\hbar]$*-linear combination of* $\{\Psi(e_h)\}_{h\in H^+}$.

The proof of Lemma 6.21 is completely analogous to those of Lemmas 3.12, 3.48, and crucially relies on the following simple computation (cf. Lemma 3.9):

Lemma 6.22 *For any* $k \geq 1$ *and* $r \in \mathbb{N}$*, the kth power of* $x^r \in \bar{W}_1^{(2)}$ *equals*

$$\underbrace{x^r \star \cdots \star x^r}_{k \text{ times}} = k! \cdot (x_1 \cdots x_k)^r. \tag{6.23}$$

Proof The proof is by induction in k and boils down to the verification of the equality

$$\sum_{p=1}^{k} \prod_{1\leq s\leq k}^{s\neq p} \frac{x_s - x_p + \hbar}{x_s - x_p} = k, \tag{6.24}$$

which is proved similarly to (3.11). □

The proof of Theorem 6.20 for $n > 2$ is completely analogous to those of Theorems 2.16, 2.20 and crucially utilizes its $n = 2$ case, established in Lemma 6.21. We refer the interested reader to [33, Proof of Theorem 3.30] for more details.

Definition 6.25 $F \in \bar{W}_{\underline{k}}^{(n)}$ is **integral** if F is divisible by $\hbar^{|\underline{k}|}$.

Remark 6.26 We note that any integral shuffle element is obviously good, cf. Definition 6.17.

Let $\mathfrak{W}_{\underline{k}}^{(n)} \subset W_{\underline{k}}^{(n)}$ denote the $\mathbb{C}[\hbar]$-submodule of all integral elements and set

$$\mathfrak{W}^{(n)} := \bigoplus_{\underline{k}\in\mathbb{N}^I} \mathfrak{W}_{\underline{k}}^{(n)}.$$

The following is our second key result of this section:

Theorem 6.27 *The* $\mathbb{C}[\hbar]$*-algebra isomorphism* $\Psi\colon Y_\hbar^>(\mathfrak{sl}_n) \xrightarrow{\sim} W^{(n)}$ *of Theorem* 6.20 *gives rise to a* $\mathbb{C}[\hbar]$*-algebra isomorphism* $\Psi\colon \mathbf{Y}_\hbar^>(\mathfrak{sl}_n) \xrightarrow{\sim} \mathfrak{W}^{(n)}$.

The proof of Theorem 6.27 is completely analogous to that of Theorem 3.40, but is much simpler. In particular, adapting Lemma 3.51 to the current setting, the key combinatorial computation from its proof is not needed, while Lemma 3.52 is adapted without any changes. We refer the interested reader to [33, Proof of Theorem 3.9] for more details.

Remark 6.28 Let us note right away that the key simplification in the proof of Theorem 6.27 (comparing to that of Theorem 3.40) and in the definition of the integral elements of Definition 6.25 (comparing to those of Definition 3.37) is due to the following rank 1 computations:

(1) $\hbar^k(x_1\cdots x_k)^r \in \Psi(\mathbf{Y}_\hbar^>(\mathfrak{sl}_2))\ \forall\, k, r \in \mathbb{N}$, due to Lemma 6.22;
(2) $(\boldsymbol{v}-\boldsymbol{v}^{-1})^k[k]_{\boldsymbol{v}}!(x_1\cdots x_k)^r \in \Psi(\mathfrak{U}_{\boldsymbol{v}}^>(L\mathfrak{sl}_2))\ \forall\, k \in \mathbb{N}, r \in \mathbb{Z}$, due to Lemma 3.9;
(3) $(\boldsymbol{v}-\boldsymbol{v}^{-1})^k(x_1\cdots x_k)^r \notin \Psi(\mathfrak{U}_{\boldsymbol{v}}^>(L\mathfrak{sl}_2))\ \forall\ k > 1, r \in \mathbb{Z}$, due to Lemma 3.46.

7 Generalizations to the super Yangian $Y_\hbar(\mathfrak{sl}(m|n))$

The super Yangian $Y_\hbar(\mathfrak{gl}(m|n))$ of the Lie superalgebra $\mathfrak{gl}(m|n)$ was first introduced in [30], following the RTT formalism of [15]. Its finite-dimensional representations were classified in [40]. Around the same time, the super Yangian $Y_\hbar(A(m,n))$ of the classical Lie superalgebra $A(m,n)$ in the new Drinfeld presentation was introduced in [32]. As shown in *loc. cit.*, these super Yangians posses most of the properties the usual Yangians have (including the PBWD bases). The explicit relation between the super Yangians of [30,32] was established in [19].

The primary goal of this section is to generalize Theorem 6.20 to the case of $Y_\hbar^>(\mathfrak{sl}(m|n))$ (we note that $Y_\hbar^>(\mathfrak{sl}(m|n)) \simeq Y_\hbar^>(\mathfrak{gl}(m|n)) \simeq Y_\hbar^>(A(m-1,n-1))$). The resulting shuffle algebra $W^{(m|n)}$ is a mixture of the shuffle algebra $S^{(m|n)}$ from Sect. 5.4 and the rational shuffle algebra $W^{(n)}$ from Sect. 6.2. We also generalize Theorem 6.27 to the case of $\mathbf{Y}_\hbar^>(\mathfrak{sl}(m|n))$.

7.1 Algebras $Y_\hbar^>(\mathfrak{sl}(m|n))$ and $\mathbf{Y}_\hbar^>(\mathfrak{sl}(m|n))$

Let $I = \{1,\ldots,m+n-1\}$, $(\bar{c}_{ij})_{i,j\in I}$ be defined as in Sect. 5.1, and $\hbar$ be a formal variable. Following [19,32] (see Remark 7.4 for a correction of the defining relations in [32, Definition 2]), define $Y_\hbar^>(\mathfrak{sl}(m|n))$ to be the associative $\mathbb{C}[\hbar]$-superalgebra generated by $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{N}}$ with the $\mathbb{Z}_2$-grading $|e_{m,r}| = \bar{1}$, $|e_{i,r}| = \bar{0}$ ($i \neq m, r \in \mathbb{N}$), and subject to the following defining relations:

$$[e_{i,r+1}, e_{j,s}] - [e_{i,r}, e_{j,s+1}] = \frac{\bar{c}_{ij}\hbar}{2}\left(e_{i,r}e_{j,s} + e_{j,s}e_{i,r}\right) \text{ if } \bar{c}_{ij} \neq 0, \tag{7.1}$$

$$\begin{aligned} &[e_{i,r}, e_{j,s}] = 0 \text{ if } \bar{c}_{ij} = 0,\\ &[e_{i,r_1}, [e_{i,r_2}, e_{j,s}]] + [e_{i,r_2}, [e_{i,r_1}, e_{j,s}]] = 0 \text{ if } \bar{c}_{ij} = \pm 1 \text{ and } i \neq m, \end{aligned} \tag{7.2}$$

as well as quartic Serre relations:

$$[[e_{m-1,s}, e_{m,r_1}], [e_{m+1,s'}, e_{m,r_2}]] + [[e_{m-1,s}, e_{m,r_2}], [e_{m+1,s'}, e_{m,r_1}]] = 0, \tag{7.3}$$

where as before, we use the super-bracket $[a,b] = ab - (-1)^{|a||b|} \cdot ba$ for $\mathbb{Z}_2$-homogeneous a, b.

Remark 7.4 (a) The first relation of (7.2) implies the validity of the second one for $i = m = j \pm 1$, which is also listed among the defining relations of [32, Definition 2].
(b) The wrong defining relation $[[e_{m-1,s}, e_{m,r_1}], [e_{m+1,s'}, e_{m,r_2}]] = 0$ of [32, Definition 2] should be replaced with the above relation (7.3).

Let $\{\alpha_i\}_{i=1}^{m+n-1}$ be as in Sect. 5.1 and Δ^+ be as in Sect. 5.2. For $\beta \in \Delta^+$, we define its parity $|\beta| \in \mathbb{Z}_2$ via (5.5). We define the *PBWD basis elements* $e_\beta(r) \in Y_\hbar^>(\mathfrak{sl}(m|n))$ via (6.3), but with $[\cdot,\cdot]$ denoting the super-bracket.

Let $\bar{H}^+$ denote the set of all functions $h\colon \Delta^+ \times \mathbb{N} \to \mathbb{N}$ with finite support and such that $h(\beta, r) \leq 1$ if $|\beta| = \bar{1}$. The monomials

$$e_h \ := \prod_{(\beta,r)\in\Delta^+\times\mathbb{N}}^{\rightarrow} e_\beta(r)^{h(\beta,r)}\,, \qquad \forall\, h \in \bar{H}^+ \tag{7.5}$$

will be called the *ordered PBWD monomials* of $Y^>_\hbar(\mathfrak{sl}(m|n))$. Analogously to [28], we have:

Theorem 7.6 ([32]) *The elements* $\{e_h\}_{h\in\bar{H}^+}$ *form a basis of the free* $\mathbb{C}[\hbar]$*-module* $Y^>_\hbar(\mathfrak{sl}(m|n))$.

For any $(\beta, r) \in \Delta^+ \times \mathbb{Z}$, define $\widetilde{e}_\beta(r) \in Y^>_\hbar(\mathfrak{sl}(m|n))$ via

$$\widetilde{e}_\beta(r) := \hbar \cdot e_\beta(r).$$

For $h \in \bar{H}^+$, we also define $\widetilde{e}_h$ via (7.5) but using $\widetilde{e}_\beta(r)$ instead of $e_\beta(r)$. Finally, let $\mathbf{Y}^>_\hbar(\mathfrak{sl}(m|n))$ denote the $\mathbb{C}[\hbar]$-subalgebra of $Y^>_\hbar(\mathfrak{sl}(m|n))$ generated by $\{\widetilde{e}_\beta(r)\}^{r\in\mathbb{N}}_{\beta\in\Delta^+}$.

The following result is analogous to [17, Theorem A.7]:

Theorem 7.7 (a) *The subalgebra* $\mathbf{Y}^>_\hbar(\mathfrak{sl}(m|n))$ *is independent of all our choices.*

(b) *The ordered PBWD monomials* $\{\widetilde{e}_h\}_{h\in\bar{H}^+}$ *form a basis of the free* $\mathbb{C}[\hbar]$*-module* $\mathbf{Y}^>_\hbar(\mathfrak{sl}(m|n))$.

7.2 Rational shuffle algebra $W^{(m|n)}$ and its integral form $\mathfrak{W}^{(m|n)}$

Define the shuffle algebra $(\bar{W}^{(m|n)}, \star)$ analogously to the shuffle algebra $(S^{(m|n)}, \star)$ of Sect. 5.4 with the following modifications:

(1) all rational functions are defined over $\mathbb{C}[\hbar]$;

(2) the rational functions $(\zeta_{i,j}(z))_{i,j\in I} \in \mathrm{Mat}_{I\times I}(\mathbb{C}[\hbar](z))$ are chosen via

$$\zeta_{i,j}(z) = 1 + \frac{\bar{c}_{ij}\hbar}{2z};$$

(3) the shuffle product $\star$ is defined via (3.1), but with the supersymmetrization SSym in place of the symmetrization Sym and $\zeta_{i,i'}(x_{i,r} - x_{i',r'})$ in place of $\zeta_{i,i'}(x_{i,r}/x_{i',r'})$;

(4) the *pole conditions* (5.14) for $F \in \bar{W}^{(m|n)}_{\underline{k}}$ are replaced with

$$F = \frac{f(x_{1,1}, \ldots, x_{n-1,k_{n-1}})}{\prod_{i=1}^{n-2}\prod_{r\leq k_i}^{r'\leq k_{i+1}}(x_{i,r} - x_{i+1,r'})}, \tag{7.8}$$

where $f \in \mathbb{C}[\hbar][\{x_{i,r}\}^{1\leq r\leq k_i}_{i\in I}]$ is a supersymmetric polynomial, that is, symmetric in $\{x_{i,r}\}^{k_i}_{r=1}$ for every $i \neq m$ and skew-symmetric in $\{x_{m,r}\}^{k_m}_{r=1}$;

(5) the *first kind wheel conditions* (5.15) for $F \in \bar{W}^{(m|n)}_{\underline{k}}$ are replaced with

$$F(\{x_{i,r}\}) = 0 \quad \text{once} \quad x_{i,r_1} = x_{i+\epsilon,s} + \hbar/2 = x_{i,r_2} + \hbar \tag{7.9}$$

for some $\epsilon \in \{\pm 1\},\ i \neq m,\ r_1 \neq r_2,\ s$;

(6) the *second kind wheel conditions* (5.16) for $F \in \bar{W}^{(m|n)}_{\underline{k}}$ are replaced with

$$F(\{x_{i,r}\}) = 0 \quad \text{once} \quad x_{m-1,s} = x_{m,r_1} + \hbar/2 = x_{m+1,s'} = x_{m,r_2} - \hbar/2 \tag{7.10}$$

for some $r_1 \neq r_2,\ s,\ s'$.

In view of Theorem 7.6, the **rational shuffle algebra** $\left(\bar{W}^{(m|n)}, \star\right)$ is related to $Y^{>}_{\hbar}(\mathfrak{sl}(m|n))$ via the following construction (cf. Propositions 6.11, 6.16):

Proposition 7.11 *The assignment* $e_{i,r} \mapsto x^{r}_{i,1}$ $(i \in I, r \in \mathbb{N})$ *gives rise to a* $\mathbb{C}[\hbar]$-*algebra embedding* $\Psi \colon Y^{>}_{\hbar}(\mathfrak{sl}(m|n)) \hookrightarrow \bar{W}^{(m|n)}$.

To describe the images of $Y^{>}_{\hbar}(\mathfrak{sl}(m|n))$ and its subalgebra $\mathbf{Y}^{>}_{\hbar}(\mathfrak{sl}(m|n))$ under the embedding Ψ, let us introduce the specialization maps in the current setting. Given a degree vector $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+} \in \mathbb{N}^{\Delta^+}$, define $\underline{\ell} \in \mathbb{N}^I$ via $\sum_{\beta\in\Delta^+} d_\beta \beta = \sum_{i\in I} \ell_i \alpha_i$. The **specialization map**

$$\phi_{\underline{d}} \colon \bar{W}^{(m|n)}_{\underline{\ell}} \longrightarrow \mathbb{C}[\hbar][\{y_{\beta,s}\}^{1\leq s\leq d_\beta}_{\beta\in\Delta^+}]$$

is defined similar to (6.14), but with the only change that the variable $x_{i,r}$ in the sth copy of the interval $[\beta]$ is specialized to $y_{\beta,s} - \frac{i\hbar}{2}$ if $i \leq m$ and to $y_{\beta,s} + \frac{(i-2m)\hbar}{2}$ if $i > m$.

Definition 7.12 (a) $F \in \bar{W}^{(m|n)}_{\underline{k}}$ is **good** if $\phi_{\underline{d}}(F)$ is divisible by $\hbar^{\sum_{\beta\in\Delta^+} d_\beta (i(\beta)-j(\beta))}$ for any degree vector $\underline{d} = \{d_\beta\}_{\beta\in\Delta^+} \in \mathbb{N}^{\Delta^+}$ such that $\sum_{\beta\in\Delta^+} d_\beta\beta = \sum_{i\in I} k_i\alpha_i$.

(b) $F \in \bar{W}^{(m|n)}_{\underline{k}}$ is **integral** if F is divisible by $\hbar^{|\underline{k}|}$.

Remark 7.13 We note that any integral shuffle element $F \in \bar{W}^{(m|n)}_{\underline{k}}$ is obviously good.

Let $W^{(m|n)}_{\underline{k}} \subset \bar{W}^{(m|n)}_{\underline{k}}$ and $\mathfrak{W}^{(m|n)}_{\underline{k}} \subset \bar{W}^{(m|n)}_{\underline{k}}$ denote the $\mathbb{C}[\hbar]$-submodules of all good and integral elements, respectively, and set

$$W^{(m|n)} := \bigoplus_{\underline{k}\in\mathbb{N}^I} W^{(m|n)}_{\underline{k}}, \qquad \mathfrak{W}^{(m|n)} := \bigoplus_{\underline{k}\in\mathbb{N}^I} \mathfrak{W}^{(m|n)}_{\underline{k}}$$

The following are the key results of this section:

Theorem 7.14 *The* $\mathbb{C}[\hbar]$-*algebra embedding* $\Psi \colon Y^{>}_{\hbar}(\mathfrak{sl}(m|n)) \hookrightarrow \bar{W}^{(m|n)}$ *of Proposition* 7.11 *gives rise to a* $\mathbb{C}[\hbar]$-*algebra isomorphism* $\Psi \colon Y^{>}_{\hbar}(\mathfrak{sl}(m|n)) \xrightarrow{\sim} W^{(m|n)}$.

Theorem 7.15 *The* $\mathbb{C}[\hbar]$*-algebra isomorphism* $\Psi\colon Y^{>}_{\hbar}(\mathfrak{sl}(m|n)) \xrightarrow{\sim} W^{(m|n)}$ *of Theorem 7.14 gives rise to a* $\mathbb{C}[\hbar]$*-algebra isomorphism* $\Psi\colon \mathbf{Y}^{>}_{\hbar}(\mathfrak{sl}(m|n)) \xrightarrow{\sim} \mathfrak{W}^{(m|n)}$.

Both Theorems 7.14, 7.15 are proved completely analogously to Theorems 5.18, 6.20, 6.27. We refer the interested reader to [33, Proofs of Theorems 3.9, 3.30] for more details.

8 Further directions

In this section, we briefly outline some of the related results that will be addressed elsewhere.

8.1 Integral forms of Grojnowski and Chari-Pressley and their PBWD bases

We shall follow the notations of Sect. 2. For $i \in I, r \in \mathbb{Z}, k \in \mathbb{N}$, define the divided power

$$\mathsf{e}^{(k)}_{i,r} := \frac{e^k_{i,r}}{[k]_{\boldsymbol{v}}!} \tag{8.1}$$

Following [20, §7.8], define the integral form $\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ as the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated by all the divided powers $\{\mathsf{e}^{(k)}_{i,r}\}^{r\in\mathbb{Z},k\in\mathbb{N}}_{i\in I}$. The main objective of this section is to construct a family of PBWD bases for $\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ as well as to provide its shuffle realization.

Recall the PBWD basis elements $\{e_\beta(r)\}^{r\in\mathbb{Z}}_{\beta\in\Delta^+}$ of (2.12), which do depend on all the choices (1)–(3) made prior to their definition. For $\beta \in \Delta^+, r \in \mathbb{Z}, k \in \mathbb{N}$, we define the divided power

$$\mathsf{e}_\beta(r)^{(k)} := \frac{e_\beta(r)^k}{[k]_{\boldsymbol{v}}!} \tag{8.2}$$

As a simple corollary of [29, Proof of Theorem 6.6], we obtain:

Lemma 8.3 $e_\beta(r)^{(k)} \in U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ *for any* $\beta \in \Delta^+, r \in \mathbb{Z}, k \in \mathbb{N}$.

The monomials

$$\mathsf{e}_h := \prod^{\rightarrow}_{(\beta,r)\in\Delta^+\times\mathbb{Z}} \mathsf{e}_\beta(r)^{(h(\beta,r))}\ , \quad \forall\, h \in H$$

will be called the *ordered PBWD monomials* of $\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$. Here, the arrow over the product sign refers to the total order (2.11).

Our first main result of this Section establishes the PBWD property for $\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$:

Theorem 8.4 *The elements* $\{\mathsf{e}_h\}_{h\in H}$ *form a basis of the free* $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$*-module* $\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

The proof of Theorem 8.4 is completely analogous to our proofs of Theorems 2.16, 2.20 and is based on (as well as used in) the description of $\Psi(\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n))$, viewed as a subspace of $S^{(n)}$. For the latter purpose, let us adapt Definition 6.17 to the current setting:

Definition 8.5 $F\in S^{(n)}_{\underline{k}}$ is **good** if it satisfies the following two properties:

(i) $F=\frac{f(x_{1,1},\ldots,x_{n-1,k_{n-1}})}{\prod_{i=1}^{n-2}\prod_{r\le k_i}^{r'\le k_{i+1}}(x_{i,r}-x_{i+1,r'})}$ with $f\in\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}][\{x_{i,r}^{\pm1}\}_{i\in I}^{1\le r\le k_i}]^{\Sigma_{\underline{k}}}$;

(ii) the specialization $\phi_{\underline{d}}(F)$ of (3.7) is divisible by $(\boldsymbol{v}-\boldsymbol{v}^{-1})^{\sum_{\beta\in\Delta^+}d_\beta(i(\beta)-j(\beta))}$ for any degree vector $\underline{d}=\{d_\beta\}_{\beta\in\Delta^+}\in\mathbb{N}^{\Delta^+}$ such that $\sum_{\beta\in\Delta^+}d_\beta\beta=\sum_{i\in I}k_i\alpha_i$.

Let $\mathsf{S}^{(n)}_{\underline{k}}\subset S^{(n)}_{\underline{k}}$ denote the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-submodule of all good elements and set

$$\mathsf{S}^{(n)}:=\bigoplus_{\underline{k}\in\mathbb{N}^I}\mathsf{S}^{(n)}_{\underline{k}}$$

Lemma 8.6 $\Psi(\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n))\subseteq\mathsf{S}^{(n)}$.

Proof As $\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is generated by $\mathsf{e}^{(k)}_{i,r}$ over $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$, it suffices to verify both properties (i, ii) of Definition 8.5 for any shuffle element $F=\Psi(\mathsf{e}^{(k_1)}_{i_1,r_1}\cdots\mathsf{e}^{(k_N)}_{i_N,r_N})$. The validity of (i) for F follows from the equality $\Psi(\mathsf{e}^{(k)}_{i,r})=\boldsymbol{v}^{-\frac{k(k-1)}{2}}(x_1\cdots x_k)^r$, due to Lemma 3.9. On the other hand, the validity of (ii) for F is established using the arguments from our proof of Lemma 6.19. □

The second key result of this section provides a shuffle realization of $\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$:

Theorem 8.7 *The* $\mathbb{C}(\boldsymbol{v})$*-algebra isomorphism* $\Psi\colon U^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)\xrightarrow{\sim}S^{(n)}$ *of Theorem 3.5 gives rise to a* $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$*-algebra isomorphism* $\Psi\colon \mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)\xrightarrow{\sim}\mathsf{S}^{(n)}$.

Define $\mathsf{U}^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ as the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $U^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated by the divided powers $\mathsf{f}^{(k)}_{i,r}:=f^k_{i,r}/[k]_{\boldsymbol{v}}!$ ($i\in I, r\in\mathbb{Z}, k\in\mathbb{N}$). Finally, consider the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra $\mathsf{U}^{0}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ of $U^{0}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ introduced in [5, §3], cf. [34, §2.3]. Following [5], define the integral form $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ as the $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$-subalgebra of $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ generated by $\mathsf{U}^{<}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, $\mathsf{U}^{0}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, $\mathsf{U}^{>}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

Remark 8.8 Identifying $U_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ with the Drinfeld–Jimbo quantum loop algebra $U^{\mathrm{DJ}}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, see [7], the form $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is identified with the Lusztig form of $U^{\mathrm{DJ}}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$, due to [5].

The following triangular decomposition of $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ is due to [5, Proposition 6.1]:

Theorem 8.9 ([5]) *The multiplication map*

$$m\colon \mathsf{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathsf{U}^0_{\boldsymbol{v}}(L\mathfrak{sl}_n) \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathsf{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n) \longrightarrow \mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$$

is an isomorphism of the free $\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]$*-modules.*

Combining Theorems 8.4 and 8.9, we obtain a family of PBWD bases for the form $\mathsf{U}_{\boldsymbol{v}}(L\mathfrak{sl}_n)$.

Remark 8.10 The results of Theorems 2.20, 3.40, 8.4, 8.7 were recently used in [34] to establish the duality between the forms $\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $\mathsf{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ (resp. $\mathfrak{U}^<_{\boldsymbol{v}}(L\mathfrak{sl}_n)$ and $\mathsf{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}_n)$) with respect to the new Drinfeld pairing. We refer the interested reader to [34] for more details.

8.2 Generalizations to all Dynkin diagrams associated with $\mathfrak{sl}(m|n)$

Recall that a novel feature of Lie superalgebras (in contrast to Lie algebras) is that they admit several non-isomorphic Dynkin diagrams. Likewise, one may consider various quantizations of universal enveloping superalgebras starting from different Dynkin diagrams. The explicit isomorphism of such algebras associated to various Dynkin diagrams of the same type is highly non-trivial: it has been established for quantum finite/affine superalgebras in [36], but seems to be an open question for general super Yangians. Furthermore, the *positive subalgebras* (those generated by $\{e_{i,r}\}$) do essentially depend on the choice of a Dynkin diagram.

In the recent paper [33], we address the above question for the Lie superalgebra $A(m,n)$ as well as generalize the results of Sects. 5, 7 to all of its Dynkin diagrams. Explicitly, given a superspace $V = V_{\bar 0} \oplus V_{\bar 1}$ with a basis $\mathsf{v}_1,\ldots,\mathsf{v}_n$ such that each v_i is either *even* ($\mathsf{v}_i \in V_{\bar 0}$) or *odd* ($\mathsf{v}_i \in V_{\bar 1}$), one may define the quantum loop superalgebras $U_{\boldsymbol{v}}(L\mathfrak{gl}(V))$, $U_{\boldsymbol{v}}(L\mathfrak{sl}(V))$ as well as the super Yangians $Y_\hbar(\mathfrak{gl}(V))$, $Y_\hbar(\mathfrak{sl}(V))$, both in the RTT presentation of [15] and the new Drinfeld presentation of [7] (their equivalence is established following the ideas of [9]). The corresponding *positive subalgebras* $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(V)) \simeq U^>_{\boldsymbol{v}}(L\mathfrak{gl}(V))$ (resp. $Y^>_\hbar(\mathfrak{sl}(V)) \simeq Y^>_\hbar(\mathfrak{gl}(V))$) are generated by $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ (resp. $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{N}}$) subject to the defining relations [33, (4.2–4.5)] (resp. [33, (2.58–2.60, 2.78)]) and with the $\mathbb{Z}_2$-grading $|e_{i,r}| = |\alpha_i|$, where

$$|\alpha_i| = \begin{cases} \bar 0, & \text{if } \mathsf{v}_i \text{ and } \mathsf{v}_{i+1} \text{ have the same parity} \\ \bar 1, & \text{otherwise} \end{cases}.$$

The construction of the PBWD bases for $U^>_{\boldsymbol{v}}(L\mathfrak{sl}(V))$, $Y^>_\hbar(\mathfrak{sl}(V))$ and their integral forms $\mathfrak{U}^>_{\boldsymbol{v}}(L\mathfrak{sl}(V))$, $\mathbf{Y}^>_\hbar(\mathfrak{sl}(V))$ is similar to Theorems 5.7, 5.11, 7.6, 7.7. The corresponding *ordered PBWD monomials* are defined analogously to (5.6, 7.5) with the indexing sets $\bar H$, $\bar H^+$ defined as before, but using a different $\mathbb{Z}_2$-grading on Δ^+:

$$|\alpha_j + \alpha_{j+1} + \cdots + \alpha_i| = |\alpha_j| + |\alpha_{j+1}| + \cdots + |\alpha_i|.$$

The associated shuffle algebras $S^{(V)}$, $W^{(V)}$ and their integral forms $\mathfrak{S}^{(V)}$, $\mathfrak{W}^{(V)}$ are defined similar to $S^{(m|n)}$, $W^{(m|n)}$, $\mathfrak{S}^{(m|n)}$, $\mathfrak{W}^{(m|n)}$. Their elements are supersymmetric rational functions in $\{x_{*,*}\}$, that is, symmetric in $\{x_{i,*}\}$ if $|\alpha_i| = \bar{0}$ and skew-symmetric in $\{x_{i,*}\}$ if $|\alpha_i| = \bar{1}$.

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