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# Lax matrices from antidominantly shifted Yangians and quantum affine algebras: A-type

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We construct a family of $GL_n$ rational and trigonometric Lax matrices $T_D(z)$ parametrized by $\Lambda^+$-valued divisors $D$ on $\mathbb{P}^1$. To this end, we study the shifted Drinfeld Yangians $Y_\mu(\mathfrak{gl}_n)$ and quantum affine algebras $U_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$, which slightly generalize their $\mathfrak{sl}_n$-counterparts of [3,18]. Our key observation is that both algebras admit the RTT type realization when $\mu$ (resp. $\mu^+$ and $\mu^-$) are antidominant coweights. We prove that $T_D(z)$ are polynomial in $z$ (up to a rational factor) and obtain explicit simple formulas for those linear in $z$. This generalizes the recent construction by the first two authors of linear rational Lax matrices [15] in both trigonometric and higher $z$-degree directions. Furthermore, we show that all $T_D(z)$ are *normalized limits* of those parametrized by $D$ supported away from $\{\infty\}$ (in the rational case) or $\{0,\infty\}$ (in the trigonometric case). The RTT approach provides conceptual and elementary proofs for the construction of the coproduct homomorphisms on shifted Yangians and quantum affine algebras of $\mathfrak{sl}_n$, previously established in [14,18] via rather tedious computations. Finally, we establish a close relation between a certain collection of explicit linear Lax matrices and the well-known parabolic Gelfand-Tsetlin formulas.



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**Contents**



# 1. Introduction

## 1.1. Summary

Let $G$ be a complex reductive group and let $(C, dz)$ be a complex projective line $\mathbb{P}^1$ with a marked point $z = \infty$, also equipped with a section $dz$ of the canonical line bundle $\mathcal{K}_C$ whose only singularity is a second order pole at $z = \infty$. Let $\langle \cdot, \cdot \rangle$ be the Killing form on the Lie algebra $\mathfrak{g}$ of $G$.

To the data $(G, C, \langle \cdot, \cdot \rangle, dz)$ one can associate in the standard way an (infinite-dimensional) Poisson-Lie group $G_1(C)$ of $G$-valued rational functions on $C$ with fixed value 1 at $\infty$. By the formal series expansion at $z = \infty$ there is a natural inclusion $G_1(C) \hookrightarrow G_1[[z^{-1}]]$, where $G_1[[z^{-1}]]$ are $G$-valued power series in $z^{-1}$ with the constant term 1. The group $G_1[[z^{-1}]]$ is the Poisson-Lie group whose Poisson structure is constructed in the standard way from the Lie bialgebra defined by the Manin triple $(\mathfrak{g}((z^{-1})), \mathfrak{g}[z], z^{-1}\mathfrak{g}[[z^{-1}]])$ and the residue pairing $\oint_\infty \langle \cdot, \cdot \rangle dz$. The quantization of the Poisson-Lie group $G_1[[z^{-1}]]$ produces the Hopf algebra called the Drinfeld Yangian $Y(\mathfrak{g})$.

Let $\Lambda^+$ be a cone of dominant coweights in the coweight lattice $\Lambda$ of $G$. A formal linear combination of points of $C$ with coefficients in $\Lambda^+$ will be called a $\Lambda^+$-valued divisor $D$ on $C$.

The symplectic leaves $\mathfrak{M}_D$ in the Poisson-Lie group $G_1(C)$ are classified by $\Lambda^+$-valued divisors $D = \sum_{x\in\mathbb{P}^1} \lambda_x[x]$ trivial at infinity [36,12], i.e. with $\lambda_\infty = 0$. Namely, for a given $D$, the symplectic leaf $\mathfrak{M}_D \subset G_1(C)$ consists of those elements in $G_1(C)$ that are regular away from $\mathrm{supp}(D)$, the support of $D$, while having a singularity of the form $G[[z_x]]z_x^{-\lambda_x}G[[z_x]]$ in a neighborhood of each $x \in \mathrm{supp}(D)$, where $z_x$ is a local coordinate near $x$ vanishing at $x$ and $\lambda_x \in \Lambda^+$ is the coefficient of $D$ at $x$.

The symplectic leaves $\mathfrak{M}_D$ of $G_1(C)$ are interesting in many aspects. A symplectic leaf $\mathfrak{M}_D$ can be identified with (I):

(1) a moduli space of $G$-multiplicative Higgs fields trivially framed at $z = \infty$ [12]
(2) a moduli space of $G_c$-monopoles on $C \times S^1$ regular at infinity and with Dirac singularities whose projection on $C$ is encoded by the $\Lambda^+$-valued divisor $D$, where $G_c$ is the compact group associated to the complex reductive group $G$ [7,6]
(3) a Coulomb branch of $\mathcal{N} = 2$ (ultraviolet fixed point) UV conformal quiver gauge theory on $\mathbb{R}^3 \times S^1$ if $G$ is of ADE type and the ADE quiver is the Dynkin diagram of $\mathfrak{g}$ [7]
(4) a phase space of an algebraic integrable system known in the quantum field theory literature as the Seiberg-Witten integrable system of $\mathcal{N} = 2$ ADE UV conformal quiver gauge theory [32]
(5) a classical limit of the GKLO-modules of $Y(\mathfrak{g})$ constructed by Gerasimov, Kharchev, Lebedev and Oblezin [21]

Let $\mu \equiv \lambda_\infty \equiv D|_\infty$ denote the coefficient of the divisor $D$ at infinity. In the constructions of the above list it was assumed that $\mu$ vanishes. In the constructions (1) and (2), the restriction $\mu = 0$ translates to the regularity either of the Higgs field at $\infty \in \mathbb{P}^1$ or to the regularity of the monopole configuration on the infinity of $\mathbb{R}^2 \times S^1$. In the points (3) and (4), for $G$ of a simple ADE type, $\mu$ encodes the UV $\beta$-function of an $\mathcal{N} = 2$ supersymmetric quiver gauge theory, and consequently, the restriction $\mu = 0$ translates to the condition that the UV $\beta$-function of the quiver theory vanishes (cf. [32]).

It is natural to explore what happens with the constructions listed above when the restriction $\mu = 0$ is lifted. The natural generalizations for not necessarily vanishing $\mu$ are (II):

(1) a moduli space of $G$-multiplicative Higgs fields with the *framed singularity* $z^\mu$ *at* $z = \infty$ *of the coweight* $\mu$
(2) a moduli space of $G_c$-monopoles on $C \times S^1$ with *a charge* $\mu$ *at infinity* and with Dirac singularities whose projection on $C$ is encoded by the $\Lambda^+$-valued divisor $D$
(3) a Coulomb branch of $\mathcal{N} = 2$ UV quiver gauge theory on $\mathbb{R}^3 \times S^1$ if $G$ is of ADE type and the ADE quiver is the Dynkin diagram of $\mathfrak{g}$ [32] with the *UV* $\beta$*-function* $-\mu$

(4) a phase space of the Seiberg-Witten algebraic integrable system of $\mathcal{N}=2$ supersymmetric ADE quiver gauge theory with the *UV $\beta$-function* $-\mu$
(5) a classical limit of the analogues of the GKLO-modules [21] but for a *shifted Yangian* $Y_{-\mu}(\mathfrak{g})$ [27,3]

In this paper, we put further details on the construction (5) focusing on $G = GL_n$ and *antidominantly shifted* Yangians, which in our notations are recorded as $Y_{-\mu}(\mathfrak{gl}_n)$ with $\mu \in \Lambda^+$. A generalization to other classical BCD types has been carried out in the follow-up paper [20].

From the perspective of Coulomb branches of the $\mathcal{N}=2$ supersymmetric ADE quiver gauge theories I (3) and II (3) there is a natural procedure to obtain the asymptotically free ADE quiver gauge theory with the non-zero UV $\beta$-function $-\mu$ with $\mu \in \Lambda^+$ from a UV conformal ADE quiver gauge theory with the vanishing UV $\beta$-function $\mu = 0$. This procedure involves:

(I) starting from the UV conformal ADE quiver gauge theory, with $\beta$-function given by $-\sum \mathbf{v}_i \alpha_i^\vee + \sum \mathbf{w}_i \omega_i^\vee = 0$ where $U(\mathbf{v_i})$ is the gauge group factor of the ADE quiver theory attached to the node $i$, the $\mathbf{w}_i$ is the number of fundamental multiplets attached to the node $i$, and their masses are $x_{i,1}, \dots x_{i,\mathbf{w_i}}$;
(II) and then switching off some of those fundamental multiplet fields from the Lagrangian. The switching off effect of a quantum field in the QFT can be achieved by sending the mass prescribed to that field in the perturbative Lagrangian to the infinity: in this way the quantum excitation of that field requires infinite energy, and therefore the correlation functions of a QFT in which some quantum fields are ascribed infinite masses are equivalent (after renormalization) to the correlation functions of the QFT where those fields have been deleted from the Lagrangian.

Therefore we can expect to recover the Coulomb branches and integrable systems associated to $\mathcal{N}=2$ supersymmetric asymptotically free ADE quiver gauge theories by taking a limit of a suitable UV conformal theory where some of the masses $x$ (corresponding to the points of the divisor in our geometrical presentation) are sent to infinity, see [32,41]. Indeed, we show explicitly in Section 2.4.2 that our construction satisfies this "normalized limit" property, expected from the physics of $\mathcal{N}=2$ ADE quiver gauge theories as described above.

Generalizing [8,4], we present the isomorphism between the Drinfeld and RTT realizations of $Y_{-\mu}(\mathfrak{gl}_n)$ and both as a consequence and a tool to prove this isomorphism we construct $GL_n$ Lax matrices $T_D(z)$ with prescribed singularities at $D$ for any $\Lambda^+$-valued divisor $D$ (with an additional property that the sum of the coefficients $\sum_{x\in\mathbb{P}^1} \lambda_x$ is in the coroot lattice of $G$).

While in the paper we implicitly assume $\hbar = 1$ (for simplicity of our exposition) and explicitly present only the quantum case, our construction can be naturally generalized to the $\mathbb{C}[\hbar]$-setup: both (antidominantly) shifted Drinfeld and shifted RTT Yangians of $\mathfrak{gl}_n$ become associative algebras over $\mathbb{C}[\hbar]$, $\hbar$ appears in the commutation relations between the canonical coordinates on $\mathfrak{M}_D$ as $[p_{i,r}, e^{q_{j,s}}] = \delta_{i,j}\delta_{r,s}\hbar e^{q_{j,s}}$, and the rational Lax matrices $T_D(z)$ obviously generalize to keep track of $\hbar$. Then, the classical limit is recovered in the usual way by sending $\hbar \to 0$ and replacing $\frac{1}{\hbar}[\cdot,\cdot]$ by the Poisson bracket $\{\cdot,\cdot\}$.

We conjecture that the classical limit of our construction describes the full family of symplectic leaves in the Poisson-Lie group obtained as the classical limit of the shifted Yangian $Y_{-\mu}(\mathfrak{g})$, and for each $\Lambda^+$-valued divisor $D$ on $C$ we obtain Darboux coordinates on the symplectic leaf $\mathfrak{M}_D$. We leave out for a future work the precise details as well as the details of the construction of the moduli space of multiplicative Higgs fields with a singularity at the framing point and moduli space of singular monopoles on $\mathbb{R}^2 \times S^1$ (cf. [13,28] for the relevant constructions of singular monopoles and Kobayashi-Hitchin correspondence in that context).

The Lax matrices $T_D(z)$ can be used to construct explicitly classical commuting Hamiltonians of the corresponding completely integrable systems on $\mathfrak{M}_D$ as well as their quantizations. The classical commuting Hamiltonians are obtained as the coefficients of the spectral curve

$$\det\Big(y - g_\infty T_D(z)\Big) = \sum_{i=0}^{n} y^{n-i}(-1)^i \mathrm{tr}_{\Lambda^i}\Big(g_\infty T_D(z)\Big). \tag{1.1}$$

Here, $g_\infty$ is a regular semi-simple element of $G$ that defines the coupling constants of the respective integrable system or encodes the gauge couplings of the respective quiver gauge theory in case when $\mathfrak{M}_D$ is interpreted as a Coulomb branch [32]. For a general $G$, the classical complete integrability can be established from the abstract cameral curve construction following [10].

In the quantum case, using that the homomorphism $\Psi_D$ of Theorem 2.35 factors through the quantized Coulomb branch, see [3, Theorem B.18], the construction of Bethe subalgebras (see [29, §1.14] or the original paper [31]) that uses a quantum version of the spectral curve gives rise to a family of *Bethe commutative subalgebras* in the quantized Coulomb branches. We note that existence of such a construction was suggested to one of the authors and Michael Finkelberg by Boris Feigin in 2017. The pre-quantized Hamiltonians are represented in the algebra of difference operators with rational coefficients on functions of $p_{*,*}$. We do not discuss in this paper the actual quantization (the choice of a polarization, the Hilbert space structure, or analytic properties of the wave-functions).

For example, the $i = n$ term in the spectral curve (1.1), the det of the Lax matrix, after a quantization is replaced by the quantum determinant and is given by the formula (2.39):

$$\mathrm{qdet}\ T_D(z) = \prod_{i=1}^{n} \prod_{x \in \mathbb{P}^1 \backslash \{\infty\}} (z - x + (i-1)\hbar)^{-\epsilon_i^\vee(\lambda_x)}.$$

The Bethe ansatz for these quantum integrable systems was constructed in [33].

The origin of the canonical coordinates $(p_*, q_*)$ of the present work goes back to the work of Atiyah-Hitchin on the moduli space of monopoles on $\mathbb{R}^3$, [1], that identified such moduli space with the moduli space of based rational maps from $C = \mathbb{P}^1$ to the flag variety $G/B$.

For example, for $G = SL_2$ the flag variety $G/B$ is $\mathbb{P}^1$, and the based rational maps from $C$ to $G/B$ are simply rational functions $f(z)$ vanishing at $z = \infty$. Given a coset representative of a based rational map from $C$ to $G/B$ in the form $\begin{pmatrix} \mathsf{A}(z) & \mathsf{B}(z) \\ \mathsf{C}(z) & \mathsf{D}(z) \end{pmatrix}$, the respective rational function is $f(z) = \mathsf{B}(z)/\mathsf{A}(z)$. For the divisor $D$ consisting only of a singularity at $\infty \in \mathbb{P}^1$, the coordinates $p_*$ are the locations of zeros of $\mathsf{A}(z)$ (i.e. poles of $f(z)$), while the coordinates $e^{q_*}$ are the values of $\mathsf{B}(z)$ at these zeros. Such canonical coordinates in the space of rational functions also appeared in the work of Sklyanin on separation of variables. Furthermore, Jarvis in his work on monopoles on $\mathbb{R}^3$, [25,26], constructed a lift of a based rational map from $C$ to $G/B$ to a rational map from $C$ to $G$. The classical limit of the formulas for the rational Lax matrices $T_D(z)$ presented in this work for $G = GL_n$ could be seen as a canonical realization of Jarvis's lift of a based rational map from $C$ to $G/B$ to a rational map from $C$ to $G$, equipped with canonical $(p_*, q_*)$-coordinates induced from the Atiyah-Hitchin construction for the based rational maps to $G/B$. We provide some more details in Remark 2.98, while referring the interested reader to [3, 2(xi, xii, xiii)] for a more detailed discussion.

In the second part of the paper we proceed to the trigonometric case by taking $C = \mathbb{P}^1 = \mathbb{C}^\times \cup \{0\} \cup \{\infty\}$ equipped with a section $dz/z$ of the canonical bundle $\mathcal{K}_C$ that has order one poles at 0 and $\infty$. Given the Borel decomposition of $\mathfrak{g}$, the section of $\mathcal{K}_C$, and the Killing form on $\mathfrak{g}$, one obtains in the usual way the Lie bialgebra structure on the loop algebra $L\mathfrak{g}$ with the trigonometric $r$-matrix and the corresponding Poisson-Lie loop group. The quantization of this Poisson-Lie group gives rise to the quantum loop algebra $U_{\boldsymbol{v}}(L\mathfrak{g})$ (also known as the quantum affine algebra with the trivial central charge).

Similar to the rational case, to each $\Lambda^+$-valued divisor $D$ on $C$ we associate a module of a shifted counterpart of $U_{\boldsymbol{v}}(L\mathfrak{g})$ in a construction analogous to [21,22]. However, in the trigonometric case there are two special framing points 0 and $\infty$ on $C$. We denote the coefficients of $D$ at these framing points by $\mu^- \equiv \lambda_0 = D|_0$ and $\mu^+ \equiv \lambda_\infty = D|_\infty$, respectively. Then, for any $\Lambda^+$-valued divisor $D$ on $C$ (with an additional property that the sum of the coefficients $\sum_{x \in \mathbb{P}^1} \lambda_x$ lies in the coroot lattice of $G$), we construct a homomorphism from the shifted quantum affine algebra $U_{-\mu^+,-\mu^-}(L\mathfrak{g})$ to the algebra of $\boldsymbol{v}$-difference operators (see Remark 3.31 and [18]), and using an isomorphism between the Drinfeld and the RTT realizations of $U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n), \mu^\pm \in \Lambda^+$, we construct and present explicitly the corresponding $GL_n$ trigonometric Lax matrices $T_D(z)$.

Conjecturally, the classical limit of our construction describes the full family of symplectic leaves in the $(-\mu^+, -\mu^-)$-shifted Poisson-Lie loop group obtained as the classical limit of the shifted quantum affine algebra $U_{-\mu^+,-\mu^-}(L\mathfrak{g})$, where $(\mu^+, \mu^-)$ are the coweights encoding the prescribed singularities at $\infty$ and 0. Conjecturally, each symplectic leaf $\mathfrak{M}_D$ is isomorphic as a symplectic variety to the moduli space of multiplicative Higgs bundles on $(\mathbb{P}^1, dz/z)$ with Borel framing at 0 and $\infty$ and with prescribed singularities on $D$. We leave out the precise definitions and details of this construction for a future work.

A subset of $GL_n$ rational Lax matrices constructed in [15] are known to be the building blocks for the transfer matrices of non-compact spin chains and Baxter $Q$-operators, see [2,11] (cf. [37] for a discussion of the trigonometric case). The matrix elements of those Lax matrices are realized as polynomials in the Heisenberg algebra generators in analogy to the free field realization. The Fock vacuum vector serves as the highest weight state and the trace in the transfer matrix construction is taken over the entire Fock space. As discussed in Section 2.7, the realization studied in this paper is closely related to the Gelfand-Tsetlin bases which are not necessarily constrained to representations of the highest/lowest weight type. In order to describe the modules that arise from the free field realization one has to impose additional conditions on the corresponding Gelfand-Tsetlin patterns. Consequently, we expect that the transfer matrices can be defined in terms of the Lax matrices presented in this article by introducing the appropriate trace over the Gelfand-Tsetlin oscillator realization. In addition to the construction of transfer matrices from Lax matrices linear in the spectral parameter, this approach should allow for the construction of the commuting family of operators with Lax matrices of higher degree in the spectral parameter. We leave the precise details of this construction as well as generalizations to Lie algebras beyond $A$-type for a future work.

Historically, the shifted Yangians $Y_\nu(\mathfrak{g})$ were first introduced for $\mathfrak{g} = \mathfrak{gl}_n$ and dominant shifts $\nu$ in [5], where their certain quotients were identified with type $A$ finite $W$-algebras, the latter being natural quantizations of type $A$ Slodowy slices. This construction was further generalized to any semisimple $\mathfrak{g}$ but still dominant $\nu \in \Lambda^+$ in [27], where it was shown that their GKLO-type quotients (called *truncated shifted Yangians*) quantize slices in the affine Grassmannians. The generalization to arbitrary shifts $\nu \in \Lambda$ was finally carried out in [3, Appendix B], where it was conjectured that their truncations quantize *generalized slices in the affine Grassmannians* introduced in [3]. The latter result was recently established in [39].

In contrast to the aforementioned original approach, we consider exactly the opposite case, with antidominant shifts, in the current paper (note that any shifted Yangian $Y_\nu(\mathfrak{g})$ may be embedded into the antidominantly shifted one $Y_{-\mu}(\mathfrak{g}),\ \mu \in \Lambda^+$, via the *shift homomorphisms* of [14]). The main technical benefit is the RTT realization of those $Y_{-\mu}(\mathfrak{gl}_n)$ (respectively $U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$), and as a result a conceptual explanation of the coproduct homomorphisms of [14] (respectively of [18]). Also, we note that the antidominant case allows to access interesting algebraic integrable systems that appear on the Coulomb branches of four-dimensional supersymmetric $\mathcal{N} = 2$ ADE quiver gauge theories of the asymptotically free type [32]; a typical representative of such an integrable system is a closed Toda chain.

*1.2. Outline of the paper*

• In Section 2.1, we introduce the *shifted Drinfeld Yangians of* $\mathfrak{gl}_n$, the algebras $Y_\mu(\mathfrak{gl}_n)$, where $\mu \in \Lambda$ is a coweight of $\mathfrak{gl}_n$. These algebras depend only on the associated coweight $\bar{\mu} \in \bar{\Lambda}$ of $\mathfrak{sl}_n$, up to an isomorphism, see Lemma 2.17. They also contain the shifted Yangians of $\mathfrak{sl}_n$ (introduced in [3]) via the natural embedding $\iota_\mu \colon Y_{\bar{\mu}}(\mathfrak{sl}_n) \hookrightarrow Y_\mu(\mathfrak{gl}_n)$ of Proposition 2.19 (generalizing the classical embedding $Y(\mathfrak{sl}_n) \hookrightarrow Y(\mathfrak{gl}_n)$). Moreover, we have the isomorphism $Y_\mu(\mathfrak{gl}_n) \simeq ZY_\mu(\mathfrak{gl}_n) \otimes_{\mathbb{C}} Y_{\bar{\mu}}(\mathfrak{sl}_n)$ with $ZY_\mu(\mathfrak{gl}_n)$ denoting the center of $Y_\mu(\mathfrak{gl}_n)$, see Corollary 2.24, Lemma 2.26 (generalizing [29, Theorem 1.8.2] in the unshifted case $\mu = 0$).

In Section 2.2, we introduce the key notion of $\Lambda$*-valued divisors on* $\mathbb{P}^1$*,* $\Lambda^+$*-valued outside* $\{\infty\} \in \mathbb{P}^1$, see (2.28), (2.29). For each such divisor $D$ satisfying an auxiliary condition (2.30) (which encodes that the sum of all the coefficients of the divisor $D$ lies in the coroot lattice), we construct in Theorem 2.35 an algebra homomorphism $\Psi_D \colon Y_{-\mu}(\mathfrak{gl}_n) \to \mathcal{A}$, where $\mu = D|_\infty$ is the coefficient of $D$ at $\infty$ and the target $\mathcal{A}$ is the algebra of difference operators (2.32), see Remark 2.33. This construction generalizes the $A_{n-1}$-case of [3, Theorem B.15] as the composition $\Psi_D \circ \iota_{-\mu} \colon Y_{-\bar{\mu}}(\mathfrak{sl}_n) \to \mathcal{A}$ is precisely the homomorphism $\Phi^{\bar{\lambda}}_{-\bar{\mu}}$ of [3, Theorem B.15] (where $\lambda$ is the sum of all coefficients of $D$ outside $\infty$).

In Section 2.3, we introduce the *(antidominantly) shifted RTT Yangians of* $\mathfrak{gl}_n$, the algebras $Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$ with $\mu \in \Lambda^+$ being a *dominant coweight of* $\mathfrak{gl}_n$. They are defined via the RTT relation (2.41) and the Gauss decomposition (2.43), (2.44). We construct the epimorphisms $\Upsilon_{-\mu} \colon Y_{-\mu}(\mathfrak{gl}_n) \twoheadrightarrow Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$ for any $\mu \in \Lambda^+$, see Theorem 2.52. The main result of this section (the proof of which is established in Section 2.4.3), Theorem 2.54, is that $\Upsilon_{-\mu}$ are actually algebra isomorphisms for any $\mu \in \Lambda^+$ (generalizing [8,4] in the unshifted case $\mu = 0$ as well as [18] in the smallest rank case $n = 2$, see Remark 2.55).

In Section 2.4, we construct $n \times n$ *rational Lax* matrices $T_D(z)$ (with coefficients in $\mathcal{A}((z^{-1}))$) for each $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying (2.30). They are explicitly defined via (2.63), (2.64) combined with (2.58), (2.60), (2.62), while arising naturally as the image of the $n \times n$ matrix $T(z)$ (encoding all the generators of $Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$) under the composition $\Psi_D \circ \Upsilon^{-1}_{-\mu} \colon Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n) \to \mathcal{A}$, assuming Theorem 2.54 has been established, see (2.56), (2.57). As Theorem 2.54 is well-known for $\mu = 0$ and any Lax matrix $T_D(z)$ is a *normalized limit* of $T_{\bar{D}}(z)$ with $\bar{D}|_\infty = 0$, see Proposition 2.75 and Corollary 2.78, we immediately derive the RTT relation (2.41) for all matrices $T_D(z)$, see Proposition 2.79 (hence, the terminology "rational Lax matrices"). Combining the latter with the key result of [40], see Theorem 2.80, we finally prove Theorem 2.54 in Section 2.4.3. We note that similar arguments may be used to prove the triviality of the centers of shifted Yangians $Y_\nu(\mathfrak{g})$ for any coweight of a semisimple Lie algebra $\mathfrak{g}$, see Remark 2.81. The key property of the rational Lax matrices $T_D(z)$ is their regularity (up to a rational factor (2.66)), see Theorem 2.67 (the proof of which is based on a certain cancelation of poles reminiscent to the one appearing in the work on $q$-characters [16] and $qq$-characters [30], see Remark 2.72). Finally, we derive simplified explicit formulas for all rational Lax matrices $T_D(z)$ which are linear in $z$, see Theorem 2.90. In the smallest rank $n = 2$ case, those recover the well-known $2 \times 2$ elementary Lax matrices for the Toda chain, the DST chain, and the Heisenberg magnet, see Remark 2.96. We conclude Section 2.4 with Remark 2.98, which is three-fold: comparing the complete monodromy matrix (2.99) of the Toda chain for $GL_N$ to the degree $N$ rational $2\times 2$ Lax matrix $T_D(z)$ with $D = N\alpha[\infty]$, identifying the phase spaces of the corresponding classical integrable systems with the $SU(2)$-*monopoles of topological charge* $N$, and generalizing the latter to $SU(2)$-*monopoles of topological charge* $N$ *with singularities*, thus providing more details to our discussion of Section 1.1.

In Section 2.5, we evaluate explicitly some linear (in $z$) rational Lax matrices $T_D(z)$ and compare them to the linear rational Lax matrices constructed by the first two authors in [15] (actually, we treat all the explicit "building blocks" of [15], the fusion of which provides the entire family of the rational Lax matrices $L_{\boldsymbol{\lambda},\underline{x},\boldsymbol{\mu}}(z)$ of [15]).

In Section 2.6, we construct coproduct homomorphisms on antidominantly shifted Yangians. We start by constructing homomorphisms $\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}\colon Y^{\mathrm{rtt}}_{-\mu_1-\mu_2}(\mathfrak{gl}_n) \to Y^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{gl}_n) \otimes Y^{\mathrm{rtt}}_{-\mu_2}(\mathfrak{gl}_n)$ defined via $\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}(T(z)) = T(z) \otimes T(z)$ for any $\mu_1, \mu_2 \in \Lambda^+$, see Proposition 2.136. Evoking the key isomorphism $Y_{-\mu}(\mathfrak{gl}_n) \simeq Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$ of Theorem 2.54, this naturally gives rise to homomorphisms $\Delta_{-\mu_1,-\mu_2}\colon Y_{-\mu_1-\mu_2}(\mathfrak{gl}_n) \to Y_{-\mu_1}(\mathfrak{gl}_n) \otimes Y_{-\mu_2}(\mathfrak{gl}_n)$, and we compute the images of the generators in Proposition 2.143. The latter, in turn, gives rise to homomorphisms $\Delta_{-\nu_1,-\nu_2}\colon Y_{-\nu_1-\nu_2}(\mathfrak{sl}_n) \to Y_{-\nu_1}(\mathfrak{sl}_n) \otimes Y_{-\nu_2}(\mathfrak{sl}_n)$ for any dominant $\mathfrak{sl}_n$-coweights $\nu_1, \nu_2 \in \bar{\Lambda}^+$, see Proposition 2.146, thus providing a conceptual and elementary proof of $A_{n-1}$-case of [14, Theorem 4.8]. Finally, we note that $\Delta_{\nu_1,\nu_2}$ with $\nu_1, \nu_2 \in -\Lambda^+$ actually give rise to homomorphisms $\Delta_{\nu_1,\nu_2}\colon Y_{\nu_1+\nu_2}(\mathfrak{sl}_n) \to Y_{\nu_1}(\mathfrak{sl}_n) \otimes Y_{\nu_2}(\mathfrak{sl}_n)$ for any $\nu_1, \nu_2 \in \bar{\Lambda}$, due to [14, Theorem 4.12], see Remark 2.150.

In Section 2.7, for any Young diagram $\boldsymbol{\lambda}$ of size $|\boldsymbol{\lambda}| = n$, we show that the homomorphism $Y^{\mathrm{rtt}}_{\varpi_0}(\mathfrak{gl}_n) \to \mathcal{A}$ determined by the rational Lax matrix $T_D(z)$ with $D = \sum_{i=1}^{\boldsymbol{\lambda}_1^t} \varpi_{n-\boldsymbol{\lambda}_i}[x_i] - \varpi_0[\infty]$ is equal (up to a gauge transformation) to a composition of the evaluation homomorphism $\widetilde{\mathrm{ev}}\colon Y^{\mathrm{rtt}}_{\varpi_0}(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$ (2.173) and the homomorphism $U(\mathfrak{gl}_n) \to \mathcal{A}$ determined by the *type* $\boldsymbol{\lambda}$ *parabolic Gelfand-Tsetlin* formulas (which arise naturally from the $\mathfrak{gl}_n$-action in the Gelfand-Tsetlin basis of the type $\boldsymbol{\lambda}$ parabolic Verma module, see (2.166)–(2.168)), see Proposition 2.175. We note that likewise choosing another standard bases of type $\boldsymbol{\lambda}$ parabolic Verma modules over $\mathfrak{gl}_n$ gives rise to all linear rational Lax matrices of [15] with $\boldsymbol{\mu} = \emptyset$ (cf. [36]), see Remark 2.176.

• In Section 3.1, we introduce the *shifted Drinfeld quantum affine algebras of* $\mathfrak{gl}_n$, the algebras $U_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$, where $\mu^+, \mu^- \in \Lambda$ are coweights of $\mathfrak{gl}_n$. These algebras depend only on the associated coweights $\bar{\mu}^+, \bar{\mu}^- \in \bar{\Lambda}$ of $\mathfrak{sl}_n$, up to an isomorphism, see Lemma 3.13. They also contain the simply-connected versions of the shifted quantum affine algebras of $\mathfrak{sl}_n$ (introduced in [18]) via the natural embedding $\iota_{\mu^+,\mu^-}: U^{\mathrm{sc}}_{\bar{\mu}^+,\bar{\mu}^-}(L\mathfrak{sl}_n) \hookrightarrow U_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$, while their centrally enlarged counterparts $U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$ of (3.15) contain the adjoint versions of the shifted quantum affine algebras of $\mathfrak{sl}_n$ via $\iota_{\mu^+,\mu^-}: U^{\mathrm{ad}}_{\bar{\mu}^+,\bar{\mu}^-}(L\mathfrak{sl}_n) \hookrightarrow U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$, see Proposition 3.16 (generalizing the classical embedding $U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \hookrightarrow U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ of quantum loop algebras). Finally, we establish the decomposition $U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n) \simeq Z \otimes_{\mathbb{C}(\boldsymbol{v})} U^{\mathrm{ad}}_{\bar{\mu}^+,\bar{\mu}^-}(L\mathfrak{sl}_n)$, see Lemma 3.22, where $Z \subset U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$ is an explicit central subalgebra (which conjecturally coincides with the center of $U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$, see Remark 3.24).

In Section 3.2, we introduce *$\Lambda$-valued divisors on $\mathbb{P}^1$, $\Lambda^+$-valued outside* $\{0,\infty\} \in \mathbb{P}^1$, see (3.26), (3.27). For each such $D$ satisfying an auxiliary condition (3.28) (which encodes that the sum of all the coefficients of the divisor $D$ lies in the coroot lattice), we construct in Theorem 3.33 an algebra homomorphism $\Psi_D: U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}$, where $\mu^+ = D|_\infty$ and $\mu^- = \mu|_0$ are the coefficients of $D$ at $\infty$ and 0, while the target $\widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}$ is the algebra of $\boldsymbol{v}$-difference operators (3.30), see Remark 3.31. This construction generalizes the $A_{n-1}$-case of [18, Theorem 7.1] as the composition $\Psi_D \circ \iota_{-\mu^+,-\mu^-}: U^{\mathrm{ad}}_{-\bar{\mu}^+,-\bar{\mu}^-}(L\mathfrak{sl}_n) \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}$ essentially coincides with the homomorphism $\widetilde{\Phi}^{\bar{\lambda}}_{-\bar{\mu}^+,-\bar{\mu}^-}: U^{\mathrm{ad}}_{-\bar{\mu}^+,-\bar{\mu}^-}(L\mathfrak{sl}_n) \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}$ of [18, Theorem 7.1] (where $\lambda$ is the sum of all coefficients of $D$ outside $0,\infty$), see Remark 3.36.

In Section 3.3, we introduce the *(antidominantly) shifted RTT quantum affine algebras of* $\mathfrak{gl}_n$, the algebras $U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$ with $\mu^+,\mu^- \in \Lambda^+$ being *dominant coweights of* $\mathfrak{gl}_n$. They are defined via the RTT relation (3.40), the Gauss decomposition (3.42), (3.43), and an additional invertibility condition (3.44). We construct the epimorphisms $\Upsilon_{-\mu^+,-\mu^-}: U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \twoheadrightarrow U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$ for any $\mu^+,\mu^- \in \Lambda^+$, similar to [9, Main Theorem], see Theorem 3.49. Modulo a trigonometric counterpart of [40, Theorem 12], see Conjecture 3.75, we establish in Theorem 3.51 that $\Upsilon_{-\mu^+,-\mu^-}$ are actually isomorphisms for any $\mu^+,\mu^- \in \Lambda^+$ (generalizing [9] in the unshifted case $\mu^+ = \mu^- = 0$ and [18] in the rank $n = 2$ case, see Remark 3.52).

In Section 3.4, we construct $n \times n$ *trigonometric Lax* matrices $T_D(z)$ (with coefficients in $\widetilde{\mathcal{A}}^{\boldsymbol{v}}(z)$) for each $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying (3.28). They are explicitly defined via (3.64), (3.65) combined with (3.56), (3.58), (3.60), while arising naturally as the image of the $n \times n$ matrices $T^\pm(z)$ (encoding all the generators of $U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$) under the composition $\Psi_D \circ \Upsilon^{-1}_{-\mu^+,-\mu^-}: U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}$, assuming Theorem 3.51 has been established, see (3.53), (3.54). As Theorem 3.51 is well-known for $\mu^+ = \mu^- = 0$ and any Lax matrix $T_D(z)$ is a *normalized limit* of $T_{\bar{D}}(z)$ with $\bar{D}|_\infty = 0 = \bar{D}|_0$, see Propositions 3.70, 3.71 and Corollary 3.73, we immediately derive the RTT relation (3.40) for all matrices $T_D(z)$, see Proposition 3.74 (hence, the terminology "trigonometric Lax matrices"). Combining the latter with the conjectural trigonometric generalization of [40, Theorem 12], see Conjecture 3.75, we finally prove Theorem 3.51 in Section 3.4.3. The key property of the trigonometric Lax matrices $T_D(z)$ is their regularity (up to a rational factor (3.67)), see Theorem 3.68. Similar to Theorem 2.67, we also derive simplified explicit formulas for all trigonometric Lax matrices $T_D(z)$ which are linear in $z$, see

Theorem 3.77. These formulas may be related to the $\boldsymbol{v}$-deformed parabolic Gelfand-Tsetlin formulas in spirit of Proposition 2.175, see Remark 3.81. Noticing that all linear trigonometric Lax matrices $T_D(z)$ are of the form $z \cdot T^+ - T^-$ with $T^+, T^-$ being $z$-independent matrices, we find a criteria on the matrices $T^+, T^-$ so that $zT^+ - T^-$ satisfies the trigonometric RTT relation (3.82), see Proposition 3.85. Finally, we explain how the trigonometric Lax matrices $T_*^{\mathrm{trig}}(z)$ of Section 3.4.1 may be degenerated into the rational Lax matrices $T_*^{\mathrm{rat}}(z)$ of Section 2.4.1, see Proposition 3.94.

In Section 3.5, we apply Theorem 3.77 to evaluate explicitly all linear trigonometric Lax matrices $T_D(z)$ for $n = 2$, thus generalizing the three Lax matrices of [18], see Remark 3.104.

In Section 3.6, we construct coproduct homomorphisms on antidominantly shifted quantum affine algebras. We start by constructing algebra homomorphisms (see Proposition 3.106)

$$\Delta^{\mathrm{rtt}}_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-} : U^{\mathrm{rtt}}_{-\mu_1^+-\mu_2^+,-\mu_1^--\mu_2^-}(L\mathfrak{gl}_n) \longrightarrow U^{\mathrm{rtt}}_{-\mu_1^+,-\mu_1^-}(L\mathfrak{gl}_n) \otimes U^{\mathrm{rtt}}_{-\mu_2^+,-\mu_2^-}(L\mathfrak{gl}_n)$$

defined via $\Delta^{\mathrm{rtt}}_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-}(T^\pm(z)) = T^\pm(z) \otimes T^\pm(z)$ for any $\mu_1^+, \mu_1^-, \mu_2^+, \mu_2^- \in \Lambda^+$. Evoking the key isomorphism $U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \xrightarrow{\sim} U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$ of Theorem 3.51, this gives rise to

$$\Delta_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-} : U_{-\mu_1^+-\mu_2^+,-\mu_1^--\mu_2^-}(L\mathfrak{gl}_n) \longrightarrow U_{-\mu_1^+,-\mu_1^-}(L\mathfrak{gl}_n) \otimes U_{-\mu_2^+,-\mu_2^-}(L\mathfrak{gl}_n).$$

The latter, in turn, gives rise to algebra homomorphisms

$$\Delta_{-\nu_1^+,-\nu_1^-,-\nu_2^+,-\nu_2^-} : U^{\mathrm{sc}}_{-\nu_1^+-\nu_2^+,-\nu_1^--\nu_2^-}(L\mathfrak{sl}_n) \longrightarrow U^{\mathrm{sc}}_{-\nu_1^+,-\nu_1^-}(L\mathfrak{sl}_n) \otimes U^{\mathrm{sc}}_{-\nu_2^+,-\nu_2^-}(L\mathfrak{sl}_n)$$

for any dominant $\mathfrak{sl}_n$-coweights $\nu_1^+, \nu_1^-, \nu_2^+, \nu_2^- \in \bar{\Lambda}^+$, see Proposition 3.113, thus recovering and providing a more conceptual and simpler proof of [18, Theorem 10.16]. The latter give rise to homomorphisms

$$\Delta_{\nu_1^+,\nu_1^-,\nu_2^+,\nu_2^-} : U^{\mathrm{sc}}_{\nu_1^++\nu_2^+,\nu_1^-+\nu_2^-}(L\mathfrak{sl}_n) \longrightarrow U^{\mathrm{sc}}_{\nu_1^+,\nu_1^-}(L\mathfrak{sl}_n) \otimes U^{\mathrm{sc}}_{\nu_2^+,\nu_2^-}(L\mathfrak{sl}_n)$$

for any $\mathfrak{sl}_n$–coweights $\nu_1^+, \nu_1^-, \nu_2^+, \nu_2^- \in \bar{\Lambda}$, due to [18, Theorem 10.20], see Remark 3.115.

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## 2. Rational Lax matrices

### 2.1. Shifted Drinfeld Yangians of $\mathfrak{gl}_n$

Consider the lattice $\Lambda^\vee = \oplus_{j=1}^n \mathbb{Z}\epsilon_j^\vee$ associated with the standard module of $\mathfrak{gl}_n$, so that $\alpha_i^\vee := \epsilon_i^\vee - \epsilon_{i+1}^\vee$ $(1 \le i < n)$ are the standard simple positive roots of $\mathfrak{sl}_n$. Let $\Lambda = \oplus_{j=1}^n \mathbb{Z}\epsilon_j$ be the dual lattice so that $\epsilon_i^\vee(\epsilon_j) = \delta_{i,j}$. We will also need its alternative $\mathbb{Z}$-basis: $\Lambda = \oplus_{i=0}^{n-1}\mathbb{Z}\varpi_i$ with $\varpi_i := -\sum_{j=i+1}^n \epsilon_j$. For $\mu \in \Lambda$, define $\underline{d} = \{d_j\}_{j=1}^n \in \mathbb{Z}^n$ and $\underline{b} = \{b_i\}_{i=1}^{n-1} \in \mathbb{Z}^{n-1}$ via

$$d_j := \epsilon_j^\vee(\mu), \quad b_i := \alpha_i^\vee(\mu) = d_i - d_{i+1}. \tag{2.1}$$

Fix a $\mathfrak{gl}_n$–coweight $\mu \in \Lambda$. Define the *shifted Drinfeld Yangian of* $\mathfrak{gl}_n$, denoted by $Y_\mu(\mathfrak{gl}_n)$, to be the associative $\mathbb{C}$-algebra generated by $\{E_i^{(r)}, F_i^{(r)}\}_{1\le i<n}^{r\ge 1} \cup \{D_i^{(s_i)}, \widetilde{D}_i^{(\widetilde{s}_i)}\}_{1\le i\le n}^{s_i \ge d_i, \widetilde{s}_i \ge -d_i}$ with the following defining relations (for all admissible $i, j, r, s, t$):

$$D_i^{(d_i)} = 1,\ \sum_{t=d_i}^{r+d_i} D_i^{(t)} \widetilde{D}_i^{(r-t)} = -\delta_{r,0},\ [D_i^{(r)}, D_j^{(s)}] = 0, \tag{2.2}$$

$$[E_i^{(r)}, F_j^{(s)}] = -\delta_{i,j} \sum_{t=-d_i}^{r+s-1-d_{i+1}} \widetilde{D}_i^{(t)} D_{i+1}^{(r+s-t-1)}, \tag{2.3}$$

$$[D_i^{(r)}, E_j^{(s)}] = (\delta_{i,j+1} - \delta_{i,j}) \sum_{t=d_i}^{r-1} D_i^{(t)} E_j^{(r+s-t-1)}, \tag{2.4}$$

$$[D_i^{(r)}, F_j^{(s)}] = (\delta_{i,j} - \delta_{i,j+1}) \sum_{t=d_i}^{r-1} F_j^{(r+s-t-1)} D_i^{(t)}, \tag{2.5}$$

$$[E_i^{(r)}, E_i^{(s)}] = \sum_{t=1}^{r-1} E_i^{(t)} E_i^{(r+s-t-1)} - \sum_{t=1}^{s-1} E_i^{(t)} E_i^{(r+s-t-1)}, \tag{2.6}$$

$$[F_i^{(r)}, F_i^{(s)}] = \sum_{t=1}^{s-1} F_i^{(r+s-t-1)} F_i^{(t)} - \sum_{t=1}^{r-1} F_i^{(r+s-t-1)} F_i^{(t)}, \tag{2.7}$$

$$[E_i^{(r+1)}, E_{i+1}^{(s)}] - [E_i^{(r)}, E_{i+1}^{(s+1)}] = -E_i^{(r)} E_{i+1}^{(s)}, \tag{2.8}$$

$$[F_i^{(r+1)}, F_{i+1}^{(s)}] - [F_i^{(r)}, F_{i+1}^{(s+1)}] = F_{i+1}^{(s)} F_i^{(r)}, \tag{2.9}$$

$$[E_i^{(r)}, E_j^{(s)}] = 0 \text{ if } |i-j| > 1, \tag{2.10}$$

$$[F_i^{(r)}, F_j^{(s)}] = 0 \text{ if } |i-j| > 1, \tag{2.11}$$

$$[E_i^{(r)}, [E_i^{(s)}, E_j^{(t)}]] + [E_i^{(s)}, [E_i^{(r)}, E_j^{(t)}]] = 0 \text{ if } |i-j| = 1, \tag{2.12}$$

$$[F_i^{(r)}, [F_i^{(s)}, F_j^{(t)}]] + [F_i^{(s)}, [F_i^{(r)}, F_j^{(t)}]] = 0 \text{ if } |i-j| = 1. \tag{2.13}$$

**Remark 2.14.** (a) For $\mu = 0$, this definition recovers the Drinfeld Yangian of $\mathfrak{gl}_n$, see [8] and [4, Theorem 5.2] (to be more precise, multiplying $E_i^{(r)}, F_i^{(r)}, D_i^{(r)}, \widetilde{D}_i^{(r)}$ by $(-1)^r$ the relations (2.2)–(2.13) transform into the defining relations (5.7–5.20) of [4], cf. Remark 2.51). We note that the conventions $r \geq 1$ instead of $r \geq 0$ are in charge of perceiving the Yangian as a *QFSHA* (quantum formal series Hopf algebra) which is related to a more standard viewpoint of it as a *QUEA* (quantum universal enveloping algebra) via the so-called Drinfeld-Gavarini quantum duality principle.
(b) Similar to [4, Remark 5.3], the relations (2.6) and (2.7) are equivalent to the relations

$$[E_i^{(r+1)}, E_i^{(s)}] - [E_i^{(r)}, E_i^{(s+1)}] = E_i^{(r)} E_i^{(s)} + E_i^{(s)} E_i^{(r)}, \tag{2.15}$$

$$[F_i^{(r+1)}, F_i^{(s)}] - [F_i^{(r)}, F_i^{(s+1)}] = -F_i^{(r)} F_i^{(s)} - F_i^{(s)} F_i^{(r)}. \tag{2.16}$$

Let $\bar{\Lambda} = \oplus_{i=1}^{n-1} \mathbb{Z}\omega_i$ be the coweight lattice of $\mathfrak{sl}_n$, where $\{\omega_i\}_{i=1}^{n-1}$ are the standard fundamental coweights of $\mathfrak{sl}_n$. There is a natural $\mathbb{Z}$-linear projection $\Lambda \to \bar{\Lambda}, \mu \mapsto \bar{\mu}$, defined via:

$$\alpha_i^\vee(\bar{\mu}) = \alpha_i^\vee(\mu) \quad \text{for } 1 \leq i \leq n-1.$$

Equivalently, we have $\bar{\varpi}_0 = 0$ and $\bar{\varpi}_i = \omega_i$ for $1 \leq i \leq n-1$.

The algebra $Y_\mu(\mathfrak{gl}_n)$ depends only on the associated $\mathfrak{sl}_n$–coweight $\bar{\mu}$, up to an isomorphism:

**Lemma 2.17.** *For $\mathfrak{gl}_n$–coweights $\mu_1, \mu_2 \in \Lambda$ such that $\bar{\mu}_1 = \bar{\mu}_2$ in $\bar{\Lambda}$, the assignment*

$$E_i^{(r)} \mapsto E_i^{(r)},\ F_i^{(r)} \mapsto F_i^{(r)},\ D_i^{(s_i)} \mapsto D_i^{(s_i - \epsilon_i^\vee(\mu_1 - \mu_2))},\ \widetilde{D}_i^{(\tilde{s}_i)} \mapsto \widetilde{D}_i^{(\tilde{s}_i + \epsilon_i^\vee(\mu_1 - \mu_2))} \tag{2.18}$$

*gives rise to a $\mathbb{C}$-algebra isomorphism $Y_{\mu_1}(\mathfrak{gl}_n) \xrightarrow{\sim} Y_{\mu_2}(\mathfrak{gl}_n)$.*

**Proof.** The assignment (2.18) is clearly compatible with the defining relations (2.2)–(2.13), thus, it gives rise to a $\mathbb{C}$-algebra homomorphism $Y_{\mu_1}(\mathfrak{gl}_n) \to Y_{\mu_2}(\mathfrak{gl}_n)$. Switching the roles of $\mu_1$ and $\mu_2$, we obtain the inverse homomorphism $Y_{\mu_2}(\mathfrak{gl}_n) \to Y_{\mu_1}(\mathfrak{gl}_n)$. Hence, the result. □

We define the generating series of the above generators as follows:

$$E_i(z) := \sum_{r\geq 1} E_i^{(r)} z^{-r}, \quad F_i(z) := \sum_{r\geq 1} F_i^{(r)} z^{-r},$$
$$D_i(z) := \sum_{r\geq d_i} D_i^{(r)} z^{-r}, \quad \widetilde{D}_i(z) := \sum_{r\geq -d_i} \widetilde{D}_i^{(r)} z^{-r} = -D_i(z)^{-1}.$$

The algebras $Y_\mu(\mathfrak{gl}_n)$ slightly generalize the *shifted (Drinfeld) Yangians of* $\mathfrak{sl}_n$, denoted by $Y_\nu(\mathfrak{sl}_n)$ in [3, Definition B.2], where $\nu \in \bar{\Lambda}$ is an $\mathfrak{sl}_n$–coweight. Recall that the latter is an associative $\mathbb{C}$-algebra generated by $\{\mathsf{E}_i^{(r)}, \mathsf{F}_i^{(r)}, \mathsf{H}_i^{(s_i)}\}_{1\leq i<n}^{r\geq 1, s_i\geq -b_i}$ with the defining relations of [3, Definition B.1] and $\mathsf{H}_i^{(-b_i)} = 1$, where $b_i := \alpha_i^\vee(\nu)$. We define the generating series

$$\mathsf{E}_i(z) := \sum_{r\geq 1} \mathsf{E}_i^{(r)} z^{-r}, \quad \mathsf{F}_i(z) := \sum_{r\geq 1} \mathsf{F}_i^{(r)} z^{-r}, \quad \mathsf{H}_i(z) := \sum_{r\geq -b_i} \mathsf{H}_i^{(r)} z^{-r}.$$

The explicit relation between the shifted Drinfeld Yangians of $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$ is as follows:

**Proposition 2.19.** *For any $\mu \in \Lambda$, there exists a $\mathbb{C}$-algebra embedding*

$$\iota_\mu \colon Y_{\bar{\mu}}(\mathfrak{sl}_n) \hookrightarrow Y_\mu(\mathfrak{gl}_n), \tag{2.20}$$

*uniquely determined by*

$$\mathsf{E}_i(z) \mapsto E_i\left(z + \frac{i}{2}\right), \ \mathsf{F}_i(z) \mapsto F_i\left(z + \frac{i}{2}\right), \ \mathsf{H}_i(z) \mapsto -\widetilde{D}_i\left(z + \frac{i}{2}\right) D_{i+1}\left(z + \frac{i}{2}\right). \tag{2.21}$$

**Remark 2.22.** For $\mu = 0$, this recovers the classical embedding $Y(\mathfrak{sl}_n) \hookrightarrow Y(\mathfrak{gl}_n)$ of Yangians.

**Proof of Proposition 2.19.** As in the $\mu = 0$ case (see Remark 2.22), it is straightforward to see that the assignment (2.21) is compatible with the defining relations of $Y_{\bar{\mu}}(\mathfrak{sl}_n)$, giving rise to a $\mathbb{C}$-algebra homomorphism $\iota_\mu \colon Y_{\bar{\mu}}(\mathfrak{sl}_n) \to Y_\mu(\mathfrak{gl}_n)$. It remains to establish the injectivity of $\iota_\mu$.

To this end, we first note that the coefficients of the series

$$C(z) = z^{-d_1-\ldots-d_n} + \sum_{s>d_1+\ldots+d_n} C_s z^{-s} := D_1(z) D_2(z+1) \cdots D_n(z+n-1) \tag{2.23}$$

are in the center of $Y_\mu(\mathfrak{gl}_n)$, due to the defining relations (2.2), (2.4), (2.5), cf. [4, Theorem 7.2].

Second, given an abstract polynomial algebra $\mathcal{B} = \mathbb{C}[\{D_i^{(r_i)}\}_{1\le i\le n}^{r_i>d_i}]$, define the elements $\{C_s\}_{s>d_1+\ldots+d_n}$ and $\{\bar{D}_i^{(s_i)}\}_{1\le i<n}^{s_i>d_{i+1}-d_i}$ of $\mathcal{B}$, respectively via the formula (2.23) and

$$\bar{D}_i(z) := z^{d_i-d_{i+1}} + \sum_{s>d_{i+1}-d_i} \bar{D}_i^{(s)} z^{-s} = D_i(z)^{-1}D_{i+1}(z),$$

where we set $D_i(z) := z^{-d_i} + \sum_{r>d_i} D_i^{(r)} z^{-r}$. It is clear that $\{\bar{D}_i^{(s_i)}\}_{1\le i<n}^{s_i>d_{i+1}-d_i} \cup \{C_s\}_{s>d_1+\ldots+d_n}$ provide an alternative collection of generators of the polynomial algebra $\mathcal{B}$, so that we have:

$$\mathcal{B} \simeq \mathbb{C}[\{C_s\}_{s>d_1+\ldots+d_n}] \otimes_{\mathbb{C}} \mathbb{C}[\{\bar{D}_i^{(s_i)}\}_{1\le i<n}^{s_i>d_{i+1}-d_i}].$$

Applying this in our setup, we get the decomposition $Y_\mu(\mathfrak{gl}_n) \simeq Z \otimes_{\mathbb{C}} Y'_\mu(\mathfrak{gl}_n)$, where $Z$ is a $\mathbb{C}$-subalgebra generated by $\{C_s\}_{s>d_1+\ldots+d_n}$ and $Y'_\mu(\mathfrak{gl}_n)$ is the $\mathbb{C}$-subalgebra generated by $\{E_i^{(r)}, F_i^{(r)}, \bar{D}_i^{(s_i)}\}_{1\le i<n}^{r\ge 1, s_i>d_{i+1}-d_i}$. Furthermore, the defining relations (2.3)–(2.5) are equivalent to the subalgebra $Z$ being central (as explained above) and the commutation relations between $\bar{D}_i^{(s)}$ and $E_j^{(r)}, F_j^{(r)}$ exactly matching those of $Y_{\bar{\mu}}(\mathfrak{sl}_n)$ through (2.21). Thus, $\iota_\mu$ is injective. □

The above proof implies the shifted version of the decomposition from [29, Theorem 1.8.2]:

**Corollary 2.24.** *There is a* $\mathbb{C}$*-algebra isomorphism*

$$Y_\mu(\mathfrak{gl}_n) \simeq \mathbb{C}[\{C_s\}_{s>d_1+\ldots+d_n}] \otimes_{\mathbb{C}} Y_{\bar{\mu}}(\mathfrak{sl}_n). \tag{2.25}$$

*In particular,* $Y_{\bar{\mu}}(\mathfrak{sl}_n)$ *may be realized both as a subalgebra of* $Y_\mu(\mathfrak{gl}_n)$ *via (2.20) and as a quotient algebra of* $Y_\mu(\mathfrak{gl}_n)$ *by the central ideal* $(C_s - b_s)_{s>d_1+\ldots+d_n}$ *for any collection of* $b_s \in \mathbb{C}$*.*

The following result provides a shifted version of the remaining part of [29, Theorem 1.8.2]:

**Lemma 2.26.** *(a) The center of the shifted Yangian* $Y_\nu(\mathfrak{sl}_n)$ *is trivial for any shift* $\nu \in \bar{\Lambda}$*.*
*(b) The center of the shifted Yangian* $Y_\mu(\mathfrak{gl}_n)$ *coincides with* $\mathbb{C}[\{C_s\}_{s>d_1+\ldots+d_n}]$ *for any* $\mu \in \Lambda$*.*

As we will not use Lemma 2.26 in the rest of this paper, we will only sketch the proof of part (a) in Remark 2.81, crucially using the result of [40] discussed below. Part (b) follows immediately from (a), the decomposition (2.25), and the centrality of $C_s$ established above.

### 2.2. Homomorphism $\Psi_D$

In this section, we generalize [3, Theorem B.15] for the type $A_{n-1}$ Dynkin diagram with arrows pointing $i \to i+1$ for $1 \le i \le n-2$ by replacing $Y_{\bar{\mu}}(\mathfrak{sl}_n)$ from Theorem B.15 of [3] with $Y_\mu(\mathfrak{gl}_n)$.

**Remark 2.27.** While similar generalizations exist for all orientations of $A_{n-1}$ Dynkin diagram, for the purposes of this paper it suffices to consider only the above orientation, see Remark 2.73.

A $\mathfrak{gl}_n$–coweight $\lambda \in \Lambda$ will be called *dominant*, which we denote by $\lambda \in \Lambda^+$, if the corresponding $\mathfrak{sl}_n$–coweight $\bar{\lambda}$ is dominant (denoted by $\bar{\lambda} \in \bar{\Lambda}^+$), that is $\alpha_i^\vee(\bar{\lambda}) \in \mathbb{N}$ for $1 \le i \le n-1$. Thus, $\sum_{i=0}^{n-1} c_i \varpi_i$ is dominant iff $c_i \in \mathbb{N}$ for $1 \le i \le n-1$.

A **$\Lambda$-valued divisor $D$ on $\mathbb{P}^1$, $\Lambda^+$-valued outside $\{\infty\} \in \mathbb{P}^1$**, is a formal sum

$$D = \sum_{1 \le s \le N} \gamma_s \varpi_{i_s}[x_s] + \mu[\infty] \tag{2.28}$$

with $N \in \mathbb{N}$, $0 \le i_s < n$, $x_s \in \mathbb{C}$, $\gamma_s = \begin{cases} 1 & \text{if } i_s \neq 0 \\ \pm 1 & \text{if } i_s = 0 \end{cases}$, and $\mu \in \Lambda$. We will write $\mu = D|_\infty$. If $\mu \in \Lambda^+$, we call $D$ a **$\Lambda^+$-valued divisor on $\mathbb{P}^1$**. It will be convenient to present

$$D = \sum_{x \in \mathbb{P}^1 \setminus \{\infty\}} \lambda_x[x] + \mu[\infty] \text{ with } \lambda_x \in \Lambda^+, \tag{2.29}$$

related to (2.28) via $\lambda_x := \sum_{1 \le s \le N}^{x_s = x} \gamma_s \varpi_{i_s}$. Set $\lambda := \sum_{s=1}^N \gamma_s \varpi_{i_s} \in \Lambda^+$. Let $\{\alpha_i\}_{i=1}^{n-1} \subset \Lambda$ be the simple coroots of $\mathfrak{sl}_n$, that is $\alpha_i = \epsilon_i - \epsilon_{i+1}$. Following [3], we make the following

$$\textbf{Assumption:} \quad \lambda + \mu = a_1\alpha_1 + \ldots + a_{n-1}\alpha_{n-1} \quad \text{with } a_i \in \mathbb{N}. \tag{2.30}$$

**Remark 2.31.** (2.30) is equivalent to $\sum_{j=1}^n \epsilon_j^\vee(\lambda + \mu) = 0$ and $\sum_{j=1}^i \epsilon_j^\vee(\lambda+\mu) \in \mathbb{N}$ for $1 \le i < n$.

Consider the associative $\mathbb{C}$-algebra

$$\mathcal{A} = \mathbb{C}\langle p_{i,r}, e^{\pm q_{i,r}}, (p_{i,r} - p_{i,s} + m)^{-1}\rangle_{1 \le i < n, m \in \mathbb{Z}}^{1 \le r \neq s \le a_i} \tag{2.32}$$

with the defining relations

$$[e^{\pm q_{i,r}}, p_{j,s}] = \mp \delta_{i,j}\delta_{r,s} e^{\pm q_{i,r}}, \quad [p_{i,r}, p_{j,s}] = 0 = [e^{q_{i,r}}, e^{q_{j,s}}], \quad e^{\pm q_{i,r}} e^{\mp q_{i,r}} = 1.$$

**Remark 2.33.** This algebra $\mathcal{A}$ can be represented in the algebra of difference operators with rational coefficients on functions of $\{p_{i,r}\}_{1\le i<n}^{1\le r\le a_i}$ by taking $e^{\mp q_{i,r}}$ to be a difference operator $\mathsf{D}_{i,r}^{\pm 1}$ that acts as

$$(\mathsf{D}_{i,r}^{\pm 1}\Psi)(p_{1,1},\ldots,p_{i,r},\ldots,p_{n-1,a_{n-1}}) = \Psi(p_{1,1},\ldots,p_{i,r}\pm 1,\ldots,p_{n-1,a_{n-1}}).$$

For $0\le i\le n-1$ and $1\le j\le n-1$, we define

$$\begin{aligned} Z_i(z) &:= \prod_{1\le s\le N}^{i_s=i}(z-x_s)^{\gamma_s} = \prod_{x\in\mathbb{P}^1\backslash\{\infty\}}(z-x)^{\alpha_i^\vee(\lambda_x)}, \\ P_j(z) &:= \prod_{r=1}^{a_j}(z-p_{j,r}), \quad P_{j,r}(z) := \prod_{1\le s\le a_j}^{s\ne r}(z-p_{j,s}), \end{aligned} \tag{2.34}$$

where $\alpha_0^\vee := -\epsilon_1^\vee$. We also define

$$a_0 := 0, \quad a_n := 0, \quad P_0(z) := 1, \quad P_n(z) := 1.$$

The following result generalizes $A_{n-1}$-case of [3, Theorem B.15] stated for semisimple Lie algebras $\mathfrak{g}$ (preceded by [21] for the trivial shift and by [27] for dominant shifts):

**Theorem 2.35.** *Let $D$ be as above and $\mu = D|_\infty$. There is a unique $\mathbb{C}$-algebra homomorphism*

$$\Psi_D\colon Y_{-\mu}(\mathfrak{gl}_n) \longrightarrow \mathcal{A} \tag{2.36}$$

*such that*

$$\begin{aligned} E_i(z) &\mapsto -\sum_{r=1}^{a_i}\frac{P_{i-1}(p_{i,r}-1)Z_i(p_{i,r})}{(z-p_{i,r})P_{i,r}(p_{i,r})}e^{q_{i,r}}, \\ F_i(z) &\mapsto \sum_{r=1}^{a_i}\frac{P_{i+1}(p_{i,r}+1)}{(z-p_{i,r}-1)P_{i,r}(p_{i,r})}e^{-q_{i,r}}, \\ D_i(z) &\mapsto \frac{P_i(z)}{P_{i-1}(z-1)}\prod_{0\le k<i}Z_k(z) = \frac{P_i(z)}{P_{i-1}(z-1)}\prod_{x\in\mathbb{P}^1\backslash\{\infty\}}(z-x)^{-\epsilon_i^\vee(\lambda_x)}. \end{aligned} \tag{2.37}$$

**Remark 2.38.** Consider a decomposition $\bar{\lambda} = \sum_{1\le s\le N}^{i_s\ne 0}\omega_{i_s}$ and assign $z_s := x_s - \frac{i_s+1}{2}\in\mathbb{C}$ to the $s$-th summand. Identifying $\mathcal{A}$ of (2.32) with $\tilde{\mathcal{A}}$ of [3, §B(ii)] (with $z_i$ of [3] specialized to complex numbers) via $p_{i,r}\leftrightarrow w_{i,r}+\frac{i}{2}$ and $e^{\pm q_{i,r}}\leftrightarrow \mathsf{u}_{i,r}^{\mp 1}$, the (restriction) composition $Y_{-\bar{\mu}}(\mathfrak{sl}_n)\xrightarrow{\iota_{-\mu}} Y_{-\mu}(\mathfrak{gl}_n)\xrightarrow{\Psi_D}\mathcal{A}$ is just the homomorphism $\Phi_{-\bar{\mu}}^{\bar{\lambda}}$ of [3, Theorem B.15].

**Proof of Theorem 2.35.** First, we need to verify that under the above assignment (2.37), the image of $D_i(z)$ is of the form $z^{d_i}$ + (lower order terms in $z$). Let $\deg_i$ denote the leading power of $z$ in the image of $D_i(z)$ (clearly the coefficient of $z^{\deg_i}$ equals 1). Then, indeed we have

$$\deg_i = a_i - a_{i-1} - \sum_{x\in\mathbb{P}^1\setminus\{\infty\}} \epsilon_i^\vee(\lambda_x) = a_i - a_{i-1} - \epsilon_i^\vee(\lambda) = a_i - a_{i-1} - (a_i - a_{i-1} - \epsilon_i^\vee(\mu)) = d_i.$$

Evoking the decomposition (2.25), it suffices to prove that the restrictions of the assignment (2.37) to the subalgebras $Y_{-\bar{\mu}}(\mathfrak{sl}_n)$ and $\mathbb{C}[\{C_s\}_{s>-(d_1+\ldots+d_n)}]$ determine algebra homomorphisms, whose images commute. The former is clear for the restriction to $Y_{-\bar{\mu}}(\mathfrak{sl}_n)$, due to Theorem B.15 of [3] combined with Remark 2.38 above. On the other hand, we have

$$\Psi_D(C(z)) = \prod_{i=1}^{n} \prod_{x\in\mathbb{P}^1\setminus\{\infty\}} (z+i-1-x)^{-\epsilon_i^\vee(\lambda_x)} = \prod_{s=1}^{N} \prod_{k=i_s}^{n-1} (z-x_s+k)^{\gamma_s}. \tag{2.39}$$

Thus, the restriction of $\Psi_D$ to the polynomial algebra $\mathbb{C}[\{C_s\}_{s>-(d_1+\ldots+d_n)}]$ defines an algebra homomorphism, whose image is central in $\mathcal{A}$. This completes our proof of Theorem 2.35. $\square$

### *2.3. Antidominantly shifted RTT Yangians of* $\mathfrak{gl}_n$

Consider the *rational* $R$-matrix $R_{\mathrm{rat}}(z) = z\mathrm{Id} + P$, where $P = \sum_{i,j=1}^n E_{ij}\otimes E_{ji} \in (\operatorname{End}\mathbb{C}^n)^{\otimes 2}$ is the permutation operator. It satisfies the Yang-Baxter equation with a spectral parameter:

$$R_{\mathrm{rat};12}(u)R_{\mathrm{rat};13}(u+v)R_{\mathrm{rat};23}(v) = R_{\mathrm{rat};23}(v)R_{\mathrm{rat};13}(u+v)R_{\mathrm{rat};12}(u). \tag{2.40}$$

Fix $\mu\in\Lambda^+$. Define the *(antidominantly) shifted RTT Yangian of* $\mathfrak{gl}_n$, denoted by $Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$, to be the associative $\mathbb{C}$-algebra generated by $\{t_{ij}^{(r)}\}_{1\le i,j\le n}^{r\in\mathbb{Z}}$ subject to the following two families of relations:

- The first family of relations may be encoded by a single RTT relation

$$R_{\mathrm{rat}}(z-w)T_1(z)T_2(w) = T_2(w)T_1(z)R_{\mathrm{rat}}(z-w), \tag{2.41}$$

  where $T(z)\in Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)[[z,z^{-1}]]\otimes_{\mathbb{C}}\operatorname{End}\mathbb{C}^n$ is defined via

$$T(z) = \sum_{i,j} t_{ij}(z)\otimes E_{ij} \quad \text{with} \quad t_{ij}(z) := \sum_{r\in\mathbb{Z}} t_{ij}^{(r)} z^{-r}. \tag{2.42}$$

  Thus, (2.41) is an equality in $Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)[[z,z^{-1},w,w^{-1}]]\otimes_{\mathbb{C}}(\operatorname{End}\mathbb{C}^n)^{\otimes 2}$.

- The second family of relations encodes the fact that $T(z)$ admits the Gauss decomposition:

$$T(z) = F(z) \cdot G(z) \cdot E(z), \tag{2.43}$$

where $F(z), G(z), E(z) \in Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)((z^{-1})) \otimes_{\mathbb{C}} \mathrm{End}\,\mathbb{C}^n$ are of the form

$$F(z) = \sum_i E_{ii} + \sum_{i<j} f_{ji}(z) \otimes E_{ji}, \quad G(z) = \sum_i g_i(z) \otimes E_{ii},$$
$$E(z) = \sum_i E_{ii} + \sum_{i<j} e_{ij}(z) \otimes E_{ij},$$

with the matrix coefficients having the following expansions in $z$:

$$e_{ij}(z) = \sum_{r\geq 1} e^{(r)}_{ij} z^{-r}, \quad f_{ji}(z) = \sum_{r\geq 1} f^{(r)}_{ji} z^{-r}, \quad g_i(z) = z^{d_i} + \sum_{r\geq 1-d_i} g^{(r)}_i z^{-r}, \tag{2.44}$$

where $\{e^{(r)}_{ij}, f^{(r)}_{ji}\}^{r\geq 1}_{1\leq i<j\leq n} \cup \{g^{(s_i)}_i\}^{s_i\geq 1-d_i}_{1\leq i\leq n} \subset Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$.

**Remark 2.45.** (a) For $\mu = 0$, the second family of relations (2.43), (2.44) is equivalent to the relations $t^{(r)}_{ij} = 0$ for $r < 0$ and $t^{(0)}_{ij} = \delta_{i,j}$. Thus, $Y^{\mathrm{rtt}}_0(\mathfrak{gl}_n)$ is the RTT Yangian of $\mathfrak{gl}_n$ of [17].
(b) Likewise, (2.44) is equivalent to a certain family of algebraic relations on $t^{(r)}_{ij}$, which can be best understood in terms of the quasi-determinants (as defined by I. Gelfand and V. Retakh in [24]) following [4, (5.2–5.4)]. In particular, we have $T(z) \in Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)((z^{-1})) \otimes_{\mathbb{C}} \mathrm{End}\,\mathbb{C}^n$. For example, (2.44) for $i = 1$ are equivalent to:

$$t^{(-d_1)}_{11} = 1 \text{ and } t^{(r)}_{11} = 0 \text{ for } r < -d_1, \qquad t^{(r)}_{1j} = 0 = t^{(r)}_{j1} \text{ for } r \leq -d_1, 1 < j \leq n.$$

(c) If $\mu \notin \Lambda^+$, then the above two families of relations are contradictive and thus the algebra $Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$ is trivial, see Remark 2.50.
(d) If $\mu_1, \mu_2 \in \Lambda^+$ satisfy $\bar{\mu}_1 = \bar{\mu}_2 \in \bar{\Lambda}$, that is, $\mu_2 = \mu_1 + c\varpi_0$ with $c \in \mathbb{Z}$, then the assignment $T(z) \mapsto z^c T(z)$ gives rise to a $\mathbb{C}$-algebra isomorphism $Y^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{gl}_n) \xrightarrow{\sim} Y^{\mathrm{rtt}}_{-\mu_2}(\mathfrak{gl}_n)$, cf. Lemma 2.17.

**Lemma 2.46.** *For any $1 \leq i < j \leq n$ and $r \geq 1$, we have the following identities:*

$$\begin{aligned} e^{(r)}_{ij} &= [e^{(1)}_{j-1,j}, [e^{(1)}_{j-2,j-1}, \cdots, [e^{(1)}_{i+1,i+2}, e^{(r)}_{i,i+1}] \cdots ]], \\ f^{(r)}_{ji} &= [[\cdots [f^{(r)}_{i+1,i}, f^{(1)}_{i+2,i+1}], \cdots, f^{(1)}_{j-1,j-2}], f^{(1)}_{j,j-1}]. \end{aligned} \tag{2.47}$$

**Proof.** The proof is analogous to that of [4, (5.5)] (see also [19, Corollary 2.23]). □

**Corollary 2.48.** *The algebra $Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$ is generated by $\{e^{(r)}_{i,i+1}, f^{(r)}_{i+1,i}, g^{(s_j)}_j\}^{r\geq 1, s_j\geq 1-d_j}_{1\leq i<n, 1\leq j\leq n}$.*

The following result is proved completely analogously to [4, Lemmas 5.4, 5.5, 5.7]:

**Lemma 2.49.** *The following identities hold:*
*(a)* $[g_i(z), g_j(w)] = 0$;
*(b)* $(z-w)[g_i(z), e_{j,j+1}(w)] = (\delta_{i,j} - \delta_{i,j+1})g_i(z)(e_{j,j+1}(z) - e_{j,j+1}(w))$;
*(c)* $(z-w)[g_i(z), f_{j+1,j}(w)] = (\delta_{i,j+1} - \delta_{i,j})(f_{j+1,j}(z) - f_{j+1,j}(w))g_i(z)$;
*(d)* $[e_{i,i+1}(z), f_{j+1,j}(w)] = 0$ *if* $i \neq j$;
*(e)* $(z-w)[e_{i,i+1}(z), f_{i+1,i}(w)] = g_i(w)^{-1}g_{i+1}(w) - g_i(z)^{-1}g_{i+1}(z)$;
*(f)* $(z-w)[e_{i,i+1}(z), e_{i,i+1}(w)] = -(e_{i,i+1}(z) - e_{i,i+1}(w))^2$;
*(g)* $(z-w)[e_{i,i+1}(z), e_{i+1,i+2}(w)] = -e_{i,i+1}(z)e_{i+1,i+2}(w) + e_{i,i+1}(w)e_{i+1,i+2}(w) - e_{i,i+2}(w) + e_{i,i+2}(z)$;
*(h)* $[e_{i,i+1}(z), e_{j,j+1}(w)] = 0$ *if* $|i-j| > 1$;
*(i)* $[e_{i,i+1}(z_1), [e_{i,i+1}(z_2), e_{j,j+1}(w)]] + [e_{i,i+1}(z_2), [e_{i,i+1}(z_1), e_{j,j+1}(w)]] = 0$ *if* $|i-j| = 1$;
*(j)* $(z-w)[f_{i+1,i}(z), f_{i+1,i}(w)] = (f_{i+1,i}(z) - f_{i+1,i}(w))^2$;
*(k)* $(z-w)[f_{i+1,i}(z), f_{i+2,i+1}(w)] = f_{i+2,i+1}(w)f_{i+1,i}(z) - f_{i+2,i+1}(w)f_{i+1,i}(w) + f_{i+2,i}(w) - f_{i+2,i}(z)$;
*(l)* $[f_{i+1,i}(z), f_{j+1,j}(w)] = 0$ *if* $|i-j| > 1$;
*(m)* $[f_{i+1,i}(z_1), [f_{i+1,i}(z_2), f_{j+1,j}(w)]] + [f_{i+1,i}(z_2), [f_{i+1,i}(z_1), f_{j+1,j}(w)]] = 0$ *if* $|i-j| = 1$.

**Remark 2.50.** If $d_i < d_{i+1}$ for some $1 \leq i < n$, then the right-hand side of the identity in Lemma 2.49(e) contains monomials $z^{d_{i+1}-d_i}$ and $w^{d_{i+1}-d_i}$, while all monomials in the left-hand side have negative degrees. Thus, the defining relations of $Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$ are contradictive unless $\mu$ is dominant (see [18, Remark 11.14] for the trigonometric $\mathfrak{sl}_2$-counterpart of this conclusion).

**Remark 2.51.** The right-hand sides in all identities of Lemma 2.49 have opposite signs to those of [4, §5], due to a different choice of the $R$-matrix $R(z) = z\mathrm{Id} - P = -R_{\mathrm{rat}}(-z)$ in [4].

Comparing the identities of Lemma 2.49 with the defining relations (2.2)–(2.13) of $Y_{-\mu}(\mathfrak{gl}_n)$ and evoking Corollary 2.48, we immediately obtain:

**Theorem 2.52.** *For any $\mu \in \Lambda^+$, there is a unique $\mathbb{C}$-algebra epimorphism*

$$\Upsilon_{-\mu} \colon Y_{-\mu}(\mathfrak{gl}_n) \twoheadrightarrow Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$$

*defined by*

$$E_i(z) \mapsto e_{i,i+1}(z), \quad F_i(z) \mapsto f_{i+1,i}(z), \quad D_j(z) \mapsto g_j(z). \tag{2.53}$$

Our first main result (the proof of which is postponed till Section 2.4.3) is:

**Theorem 2.54.** $\Upsilon_{-\mu} \colon Y_{-\mu}(\mathfrak{gl}_n) \xrightarrow{\sim} Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$ *is a $\mathbb{C}$-algebra isomorphism for any $\mu \in \Lambda^+$.*

**Remark 2.55.** (a) For $\mu = 0$ and any $n$, the isomorphism $\Upsilon_0 \colon Y(\mathfrak{gl}_n) \xrightarrow{\sim} Y^{\mathrm{rtt}}(\mathfrak{gl}_n)$ of Theorem 2.54 was stated in [8], but was properly established only in [4, Theorem 5.2].
(b) For $n = 2$ and $\mu \in \Lambda^+$, a long straightforward verification shows (see [18, Remark 11.17]) that the assignment

$$\begin{aligned} &t_{11}(z) \mapsto D_1(z), \quad t_{22}(z) \mapsto F_1(z)D_1(z)E_1(z) + D_2(z), \\ &t_{12}(z) \mapsto D_1(z)E_1(z), \quad t_{21}(z) \mapsto F_1(z)D_1(z), \end{aligned}$$

gives rise to an algebra homomorphism $Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_2) \to Y_{-\mu}(\mathfrak{gl}_2)$ (the trigonometric $\mathfrak{sl}_2$-counterpart of this result has been properly established in [18, Theorem 11.11]), which is clearly the inverse of $\Upsilon_{-\mu}$. Thus, Theorem 2.54 for $n = 2$ is essentially due to [18].

## 2.4. Rational Lax matrices via antidominantly shifted Yangians of $\mathfrak{gl}_n$

In this section, we construct $n \times n$ *rational Lax* matrices $T_D(z)$ (with coefficients in $\mathcal{A}((z^{-1}))$) for each $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying (2.30). They are explicitly defined via (2.63), (2.64) combined with (2.58), (2.60), (2.62). We note that these long formulas arise naturally as the image of $T(z) \in Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)((z^{-1})) \otimes_{\mathbb{C}} \mathrm{End}\,\mathbb{C}^n$ under the composition $\Psi_D \circ \Upsilon^{-1}_{-\mu} \colon Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n) \to \mathcal{A}$, assuming Theorem 2.54 has been established, see (2.56), (2.57). As the name indicates, these $T_D(z)$ satisfy the RTT relation (2.41), which is derived in Proposition 2.79. Combining the latter with the results of [40], see Theorem 2.80, we finally prove Theorem 2.54 in Section 2.4.3. We also establish the regularity (up to a rational factor (2.66)) of $T_D(z)$ in Theorem 2.67, and find simplified explicit formulas for those $T_D(z)$ which are linear in $z$ in Theorem 2.90.

### 2.4.1. Construction of $T_D(z)$ and their regularity

Consider a $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$, see (2.28), satisfying the assumption (2.30). Note that $\mu := D|_\infty \in \Lambda^+$. Assuming the validity of Theorem 2.54, let us compose $\Psi_D \colon Y_{-\mu}(\mathfrak{gl}_n) \to \mathcal{A}$ of (2.36) with $\Upsilon^{-1}_{-\mu} \colon Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n) \xrightarrow{\sim} Y_{-\mu}(\mathfrak{gl}_n)$ to obtain an algebra homomorphism

$$\Theta_D = \Psi_D \circ \Upsilon^{-1}_{-\mu} \colon Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n) \longrightarrow \mathcal{A}. \tag{2.56}$$

Such a homomorphism is uniquely determined by $T_D(z) \in \mathcal{A}((z^{-1})) \otimes_{\mathbb{C}} \mathrm{End}\,\mathbb{C}^n$ defined via

$$T_D(z) := \Theta_D(T(z)) = \Theta_D(F(z)) \cdot \Theta_D(G(z)) \cdot \Theta_D(E(z)). \tag{2.57}$$

Let us compute explicitly the images of the matrices $F(z), G(z), E(z)$ under $\Theta_D$, which shall provide an explicit formula for the matrix $T_D(z)$ via (2.57).

Combining $\Upsilon^{-1}_{-\mu}(g_i(z)) = D_i(z)$ with the formula for $\Psi_D(D_i(z))$, we obtain:

$$\Theta_D(g_i(z)) = \frac{P_i(z)}{P_{i-1}(z-1)} \prod_{0 \le k < i} Z_k(z) = \frac{P_i(z)}{P_{i-1}(z-1)} \prod_{x \in \mathbb{P}^1 \setminus \{\infty\}} (z-x)^{-\epsilon_i^\vee(\lambda_x)}. \tag{2.58}$$

Combining $\Upsilon_{-\mu}^{-1}(e_{i,i+1}(z)) = E_i(z)$ with the formula for $\Psi_D(E_i(z))$, we obtain:

$$\Theta_D(e_{i,i+1}(z)) = -\sum_{r=1}^{a_i} \frac{P_{i-1}(p_{i,r}-1)Z_i(p_{i,r})}{(z-p_{i,r})P_{i,r}(p_{i,r})} e^{q_{i,r}}. \tag{2.59}$$

As $e_{ij}(z) = [e^{(1)}_{j-1,j}, \cdots, [e^{(1)}_{i+1,i+2}, e_{i,i+1}(z)] \cdots]$ due to (2.47), we thus get (cf. [19, (2.29)]):

$$\Theta_D(e_{ij}(z)) = \\ - \sum_{\substack{1 \le r_i \le a_i \\ \cdots \\ 1 \le r_{j-1} \le a_{j-1}}} \frac{P_{i-1}(p_{i,r_i}-1)\prod_{k=i}^{j-2} P_{k,r_k}(p_{k+1,r_{k+1}}-1)}{(z-p_{i,r_i})\prod_{k=i}^{j-1} P_{k,r_k}(p_{k,r_k})} \cdot \prod_{k=i}^{j-1} Z_k(p_{k,r_k}) \cdot e^{\sum_{k=i}^{j-1} q_{k,r_k}}. \tag{2.60}$$

Combining $\Upsilon_{-\mu}^{-1}(f_{i+1,i}(z)) = F_i(z)$ with the formula for $\Psi_D(F_i(z))$, we obtain:

$$\Theta_D(f_{i+1,i}(z)) = \sum_{r=1}^{a_i} \frac{P_{i+1}(p_{i,r}+1)}{(z-p_{i,r}-1)P_{i,r}(p_{i,r})} e^{-q_{i,r}}. \tag{2.61}$$

As $f_{ji}(z) = [\cdots [f_{i+1,i}(z), f^{(1)}_{i+2,i+1}], \cdots, f^{(1)}_{j,j-1}]$ due to (2.47), we thus get (cf. [19, (2.30)]):

$$\Theta_D(f_{ji}(z)) = \\ \sum_{\substack{1 \le r_i \le a_i \\ \cdots \\ 1 \le r_{j-1} \le a_{j-1}}} \frac{P_j(p_{j-1,r_{j-1}}+1)\prod_{k=i+1}^{j-1} P_{k,r_k}(p_{k-1,r_{k-1}}+1)}{(z-p_{i,r_i}-1)\prod_{k=i}^{j-1} P_{k,r_k}(p_{k,r_k})} \cdot e^{-\sum_{k=i}^{j-1} q_{k,r_k}}. \tag{2.62}$$

While the above derivation of the formulas (2.58), (2.60), (2.62) is based on yet unproved Theorem 2.54, we shall use their explicit right-hand sides from now on, without any direct referral to Theorem 2.54. More precisely, let us define $\mathcal{A}((z^{-1}))$-valued $n \times n$ diagonal matrix $G_D(z)$, an upper-triangular matrix $E_D(z)$, and a lower-triangular matrix $F_D(z)$, whose matrix coefficients $g^D_i(z), e^D_{ij}(z), f^D_{ji}(z)$ are given by the right-hand sides of (2.58), (2.60), (2.62) expanded in $z^{-1}$, respectively. Thus, we amend (2.57) and define

$$T_D(z) := F_D(z)G_D(z)E_D(z), \tag{2.63}$$

so that the matrix coefficients of $T_D(z)$ are given by

$$T_D(z)_{\alpha,\beta} = \sum_{i=1}^{\min\{\alpha,\beta\}} f^D_{\alpha,i}(z) \cdot g^D_i(z) \cdot e^D_{i,\beta}(z) \tag{2.64}$$

for any $1 \le \alpha, \beta \le n$, where the three factors in the right-hand side of (2.64) are determined via (2.62), (2.58), (2.60), respectively, with the conventions $f^D_{\alpha,\alpha}(z) = 1 = e^D_{\beta,\beta}(z)$.

**Remark 2.65.** We note that $T_D(z)$ is singular at $x \in \mathbb{C}$ if and only if $\lambda_x \neq 0$. As $F_D(z)$ and $E_D(z)$ are regular in the neighborhood of $x$, while $G_D(z) = (\text{regular part}) \cdot (z-x)^{-\lambda_x}$, we see that in the classical limit $T_D(z)$ represents a $GL_n$-multiplicative Higgs field on $\mathbb{P}^1$ with a framing at $\infty \in \mathbb{P}^1$ (rational type) and with prescribed singularities on $D$, cf. [12].

We shall also need the following normalized rational Lax matrices:

$$\mathsf{T}_D(z) := \frac{T_D(z)}{Z_0(z)}, \tag{2.66}$$

with the normalization factor determined via (2.34):

$$\frac{1}{Z_0(z)} = \prod_{1\le s\le N}^{i_s=0} (z-x_s)^{-\gamma_s} = \prod_{x\in\mathbb{P}^1\backslash\{\infty\}} (z-x)^{-\alpha_0^\vee(\lambda_x)}.$$

The first main result of this section establishes the regularity of these matrices:

**Theorem 2.67.** *We have* $\mathsf{T}_D(z) \in \mathcal{A}[z] \otimes_{\mathbb{C}} \operatorname{End}\mathbb{C}^n$.

**Proof.** In view of (2.64), it suffices to prove for any $1 \le \alpha, \beta \le n$ that

$$\frac{1}{Z_0(z)} \sum_{i=1}^{\min\{\alpha,\beta\}} f^D_{\alpha,i}(z) \cdot g^D_i(z) \cdot e^D_{i,\beta}(z) \text{ is polynomial in } z, \tag{2.68}$$

where the factors in the right-hand side are determined via (2.62), (2.58), (2.60), respectively.

The $i$-th summand in (2.68) is explicitly given by

$$\begin{aligned}
&Z_0(z)^{-1} \cdot f^D_{\alpha,i}(z) \cdot g^D_i(z) \cdot e^D_{i,\beta}(z) = \\
&- \sum_{\substack{1\le r_i\le a_i \\ \cdots \\ 1\le r_{\alpha-1}\le a_{\alpha-1}}} \frac{P_{i+1,r_{i+1}}(p_{i,r_i}+1)\cdots P_{\alpha-1,r_{\alpha-1}}(p_{\alpha-2,r_{\alpha-2}}+1)P_\alpha(p_{\alpha-1,r_{\alpha-1}}+1)}{(z-p_{i,r_i}-1)P_{i,r_i}(p_{i,r_i})\cdots P_{\alpha-1,r_{\alpha-1}}(p_{\alpha-1,r_{\alpha-1}})} \\
&\times e^{-q_{i,r_i}-q_{i+1,r_{i+1}}-\ldots-q_{\alpha-1,r_{\alpha-1}}} \cdot \frac{P_i(z)}{P_{i-1}(z-1)} \cdot Z_1(z)\cdots Z_{i-1}(z) \\
&\times \sum_{\substack{1\le s_i\le a_i \\ \cdots \\ 1\le s_{\beta-1}\le a_{\beta-1}}} \frac{P_{i-1}(p_{i,s_i}-1)P_{i,s_i}(p_{i+1,s_{i+1}}-1)\cdots P_{\beta-2,s_{\beta-2}}(p_{\beta-1,s_{\beta-1}}-1)}{(z-p_{i,s_i})P_{i,s_i}(p_{i,s_i})\cdots P_{\beta-1,s_{\beta-1}}(p_{\beta-1,s_{\beta-1}})} \\
&\times Z_i(p_{i,s_i})\cdots Z_{\beta-1}(p_{\beta-1,s_{\beta-1}}) \cdot e^{q_{i,s_i}+q_{i+1,s_{i+1}}+\ldots+q_{\beta-1,s_{\beta-1}}}.
\end{aligned} \tag{2.69}$$

Moving $e^{-q_{i,r_i}-\ldots-q_{\alpha-1,r_{\alpha-1}}}$ to the rightmost side, we rewrite the right-hand side of (2.69) as

$$\sum_{\substack{1\le s_i\le a_i\\ \cdots\\ 1\le s_{\beta-1}\le a_{\beta-1}\\ 1\le r_i\le a_i\\ \cdots\\ 1\le r_{\alpha-1}\le a_{\alpha-1}}} Q^{s_i,\ldots,s_{\beta-1}}_{r_i,\ldots,r_{\alpha-1}}(z)\cdot e^{-q_{i,r_i}-\ldots-q_{\alpha-1,r_{\alpha-1}}}\cdot e^{q_{i,s_i}+\ldots+q_{\beta-1,s_{\beta-1}}}.$$

The coefficient $Q^{s_i,\ldots,s_{\beta-1}}_{r_i,\ldots,r_{\alpha-1}}(z)$ is a rational function in $z$ with simple poles at:

- $\{1+p_{i-1,s}\,|\,1\le s\le a_{i-1}\}$ if $r_i\ne s_i$;
- $\{1+p_{i-1,s}\,|\,1\le s\le a_{i-1}\}\cup\{1+p_{i,r_i}\}$ if $r_i=s_i$.

Thus, the only (at most simple) poles of (2.68) are at $\{1+p_{i,r}|1\le i<\min\{\alpha,\beta\},1\le r\le a_i\}$. The following straightforward result actually shows that the residues at these points vanish:

**Lemma 2.70.** *For any $1\le i<\min\{\alpha,\beta\}, 1\le r\le a_i$, and any admissible collection of indices $r_{i+1},\ldots,r_{\alpha-1},s_{i+1},\ldots,s_{\beta-1}$, we have the equality*

$$\mathrm{Res}_{z=1+p_{i,r}}\left(Q^{s_{i+1},\ldots,s_{\beta-1}}_{r_{i+1},\ldots,r_{\alpha-1}}(z)\mathrm{d}z\right)+\mathrm{Res}_{z=1+p_{i,r}}\left(Q^{r,s_{i+1},\ldots,s_{\beta-1}}_{r,r_{i+1},\ldots,r_{\alpha-1}}(z)\mathrm{d}z\right)=0. \qquad (2.71)$$

**Proof of Lemma 2.70.** Applying the explicit formula (2.69), we find

$$\begin{aligned}&Q^{s_{i+1},\ldots,s_{\beta-1}}_{r_{i+1},\ldots,r_{\alpha-1}}(z)e^{-q_{i+1,r_{i+1}}}\\ &\quad=\frac{A}{z-p_{i+1,r_{i+1}}-1}\cdot e^{-q_{i+1,r_{i+1}}}\cdot\frac{P_{i+1}(z)}{P_i(z-1)}\cdot Z_i(z)\cdot\frac{P_i(p_{i+1,s_{i+1}}-1)}{z-p_{i+1,s_{i+1}}}\end{aligned}$$

and

$$\begin{aligned}Q^{r,s_{i+1},\ldots,s_{\beta-1}}_{r,r_{i+1},\ldots,r_{\alpha-1}}(z)e^{-q_{i+1,r_{i+1}}}&=\frac{A\cdot P_{i+1,r_{i+1}}(p_{i,r}+1)}{(z-p_{i,r}-1)P_{i,r}(p_{i,r})}\cdot e^{-q_{i,r}-q_{i+1,r_{i+1}}}\\ &\quad\times\frac{P_i(z)P_{i-1}(p_{i,r}-1)P_{i,r}(p_{i+1,s_{i+1}}-1)Z_i(p_{i,r})}{P_{i-1}(z-1)(z-p_{i,r})P_{i,r}(p_{i,r})}\cdot e^{q_{i,r}},\end{aligned}$$

where $A$ is a common $(z,p_{i+1,r_{i+1}})$-independent factor (its explicit form is irrelevant for us). Hence,

$$Q^{s_{i+1},\ldots,s_{\beta-1}}_{r_{i+1},\ldots,r_{\alpha-1}}(z)=A\cdot\frac{P_{i+1,r_{i+1}}(z)}{P_{i,r}(z-1)(z-p_{i,r}-1)}\cdot Z_i(z)\cdot\frac{P_i(p_{i+1,s_{i+1}}-1+\delta_{r_{i+1},s_{i+1}})}{z-p_{i+1,s_{i+1}}-\delta_{r_{i+1},s_{i+1}}}$$

and

$$Q^{r,s_{i+1},\ldots,s_{\beta-1}}_{r,r_{i+1},\ldots,r_{\alpha-1}}(z) = A \cdot \frac{P_{i+1,r_{i+1}}(p_{i,r}+1)}{P_{i,r}(p_{i,r})} \cdot \frac{P_{i,r}(z)}{P_{i-1}(z-1)} \\ \times \frac{P_{i-1}(p_{i,r})P_{i,r}(p_{i+1,s_{i+1}}-1+\delta_{r_{i+1},s_{i+1}})}{P_{i,r}(p_{i,r}+1)(z-p_{i,r}-1)} Z_i(p_{i,r}+1).$$

Therefore, the corresponding residues are given by

$$\mathrm{Res}_{z=1+p_{i,r}}\left(Q^{s_{i+1},\ldots,s_{\beta-1}}_{r_{i+1},\ldots,r_{\alpha-1}}(z)\mathrm{d}z\right) = \\ A \cdot \frac{P_{i+1,r_{i+1}}(p_{i,r}+1)}{P_{i,r}(p_{i,r})} \cdot Z_i(p_{i,r}+1) \cdot \left(-P_{i,r}(p_{i+1,s_{i+1}}-1+\delta_{r_{i+1},s_{i+1}})\right),$$

$$\mathrm{Res}_{z=1+p_{i,r}}\left(Q^{r,s_{i+1},\ldots,s_{\beta-1}}_{r,r_{i+1},\ldots,r_{\alpha-1}}(z)\mathrm{d}z\right) = \\ A \cdot \frac{P_{i+1,r_{i+1}}(p_{i,r}+1)}{P_{i,r}(p_{i,r})} \cdot \frac{P_{i,r}(p_{i,r}+1)}{P_{i-1}(p_{i,r})} \cdot \frac{P_{i-1}(p_{i,r})P_{i,r}(p_{i+1,s_{i+1}}-1+\delta_{r_{i+1},s_{i+1}})}{P_{i,r}(p_{i,r}+1)} \\ \times Z_i(p_{i,r}+1),$$

thus summing up to zero and implying (2.71). □

This completes our proof of (2.68) and, hence, of Theorem 2.67. □

**Remark 2.72.** We note that a similar cancelation of poles appeared in the work on $q$-characters [16] and $qq$-characters [30].

**Remark 2.73.** Similar to [3, Theorem B.15], one can generalize Theorem 2.35 by constructing the homomorphisms $\Psi_D \colon Y_{-\mu}(\mathfrak{gl}_n) \to \mathcal{A}$ for any orientation of $A_{n-1}$ Dynkin diagram (so that $\Psi_D \circ \iota_{-\mu} = \Phi^{\bar\lambda}_{-\bar\mu}$ as in Remark 2.38, while the images of $D_i(z)$ are given by the same formulas as in (2.37)). However, extending $\mathcal{A}$ to its localization $\mathcal{A}_{\mathrm{loc}}$ by the multiplicative set generated by $\{p_{i,r} - p_{i+1,s} + m\}^{m\in\mathbb{Z}}_{r\le a_i, s\le a_{i+1}}$, all such homomorphisms are compositions of the one from (2.36) with algebra automorphisms of $\mathcal{A}_{\mathrm{loc}}$. Thus, the resulting rational Lax matrices are equivalent to $T_D(z)$ constructed above via algebra automorphisms of $\mathcal{A}_{\mathrm{loc}}$, cf. Remark 2.27.

*2.4.2. Normalized limit description and the RTT relation for $T_D(z)$*

Consider a $\Lambda^+$-valued divisor $D = \sum_{s=1}^N \gamma_s \varpi_{i_s}[x_s] + \mu[\infty]$. As $x_N \to \infty$, we obtain another $\Lambda^+$-valued divisor $D' = \sum_{s=1}^{N-1} \gamma_s \varpi_{i_s}[x_s] + (\mu + \gamma_N \varpi_{i_N})[\infty]$. We will relate $T_{D'}(z)$ to $T_D(z)$.

If $i_N = 0$, then

$$T_{D'}(z) = (z - x_N)^{-\gamma_N} T_D(z), \tag{2.74}$$

due to $F_D(z) = F_{D'}(z), E_D(z) = E_{D'}(z), G_D(z) = (z - x_N)^{\gamma_N} G_{D'}(z)$ and (2.63).

Let us now consider the case $1 \leq i_N \leq n-1$ (note that $\gamma_N = 1$), so that $(-x_N)^{\varpi_{i_N}} = \mathrm{diag}(1^{i_N}, (-x_N^{-1})^{n-i_N})$ is the diagonal $n \times n$ matrix with the first $i_N$ diagonal entries equal to 1 and the remaining $n - i_N$ entries equal to $-x_N^{-1}$.

**Proposition 2.75.** *The $x_N \to \infty$ limit of $T_D(z) \cdot (-x_N)^{\varpi_{i_N}}$ equals $T_{D'}(z)$.*

**Proof.** According to (2.63), $T_D(z) = F_D(z)G_D(z)E_D(z)$ and $T_{D'}(z) = F_{D'}(z)G_{D'}(z) \times E_{D'}(z)$ with the three factors determined explicitly via (2.62), (2.58), (2.60). Hence, $T_D(z) \cdot (-x_N)^{\varpi_{i_N}}$ has the following Gauss decomposition:

$$T_D(z) \cdot (-x_N)^{\varpi_{i_N}} = F_D(z) \cdot (G_D(z)(-x_N)^{\varpi_{i_N}}) \cdot \left((-x_N)^{-\varpi_{i_N}} E_D(z)(-x_N)^{\varpi_{i_N}}\right). \quad (2.76)$$

The leftmost factor in the right-hand side of (2.76) does not depend on $\{x_s\}_{s=1}^N$ and coincides with $F_{D'}(z)$. As $G_D(z) = (z - x_N)^{-\varpi_{i_N}} G_{D'}(z)$ and $\lim_{x_N \to \infty} \frac{z-x_N}{-x_N} = 1$, it is clear that the $x_N \to \infty$ limit of the diagonal factor $G_D(z)(-x_N)^{\varpi_{i_N}}$ in (2.76) coincides with $G_{D'}(z)$. Finally, the matrix coefficients of the upper-triangular factor in (2.76) are $((-x_N)^{-\varpi_{i_N}} E_D(z)(-x_N)^{\varpi_{i_N}})_{\alpha,\beta} = e^D_{\alpha,\beta}(z) \cdot (-x_N)^{-\delta_{\alpha \leq i_N < \beta}}$ and their $x_N \to \infty$ limits exactly coincide with $e^{D'}_{\alpha,\beta}(z)$, the matrix coefficients of $E_{D'}(z)$.

This completes our proof of Proposition 2.75. $\square$

**Corollary 2.77.** *$T_{D'}(z)$ is a* normalized limit *of $T_D(z)$.*

For $D$ as above, we can pick a $\Lambda^+$-valued divisor $\bar{D} = \sum_{s=1}^{N+M} \gamma_s \varpi_{i_s}[x_s]$, so that $\{x_s\}_{s=N+1}^{N+M}$ are some points on $\mathbb{P}^1 \backslash \{\infty\}$ while $\sum_{s=N+1}^{N+M} \gamma_s \varpi_{i_s} = \mu$. Note that $\infty \notin \mathrm{supp}(\bar{D})$, i.e. $\bar{D}|_\infty = 0$.

**Corollary 2.78.** *For any $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying (2.30), the matrix $T_D(z)$ of (2.64) is a normalized limit of $T_{\bar{D}}(z)$ with a $\Lambda^+$-valued divisor $\bar{D}$ satisfying $\bar{D}|_\infty = 0$.*

Evoking Remark 2.55(a), we see that the original definition of $T_{\bar{D}}(z)$ via (2.56), (2.57) is valid. Hence, $T_{\bar{D}}(z)$ defined via (2.64) indeed satisfies the RTT relation (2.41). As a multiplication by diagonal $z$-independent matrices preserves (2.41), we obtain the main result of this section:

**Proposition 2.79.** *For any $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying the assumption (2.30), the matrix $T_D(z)$ defined via (2.63), (2.64) is Lax, i.e. it satisfies the RTT relation (2.41).*

*2.4.3. Proof of Theorem 2.54*

Due to Proposition 2.79 and the Gauss decomposition (2.63), (2.64) of $T_D(z)$ with the factors defined via (2.58), (2.60), (2.62), we see that $T_D(z)$ indeed gives rise to the algebra homomorphism $\Theta_D \colon Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n) \to \mathcal{A}$, whose composition with the epimorphism $\Upsilon_{-\mu} \colon Y_{-\mu}(\mathfrak{gl}_n) \twoheadrightarrow Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$ of Theorem 2.52 coincides with $\Psi_D$ of (2.36). Thus, for $\mu \in \Lambda^+$ and any $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$, satisfying (2.30) and $D|_\infty = \mu$, the homomorphism $\Psi_D$ does factor through $\Upsilon_{-\mu}$.

This observation immediately implies the injectivity of $\Upsilon_{-\mu}$, due to the following recent result of Alex Weekes (actually, we need its $\mathfrak{gl}_n$-counterpart that follows from (2.25) and (2.39)):

**Theorem 2.80** *([40]). For any coweight $\nu$ of a semisimple Lie algebra $\mathfrak{g}$, the intersection of kernels of the homomorphisms $\Phi^*_{-\nu}$ of [3, Theorem B.15] is zero: $\bigcap_\lambda \operatorname{Ker}(\Phi^\lambda_{-\nu}) = 0$, where $\lambda$ ranges through all dominant coweights of $\mathfrak{g}$ such that $\lambda + \nu = \sum a_i\alpha_i$ with $a_i \in \mathbb{N}$, $\alpha_i$ being simple coroots of $\mathfrak{g}$, and points $\{z_i\}$ of [3] specialized to arbitrary complex parameters.*

This completes our proof of Theorem 2.54.

**Remark 2.81.** (A. Weekes) Using similar arguments, one can show that the center of the shifted Yangian $Y_\nu(\mathfrak{g})$ is trivial (thus implying Lemma 2.26(a)) for any coweight $\nu$ of a semisimple Lie algebra $\mathfrak{g}$. Indeed, due to Theorem 2.80, it suffices to show that the $\Phi^\lambda_\nu$-images have no nonconstant central elements. Assuming $x$ is central, one can show it is a symmetric rational function in $p_{*,*}$ (as $\operatorname{Im}(\Phi^\lambda_\nu)$ contains all symmetric polynomials in $p_{*,*}$), and then show that it is actually $p_{*,*}$-independent (using the commutativity with the images of $E_i(z), F_i(z)$).

The above argument can also be used to identify the image of the central series $C(z)$ (2.23) under the isomorphism $\Upsilon_{-\mu}$ with the *quantum determinant* $\operatorname{qdet} T(z)$ of $Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$ defined via:

$$\operatorname{qdet} T(z) := \sum_{\sigma \in S_n} (-1)^{\ell(\sigma)} t_{1,\sigma(1)}(z+n-1)t_{2,\sigma(2)}(z+n-2)\cdots t_{n-1,\sigma(n-1)}(z+1)t_{n,\sigma(n)}(z). \tag{2.82}$$

**Proposition 2.83.** *For any $\mu \in \Lambda^+$, the series (2.23) and (2.82) are related via*

$$\Upsilon_{-\mu}(C(z)) = \operatorname{qdet} T(z). \tag{2.84}$$

**Proof.** According to (the $\mathfrak{gl}_n$-counterpart of) Theorem 2.80 and (2.39), it suffices to verify:

$$\operatorname{qdet} T_D(z) = \prod_{s=1}^{N} \prod_{k=i_s}^{n-1} (z - x_s + k)^{\gamma_s} \tag{2.85}$$

for any $\Lambda^+$-valued divisor $D = \sum_{s=1}^N \gamma_s \varpi_{i_s}[x_s] + \mu[\infty]$ on $\mathbb{P}^1$ with $x_s \in \mathbb{C}$, as in (2.28), satisfying the assumption (2.30) and $D|_\infty = \mu$. According to [4, Theorem 8.6], the equality (2.84) holds for $\mu = 0$, and consequently the equality (2.85) holds for those $D$ such that $D|_\infty = 0$.

Next, using the notations of Section 2.4.2, we note that the validity of (2.85) for $D$ implies the one for $D'$ as follows from the following equalities:

$$\operatorname{qdet} T_{D'}(z) = \lim_{x_N \to \infty} \operatorname{qdet}\left(T_D(z) \cdot (-x_N)^{\gamma_N \varpi_{i_N}}\right) =$$

$$\lim_{x_N\to\infty}\left(\prod_{s=1}^{N}\prod_{k=i_s}^{n-1}(z-x_s+k)^{\gamma_s}\cdot(-x_N)^{-\gamma_N(n-i_N)}\right) = \prod_{s=1}^{N-1}\prod_{k=i_s}^{n-1}(z-x_s+k)^{\gamma_s}.$$

Therefore, the validity of (2.85) for any $D$ follows from its special case $D|_\infty = 0$ (established above) combined with Corollary 2.78. $\square$

Combining this result with Lemma 2.26(b), we obtain:

**Corollary 2.86.** *For any $\mu \in \Lambda^+$, the center of the shifted RTT Yangian $Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$ is a polynomial algebra in the coefficients of the quantum determinant* $\operatorname{qdet} T(z)$ *defined via* (2.82)*.*

*2.4.4. Linear rational Lax matrices*

In this section, we will obtain simplified explicit formulas for all $\mathsf{T}_D(z)$ that are linear in $z$.

First, let us note that elements of $\Lambda^+$ may be encoded by weakly decreasing sequences $\boldsymbol{\lambda}$ of $n$ integers $\boldsymbol{\lambda}_1 \geq \cdots \geq \boldsymbol{\lambda}_n$, which we call *pseudo Young diagrams* with $n$ rows (in mathematical literature, they are also called *signatures* of length $n$, following Hermann Weyl). Explicitly, such a pseudo Young diagram $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_n)$ encodes a dominant coweight $\lambda \in \Lambda^+$ via

$$\lambda := -\sum_{1\leq i\leq n} \boldsymbol{\lambda}_{n-i+1}\epsilon_i = \boldsymbol{\lambda}_n\varpi_0 + \sum_{1\leq i\leq n-1}(\boldsymbol{\lambda}_{n-i} - \boldsymbol{\lambda}_{n-i+1})\varpi_i. \tag{2.87}$$

We denote $|\boldsymbol{\lambda}| := \sum_{i=1}^n \boldsymbol{\lambda}_i$. If $\boldsymbol{\lambda}_n \geq 0$, then $\boldsymbol{\lambda}$ is a standard Young diagram of length $\leq n$.

Fix a pair of pseudo Young diagrams $\boldsymbol{\lambda}, \boldsymbol{\mu}$. Then, $\lambda+\mu$ is of the form $\lambda+\mu = \sum_{i=1}^{n-1} a_i\alpha_i$ for some $a_i \in \mathbb{C}$ iff $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}| = 0$. Let us establish the key properties of $a_i$ in the latter case:

**Lemma 2.88.** *(a) $a_i = -\sum_{j=n-i+1}^{n}(\boldsymbol{\lambda}_j + \boldsymbol{\mu}_j)$ for any $1 \leq i \leq n-1$.*
*(b) $a_i \in \mathbb{N}$ for any $1 \leq i \leq n-1$.*
*(c) $a_j - a_{j-1} = -\boldsymbol{\lambda}_{n-j+1} - \boldsymbol{\mu}_{n-j+1}$ for any $1 \leq j \leq n$, where we set $a_0 := 0, a_n := 0$.*

**Proof.** (c) Follows from the equality

$$\sum_{1\leq j\leq n}(a_j - a_{j-1})\epsilon_j = \sum_{1\leq i\leq n-1} a_i\alpha_i = \lambda+\mu = \sum_{1\leq j\leq n}(-\boldsymbol{\lambda}_{n-j+1} - \boldsymbol{\mu}_{n-j+1})\epsilon_j.$$

(a) Follows by summing the equalities of part (c) for $j = 1, \ldots, i$.

(b) As $-\boldsymbol{\lambda}_n - \boldsymbol{\mu}_n \geq -\boldsymbol{\lambda}_{n-1} - \boldsymbol{\mu}_{n-1} \geq \ldots \geq -\boldsymbol{\lambda}_1 - \boldsymbol{\mu}_1$, we have an obvious inequality $\sum_{j=n-i+1}^{n}(-\boldsymbol{\lambda}_j - \boldsymbol{\mu}_j) \geq \frac{i}{n}\sum_{j=1}^{n}(-\boldsymbol{\lambda}_j - \boldsymbol{\mu}_j) = \frac{-i}{n}(|\boldsymbol{\lambda}| + |\boldsymbol{\mu}|) = 0$. Hence, $a_i \in \mathbb{N}$ by part (a). □

Thus, $\Lambda^+$-valued divisors on $\mathbb{P}^1$ satisfying (2.30) and without summands $\{-\varpi_0[x]\}_{x\in\mathbb{C}}$ may be encoded by pairs $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ of a Young diagram $\boldsymbol{\lambda}$ of length $\leq n$ and a pseudo Young diagram $\boldsymbol{\mu}$ with $n$ rows and of total size $|\boldsymbol{\lambda}|+|\boldsymbol{\mu}| = 0$, together with a collection of points $\underline{x} = \{x_i\}_{i=1}^{\boldsymbol{\lambda}_1}$ of $\mathbb{C}$ (so that $x_i$ is assigned to the $i$-th column of $\boldsymbol{\lambda}$). Explicitly, given $\boldsymbol{\lambda}, \boldsymbol{\mu}, \underline{x}$ as above, we set $D = D(\boldsymbol{\lambda}, \underline{x}, \boldsymbol{\mu}) := \sum_{i=1}^{\boldsymbol{\lambda}_1} \varpi_{n-\boldsymbol{\lambda}_i^t}[x_i] + \mu[\infty]$, where $\boldsymbol{\lambda}_i^t$ is the height of the $i$-th column of $\boldsymbol{\lambda}$.

Due to (2.74), we shall assume that $D$ does not contain summands $\{\pm\varpi_0[x]\}_{x\in\mathbb{C}}$. Thus, $\boldsymbol{\lambda}_n = 0$ so that $Z_0(z) = 1$, and $T_D(z) = \mathsf{T}_D(z)$ is polynomial in $z$ by Theorem 2.67. Moreover, $T_D(z)_{11} = g_1^D(z)$ is a polynomial in $z$ of degree $a_1 = -(\boldsymbol{\lambda}_n + \boldsymbol{\mu}_n) = -\boldsymbol{\mu}_n \geq 0$. Hence, we have $-\boldsymbol{\mu}_n \leq 1$ for linear Lax matrices $T_D(z)$. If $\boldsymbol{\mu}_n = 0$, then $\boldsymbol{\lambda}_i = \boldsymbol{\mu}_i = 0$ for all $i$, since $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}| = 0$, and so $T_D(z) = \mathsf{T}_D(z) = I_n$, the identity matrix. Therefore, it remains to treat the case when $\boldsymbol{\lambda}_n = 0$ and $\boldsymbol{\mu}_n = -1$, which constitutes the key result of this section.

**Remark 2.89.** If $|\boldsymbol{\lambda}|+|\boldsymbol{\mu}| = 0, \boldsymbol{\lambda}_n = 0, \boldsymbol{\mu}_n = -1$, then $\boldsymbol{\lambda}$ and $\widetilde{\boldsymbol{\mu}} = (\boldsymbol{\mu}_1+1, \ldots, \boldsymbol{\mu}_n+1)$ form a pair of Young diagrams of total size $|\boldsymbol{\lambda}| + |\widetilde{\boldsymbol{\mu}}| = n$. In that setup, an alternative construction of rational Lax matrices $L_{\boldsymbol{\lambda},\underline{x},\widetilde{\boldsymbol{\mu}}}(z)$ was recently proposed in [15]. In Section 2.5, we shall compare all explicit Lax matrices $L_{\boldsymbol{\lambda},\underline{x},\widetilde{\boldsymbol{\mu}}}(z)$ of [15] to the corresponding Lax matrices $T_D(z)$. However, we do not have an interpretation of the “fusion procedure” of [15] (used to construct all $L_{\boldsymbol{\lambda},\underline{x},\widetilde{\boldsymbol{\mu}}}(z)$ from the aforementioned explicit “building blocks”) in the present approach.

**Theorem 2.90.** *Following the above notations, assume further that $\boldsymbol{\lambda}_n = 0$ and $\boldsymbol{\mu}_n = -1$. Define $m := \max\{i \,|\, \boldsymbol{\mu}_{n-i+1} = -1\}$ and $m' := \max\{i \,|\, \boldsymbol{\mu}_{n-i+1} \leq 0\}$.*
*(a) The rational Lax matrix $\mathsf{T}_D(z)$ is explicitly determined as follows:*
*(I) The matrix coefficients on the main diagonal are:*

$$\mathsf{T}_D(z)_{ii} = \begin{cases} z + \sum_{r=1}^{a_{i-1}}(p_{i-1,r}+1) - \sum_{r=1}^{a_i} p_{i,r} + \sum_{x\in\mathbb{P}^1\backslash\{\infty\}} \epsilon_i^\vee(\lambda_x)x & \text{if } \ i \leq m \\ 1 & \text{if } \ m < i \leq m' \\ 0 & \text{if } \ i > m' \end{cases}. \tag{2.91}$$

*(II) The matrix coefficients above the main diagonal are:*

$$\mathsf{T}_D(z)_{ij} = 0 \quad \text{if } \ m < i < j, \tag{2.92}$$

$$\mathsf{T}_D(z)_{ij} = \\ -\sum_{\substack{1\le r_i\le a_i\\ \cdots\\ 1\le r_{j-1}\le a_{j-1}}} \frac{P_{i-1}(p_{i,r_i}-1)\prod_{k=i}^{j-2}P_{k,r_k}(p_{k+1,r_{k+1}}-1)}{\prod_{k=i}^{j-1}P_{k,r_k}(p_{k,r_k})}\cdot\prod_{k=i}^{j-1}Z_k(p_{k,r_k})\cdot e^{\sum_{k=i}^{j-1}q_{k,r_k}}$$

*if* $i\le m$ and $i<j$.

$$(2.93)$$

*(III) The matrix coefficients below the main diagonal are:*

$$\mathsf{T}_D(z)_{ji}=0 \quad \textit{if}\;\; m<i<j, \tag{2.94}$$

$$\mathsf{T}_D(z)_{ji} = \sum_{\substack{1\le r_i\le a_i\\ \cdots\\ 1\le r_{j-1}\le a_{j-1}}} \frac{P_j(p_{j-1,r_{j-1}}+1)\prod_{k=i+1}^{j-1}P_{k,r_k}(p_{k-1,r_{k-1}}+1)}{\prod_{k=i}^{j-1}P_{k,r_k}(p_{k,r_k})}\cdot e^{-\sum_{k=i}^{j-1}q_{k,r_k}}$$

*if* $i\le m$ *and* $i<j$.

$$(2.95)$$

*(b)* $\mathsf{T}_D(z)=T_D(z)$ *is polynomial of degree* 1 *in* $z$*, and the coefficient of* $z$ *equals* $\sum_{i=1}^m E_{ii}$*.*

**Proof.** (a) Combining the explicit formulas (2.64), (2.66) for the matrix coefficients $\mathsf{T}_D(z)_{\alpha,\beta}$ with their polynomiality of Theorem 2.67, we may immediately determine all of them explicitly. The latter is based on the following observations:

- The leading power of $z$ in $e^D_{ij}(z)$ given by the right-hand side of (2.60) expanded in $z^{-1}$ equals $-1$, while the coefficient of $z^{-1}$ is exactly the right-hand side of (2.93) for any $i<j$.
- The leading power of $z$ in $f^D_{ji}(z)$ given by the right-hand side of (2.62) expanded in $z^{-1}$ equals $-1$, while the coefficient of $z^{-1}$ is exactly the right-hand side of (2.95) for any $i<j$.
- The leading power of $z$ in $Z_0(z)^{-1}g^D_i(z)=g^D_i(z)$ expanded in $z^{-1}$, cf. (2.58), equals

$$a_i-a_{i-1}+(\epsilon_1^\vee-\epsilon_i^\vee)(\lambda)=(-\boldsymbol{\lambda}_{n-i+1}-\boldsymbol{\mu}_{n-i+1})+(-\boldsymbol{\lambda}_n+\boldsymbol{\lambda}_{n-i+1})=-\boldsymbol{\mu}_{n-i+1},$$

  due to Lemma 2.88(c) and the assumption $\boldsymbol{\lambda}_n=0$. By the definition of $m$ and $m'$, we note that $-\boldsymbol{\mu}_{n-i+1}$ is negative if $i>m'$, is zero if $m<i\le m'$, and equals 1 if $i\le m$, while the corresponding coefficient of $z^{-\boldsymbol{\mu}_{n-i+1}}$ equals 1. Finally, for $i\le m$, the coefficient of $z^0$ in $Z_0(z)^{-1}g^D_i(z)$ equals $\sum_{r=1}^{a_{i-1}}(p_{i-1,r}+1)-\sum_{r=1}^{a_i}p_{i,r}+\sum_{x\in\mathbb{P}^1\backslash\{\infty\}}\epsilon_i^\vee(\lambda_x)x$.

Part (b) follows immediately from part (a). □

**Remark 2.96.** Applying Theorem 2.90 for $n=2$, we obtain three $2\times 2$ rational Lax matrices

$$\begin{pmatrix} z-p & -e^{q} \\ e^{-q} & 0 \end{pmatrix}, \quad \begin{pmatrix} z-p & -(p-x_1)e^{q} \\ e^{-q} & 1 \end{pmatrix}, \quad \begin{pmatrix} z-p & -(p-x_1)(p-x_2)e^{q} \\ e^{-q} & z+p+1-x_1-x_2 \end{pmatrix}, \tag{2.97}$$

corresponding to $\boldsymbol{\lambda} = (0,0)$ and $\boldsymbol{\mu} = (1,-1)$, $\boldsymbol{\lambda} = (1,0)$ and $\boldsymbol{\mu} = (0,-1)$, $\boldsymbol{\lambda} = (2,0)$ and $\boldsymbol{\mu} = (-1,-1)$, respectively (as $a_1 = 1$, we relabeled $p_1, q_1$ by $p, q$). These are the well-known $2 \times 2$ elementary Lax matrices for the Toda chain, the DST chain, and the Heisenberg magnet.

**Remark 2.98.** At this point, it is instructive to discuss higher $z$-degree Lax matrices for $n = 2$. Fix a positive integer $N$ and let $\mathcal{A}_N$ denote the algebra $\mathcal{A}$ of (2.32) with $n = 2, a_1 = N$. To simplify our notations, we shall denote the generators $\{p_{1,r}, e^{\pm q_{1,r}}\}_{r=1}^{N}$ simply by $\{p_r, e^{\pm q_r}\}_{r=1}^{N}$.

Let $L_r(z) = \begin{pmatrix} z-p_r & -e^{q_r} \\ e^{-q_r} & 0 \end{pmatrix}, 1 \leq r \leq N$, be the $2 \times 2$ elementary Lax matrices for the Toda chain, and consider the *complete monodromy matrix*

$$T_N(z) := L_1(z) \cdots L_N(z) = \begin{pmatrix} A_N(z) & B_N(z) \\ C_N(z) & D_N(z) \end{pmatrix}. \tag{2.99}$$

Note that the matrix coefficients $A_N(z), B_N(z), C_N(z), D_N(z)$ are polynomials in $z$ with coefficients in the algebra $\mathcal{A}_1^{\otimes N}$ of degrees $N, N-1, N-1, N-2$, respectively. For any $\epsilon \in \mathbb{C}$, the coefficients in powers of $z$ of the linear combination $A_N(z) + \epsilon D_N(z)$ pairwise commute and coincide with Hamiltonians of the quantum closed Toda system of $GL_N$, due to [38].

Following Remark 2.96 and our construction (2.56), (2.57) of rational Lax matrices $T_*(z)$, we note that local Lax matrices $L_r(z)$ encode the homomorphisms $\Psi_{\alpha[\infty]} \colon Y_{-\alpha}(\mathfrak{gl}_2) \to \mathcal{A}_1$ of (2.36), where $\alpha := \alpha_1 = -\varpi_0 + 2\varpi_1$ is a simple coroot of $\mathfrak{sl}_2$. Furthermore, evoking the coproduct homomorphisms of Propositions 2.136 and 2.143 below, we see that the complete monodromy matrix $T_N(z)$ of (2.99) encodes the homomorphism $Y_{-N\alpha}(\mathfrak{gl}_2) \to \mathcal{A}_1^{\otimes N}$ obtained as a composition of the iterated coproduct homomorphism $Y_{-N\alpha}(\mathfrak{gl}_2) \to Y_{-\alpha}(\mathfrak{gl}_2)^{\otimes N}$ and the homomorphism $\Psi_{\alpha[\infty]}^{\otimes N} \colon Y_{-\alpha}(\mathfrak{gl}_2)^{\otimes N} \to \mathcal{A}_1^{\otimes N}$.

On the other hand, consider the rational Lax matrix $T_D(z)$ for the $\Lambda^+$-valued divisor $D = N\alpha[\infty]$ on $\mathbb{P}^1$. According to Theorem 2.67, the matrix coefficients of $T_D(z)$ are polynomials in $z$ with coefficients in the algebra $\mathcal{A}_N$. Moreover, evoking formulas (2.58), (2.60), (2.62), we find:

$$T_D(z)_{11} = P(z), \quad T_D(z)_{12} = -\sum_{r=1}^{N} \frac{P_r(z)}{P_r(p_r)} e^{q_r}, \quad T_D(z)_{21} = \sum_{r=1}^{N} \frac{P_r(z)}{P_r(p_r)} e^{-q_r},$$

$$T_D(z)_{22} = \frac{1}{P(z-1)} - \sum_{1 \leq r \leq N} \frac{P_r(z)}{(z-p_r-1) P_r(p_r) P_r(p_r+1)} - \sum_{1 \leq r \neq s \leq N} \frac{P_{r,s}(z)}{P_{r,s}(p_r) P_{r,s}(p_s)(p_r-p_s)(p_s-p_r-1)} e^{q_s - q_r},$$

where

$$P(z):=\prod_{r=1}^{N}(z-p_r),\quad P_r(z):=\prod_{1\le s\le N}^{s\neq r}(z-p_s),\quad P_{r,s}(z):=\prod_{1\le t\le N}^{t\neq r,s}(z-p_t),$$

cf. (2.34). Due to the RTT relation (2.41) for $T_D(z)$, the coefficients in powers of $z$ of the linear combination $T_D(z)_{11}+\epsilon T_D(z)_{22}$ pairwise commute and define a quantum integrable system. These commuting Hamiltonians can be constructed by applying (1.1) to the Lax matrix $T_D(z)$ with $g_\infty=\begin{pmatrix}1&0\\0&\epsilon\end{pmatrix}$, where $\epsilon$ is known as the *coupling constant*.

The classical limits of the above two quantum integrable systems coincide and recover the well-known *Atiyah-Hitchin integrable system*, see [1] (we note that the identification of the corresponding quantum integrable systems was established in [14, Theorem 6.12]). Its phase space $Z^N$, known as the space of *$SU(2)$-monopoles of topological charge $N$*, consists of degree $N$ based rational maps from $\mathbb{P}^1$ to the flag variety $\mathcal{B}$ of $SL_2$ (note that $\mathcal{B}\simeq\mathbb{P}^1$). Explicitly, $Z^N$ consists of pairs of relatively prime polynomials of degrees $N$ and $N-1$ (and the former is monic):

$$Z^N=\big\{(\mathsf{A}(z)=z^N+\mathsf{a}_1z^{N-1}+\ldots+\mathsf{a}_N,\mathsf{B}(z)=\mathsf{b}_1z^{N-1}+\ldots+\mathsf{b}_N)\,|\,\gcd(\mathsf{A}(z),\mathsf{B}(z))=1\big\}.$$

To see $Z^N$ as the classical limit of the above quantum integrable systems, recall an important embedding $Z^N\hookrightarrow SL(2,\mathbb{C}[z])$ taking $(\mathsf{A}(z),\mathsf{B}(z))$ to a unique matrix (known as *the scattering matrix of the $SU(2)$-monopole*) $\begin{pmatrix}\mathsf{A}(z)&\mathsf{B}(z)\\\mathsf{C}(z)&\mathsf{D}(z)\end{pmatrix}$ such that $\deg\mathsf{C}(z)\le N-1>\deg\mathsf{D}(z)$ (such $\mathsf{C}(z),\mathsf{D}(z)$ exist due to the Euclidean algorithm). Identifying $Z^N$ with its image in $SL(2,\mathbb{C}[z])$, we note that the matrix multiplication gives rise to the multiplication homomorphisms

$$Z^N\times Z^{N'}\longrightarrow Z^{N+N'}.$$

From that perspective, the classical limit of the $p_*$-generators appearing in $T_D(z)$ are the roots of $\mathsf{A}(z)$, while the classical limit of $e^{q_*}$-generators are the values of $-\mathsf{B}(z)$ at these roots.

In the smallest rank $n=2$ case, our construction of $T_D(z)$ is a generalization of the above one as we may add some points $x_i\in\mathbb{C}$ to the support of $D$. Given $k\le 2N$ and a collection of points $\underline{x}=\{x_i\}_{i=1}^k$ on $\mathbb{C}$, consider the $\Lambda^+$-valued divisor $D:=\sum_{i=1}^k\varpi_1[x_i]+(N\alpha-k\varpi_1)[\infty]$. The phase space $Z^N_{k,\underline{x}}$ of the classical limit of the quantum integrable system determined by $T_D(z)$ is known as the space of $SU(2)$-*monopoles of topological charge $N$ with singularity $k$*. Similar to $Z^N$, it may be identified with a closed subvariety of $\mathrm{Mat}(2,\mathbb{C}[z])$ consisting of

$$M(z) = \begin{pmatrix} \mathsf{A}(z) & \mathsf{B}(z) \\ \mathsf{C}(z) & \mathsf{D}(z) \end{pmatrix} \text{ such that}$$

$$\mathsf{A}(z) = z^N + \mathsf{a}_1 z^{N-1} + \ldots + \mathsf{a}_N, \quad \deg \mathsf{B}(z) < N > \deg \mathsf{C}(z), \quad \det M(z) = \prod_{i=1}^{k} (z - x_i).$$

Let us note that the condition $k \leq 2N$ guarantees that the matrix multiplication gives rise to the multiplication homomorphisms (closely related to [3, §2(vi)] and [14, §5.9])

$$Z^N_{k,\underline{x}} \times Z^{N'}_{k',\underline{x}'} \longrightarrow Z^{N+N'}_{k+k',\underline{x}\cup\underline{x}'}.$$

*2.5. Examples and comparison to the rational Lax matrices of [15]*

In this section, we consider some examples of the Lax matrices $\mathsf{T}_D(z)$ of Theorem 2.90 and compare them to the corresponding Lax matrices $L_{\boldsymbol{\lambda},\underline{x},\widetilde{\boldsymbol{\mu}}}(z)$ (cf. Remark 2.89) of [15].

- *Example 1:* $\boldsymbol{\lambda} = (0^n), \boldsymbol{\mu} = (1, 0^{n-2}, -1)$.

Then $a_1 = \ldots = a_{n-1} = 1$ and $D = D(\boldsymbol{\lambda}, \emptyset, \boldsymbol{\mu}) = (\varpi_1 + \varpi_{n-1} - \varpi_0)[\infty]$. To simplify our notations, let us relabel $\{p_{i,1}, e^{\pm q_{i,1}}\}_{i=1}^{n-1}$ by $\{p_i, e^{\pm q_i}\}_{i=1}^{n-1}$. Due to Theorem 2.90, we have:

$$\mathsf{T}_D(z) =$$

$$\begin{pmatrix} z - p_1 & -e^{q_1} & -e^{q_1+q_2} & \cdots & -e^{q_1+\ldots+q_{n-2}} & -e^{q_1+\ldots+q_{n-1}} \\ (p_1 + 1 - p_2)e^{-q_1} & 1 & 0 & \cdots & 0 & 0 \\ (p_2 + 1 - p_3)e^{-q_1-q_2} & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (p_{n-2} + 1 - p_{n-1})e^{-q_1-\ldots-q_{n-2}} & 0 & 0 & \cdots & 1 & 0 \\ e^{-q_1-\ldots-q_{n-1}} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.100}$$

Let us compare this Lax matrix $\mathsf{T}_D(z)$ to the Lax matrix $L_{\boldsymbol{\lambda},\widetilde{\boldsymbol{\mu}}}(z)$ of [15, (4.7)] with $\widetilde{\boldsymbol{\mu}} = (2, 1^{n-2}, 0) = \boldsymbol{\mu} + (1^n)$, cf. Remark 2.89, given by

$$L_{\boldsymbol{\lambda},\widetilde{\boldsymbol{\mu}}}(z) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -e^{-q_{n,n}} \\ 0 & 1 & \cdots & 0 & -p_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -p_{n-1,n} \\ e^{q_{n,n}} & q_{n,2} & \cdots & q_{n,n-1} & z - p_{n,n} - q_{n,2}p_{2,n} - \ldots - q_{n,n-1}p_{n-1,n} \end{pmatrix}. \tag{2.101}$$

Conjugating (2.101) by the permutation matrix $\sum_{i=1}^n E_{i,n-i+1}$ (which clearly preserves the RTT relation (2.41)), and making the canonical transformation (preserving commutation relations)

$$q_{n,n-i} = -e^{\mathbf{q}_i}, \quad p_{n-i,n} = -\mathbf{p}_i e^{-\mathbf{q}_i}, \quad e^{q_{n,n}} = -e^{\mathbf{q}_{n-1}}, \quad p_{n,n} = \mathbf{p}_{n-1} \quad \text{for } 1 \leq i \leq n-2,$$

we obtain the following rational Lax matrix:

$$\widetilde{L}_{\boldsymbol{\lambda},\widetilde{\boldsymbol{\mu}}}(z) = \begin{pmatrix} z - \mathbf{p}_{n-1} - (\mathbf{p}_1 - 1) - \ldots - (\mathbf{p}_{n-2} - 1) & -e^{\mathbf{q}_1} & \cdots & -e^{\mathbf{q}_{n-2}} & -e^{\mathbf{q}_{n-1}} \\ \mathbf{p}_1 e^{-\mathbf{q}_1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{p}_{n-2} e^{-\mathbf{q}_{n-2}} & 0 & \cdots & 1 & 0 \\ e^{-\mathbf{q}_{n-1}} & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{2.102}$$

Thus $\mathsf{T}_D(z)$ of (2.100) and $\widetilde{L}_{\boldsymbol{\lambda},\widetilde{\boldsymbol{\mu}}}(z)$ of (2.102) coincide upon the canonical transformation:

$$\mathbf{q}_j = q_1 + \ldots + q_j, \quad \mathbf{p}_i = p_i - p_{i+1} + 1, \quad \mathbf{p}_{n-1} = p_{n-1} \quad \text{for } 1 \le i \le n-2, 1 \le j \le n-1.$$

- <u>*Example 2:*</u> $\boldsymbol{\lambda} = (0^{2\mathsf{r}}), \boldsymbol{\mu} = (1^{\mathsf{r}}, (-1)^{\mathsf{r}}), n = 2\mathsf{r}$.
  Then $D = D(\boldsymbol{\lambda}, \emptyset, \boldsymbol{\mu}) = (2\varpi_{\mathsf{r}} - \varpi_0)[\infty]$ and the coefficients $\{a_i\}_{i=1}^{n-1}$ are given by:

$$a_1 = 1,\ a_2 = 2,\ \ldots\ ,\ a_{\mathsf{r}-1} = \mathsf{r} - 1,\ a_{\mathsf{r}} = \mathsf{r},\ a_{\mathsf{r}+1} = \mathsf{r} - 1,\ \ldots\ ,\ a_{2\mathsf{r}-2} = 2,\ a_{2\mathsf{r}-1} = 1.$$

According to Theorem 2.90, $\mathsf{T}_D(z)$ is a block matrix of the form

$$\mathsf{T}_D(z) = \begin{pmatrix} zI_{\mathsf{r}} - F & \bar{K} \\ K & 0 \end{pmatrix}, \tag{2.103}$$

where $F, K, \bar{K}$ are $z$-independent $\mathsf{r} \times \mathsf{r}$ matrices, and $I_{\mathsf{r}}$ is the identity $\mathsf{r} \times \mathsf{r}$ matrix.

The first simple property of the matrices $F, K, \bar{K}$ is:

**Lemma 2.104.** *(a) The matrix elements $\{K_{ij}\}_{i,j=1}^{\mathsf{r}}$ of the matrix $K$ pairwise commute.*
*(b) The matrix elements $\{\bar{K}_{ij}\}_{i,j=1}^{\mathsf{r}}$ of the matrix $\bar{K}$ pairwise commute.*
*(c) The matrix elements of $K$ commute with the matrix elements of $\bar{K}$, that is $[K_{ij}, \bar{K}_{k\ell}] = 0$.*
*(d) The matrix elements $\{F_{ij}\}_{i,j=1}^{\mathsf{r}}$ of the matrix $F$ satisfy the following commutation relations:*

$$[F_{ij}, \bar{K}_{k\ell}] = \delta_{j,k} \bar{K}_{i\ell}, \quad [F_{ij}, K_{k\ell}] = -\delta_{\ell,i} K_{kj}, \quad [F_{ij}, F_{k\ell}] = \delta_{j,k} F_{i\ell} - \delta_{\ell,i} F_{kj}. \tag{2.105}$$

**Proof.** It is a direct consequence of the RTT relation (2.41) for $\mathsf{T}_D(z)$ and the ansatz (2.103). □

A much deeper relation between $K$ and $\bar{K}$ is established in the following result:

**Theorem 2.106.** *We have $K \cdot \bar{K} = -I_{\mathsf{r}}$.*

**Proof.** Due to (2.93), (2.95), it suffices to prove the following equality:

$$\sum_{\gamma=1}^{\mathsf{r}} \sum_{\substack{1\le r_\alpha\le a_\alpha \\ \cdots \\ 1\le r_{\mathsf{r}+\gamma-1}\le a_{\mathsf{r}+\gamma-1}}} \frac{P_{\alpha-1}(p_{\alpha,r_\alpha}-1)\prod_{k=\alpha}^{\mathsf{r}+\gamma-2} P_{k,r_k}(p_{k+1,r_{k+1}}-1)}{\prod_{k=\alpha}^{\mathsf{r}+\gamma-1} P_{k,r_k}(p_{k,r_k})} \cdot e^{q_{\alpha,r_\alpha}+\ldots+q_{\mathsf{r}+\gamma-1,r_{\mathsf{r}+\gamma-1}}} \times$$

$$\sum_{\substack{1\le s_\beta\le a_\beta \\ \cdots \\ 1\le s_{\mathsf{r}+\gamma-1}\le a_{\mathsf{r}+\gamma-1}}} \frac{P_{\mathsf{r}+\gamma}(p_{\mathsf{r}+\gamma-1,s_{\mathsf{r}+\gamma-1}}+1)\prod_{k=\beta+1}^{\mathsf{r}+\gamma-1} P_{k,s_k}(p_{k-1,s_{k-1}}+1)}{\prod_{k=\beta}^{\mathsf{r}+\gamma-1} P_{k,s_k}(p_{k,s_k})} \times$$

$$e^{-q_{\beta,s_\beta}-\ldots-q_{\mathsf{r}+\gamma-1,s_{\mathsf{r}+\gamma-1}}} = \delta_{\alpha,\beta} \quad (2.107)$$

for any $1 \le \alpha, \beta \le \mathsf{r}$.

To evaluate the sum in the left-hand side of (2.107), we first move $e^{q_{\alpha,r_\alpha}+\ldots+q_{\mathsf{r}+\gamma-1,r_{\mathsf{r}+\gamma-1}}}$ to the right of $p_{*,*}$-terms, then simplify $e^{q_{\iota,r_\iota}}e^{-q_{\iota,s_\iota}} \rightsquigarrow 1$ once $r_\iota = s_\iota$, and finally group together the summands which have the common $e^{q_{*,*}}$-factor. For each such group, pick the maximal $k$ (if such exists) such that $e^{q_{k,*}}$ does appear. If $k$ exists, then $1 \le k \le 2\mathsf{r}-2$ as $a_{2\mathsf{r}-1} = 1$, while $k$ does not exist if and only if $\alpha = \beta$ and $r_\iota = s_\iota$ for each $\alpha \le \iota \le \mathsf{r}+\gamma-1$.

The equality (2.107) follows from the following result:

**Proposition 2.108.** *Pick any of the above groups and consider the associated $k$ (if it exists).*
*(a) If $\mathsf{r} \le k \le 2\mathsf{r}-2$, then the sum of terms in the corresponding group is zero.*
*(b) If $1 \le k < \mathsf{r}$, then the sum of terms in the corresponding group is zero.*
*(c) If $k$ does not exist, then the sum of terms in the corresponding group equals 1.*

**Proof of Proposition 2.108.** (a) Fix any admissible collections $r_\alpha, \ldots, r_k$ and $s_\beta, \ldots, s_k$ with $r_k \ne s_k$. Then, the terms in the corresponding group are parametrized by $k+1-\mathsf{r} \le \gamma \le \mathsf{r}$ and all admissible collections $r_{k+1} = s_{k+1}, \ldots, r_{\mathsf{r}+\gamma-1} = s_{\mathsf{r}+\gamma-1}$. Ignoring the common factor, the total sum of terms in this group equals $\sum_{\gamma=k+1-\mathsf{r}}^{\mathsf{r}} S_\gamma$, where each summand is given by

$$S_\gamma := \sum_{\substack{1\le r_{k+1}\le a_{k+1} \\ \cdots \\ 1\le r_{\mathsf{r}+\gamma-1}\le a_{\mathsf{r}+\gamma-1}}} \frac{P_{k,r_k}(p_{k+1,r_{k+1}}-1)\cdots P_{\mathsf{r}+\gamma-2,r_{\mathsf{r}+\gamma-2}}(p_{\mathsf{r}+\gamma-1,r_{\mathsf{r}+\gamma-1}}-1)}{P_{k+1,r_{k+1}}(p_{k+1,r_{k+1}})\cdots P_{\mathsf{r}+\gamma-1,r_{\mathsf{r}+\gamma-1}}(p_{\mathsf{r}+\gamma-1,r_{\mathsf{r}+\gamma-1}})} \times$$

$$\frac{P_{k+1,r_{k+1}}(p_{k,s_k}+1)P_{k+2,r_{k+2}}(p_{k+1,r_{k+1}})\cdots P_{\mathsf{r}+\gamma}(p_{\mathsf{r}+\gamma-1,r_{\mathsf{r}+\gamma-1}})}{P_{k+1,r_{k+1}}(p_{k+1,r_{k+1}}-1)\cdots P_{\mathsf{r}+\gamma-1,r_{\mathsf{r}+\gamma-1}}(p_{\mathsf{r}+\gamma-1,r_{\mathsf{r}+\gamma-1}}-1)}. \quad (2.109)$$

It remains to prove $\sum_{\gamma=k+1-\mathsf{r}}^{\mathsf{r}} S_\gamma = 0$. For the latter, we need the following simple result:

**Lemma 2.110.** *Fix $\mathsf{r} < l \le 2\mathsf{r}-1$ and $1 \le r_{l-1} \ne s_{l-1} \le a_{l-1}$. Then, we have*

$$1 + \sum_{1\le r_l\le a_l} \frac{P_{l-1,r_{l-1}}(p_{l,r_l}-1)}{P_{l,r_l}(p_{l,r_l})} \cdot \frac{1}{1+p_{l-1,s_{l-1}}-p_{l,r_l}} = 0, \quad (2.111)$$

$$1+\sum_{1\le r_l\le a_l}\frac{P_{l-1,r_{l-1}}(p_{l,r_l}-1)}{P_{l,r_l}(p_{l,r_l})}\cdot\frac{1}{p_{l-1,r_{l-1}}-p_{l,r_l}}=\frac{P_{l-1,r_{l-1}}(p_{l-1,r_{l-1}}-1)}{P_l(p_{l-1,r_{l-1}})}. \tag{2.112}$$

**Proof of Lemma 2.110.** Recall that $a_l=2\mathsf{r}-l, a_{l-1}=2\mathsf{r}-l+1$. Without loss of generality, we may assume that $r_{l-1}=2\mathsf{r}-l+1$ and $s_{l-1}=2\mathsf{r}-l$. To simplify the formulas below, let us relabel $\{p_{l,i}\}_{i=1}^{2\mathsf{r}-l}$ by $\{c_i\}_{i=1}^{2\mathsf{r}-l}$ and $\{p_{l-1,i}\}_{i=1}^{2\mathsf{r}-l+1}$ by $\{b_i\}_{i=1}^{2\mathsf{r}-l+1}$, respectively.

Then, the left-hand side of (2.111) becomes

$$1-\sum_{i=1}^{2\mathsf{r}-l}\frac{(c_i-1-b_1)\cdots(c_i-1-b_{2\mathsf{r}-l-1})}{(c_i-c_1)\cdots(c_i-c_{i-1})(c_i-c_{i+1})\cdots(c_i-c_{2\mathsf{r}-l})}.$$

This is a symmetric rational function in $\{c_i\}_{i=1}^{2\mathsf{r}-l}$ without poles (as symmetric functions may not have simple poles at $c_i=c_j$ with $i\neq j$), hence, it is polynomial in $\{c_i\}_{i=1}^{2\mathsf{r}-l}$. However, being of degree $\le 0$, this polynomial must be a constant (depending on $\{b_i\}_{i=1}^{2\mathsf{r}-l+1}$). To determine the latter, let $c_1\to\infty$, in which case the sum tends to 0. This completes our proof of (2.111).

Likewise, the left-hand side of (2.112) becomes

$$1+\sum_{i=1}^{2\mathsf{r}-l}\frac{(c_i-1-b_1)\cdots(c_i-1-b_{2\mathsf{r}-l})}{(c_i-c_1)\cdots(c_i-c_{i-1})(c_i-c_{i+1})\cdots(c_i-c_{2\mathsf{r}-l})}\cdot\frac{1}{b_{2\mathsf{r}-l+1}-c_i}.$$

This is a symmetric rational function in $\{c_i\}_{i=1}^{2\mathsf{r}-l}$ with the only poles (which are at most simple) at $c_i=b_{2\mathsf{r}-l+1}$ $(1\le i\le 2\mathsf{r}-l)$. Hence, it is of the form $\frac{R(\{b_i\}_{i=1}^{2\mathsf{r}-l+1},\{c_i\}_{i=1}^{2\mathsf{r}-l})}{\prod_{i=1}^{2\mathsf{r}-l}(b_{2\mathsf{r}-l+1}-c_i)}$ for some polynomial $R$ of total degree $\deg(R)\le 2\mathsf{r}-l$. Due to (2.111), $R$ must be divisible by $\prod_{i=1}^{2\mathsf{r}-l}(b_{2\mathsf{r}-l+1}-1-b_i)$, and thus, for degree reasons, we have $R(\{b_i\},\{c_i\})=t\cdot\prod_{i=1}^{2\mathsf{r}-l}(b_{2\mathsf{r}-l+1}-1-b_i)$ with $t\in\mathbb{C}$. Letting $b_{2\mathsf{r}-l+1}\to\infty$, we find $t=1$. This completes our proof of (2.112). □

Applying (2.112) to simplify $S_{\mathsf{r}-1}+S_{\mathsf{r}}$, we find

$$S_{\mathsf{r}-1}+S_{\mathsf{r}}=\sum_{\substack{1\le r_{k+1}\le a_{k+1}\\ \cdots\\ 1\le r_{2\mathsf{r}-2}\le a_{2\mathsf{r}-2}}}\frac{P_{k,r_k}(p_{k+1,r_{k+1}}-1)\cdots P_{2\mathsf{r}-3,r_{2\mathsf{r}-3}}(p_{2\mathsf{r}-2,r_{2\mathsf{r}-2}}-1)}{P_{k+1,r_{k+1}}(p_{k+1,r_{k+1}})\cdots P_{2\mathsf{r}-2,r_{2\mathsf{r}-2}}(p_{2\mathsf{r}-2,r_{2\mathsf{r}-2}})}\times$$
$$\frac{P_{k+1,r_{k+1}}(p_{k,s_k}+1)P_{k+2,r_{k+2}}(p_{k+1,r_{k+1}})\cdots P_{2\mathsf{r}-2,r_{2\mathsf{r}-2}}(p_{2\mathsf{r}-3,r_{2\mathsf{r}-3}})}{P_{k+1,r_{k+1}}(p_{k+1,r_{k+1}}-1)\cdots P_{2\mathsf{r}-3,r_{2\mathsf{r}-3}}(p_{2\mathsf{r}-3,r_{2\mathsf{r}-3}}-1)}.$$

Applying (2.112) once again, we can now simplify the sum of the above expression and $S_{\mathsf{r}-2}$. Proceeding in the same way and applying (2.112) at each step, we eventually get

$$\sum_{k+1-\mathsf{r}\le\gamma\le\mathsf{r}}S_\gamma=1+\sum_{r_{k+1}}\frac{P_{k,r_k}(p_{k+1,r_{k+1}}-1)}{P_{k+1,r_{k+1}}(p_{k+1,r_{k+1}})}\cdot\frac{1}{1+p_{k,s_k}-p_{k+1,r_{k+1}}}=0,$$

due to (2.111) as $r_k\neq s_k$. This completes our proof of Proposition 2.108(a).

(b) The proof of Proposition 2.108(b) is completely analogous to the above proof of part (a) and is crucially based both on Lemma 2.110 and its following counterpart:

**Lemma 2.113.** *Fix* $1 < l \leq \mathsf{r}$ *and* $1 \leq r_{l-1} \neq s_{l-1} \leq a_{l-1}$*. Then, we have*

$$\sum_{1 \leq r_l \leq a_l} \frac{P_{l-1,r_{l-1}}(p_{l,r_l}-1)}{P_{l,r_l}(p_{l,r_l})} \cdot \frac{1}{1+p_{l-1,s_{l-1}}-p_{l,r_l}} = 0, \tag{2.114}$$

$$\sum_{1 \leq r_l \leq a_l} \frac{P_{l-1,r_{l-1}}(p_{l,r_l}-1)}{P_{l,r_l}(p_{l,r_l})} \cdot \frac{1}{p_{l-1,r_{l-1}}-p_{l,r_l}} = \frac{P_{l-1,r_{l-1}}(p_{l-1,r_{l-1}}-1)}{P_l(p_{l-1,r_{l-1}})}. \tag{2.115}$$

**Proof.** The proof is similar to that of (2.111), (2.112); we leave details to the interested reader. □

(c) The proof of Proposition 2.108(c) is completely analogous to the above proofs of parts (a,b) and is crucially based both on Lemmas 2.110, 2.113 and their following counterpart:

**Lemma 2.116.** *Fix* $1 < l \leq \mathsf{r}$ *and* $1 \leq r_{l-1} \leq a_{l-1}$*. Then, we have*

$$\sum_{1 \leq r_l \leq a_l} \frac{P_{l-1,r_{l-1}}(p_{l,r_l}-1)}{P_{l,r_l}(p_{l,r_l})} = 0, \tag{2.117}$$

$$\sum_{1 \leq r_l \leq a_l} \frac{P_{l-1}(p_{l,r_l}-1)}{P_{l,r_l}(p_{l,r_l})} = 1. \tag{2.118}$$

**Proof.** The proof is similar to that of (2.111), (2.112); we leave details to the interested reader. □

This completes our proof of Proposition 2.108. □

As Proposition 2.108 implies the equality (2.107), the proof of Theorem 2.106 is completed. □

It is instructive to compare this Lax matrix $\mathsf{T}_D(z)$ to the Lax matrix $L_{\boldsymbol{\lambda},\widetilde{\boldsymbol{\mu}}}(z)$ of [15, (4.2)] with $\widetilde{\boldsymbol{\mu}} = (2^{\mathsf{r}}, 0^{\mathsf{r}}) = \boldsymbol{\mu} + (1^n)$, cf. Remark 2.89. Conjugating the latter by the permutation matrix $\begin{pmatrix} 0 & I_{\mathsf{r}} \\ I_{\mathsf{r}} & 0 \end{pmatrix}$, we obtain the following rational Lax matrix

$$\widetilde{L}_{\boldsymbol{\lambda},\widetilde{\boldsymbol{\mu}}}(z) = \begin{pmatrix} zI_{\mathsf{r}} - \mathbf{F} & \bar{\mathbf{K}} \\ \mathbf{K} & 0 \end{pmatrix}, \tag{2.119}$$

where $\mathbf{K}\bar{\mathbf{K}} = -I_{\mathsf{r}}$ and $\mathbf{K}$ encodes all the $\mathbf{q}_{*,*}$-variables via [15, (4.4)].

- *Example 3:* $\boldsymbol{\lambda} = (0^{2\mathsf{r}+\mathsf{s}}), \boldsymbol{\mu} = (1^{\mathsf{r}}, 0^{\mathsf{s}}, (-1)^{\mathsf{r}}), n = 2\mathsf{r}+\mathsf{s}$ *with* $\mathsf{r}, \mathsf{s} > 0$.

Then $D = D(\boldsymbol{\lambda}, \emptyset, \boldsymbol{\mu}) = (\varpi_{\mathsf{r}} + \varpi_{\mathsf{r+s}} - \varpi_0)[\infty]$ and the coefficients $\{a_i\}_{i=1}^{n-1}$ are given by:

$$a_1 = 1,\ \dots\ ,\ a_{\mathsf{r}-1} = \mathsf{r}-1,\ a_{\mathsf{r}} = a_{\mathsf{r}+1} = \ldots = a_{\mathsf{r+s}} = \mathsf{r},\ a_{\mathsf{r+s}+1} = \mathsf{r}-1,\ \dots\ ,\ a_{2\mathsf{r+s}-1} = 1.$$

According to Theorem 2.90, $\mathsf{T}_D(z)$ is a block matrix of the form

$$\mathsf{T}_D(z) = \begin{pmatrix} zI_{\mathsf{r}} - F & Q & \bar{K} \\ -P & I_{\mathsf{s}} & 0 \\ K & 0 & 0 \end{pmatrix}, \tag{2.120}$$

where $F = (F_{ij})_{i,j=1}^{\mathsf{r}}, K = (K_{ij})_{i,j=1}^{\mathsf{r}}, \bar{K} = (\bar{K}_{ij})_{i,j=1}^{\mathsf{r}}$ are $\mathsf{r} \times \mathsf{r}$ matrices, $P = (P_{ij})_{1 \le i \le \mathsf{s}}^{1 \le j \le \mathsf{r}}$ is an $\mathsf{s} \times \mathsf{r}$ matrix, $Q = (Q_{ji})_{1 \le i \le \mathsf{s}}^{1 \le j \le \mathsf{r}}$ is an $\mathsf{r} \times \mathsf{s}$ matrix, and all of them are $z$-independent.

The first simple property of the matrices $P, Q, K, \bar{K}$ is:

**Lemma 2.121.** *(a) The matrix elements $\{K_{ij}\}_{i,j=1}^{\mathsf{r}} \cup \{P_{ij}\}_{1 \le i \le \mathsf{s}}^{1 \le j \le \mathsf{r}}$ pairwise commute.*
*(b) The matrix elements $\{\bar{K}_{ij}\}_{i,j=1}^{\mathsf{r}} \cup \{Q_{ji}\}_{1 \le i \le \mathsf{s}}^{1 \le j \le \mathsf{r}}$ pairwise commute.*
*(c) We have $[K_{ij}, \bar{K}_{k\ell}] = 0$, $[P_{ij}, \bar{K}_{k\ell}] = 0$, $[Q_{ji}, K_{k\ell}] = 0$, $[P_{ij}, Q_{\ell k}] = \delta_{i,k}\delta_{j,\ell}$.*
*(d) The matrix elements $\{F_{ij}\}_{i,j=1}^{\mathsf{r}}$ of the matrix $F$ satisfy (2.105) as well as:*

$$[F_{ij}, Q_{k\ell}] = \delta_{j,k} Q_{i\ell}, \quad [F_{ij}, P_{k\ell}] = -\delta_{\ell,i} P_{kj}.$$

**Proof.** These results follow from the RTT relation (2.41) for $\mathsf{T}_D(z)$ and the ansatz (2.120). $\square$

Similar to Theorem 2.106, there is also a much deeper relation between $K$ and $\bar{K}$:

**Theorem 2.122.** *We have $K \cdot \bar{K} = -I_{\mathsf{r}}$.*

**Proof.** The proof of Theorem 2.122 is completely analogous to the above proof of Theorem 2.106. The only extra technical result needed is the following counterpart of Lemma 2.113:

**Lemma 2.123.** *For $\mathsf{r} < l \le \mathsf{r+s}$ and $1 \le r_{l-1} \ne s_{l-1} \le a_{l-1}$, both (2.114), (2.115) hold.*

We leave details to the interested reader. $\square$

It is instructive to compare the Lax matrix $\mathsf{T}_D(z)$ of (2.120) to the Lax matrix $L_{\boldsymbol{\lambda},\widetilde{\boldsymbol{\mu}}}(z)$ of [15, (4.7)] with $\widetilde{\boldsymbol{\mu}} = (2^{\mathsf{r}}, 1^{\mathsf{s}}, 0^{\mathsf{r}}) = \boldsymbol{\mu} + (1^n)$, cf. Remark 2.89. Conjugating the latter by the permutation matrix $\begin{pmatrix} 0 & 0 & I_{\mathsf{r}} \\ 0 & I_{\mathsf{s}} & 0 \\ I_{\mathsf{r}} & 0 & 0 \end{pmatrix}$, we obtain the following rational Lax matrix

$$\widetilde{L}_{\boldsymbol{\lambda},\widetilde{\boldsymbol{\mu}}}(z) = \begin{pmatrix} zI_{\mathsf{r}} - \widetilde{\mathbf{F}} & \mathbf{Q} & \bar{\mathbf{K}} \\ -\mathbf{P} & I_{\mathsf{s}} & 0 \\ \mathbf{K} & 0 & 0 \end{pmatrix}, \tag{2.124}$$

where $\mathbf{K}\bar{\mathbf{K}} = -I_{\mathsf{r}}$ and the matrices $\mathbf{K}, \mathbf{Q}$ encode all the $\mathbf{q}_{*,*}$-variables via [15, (4.4, 4.8)].

**Remark 2.125.** We note that $\mathbf{K}, \bar{\mathbf{K}}$ of (2.124) coincide with $\mathbf{K}, \bar{\mathbf{K}}$ of (2.119), while $K, \bar{K}$ of (2.120) are not the same as $K, \bar{K}$ of (2.103).

- *Example 4:* $\boldsymbol{\lambda} = (1, 0^{n-1}), \boldsymbol{\mu} = (0^{n-1}, -1), \underline{x} = \{x_1\}$.

The corresponding divisor is $D = D(\boldsymbol{\lambda}, \{x_1\}, \boldsymbol{\mu}) = \varpi_{n-1}[x_1] + (\varpi_1 - \varpi_0)[\infty]$ with $x_1 \in \mathbb{C}$. This example is similar to the above Example 1 since the coefficients $a_i$ are the same: $a_1 = \ldots = a_{n-1} = 1$. To simplify our notations, let us relabel $\{p_{i,1}, e^{\pm q_{i,1}}\}_{i=1}^{n-1}$ by $\{p_i, e^{\pm q_i}\}_{i=1}^{n-1}$. Due to Theorem 2.90, the matrix $\mathsf{T}_D(z)$ equals:

$$\mathsf{T}_D(z) = \begin{pmatrix} z - p_1 & -e^{q_1} & \cdots & -e^{q_1+\ldots+q_{n-2}} & -(p_{n-1}-x_1)e^{q_1+\ldots+q_{n-1}} \\ (p_1+1-p_2)e^{-q_1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (p_{n-2}+1-p_{n-1})e^{-q_1-\ldots-q_{n-2}} & 0 & \cdots & 1 & 0 \\ e^{-q_1-\ldots-q_{n-1}} & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{2.126}$$

Let us compare this Lax matrix $\mathsf{T}_D(z)$ to the rational Lax matrix $L_{\boldsymbol{\lambda},x_1,\widetilde{\boldsymbol{\mu}}}(z)$ of [15, (3.1)] with $\widetilde{\boldsymbol{\mu}} = (1^{n-1}, 0) = \boldsymbol{\mu} + (1^n)$, cf. Remark 2.89. Conjugating the latter by the permutation matrix $\sum_{i=1}^n E_{i,n-i+1}$, we obtain the following rational Lax matrix:

$$\widetilde{L}_{\boldsymbol{\lambda},x_1,\widetilde{\boldsymbol{\mu}}}(z) = \begin{pmatrix} z - x_1 - \mathbf{q}_{n,1}\mathbf{p}_{1,n} - \ldots - \mathbf{q}_{n,n-1}\mathbf{p}_{n-1,n} & \mathbf{q}_{n,n-1} & \cdots & \mathbf{q}_{n,2} & \mathbf{q}_{n,1} \\ -\mathbf{p}_{n-1,n} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\mathbf{p}_{2,n} & 0 & \cdots & 1 & 0 \\ -\mathbf{p}_{1,n} & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{2.127}$$

Thus $\mathsf{T}_D(z)$ of (2.126) and $\widetilde{L}_{\boldsymbol{\lambda},x_1,\widetilde{\boldsymbol{\mu}}}(z)$ of (2.127) coincide upon the canonical transformation:

$$\begin{aligned} &\mathbf{q}_{n,n-1} = -e^{q_1}, \ \ldots \ , \ \mathbf{q}_{n,2} = -e^{q_1+\ldots+q_{n-2}}, \ \mathbf{q}_{n,1} = -(p_{n-1}-x_1)e^{q_1+\ldots+q_{n-1}}, \\ &\mathbf{p}_{n-1,n} = (p_2 - p_1 - 1)e^{-q_1}, \ldots, \mathbf{p}_{2,n} = (p_{n-1} - p_{n-2} - 1)e^{-q_1-\ldots-q_{n-2}}, \\ &\mathbf{p}_{1,n} = -e^{-q_1-\ldots-q_{n-1}}. \end{aligned}$$

- *Example 5:* $\boldsymbol{\lambda} = (1^{n-1}, 0), \boldsymbol{\mu} = (0, (-1)^{n-1}), \underline{x} = \{x_1\}$.

The corresponding divisor is $D = D(\boldsymbol{\lambda}, \{x_1\}, \boldsymbol{\mu}) = \varpi_1[x_1] + (\varpi_{n-1} - \varpi_0)[\infty]$ with $x_1 \in \mathbb{C}$. This example is similar to the previous one as $a_1 = \ldots = a_{n-1} = 1$, and we shall still relabel $\{p_{i,1}, e^{\pm q_{i,1}}\}_{i=1}^{n-1}$ by $\{p_i, e^{\pm q_i}\}_{i=1}^{n-1}$. Due to Theorem 2.90, the matrix coefficients of $\mathsf{T}_D(z)$ are:

$$\mathsf{T}_D(z)_{ii} = \begin{cases} z - p_1 & \text{if } i = 1 \\ z + p_{i-1} - p_i + 1 - x_1 & \text{if } 1 < i < n \\ 1 & \text{if } i = n \end{cases},$$

$$\mathsf{T}_D(z)_{ij} = \begin{cases} -(p_1 - x_1)e^{q_1+\ldots+q_{j-1}} & \text{if } 1 = i < j \\ -(p_i - 1 - p_{i-1})e^{q_i+\ldots+q_{j-1}} & \text{if } 1 < i < j \end{cases}, \tag{2.128}$$

$$\mathsf{T}_D(z)_{ji} = \begin{cases} (p_{j-1} + 1 - p_j)e^{-q_i-\ldots-q_{j-1}} & \text{if } i < j < n \\ e^{-q_i-\ldots-q_{n-1}} & \text{if } i < j = n \end{cases}.$$

The following is straightforward:

**Lemma 2.129.** *For any* $1 \le i, j \le n-1$, *we have* $\mathsf{T}_D(z)_{ij} = \delta_{i,j}(z-x_1)+\mathsf{T}_D(z)_{in}\mathsf{T}_D(z)_{nj}$.

Let us compare this Lax matrix $\mathsf{T}_D(z)$ to the rational Lax matrix $L_{\boldsymbol{\lambda},x_1,\widetilde{\boldsymbol{\mu}}}(z)$ of [15, (3.1)] with $\widetilde{\boldsymbol{\mu}} = (1, 0^{n-1}) = \boldsymbol{\mu} + (1^n)$, cf. Remark 2.89. Conjugating the latter by the permutation matrix $E_{12} + \ldots + E_{n-1,n} + E_{n,1}$, we obtain the rational Lax matrix $\widetilde{L}_{\boldsymbol{\lambda},x_1,\widetilde{\boldsymbol{\mu}}}(z)$ with the following matrix coefficients:

$$\begin{gathered}\widetilde{L}_{\boldsymbol{\lambda},x_1,\widetilde{\boldsymbol{\mu}}}(z)_{ij} = \delta_{i,j}(z - x_1) - \mathbf{q}_{i+1,1}\mathbf{p}_{1,j+1} \quad \text{if } 1 \le i, j < n, \\ \widetilde{L}_{\boldsymbol{\lambda},x_1,\widetilde{\boldsymbol{\mu}}}(z)_{in} = \mathbf{q}_{i+1,1}, \quad \widetilde{L}_{\boldsymbol{\lambda},x_1,\widetilde{\boldsymbol{\mu}}}(z)_{ni} = -\mathbf{p}_{1,i+1}, \quad \widetilde{L}_{\boldsymbol{\lambda},x_1,\widetilde{\boldsymbol{\mu}}}(z)_{nn} = 1 \quad \text{if } 1 \le i < n.\end{gathered} \tag{2.130}$$

Thus $\mathsf{T}_D(z)$ of (2.128) and $\widetilde{L}_{\boldsymbol{\lambda},x_1,\widetilde{\boldsymbol{\mu}}}(z)$ of (2.130) coincide upon the canonical transformation:

$$\begin{gathered}\mathbf{q}_{2,1} = (x_1 - p_1)e^{q_1+\ldots+q_{n-1}}, \mathbf{q}_{3,1} = (p_1 - p_2 + 1)e^{q_2+\ldots+q_{n-1}}, \ldots, \\ \mathbf{q}_{n,1} = (p_{n-2} - p_{n-1} + 1)e^{q_{n-1}}, \\ \mathbf{p}_{1,2} = -e^{-q_1-\ldots-q_{n-1}},\ \mathbf{p}_{1,3} = -e^{-q_2-\ldots-q_{n-1}},\ \ldots\ ,\ \mathbf{p}_{1,n} = -e^{-q_{n-1}}.\end{gathered}$$

- *Example 6:* $\boldsymbol{\lambda} = (1^{\mathsf{r}}, 0^{\mathsf{s}}), \boldsymbol{\mu} = (0^{\mathsf{s}}, (-1)^{\mathsf{r}}), \underline{x} = \{x_1\}, n = \mathsf{r} + \mathsf{s}$ *with* $\mathsf{r}, \mathsf{s} > 0$.

This example naturally generalizes Example 4 ($\mathsf{r} = 1$ case) and Example 5 ($\mathsf{s} = 1$ case) above. The corresponding divisor is $D = D(\boldsymbol{\lambda}, \{x_1\}, \boldsymbol{\mu}) = \varpi_{\mathsf{s}}[x_1] + (\varpi_{\mathsf{r}} - \varpi_0)[\infty]$ with $x_1 \in \mathbb{C}$. According to Theorem 2.90, $\mathsf{T}_D(z)$ is a block matrix of the form

$$\mathsf{T}_D(z) = \begin{pmatrix} zI_{\mathsf{r}} - F & Q \\ -P & I_{\mathsf{s}} \end{pmatrix}, \tag{2.131}$$

where $F = (F_{ij})_{i,j=1}^{\mathsf{r}}$ is an $\mathsf{r}\times\mathsf{r}$ matrix, $P = (P_{ij})_{1\le i\le \mathsf{s}}^{1\le j\le \mathsf{r}}$ is an $\mathsf{s}\times\mathsf{r}$ matrix, $Q = (Q_{ji})_{1\le i\le \mathsf{s}}^{1\le j\le \mathsf{r}}$ is an $\mathsf{r} \times \mathsf{s}$ matrix, and all of them are $z$-independent.

The first simple property of the matrices $P, Q$ is:

**Lemma 2.132.** *(a) The matrix elements* $\{P_{ij}\}_{1\leq i\leq \mathsf{s}}^{1\leq j\leq \mathsf{r}}$ *pairwise commute.*
*(b) The matrix elements* $\{Q_{ji}\}_{1\leq i\leq \mathsf{s}}^{1\leq j\leq \mathsf{r}}$ *pairwise commute.*
*(c) The commutation relation between the matrix elements of* $P, Q$ *is* $[P_{ij}, Q_{\ell k}] = \delta_{i,k}\delta_{j,\ell}$.
*(d) The matrix elements* $\{F_{ij}\}_{i,j=1}^{\mathsf{r}}$ *of the matrix* $F$ *satisfy the following commutation relations:*

$$[F_{ij}, Q_{k\ell}] = \delta_{j,k}Q_{i\ell}, \quad [F_{ij}, P_{k\ell}] = -\delta_{\ell,i}P_{kj}, \quad [F_{ij}, F_{k\ell}] = \delta_{j,k}F_{i\ell} - \delta_{\ell,i}F_{kj}.$$

**Proof.** These results follow from the RTT relation (2.41) for $\mathsf{T}_D(z)$ and the ansatz (2.131). $\square$

A much deeper relation between $P, Q$, and $F$ is established in the following result:

**Theorem 2.133.** *We have* $F = x_1 I_{\mathsf{r}} + QP$.

**Proof.** The proof of Theorem 2.133 is completely analogous to the above proof of Theorem 2.106. We leave details to the interested reader. $\square$

Let us compare this Lax matrix $\mathsf{T}_D(z)$ to the rational Lax matrix $L_{\boldsymbol{\lambda},x_1,\widetilde{\boldsymbol{\mu}}}(z)$ of [15, (3.1)] with $\widetilde{\boldsymbol{\mu}} = (1^{\mathsf{s}}, 0^{\mathsf{r}}) = \boldsymbol{\mu} + (1^n)$, cf. Remark 2.89. Conjugating the latter by the permutation matrix $\sum_{i=1}^{\mathsf{r}} E_{i,\mathsf{s}+i} + \sum_{i=1}^{\mathsf{s}} E_{\mathsf{r}+i,i}$, we obtain the following rational Lax matrix

$$\widetilde{L}_{\boldsymbol{\lambda},x_1,\widetilde{\boldsymbol{\mu}}}(z) = \begin{pmatrix} (z-x_1)I_{\mathsf{r}} - \mathbf{QP} & \mathbf{Q} \\ -\mathbf{P} & I_{\mathsf{s}} \end{pmatrix}, \tag{2.134}$$

where $\mathbf{P} = (\mathbf{p}_{i,\mathsf{s}+j})_{1\leq i\leq \mathsf{s}}^{1\leq j\leq \mathsf{r}}$ and $\mathbf{Q} = (\mathbf{q}_{\mathsf{s}+j,i})_{1\leq i\leq \mathsf{s}}^{1\leq j\leq \mathsf{r}}$ encode all the variables $\mathbf{p}_{*,*}, \mathbf{q}_{*,*}$ of [15].

Thus $\mathsf{T}_D(z)$ of (2.131) and $\widetilde{L}_{\boldsymbol{\lambda},x_1,\widetilde{\boldsymbol{\mu}}}(z)$ of (2.134) coincide upon the canonical transformation:

$$\mathbf{q}_{\mathsf{s}+j,i} = \mathsf{T}_D(z)_{j,\mathsf{r}+i}, \quad \mathbf{p}_{i,\mathsf{s}+j} = -\mathsf{T}_D(z)_{\mathsf{r}+i,j},$$

with $\mathsf{T}_D(z)_{j,\mathsf{r}+i}$ and $\mathsf{T}_D(z)_{\mathsf{r}+i,j}$ evaluated via (2.93) and (2.95), respectively.

### 2.6. Coproduct homomorphisms for shifted Yangians

One of the crucial benefits of the RTT realization is that it immediately endows the Yangian of $\mathfrak{gl}_n$ with the Hopf algebra structure, in particular, the coproduct homomorphism

$$\Delta^{\mathrm{rtt}} \colon Y^{\mathrm{rtt}}(\mathfrak{gl}_n) \longrightarrow Y^{\mathrm{rtt}}(\mathfrak{gl}_n) \otimes Y^{\mathrm{rtt}}(\mathfrak{gl}_n), \qquad T(z) \mapsto T(z) \otimes T(z). \tag{2.135}$$

The main observation of this section is that (2.135) naturally admits a shifted version:

**Proposition 2.136.** *For any $\mu_1, \mu_2 \in \Lambda^+$, there is a unique $\mathbb{C}$-algebra homomorphism*

$$\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2} \colon Y^{\mathrm{rtt}}_{-\mu_1-\mu_2}(\mathfrak{gl}_n) \longrightarrow Y^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{gl}_n) \otimes Y^{\mathrm{rtt}}_{-\mu_2}(\mathfrak{gl}_n)$$

*defined by*

$$\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}(T(z)) = T(z) \otimes T(z). \tag{2.137}$$

**Proof.** We need to prove that $T(z) \otimes T(z)$, the $n \times n$ matrix with values in the algebra $(Y^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{gl}_n) \otimes Y^{\mathrm{rtt}}_{-\mu_2}(\mathfrak{gl}_n))((z^{-1}))$, satisfies the defining relations of $Y^{\mathrm{rtt}}_{-\mu_1-\mu_2}(\mathfrak{gl}_n)$. The first of those, the RTT relation (2.41), follows immediately from the fact that both factors $T(z)$ satisfy it. Let us now deduce the second relation, the particular form of the Gauss decomposition (2.43), (2.44), from $\mu_1, \mu_2 \in \Lambda^+$ and the corresponding relations for both factors $T(z)$.

We start from the following simple observation. Let $\mathcal{C}$ be an associative algebra and consider a collection of its elements $\{\mathsf{f}^{(r)}_{ji}, \mathsf{e}^{(r)}_{ij}\}^{r\geq 1}_{1\leq i<j\leq n}$, which are encoded via a lower-triangular matrix $\mathsf{F}(z) = \sum_i E_{ii} + \sum_{i<j} \mathsf{f}_{ji}(z) \otimes E_{ji}$ with $\mathsf{f}_{ji}(z) = \sum_{r\geq 1} \mathsf{f}^{(r)}_{ji} z^{-r}$ and an upper-triangular matrix $\mathsf{E}(z) = \sum_i E_{ii} + \sum_{i<j} \mathsf{e}_{ij}(z) \otimes E_{ij}$ with $\mathsf{e}_{ij}(z) = \sum_{r\geq 1} \mathsf{e}^{(r)}_{ij} z^{-r}$. Then, the product $\mathsf{E}(z) \cdot \mathsf{F}(z)$ admits a Gauss decomposition

$$\mathsf{E}(z) \cdot \mathsf{F}(z) = \bar{\mathsf{F}}(z) \cdot \bar{\mathsf{G}}(z) \cdot \bar{\mathsf{E}}(z), \tag{2.138}$$

$$\bar{\mathsf{F}}(z) = \sum_i E_{ii} + \sum_{i<j} \bar{\mathsf{f}}_{ji}(z) \otimes E_{ji},\ \bar{\mathsf{G}}(z) = \sum_i \bar{\mathsf{g}}_i(z) \otimes E_{ii},\ \bar{\mathsf{E}}(z) = \sum_i E_{ii} + \sum_{i<j} \bar{\mathsf{e}}_{ij}(z) \otimes E_{ij},$$

with the matrix coefficients having the following expansions in $z$:

$$\bar{\mathsf{e}}_{ij}(z) = \sum_{r\geq 1} \bar{\mathsf{e}}^{(r)}_{ij} z^{-r}, \quad \bar{\mathsf{f}}_{ji}(z) = \sum_{r\geq 1} \bar{\mathsf{f}}^{(r)}_{ji} z^{-r}, \quad \bar{\mathsf{g}}_i(z) = 1 + \sum_{r\geq 1} \bar{\mathsf{g}}^{(r)}_i z^{-r}$$

for some elements $\{\bar{\mathsf{f}}^{(r)}_{ji}, \bar{\mathsf{e}}^{(r)}_{ij}\}^{r\geq 1}_{1\leq i<j\leq n} \cup \{\bar{\mathsf{g}}^{(r)}_i\}^{r\geq 1}_{1\leq i\leq n}$ of $\mathcal{C}$.

Moreover, if $z^{\underline{d}} = \mathrm{diag}(z^{d_1}, \cdots, z^{d_n})$ with $d_1 \geq \cdots \geq d_n$, then

$$z^{\underline{d}} \bar{\mathsf{F}}(z) (z^{\underline{d}})^{-1} = \sum_i E_{ii} + \sum_{i<j} \widetilde{\mathsf{f}}_{ji}(z) \otimes E_{ji} \quad \text{with} \quad \widetilde{\mathsf{f}}_{ji}(z) = \sum_{r\geq 1} \widetilde{\mathsf{f}}^{(r)}_{ji} z^{-r} = z^{d_j - d_i} \bar{\mathsf{f}}_{ji}(z) \tag{2.139}$$

and

$$(z^{\underline{d}})^{-1} \bar{\mathsf{E}}(z) z^{\underline{d}} = \sum_i E_{ii} + \sum_{i<j} \widetilde{\mathsf{e}}_{ij}(z) \otimes E_{ij} \quad \text{with} \quad \widetilde{\mathsf{e}}_{ij}(z) = \sum_{r\geq 1} \widetilde{\mathsf{e}}^{(r)}_{ij} z^{-r} = z^{d_j - d_i} \bar{\mathsf{e}}_{ij}(z) \tag{2.140}$$

for some elements $\{\widetilde{\mathsf{f}}^{(r)}_{ji}, \widetilde{\mathsf{e}}^{(r)}_{ij}\}^{r\geq 1}_{1\leq i<j\leq n}$ of $\mathcal{C}$.

Finally, let us consider the Gauss decompositions of both factors $T(z)$:

$$T(z) \otimes 1 = F^{(1)}(z)G^{(1)}(z)E^{(1)}(z) = F^{(1)}(z)D^{(1)}(z)z^{\mu_1}E^{(1)}(z),$$
$$1 \otimes T(z) = F^{(2)}(z)G^{(2)}(z)E^{(2)}(z) = F^{(2)}(z)z^{\mu_2}D^{(2)}(z)E^{(2)}(z),$$

where $z^{\mu_a} := \mathrm{diag}(z^{d_1^{(a)}}, \cdots, z^{d_n^{(a)}}), D^{(a)}(z) := z^{-\mu_a}G^{(a)}(z)$ with $d_i^{(a)} := \epsilon_i^\vee(\mu_a)$ and $a = 1, 2$. To obtain the Gauss decomposition of

$$T(z) \otimes T(z) = F^{(1)}(z)D^{(1)}(z)z^{\mu_1}E^{(1)}(z)F^{(2)}(z)z^{\mu_2}D^{(2)}(z)E^{(2)}(z),$$

we apply the above general observation with $\mathcal{C} = Y^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{gl}_n) \otimes Y^{\mathrm{rtt}}_{-\mu_2}(\mathfrak{gl}_n)$ and $\mathsf{e}_{ij}^{(r)} = e_{ij}^{(r)} \otimes 1$, $\mathsf{f}_{ji}^{(r)} = 1 \otimes f_{ji}^{(r)}$ to get the Gauss decomposition of $E^{(1)}(z)F^{(2)}(z)$ first. As conjugating by $D^{(a)}(z)$ does not change the leading $z$-modes, matrix coefficients appearing in the Gauss decomposition of $T(z) \otimes T(z)$ have the desired form, due to (2.139), (2.140).

This completes our proof of Proposition 2.136. $\square$

The following basic property of $\Delta^{\mathrm{rtt}}_{*,*}$ is straightforward:

**Corollary 2.141.** *For any $\mu_1, \mu_2, \mu_3 \in \Lambda^+$, the following diagram is commutative:*

$$\begin{tikzcd}
Y^{\mathrm{rtt}}_{-\mu_1-\mu_2-\mu_3}(\mathfrak{gl}_n) \arrow[r, "\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2-\mu_3}"] \arrow[d, "\Delta^{\mathrm{rtt}}_{-\mu_1-\mu_2,-\mu_3}"'] & Y^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{gl}_n) \otimes Y^{\mathrm{rtt}}_{-\mu_2-\mu_3}(\mathfrak{gl}_n) \arrow[d, "\mathrm{Id} \otimes \Delta^{\mathrm{rtt}}_{-\mu_2,-\mu_3}"] \\
Y^{\mathrm{rtt}}_{-\mu_1-\mu_2}(\mathfrak{gl}_n) \otimes Y^{\mathrm{rtt}}_{-\mu_3}(\mathfrak{gl}_n) \arrow[r, "\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2} \otimes \mathrm{Id}"'] & Y^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{gl}_n) \otimes Y^{\mathrm{rtt}}_{-\mu_2}(\mathfrak{gl}_n) \otimes Y^{\mathrm{rtt}}_{-\mu_3}(\mathfrak{gl}_n)
\end{tikzcd}$$

Evoking the key isomorphisms $\Upsilon_{-\mu} \colon Y_{-\mu}(\mathfrak{gl}_n) \xrightarrow{\sim} Y^{\mathrm{rtt}}_{-\mu}(\mathfrak{gl}_n)$ of Theorem 2.54 for $\mu = \mu_1, \mu_2, \mu_1 + \mu_2$, we conclude that $\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}$ gives rise to the $\mathbb{C}$-algebra homomorphism

$$\Delta_{-\mu_1,-\mu_2} \colon Y_{-\mu_1-\mu_2}(\mathfrak{gl}_n) \longrightarrow Y_{-\mu_1}(\mathfrak{gl}_n) \otimes Y_{-\mu_2}(\mathfrak{gl}_n). \tag{2.142}$$

**Proposition 2.143.** *For any $\mu_1, \mu_2 \in \Lambda^+$, the above $\mathbb{C}$-algebra homomorphism* (2.142)

$$\Delta_{-\mu_1,-\mu_2} \colon Y_{-\mu_1-\mu_2}(\mathfrak{gl}_n) \longrightarrow Y_{-\mu_1}(\mathfrak{gl}_n) \otimes Y_{-\mu_2}(\mathfrak{gl}_n)$$

*is uniquely determined by specifying the image of the central series $C(z)$ of* (2.23) *via*

$$C(z) \mapsto C(z) \otimes C(z), \tag{2.144}$$

*and the following formulas (for any $1 \le i \le n-1, 1 \le j \le n$):*

$$
\begin{aligned}
&F_i^{(r)} \mapsto F_i^{(r)} \otimes 1 \quad \text{for } 1 \leq r \leq \alpha_i^\vee(\mu_1),\\
&F_i^{(\alpha_i^\vee(\mu_1)+1)} \mapsto F_i^{(\alpha_i^\vee(\mu_1)+1)} \otimes 1 + 1 \otimes F_i^{(1)},\\
&E_i^{(r)} \mapsto 1 \otimes E_i^{(r)} \quad \text{for } 1 \leq r \leq \alpha_i^\vee(\mu_2),\\
&E_i^{(\alpha_i^\vee(\mu_2)+1)} \mapsto 1 \otimes E_i^{(\alpha_i^\vee(\mu_2)+1)} + E_i^{(1)} \otimes 1,\\
&D_j^{(-\epsilon_j^\vee(\mu_1+\mu_2)+1)} \mapsto D_j^{(-\epsilon_j^\vee(\mu_1)+1)} \otimes 1 + 1 \otimes D_j^{(-\epsilon_j^\vee(\mu_2)+1)},\\
&D_j^{(-\epsilon_j^\vee(\mu_1+\mu_2)+2)} \mapsto D_j^{(-\epsilon_j^\vee(\mu_1)+2)} \otimes 1 + 1 \otimes D_j^{(-\epsilon_j^\vee(\mu_2)+2)} +\\
&\qquad\qquad D_j^{(-\epsilon_j^\vee(\mu_1)+1)} \otimes D_j^{(-\epsilon_j^\vee(\mu_2)+1)} + \sum_{\gamma^\vee \in \Delta^+} \epsilon_j(\gamma^\vee) E_{\gamma^\vee}^{(1)} \otimes F_{\gamma^\vee}^{(1)},
\end{aligned}
\tag{2.145}
$$

*where the last sum is taken over the set* $\Delta^+ = \{\alpha_a^\vee + \ldots + \alpha_{b-1}^\vee \,|\, 1 \leq a < b \leq n\}$ *of positive roots of* $\mathfrak{sl}_n$*, and the root generators* $\{E_{\gamma^\vee}^{(1)}, F_{\gamma^\vee}^{(1)}\}_{\gamma^\vee \in \Delta^+}$ *are defined via (cf. (2.47)):*

$$
\begin{aligned}
E^{(1)}_{\alpha_a^\vee + \ldots + \alpha_{b-1}^\vee} &:= [E_{b-1}^{(1)}, \cdots, [E_{a+1}^{(1)}, E_a^{(1)}] \cdots],\\
F^{(1)}_{\alpha_a^\vee + \ldots + \alpha_{b-1}^\vee} &:= [\cdots [F_a^{(1)}, F_{a+1}^{(1)}], \cdots, F_{b-1}^{(1)}].
\end{aligned}
$$

**Proof.** Since $Y_{-\mu_1-\mu_2}(\mathfrak{gl}_n)$ is generated (as an algebra) by the coefficients of the central series $C(z)$ and the elements $\{E_i^{(1)}, F_i^{(1)}, D_j^{(-\epsilon_j^\vee(\mu_1+\mu_2)+1)}, D_j^{(-\epsilon_j^\vee(\mu_1+\mu_2)+2)}\}_{1\leq i<n}^{1\leq j\leq n}$, as follows from Corollary 2.24, it suffices to show that (2.142) satisfies the above formulas (2.144) and (2.145).

Using the standard arguments (see [29, Corollary 1.6.10] or [4, Lemma 8.1] and the references therein), we have $\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}(\mathrm{qdet}\, T(z)) = \mathrm{qdet}\, T(z) \otimes \mathrm{qdet}\, T(z)$. Combining this formula with $\Upsilon_{-\mu}^{-1}(\mathrm{qdet}\, T(z)) = C(z)$ of Proposition 2.83, we obtain the desired formula (2.144).

Following our notations from the above proof of Proposition 2.136, we note that

$$
\begin{aligned}
&\widetilde{\mathsf{f}}_{ji}^{(1)} = \ldots = \widetilde{\mathsf{f}}_{ji}^{(d_i^{(1)}-d_j^{(1)})} = 0, \quad \widetilde{\mathsf{f}}_{ji}^{(d_i^{(1)}-d_j^{(1)}+1)} = \bar{\mathsf{f}}_{ji}^{(1)} = \mathsf{f}_{ji}^{(1)},\\
&\widetilde{\mathsf{e}}_{ij}^{(1)} = \ldots = \widetilde{\mathsf{e}}_{ij}^{(d_i^{(2)}-d_j^{(2)})} = 0, \quad \widetilde{\mathsf{e}}_{ij}^{(d_i^{(2)}-d_j^{(2)}+1)} = \bar{\mathsf{e}}_{ij}^{(1)} = \mathsf{e}_{ij}^{(1)}.
\end{aligned}
$$

Thus, following the proof of Proposition 2.136, we immediately get

$$
\begin{aligned}
&\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}(f_{i+1,i}^{(r)}) = f_{i+1,i}^{(r)} \otimes 1 \quad \text{for } 1 \leq r \leq \alpha_i^\vee(\mu_1),\\
&\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}(f_{i+1,i}^{(\alpha_i^\vee(\mu_1)+1)}) = f_{i+1,i}^{(\alpha_i^\vee(\mu_1)+1)} \otimes 1 + 1 \otimes f_{i+1,i}^{(1)},\\
&\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}(e_{i,i+1}^{(r)}) = 1 \otimes e_{i,i+1}^{(r)} \quad \text{for } 1 \leq r \leq \alpha_i^\vee(\mu_2),\\
&\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}(e_{i,i+1}^{(\alpha_i^\vee(\mu_2)+1)}) = 1 \otimes e_{i,i+1}^{(\alpha_i^\vee(\mu_2)+1)} + e_{i,i+1}^{(1)} \otimes 1,
\end{aligned}
$$

which give rise to the first four formulas of (2.145) by evoking the construction of $\Upsilon_{-\mu}$.

To deduce the last two formulas of (2.145), it remains to use obvious equalities

$$\bar{\mathsf{g}}_i^{(1)} = 0, \quad \bar{\mathsf{g}}_i^{(2)} = \sum_{j>i} \mathsf{e}_{ij}^{(1)} \cdot \mathsf{f}_{ji}^{(1)} - \sum_{j<i} \mathsf{f}_{ji}^{(1)} \cdot \mathsf{e}_{ji}^{(1)} = \sum_{1\leq a<b\leq n} \epsilon_i(\alpha_a^\vee + \ldots + \alpha_{b-1}^\vee) e_{ab}^{(1)} \otimes f_{ba}^{(1)}.$$

This completes our proof of Proposition 2.143. □

Proposition 2.143 provides a conceptual and elementary proof of [14, Theorem 4.8]:

**Proposition 2.146.** *(a) For any $\nu_1, \nu_2 \in \bar{\Lambda}^+$, there is a unique $\mathbb{C}$-algebra homomorphism*

$$\Delta_{-\nu_1,-\nu_2} \colon Y_{-\nu_1-\nu_2}(\mathfrak{sl}_n) \longrightarrow Y_{-\nu_1}(\mathfrak{sl}_n) \otimes Y_{-\nu_2}(\mathfrak{sl}_n) \tag{2.147}$$

*such that the following diagram is commutative*

$$\begin{array}{ccc} Y_{-\bar{\mu}_1-\bar{\mu}_2}(\mathfrak{sl}_n) & \xrightarrow{\Delta_{-\bar{\mu}_1,-\bar{\mu}_2}} & Y_{-\bar{\mu}_1}(\mathfrak{sl}_n) \otimes Y_{-\bar{\mu}_2}(\mathfrak{sl}_n) \\ {\scriptstyle \iota_{-\mu_1-\mu_2}}\big\downarrow & & \big\downarrow{\scriptstyle \iota_{-\mu_1}\otimes\iota_{-\mu_2}} \\ Y_{-\mu_1-\mu_2}(\mathfrak{gl}_n) & \xrightarrow{\Delta_{-\mu_1,-\mu_2}} & Y_{-\mu_1}(\mathfrak{gl}_n) \otimes Y_{-\mu_2}(\mathfrak{gl}_n) \end{array} \tag{2.148}$$

*for any $\mu_1, \mu_2 \in \Lambda^+$.*
*(b) The homomorphism $\Delta_{-\nu_1,-\nu_2}$ is uniquely determined by the following formulas:*

$$\begin{aligned}
&\mathsf{F}_i^{(r)} \mapsto \mathsf{F}_i^{(r)} \otimes 1 \quad \text{for } 1 \leq r \leq \alpha_i^\vee(\nu_1), \\
&\mathsf{F}_i^{(\alpha_i^\vee(\nu_1)+1)} \mapsto \mathsf{F}_i^{(\alpha_i^\vee(\nu_1)+1)} \otimes 1 + 1 \otimes \mathsf{F}_i^{(1)}, \\
&\mathsf{E}_i^{(r)} \mapsto 1 \otimes \mathsf{E}_i^{(r)} \quad \text{for } 1 \leq r \leq \alpha_i^\vee(\nu_2), \\
&\mathsf{E}_i^{(\alpha_i^\vee(\nu_2)+1)} \mapsto 1 \otimes \mathsf{E}_i^{(\alpha_i^\vee(\nu_2)+1)} + \mathsf{E}_i^{(1)} \otimes 1, \\
&\mathsf{H}_i^{(\alpha_i^\vee(\nu_1+\nu_2)+1)} \mapsto \mathsf{H}_i^{(\alpha_i^\vee(\nu_1)+1)} \otimes 1 + 1 \otimes \mathsf{H}_i^{(\alpha_i^\vee(\nu_2)+1)}, \\
&\mathsf{H}_i^{(\alpha_i^\vee(\nu_1+\nu_2)+2)} \mapsto \mathsf{H}_i^{(\alpha_i^\vee(\nu_1)+2)} \otimes 1 + 1 \otimes \mathsf{H}_i^{(\alpha_i^\vee(\nu_2)+2)} + \\
&\qquad\qquad \mathsf{H}_i^{(\alpha_i^\vee(\nu_1)+1)} \otimes \mathsf{H}_i^{(\alpha_i^\vee(\nu_2)+1)} - \sum_{\gamma^\vee\in\Delta^+} \alpha_i(\gamma^\vee) \mathsf{E}_{\gamma^\vee}^{(1)} \otimes \mathsf{F}_{\gamma^\vee}^{(1)},
\end{aligned} \tag{2.149}$$

*where* $\mathsf{E}^{(1)}_{\alpha_a^\vee+\ldots+\alpha_{b-1}^\vee} := [\mathsf{E}^{(1)}_{b-1}, \cdots, [\mathsf{E}^{(1)}_{a+1}, \mathsf{E}^{(1)}_a] \cdots]$ *and* $\mathsf{F}^{(1)}_{\alpha_a^\vee+\ldots+\alpha_{b-1}^\vee} := [\cdots [\mathsf{F}^{(1)}_a, \mathsf{F}^{(1)}_{a+1}], \cdots, \mathsf{F}^{(1)}_{b-1}]$.

**Proof.** Follows immediately from the formulas (2.145) of Proposition 2.143 combined with the defining formulas (2.21) for the embedding $\iota_{-\mu} \colon Y_{-\bar{\mu}}(\mathfrak{sl}_n) \hookrightarrow Y_{-\mu}(\mathfrak{gl}_n)$ of Proposition 2.19. □

**Remark 2.150.** Due to [14, Theorem 4.12], $\Delta_{-\nu_1,-\nu_2}$ with $\nu_1, \nu_2 \in \bar{\Lambda}^+$ give rise to algebra homomorphisms $\Delta_{\nu_1,\nu_2} \colon Y_{\nu_1+\nu_2}(\mathfrak{sl}_n) \to Y_{\nu_1}(\mathfrak{sl}_n) \otimes Y_{\nu_2}(\mathfrak{sl}_n)$ for any $\mathfrak{sl}_n$–coweights

$\nu_1, \nu_2 \in \bar{\Lambda}$. However, we note that $\Delta_{\nu_1,\nu_2}$ $(\nu_1, \nu_2 \in \Lambda)$ are not coassociative, in contrast to Corollary 2.141.

**Remark 2.151.** We note that [35, §2.4] contains an attempt to construct the simplest coproduct homomorphism $Y_{-\alpha}(\mathfrak{sl}_2) \to Y_{-\alpha/2}(\mathfrak{sl}_2) \otimes Y_{-\alpha/2}(\mathfrak{sl}_2)$ from Proposition 2.146.

### *2.7. Relation to Gelfand-Tsetlin bases of parabolic Verma modules of $\mathfrak{gl}_n$*

Evoking the setup of Section 2.4.4, assume $\boldsymbol{\mu} = ((-1)^n)$ while $\boldsymbol{\lambda}$ is a Young diagram of size $n$ and length $< n$, i.e. $|\boldsymbol{\lambda}| = n$ and $\boldsymbol{\lambda}_n = 0$, and consider the corresponding $\Lambda^+$-valued divisor on $\mathbb{P}^1$: $D = \sum_{k=1}^{\boldsymbol{\lambda}_1} \varpi_{i_k}[x_k] - \varpi_0[\infty]$ with $x_k \in \mathbb{C}$ (note that $i_k = n - \boldsymbol{\lambda}_k^t$). In this section, we show that the homomorphism $\Theta_D \colon Y^{\mathrm{rtt}}_{\varpi_0}(\mathfrak{gl}_n) \to \mathcal{A}$ of (2.56) may be viewed (up to a gauge transformation) as a composition of the *evaluation* homomorphism $\widetilde{\mathrm{ev}} \colon Y^{\mathrm{rtt}}_{\varpi_0}(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$ and the homomorphism $U(\mathfrak{gl}_n) \to \mathcal{A}$ determined by the *parabolic Gelfand-Tsetlin* formulas.

Let us recall the explicit formulas for the matrix coefficients $T_D(z)_{i,i}$, $T_D(z)_{i,i+1}$, $T_D(z)_{i+1,i}$ of Theorem 2.90 (note that $T_D(z) = \mathsf{T}_D(z)$ in the present setup):

$$T_D(z)_{i,i} = z + \sum_{r=1}^{a_{i-1}} (p_{i-1,r} + 1) - \sum_{r=1}^{a_i} p_{i,r} - \sum_{k: i_k \le i-1} x_k, \tag{2.152}$$

$$T_D(z)_{i,i+1} = -\sum_{r=1}^{a_i} \frac{\prod_{s=1}^{a_{i-1}} (p_{i,r} - 1 - p_{i-1,s})}{\prod_{1\le s\le a_i}^{s\ne r} (p_{i,r} - p_{i,s})} \cdot \prod_{k: i_k = i} (p_{i,r} - x_k) \cdot e^{q_{i,r}}, \tag{2.153}$$

$$T_D(z)_{i+1,i} = \sum_{r=1}^{a_i} \frac{\prod_{s=1}^{a_{i+1}} (p_{i,r} + 1 - p_{i+1,s})}{\prod_{1\le s\le a_i}^{s\ne r} (p_{i,r} - p_{i,s})} \cdot e^{-q_{i,r}}. \tag{2.154}$$

Consider the following factor

$$S = \frac{\prod_{i=1}^{n-2} \prod_{r \le a_i}^{s \le a_{i+1}} \Gamma(p_{i,r} - p_{i+1,s} + 1) \cdot \prod_{i=1}^{n-1} \prod_{r=1}^{a_i} \prod_{k: i_k \le i-1} \Gamma(p_{i,r} - x_k + 1)}{\prod_{i=1}^{n-1} \prod_{1\le r,s\le a_i}^{r\ne s} \Gamma(p_{i,s} - p_{i,r})}, \tag{2.155}$$

where $\Gamma(\cdot)$ denotes the classical Gamma function. Then, $\mathrm{Ad}(S)$ is a well-defined automorphism of $\mathcal{A}$, which shall be referred to as the *gauge transformation with respect to $S$*. Applying $\mathrm{Ad}(S)$ to $T_D(z)$ partially described by the formulas (2.152), (2.153), (2.154), we obtain

$$\mathrm{Ad}(S) T_D(z)_{i,i} = z + \sum_{r=1}^{a_{i-1}} (p_{i-1,r} + 1) - \sum_{r=1}^{a_i} p_{i,r} - \sum_{k: i_k \le i-1} x_k, \tag{2.156}$$

$$\mathrm{Ad}(S) T_D(z)_{i,i+1} = \sum_{r=1}^{a_i} (-1)^{a_i + a_{i-1}} \frac{\prod_{s=1}^{a_{i+1}} (p_{i,r} - p_{i+1,s})}{\prod_{1\le s\le a_i}^{s\ne r} (p_{i,r} - 1 - p_{i,s})} \prod_{k: i_k \le i} (p_{i,r} - x_k) \cdot e^{q_{i,r}}, \tag{2.157}$$

$$\begin{aligned}&\mathrm{Ad}(S)T_D(z)_{i+1,i}\\&\quad=\sum_{r=1}^{a_i}(-1)^{a_i+a_{i-1}-1}\frac{\prod_{s=1}^{a_{i-1}}(p_{i,r}-p_{i-1,s})}{\prod_{1\le s\le a_i}^{s\neq r}(p_{i,r}+1-p_{i,s})}\prod_{k:i_k\le i-1}\frac{1}{p_{i,r}-x_k+1}\cdot e^{-q_{i,r}}.\end{aligned} \tag{2.158}$$

We also consider the factor

$$U:=\prod_{i=1}^{n-1}\prod_{r=1}^{a_i}\left((-1)^{\boldsymbol{\lambda}_{n-i+1}p_{i,r}}\cdot e^{-iq_{i,r}}\right), \tag{2.159}$$

so that $\mathrm{Ad}(U)$ is a well-defined automorphism of $\mathcal{A}$ which maps

$$p_{i,r}\mapsto p_{i,r}+i,\quad e^{q_{i,r}}\mapsto(-1)^{\boldsymbol{\lambda}_{n-i+1}}e^{q_{i,r}}.$$

Applying this automorphism to (2.156), (2.157), (2.158), we obtain

$$\mathrm{Ad}(US)T_D(z)_{i,i}=z+\sum_{r=1}^{a_{i-1}}p_{i-1,r}-\sum_{r=1}^{a_i}p_{i,r}+i(a_{i-1}-a_i)-\sum_{k:i_k\le i-1}x_k, \tag{2.160}$$

$$\mathrm{Ad}(US)T_D(z)_{i,i+1}=\sum_{r=1}^{a_i}(-1)^{\beta_i}\frac{\prod_{s=1}^{a_{i+1}}(p_{i+1,s}-p_{i,r}+1)}{\prod_{1\le s\le a_i}^{s\neq r}(p_{i,s}-p_{i,r}+1)}\prod_{k:i_k\le i}(x_k-p_{i,r}-i)\cdot e^{q_{i,r}}, \tag{2.161}$$

$$\mathrm{Ad}(US)T_D(z)_{i+1,i}=\sum_{r=1}^{a_i}\frac{\prod_{s=1}^{a_{i-1}}(p_{i-1,s}-p_{i,r}-1)}{\prod_{1\le s\le a_i}^{s\neq r}(p_{i,s}-p_{i,r}-1)}\prod_{k:i_k\le i-1}\frac{1}{x_k-p_{i,r}-i-1}\cdot e^{-q_{i,r}}, \tag{2.162}$$

where $\beta_i:=a_{i-1}+a_{i+1}+1+\boldsymbol{\lambda}_{n-i}+\boldsymbol{\lambda}_{n-i+1}$. Evoking $a_i-a_{i-1}=1-\boldsymbol{\lambda}_{n-i+1}$, we see that $\beta_i$ is odd. Thus, the formulas (2.160), (2.161), (2.162) may be written as follows:

$$\mathrm{Ad}(US)T_D(z)_{i,i}=z+\sum_{r=1}^{a_{i-1}}p_{i-1,r}-\sum_{r=1}^{a_i}p_{i,r}+i(\boldsymbol{\lambda}_{n-i+1}-1)-\sum_{k:i_k\le i-1}x_k, \tag{2.163}$$

$$\mathrm{Ad}(US)T_D(z)_{i,i+1}=-\sum_{r=1}^{a_i}\frac{\prod_{s=1}^{a_{i+1}}(p_{i+1,s}-p_{i,r}+1)}{\prod_{1\le s\le a_i}^{s\neq r}(p_{i,s}-p_{i,r}+1)}\prod_{k:i_k\le i}(x_k-p_{i,r}-i)\cdot e^{q_{i,r}}, \tag{2.164}$$

$$\mathrm{Ad}(US)T_D(z)_{i+1,i}=\sum_{r=1}^{a_i}\frac{\prod_{s=1}^{a_{i-1}}(p_{i-1,s}-p_{i,r}-1)}{\prod_{1\le s\le a_i}^{s\neq r}(p_{i,s}-p_{i,r}-1)}\prod_{k:i_k\le i-1}\frac{1}{x_k-p_{i,r}-i-1}\cdot e^{-q_{i,r}}. \tag{2.165}$$

Let us now relate formulas (2.163), (2.164), (2.165) to the parabolic Gelfand-Tsetlin formulas. Let $\mathfrak{p}\subseteq\mathfrak{gl}_n$ be a parabolic subalgebra with the Levi factor $\mathfrak{l}\simeq\mathfrak{gl}_{\boldsymbol{\lambda}_1^t}\oplus\mathfrak{gl}_{\boldsymbol{\lambda}_2^t}\oplus\cdots\oplus\mathfrak{gl}_{\boldsymbol{\lambda}^t_{\boldsymbol{\lambda}_1}}$ embedded block-diagonally into $\mathfrak{gl}_n$. For $\underline{y}=(y_1,\ldots,y_{\boldsymbol{\lambda}_1})\in\mathbb{C}^{\boldsymbol{\lambda}_1}$, let $\mathbb{C}_{\underline{y}}$ be the 1-dimensional $\mathfrak{p}$-module obtained as a pull-back (along the natural projection $\mathfrak{p}\twoheadrightarrow\mathfrak{l}$) of the 1-dimensional $\mathfrak{l}$-module with $\mathfrak{gl}_{\boldsymbol{\lambda}_i^t}$-factor acting via $y_i\mathrm{tr}$. We also assume that

$y_i - y_j \notin \mathbb{Z}$ for $i \neq j$. Consider the *parabolic Verma module* $M_{\underline{y}} := \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{gl}_n} \mathbb{C}_{\underline{y}}$. It has a distinguished basis $\{\xi_\Lambda\}$, called the *Gelfand-Tsetlin basis*, parametrized by $\Lambda = (\Lambda_{i,j})_{1\leq j\leq i\leq n}$ subject to the following conditions:

(a) $\Lambda_{n,\boldsymbol{\lambda}_1^t+\ldots+\boldsymbol{\lambda}_{i-1}^t+a} = y_a$ for $1 \leq a \leq \boldsymbol{\lambda}_i^t$;

(b) $\Lambda_{i+1,j} - \Lambda_{i,j} \in \mathbb{N}$;

(c) if $\Lambda_{i,j} - \Lambda_{i+1,j+1} \in \mathbb{Z}$, then actually $\Lambda_{i,j} - \Lambda_{i+1,j+1} \in \mathbb{N}$.

Note that the conditions (b,c) imply $\Lambda_{i,k} = y_a$ if $\boldsymbol{\lambda}_1^t + \ldots + \boldsymbol{\lambda}_{a-1}^t < k \leq \boldsymbol{\lambda}_1^t + \ldots + \boldsymbol{\lambda}_a^t - (n-i)$. We call such coordinates $(i,k)$ *frozen*. For $1 \leq i \leq n-1$, let $J_i \subset \{1, \cdots, i\}$ denote the set of non-frozen coordinates among $\{(i,*)\}$. It is easy to see that $|J_i| = a_i$. Set $l_{i,j} := \Lambda_{i,j} - j + 1$.

Then, the *classical Gelfand-Tsetlin formulas* [34] (corresponding to the case $\mathfrak{l} \simeq \mathfrak{gl}_1^{\oplus n}$) give rise to the *parabolic Gelfand-Tsetlin formulas* for the action of $\mathfrak{gl}_n$ in the basis $\xi_\Lambda$ of $M_{\underline{y}}$:

$$E_{i,i}(\xi_\Lambda) = \left( \sum_{k\in J_i} l_{i,k} - \sum_{k\in J_{i-1}} l_{i-1,k} + \sum_{a:\boldsymbol{\lambda}_a^t \geq n-i+1} (y'_a - i) + (i-1) \right) \cdot \xi_\Lambda, \tag{2.166}$$

$$E_{i,i+1}(\xi_\Lambda) = -\sum_{k\in J_i} \frac{\prod_{m\in J_{i+1}} (l_{i+1,m} - l_{i,k})}{\prod_{m\in J_i\setminus\{k\}} (l_{i,m} - l_{i,k})} \prod_{a:\boldsymbol{\lambda}_a^t \geq n-i} (y'_a - l_{i,k} - i - 1) \cdot \xi_{\Lambda+\delta_{i,k}}, \tag{2.167}$$

$$E_{i+1,i}(\xi_\Lambda) = \sum_{k\in J_i} \frac{\prod_{m\in J_{i-1}} (l_{i-1,m} - l_{i,k})}{\prod_{m\in J_i\setminus\{k\}} (l_{i,m} - l_{i,k})} \prod_{a:\boldsymbol{\lambda}_a^t \geq n-i+1} \frac{1}{y'_a - l_{i,k} - i} \cdot \xi_{\Lambda-\delta_{i,k}}, \tag{2.168}$$

where $y'_a := y_a - (\boldsymbol{\lambda}_1^t + \ldots + \boldsymbol{\lambda}_a^t) + (n+1)$ and $\Lambda \pm \delta_{i,k}$ is obtained from $\Lambda$ by adding $\pm 1$ to its $(i,k)$-th entry (if $\Lambda \pm \delta_{i,k}$ does not satisfy (b) or (c), then the corresponding coefficient in front of $\xi_{\Lambda\pm\delta_{i,k}}$ in (2.167) or (2.168), respectively, is actually zero).

These formulas naturally give rise to the algebra homomorphism $\varrho\colon U(\mathfrak{gl}_n) \to \mathcal{A}$ with

$$E_{i,i} \mapsto \sum_{k\in J_i} p_{i,k} - \sum_{k\in J_{i-1}} p_{i-1,k} + \sum_{a:\boldsymbol{\lambda}_a^t \geq n-i+1} (y'_a - i) + (i-1), \tag{2.169}$$

$$\begin{aligned} E_{i,i+1} \mapsto & -\sum_{k\in J_i} e^{q_{i,k}} \frac{\prod_{m\in J_{i+1}} (p_{i+1,m} - p_{i,k})}{\prod_{m\in J_i\setminus\{k\}} (p_{i,m} - p_{i,k})} \prod_{a:\boldsymbol{\lambda}_a^t \geq n-i} (y'_a - p_{i,k} - i - 1) = \\ & -\sum_{k\in J_i} \frac{\prod_{m\in J_{i+1}} (p_{i+1,m} - p_{i,k} + 1)}{\prod_{m\in J_i\setminus\{k\}} (p_{i,m} - p_{i,k} + 1)} \prod_{a:\boldsymbol{\lambda}_a^t \geq n-i} (y'_a - p_{i,k} - i) \cdot e^{q_{i,k}}, \end{aligned} \tag{2.170}$$

$$\begin{aligned} E_{i+1,i} \mapsto & \sum_{k\in J_i} e^{-q_{i,k}} \frac{\prod_{m\in J_{i-1}} (p_{i-1,m} - p_{i,k})}{\prod_{m\in J_i\setminus\{k\}} (p_{i,m} - p_{i,k})} \prod_{a:\boldsymbol{\lambda}_a^t \geq n-i+1} \frac{1}{y'_a - p_{i,k} - i} = \\ & \sum_{k\in J_i} \frac{\prod_{m\in J_{i-1}} (p_{i-1,m} - p_{i,k} - 1)}{\prod_{m\in J_i\setminus\{k\}} (p_{i,m} - p_{i,k} - 1)} \prod_{a:\boldsymbol{\lambda}_a^t \geq n-i+1} \frac{1}{y'_a - p_{i,k} - i - 1} \cdot e^{-q_{i,k}}. \end{aligned} \tag{2.171}$$

**Remark 2.172.** We note that the algebra $\mathcal{A}$ acts on the bigger space $\widetilde{M}_{\underline{y}}$ parametrized by $\Lambda = (\Lambda_{i,j})_{1\leq j\leq i\leq n}$ satisfying only the condition (a) via $p_{i,k}\colon \xi_\Lambda \mapsto l_{i,k}\xi_\Lambda$, $e^{\pm q_{i,k}}\colon \xi_\Lambda \mapsto \xi_{\Lambda\pm\delta_{i,k}}$. Meanwhile, the same formulas actually define the action of the subalgebra $\mathrm{Im}(\varrho) \subset \mathcal{A}$ on $M_{\underline{y}}$, composing which with $\varrho$ recovers the action of $U(\mathfrak{gl}_n)$ on $M_{\underline{y}}$ defined via (2.166), (2.167), (2.168).

Consider the *evaluation homomorphism* $\widetilde{\mathrm{ev}}\colon Y^{\mathrm{rtt}}_{\varpi_0}(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$ such that

$$T(z)_{i,i} \mapsto z - (E_{i,i}+1), \quad T(z)_{i,i+1} \mapsto E_{i,i+1}, \quad T(z)_{i+1,i} \mapsto E_{i+1,i}. \tag{2.173}$$

**Remark 2.174.** $\widetilde{\mathrm{ev}}$ is a composition of the isomorphism $Y^{\mathrm{rtt}}_{\varpi_0}(\mathfrak{gl}_n) \xrightarrow{\sim} Y^{\mathrm{rtt}}_0(\mathfrak{gl}_n)$ given by $T(z) \mapsto zT(z)$ of Remark 2.45(d), the evaluation homomorphism $\mathrm{ev}\colon Y^{\mathrm{rtt}}_0(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$ given by $t_{ij}(z) \mapsto \delta_{ij} - E_{ij}z^{-1}$, and the isomorphism $U(\mathfrak{gl}_n) \xrightarrow{\sim} U(\mathfrak{gl}_n)$ determined by $E_{ii} \mapsto E_{ii}+1, E_{i,i\pm1} \mapsto -E_{i,i\pm1}$.

The key result of this section is:

**Proposition 2.175.** *The homomorphism* $\mathrm{Ad}(US)\circ\Theta_D\colon Y^{\mathrm{rtt}}_{\varpi_0}(\mathfrak{gl}_n) \to \mathcal{A}$ *(the gauge transformation of* $\Theta_D$*) coincides with the composition* $\varrho\circ\widetilde{\mathrm{ev}}\colon Y^{\mathrm{rtt}}_{\varpi_0}(\mathfrak{gl}_n) \to \mathcal{A}$*, under the identification* $x_k = y'_k$ *for* $1 \leq k \leq \boldsymbol{\lambda}_1$.

**Proof.** The proof immediately follows by comparing the formulas (2.163), (2.164), (2.165) with the formulas (2.169), (2.170), (2.171) via (2.173) (as well as recalling that $i_k = n - \boldsymbol{\lambda}^t_k$, hence, for example $\sum_{k: i_k\leq i-1} x_k$ of (2.163) coincides with $\sum_{a:\boldsymbol{\lambda}^t_a\geq n-i+1} y'_a$ of (2.169)). $\square$

**Remark 2.176.** Choosing a basis of a Lie subalgebra $\mathfrak{n}_- \subseteq \mathfrak{gl}_n$ such that $\mathfrak{gl}_n \simeq \mathfrak{p}\oplus\mathfrak{n}_-$, yields another standard basis of $M_{\underline{y}}$ via the vector space isomorphisms $M_{\underline{y}} \simeq U(\mathfrak{n}_-) \simeq S(\mathfrak{n}_-)$, which similar to Proposition 2.175 gives rise to the rational Lax matrices $L_{\boldsymbol{\lambda},\underline{x},\widetilde{\boldsymbol{\mu}}=\emptyset}(z)$ of [15, §3.2].

## 3. Trigonometric Lax matrices

In this section, we generalize previous results to the trigonometric case.

### *3.1. Shifted Drinfeld quantum affine algebras of $\mathfrak{gl}_n$*

For a pair of $\mathfrak{gl}_n$–coweights $\mu^+,\mu^-\in\Lambda$, define $\underline{d}^\pm = \{d^\pm_j\}_{j=1}^n \in \mathbb{Z}^n, \underline{b}^\pm = \{b^\pm_i\}_{i=1}^{n-1} \in \mathbb{Z}^{n-1}$ via

$$d^\pm_j := \epsilon^\vee_j(\mu^\pm), \quad b^\pm_i := \alpha^\vee_i(\mu^\pm) = d^\pm_i - d^\pm_{i+1}. \tag{3.1}$$

Then, we define the *shifted Drinfeld quantum affine algebra of* $\mathfrak{gl}_n$, denoted by $U_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$, to be the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by

$$\{E_{i,r}, F_{i,r}\}_{1\le i<n}^{r\in\mathbb{Z}} \cup \{\varphi^{\pm}_{i,\pm s_i^{\pm}}, (\varphi^{\pm}_{i,\pm d_i^{\pm}})^{-1}\}_{1\le i\le n}^{s_i^{\pm}\ge d_i^{\pm}}$$

with the following defining relations (for all admissible $i,j$ and $\epsilon,\epsilon'\in\{\pm\}$):

$$[\varphi_i^{\epsilon}(z), \varphi_j^{\epsilon'}(w)] = 0, \quad \varphi^{\pm}_{i,\pm d_i^{\pm}} \cdot (\varphi^{\pm}_{i,\pm d_i^{\pm}})^{-1} = (\varphi^{\pm}_{i,\pm d_i^{\pm}})^{-1} \cdot \varphi^{\pm}_{i,\pm d_i^{\pm}} = 1, \tag{3.2}$$

$$[E_i(z), F_j(w)] = (\boldsymbol{v}-\boldsymbol{v}^{-1})\delta_{i,j}\delta\left(\frac{z}{w}\right)\Big((\varphi_i^+(z))^{-1}\varphi_{i+1}^+(z) - (\varphi_i^-(z))^{-1}\varphi_{i+1}^-(z)\Big), \tag{3.3}$$

$$\varphi_i^{\epsilon}(z)E_j(w) = \left(\frac{z-w}{\boldsymbol{v}^{-1}z-\boldsymbol{v}w}\right)^{\delta_{i,j+1}}\left(\frac{z-w}{\boldsymbol{v}z-\boldsymbol{v}^{-1}w}\right)^{\delta_{i,j}} E_j(w)\varphi_i^{\epsilon}(z), \tag{3.4}$$

$$\varphi_i^{\epsilon}(z)F_j(w) = \left(\frac{\boldsymbol{v}^{-1}z-\boldsymbol{v}w}{z-w}\right)^{\delta_{i,j+1}}\left(\frac{\boldsymbol{v}z-\boldsymbol{v}^{-1}w}{z-w}\right)^{\delta_{i,j}} F_j(w)\varphi_i^{\epsilon}(z), \tag{3.5}$$

$$E_i(z)E_j(w) = \left(\frac{\boldsymbol{v}z-\boldsymbol{v}^{-1}w}{\boldsymbol{v}^{-1}z-\boldsymbol{v}w}\right)^{\delta_{i,j}}\left(\frac{z-w}{\boldsymbol{v}z-\boldsymbol{v}^{-1}w}\right)^{\delta_{i,j-1}}\left(\frac{\boldsymbol{v}^{-1}z-\boldsymbol{v}w}{z-w}\right)^{\delta_{i,j+1}} E_j(w)E_i(z), \tag{3.6}$$

$$F_i(z)F_j(w) = \left(\frac{\boldsymbol{v}^{-1}z-\boldsymbol{v}w}{\boldsymbol{v}z-\boldsymbol{v}^{-1}w}\right)^{\delta_{i,j}}\left(\frac{\boldsymbol{v}z-\boldsymbol{v}^{-1}w}{z-w}\right)^{\delta_{i,j-1}}\left(\frac{z-w}{\boldsymbol{v}^{-1}z-\boldsymbol{v}w}\right)^{\delta_{i,j+1}} F_j(w)F_i(z), \tag{3.7}$$

$$[E_i(z_1),[E_i(z_2),E_j(w)]_{\boldsymbol{v}}]_{\boldsymbol{v}^{-1}} + [E_i(z_2),[E_i(z_1),E_j(w)]_{\boldsymbol{v}}]_{\boldsymbol{v}^{-1}} = 0 \text{ if } |i-j|=1, \tag{3.8}$$

$$[F_i(z_1),[F_i(z_2),F_j(w)]_{\boldsymbol{v}}]_{\boldsymbol{v}^{-1}} + [F_i(z_2),[F_i(z_1),F_j(w)]_{\boldsymbol{v}}]_{\boldsymbol{v}^{-1}} = 0 \text{ if } |i-j|=1, \tag{3.9}$$

where $[a,b]_x := ab - x\cdot ba$ and the generating series are defined as follows:

$$E_i(z) := \sum_{r\in\mathbb{Z}} E_{i,r}z^{-r},\ F_i(z) := \sum_{r\in\mathbb{Z}} F_{i,r}z^{-r},\ \varphi_i^{\pm}(z) := \sum_{r\ge d_i^{\pm}} \varphi^{\pm}_{i,\pm r}z^{\mp r},\ \delta(z) := \sum_{r\in\mathbb{Z}} z^r. \tag{3.10}$$

We will also need *Drinfeld half-currents* $E_i^{\pm}(z), F_i^{\pm}(z)$ defined via

$$\begin{aligned} E_i^+(z) &:= \sum_{r\ge 0} E_{i,r}z^{-r}, \quad E_i^-(z) := -\sum_{r<0} E_{i,r}z^{-r}, \\ F_i^+(z) &:= \sum_{r>0} F_{i,r}z^{-r}, \quad F_i^-(z) := -\sum_{r\le 0} F_{i,r}z^{-r}, \end{aligned} \tag{3.11}$$

so that $E_i(z) = E_i^+(z) - E_i^-(z)$ and $F_i(z) = F_i^+(z) - F_i^-(z)$.

**Remark 3.12.** For $\mu^+ = \mu^- = 0$, we have $U_{0,0}(L\mathfrak{gl}_n)/(\varphi^+_{i,0}\varphi^-_{i,0} - 1) \simeq U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$–the standard quantum loop (the quantum affine with the trivial central charge) algebra of $\mathfrak{gl}_n$ as defined in [9, Definition 3.1]. More precisely, the generating series $X_i^-(z), X_i^+(z), k_j^{\pm}(z)$ of [9] correspond to $E_i(z), F_i(z), \varphi_j^{\mp}(z)$ of (3.10), respectively.

Similarly to Lemma 2.17, the algebra $U_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$ depends only on the associated $\mathfrak{sl}_n$–coweights $\bar{\mu}^+, \bar{\mu}^- \in \bar{\Lambda}$, up to an isomorphism:

**Lemma 3.13.** *For $\mathfrak{gl}_n$–coweights $\mu_1^+, \mu_1^-, \mu_2^+, \mu_2^- \in \Lambda$ such that $\bar{\mu}_1^+ = \bar{\mu}_2^+, \bar{\mu}_1^- = \bar{\mu}_2^-$ in $\bar{\Lambda}$, the assignment*

$$E_i^{(r)} \mapsto E_i^{(r)}, \quad F_i^{(r)} \mapsto F_i^{(r)}, \quad \varphi^\pm_{i,\pm s_i^\pm} \mapsto \varphi^\pm_{i,\pm s_i^\pm \mp \epsilon_i^\vee(\mu_1^\pm - \mu_2^\pm)} \tag{3.14}$$

*gives rise to a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $U_{\mu_1^+,\mu_1^-}(L\mathfrak{gl}_n) \xrightarrow{\sim} U_{\mu_2^+,\mu_2^-}(L\mathfrak{gl}_n)$.*

Let $U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$ be the associative $\mathbb{C}(\boldsymbol{v})$-algebra obtained from $U_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$ by formally adjoining $n$-th roots of its central elements $\varphi^\pm := \varphi^\pm_{1,\pm d_1^\pm}\varphi^\pm_{2,\pm d_2^\pm}\cdots\varphi^\pm_{n,\pm d_n^\pm}$, that is,

$$U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n) := U_{\mu^+,\mu^-}(L\mathfrak{gl}_n)\left[(\varphi^+)^{\pm 1/n}, (\varphi^-)^{\pm 1/n}\right]. \tag{3.15}$$

The algebras $U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$ slightly generalize the *shifted (Drinfeld) quantum affine algebras of* $\mathfrak{sl}_n$, denoted by $U^{\mathrm{sc}}_{\nu^+,\nu^-}(L\mathfrak{sl}_n)$ (the *simply-connected version*) and $U^{\mathrm{ad}}_{\nu^+,\nu^-}(L\mathfrak{sl}_n)$ (the *adjoint version*) in [18, §5], where $\nu^+, \nu^- \in \bar{\Lambda}$ are $\mathfrak{sl}_n$–coweights. Recall that the latter, the algebra $U^{\mathrm{ad}}_{\nu^+,\nu^-}(L\mathfrak{sl}_n)$, is an associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by $\{e_{i,r}, f_{i,r}, \psi^\pm_{i,\pm s_i^\pm}, (\phi_i^\pm)^{\pm 1}\}_{1\le i<n}^{r\in\mathbb{Z}, s_i^\pm \ge -b_i^\pm}$ with the defining relations [18, (U1–U10)], where $b_i^\pm := \alpha_i^\vee(\nu_i^\pm)$. Define the generating series

$$e_i(z) := \sum_{r\in\mathbb{Z}} e_{i,r}z^{-r}, \quad f_i(z) := \sum_{r\in\mathbb{Z}} f_{i,r}z^{-r}, \quad \psi_i^\pm(z) := \sum_{r\ge -b_i^\pm} \psi^\pm_{i,\pm r}z^{\mp r}.$$

The explicit relation between the shifted Drinfeld quantum affine algebras of $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$ is:

**Proposition 3.16.** *For any $\mu^+, \mu^- \in \Lambda$, there exists a $\mathbb{C}(\boldsymbol{v})$-algebra embedding*

$$\iota_{\mu^+,\mu^-} : U^{\mathrm{ad}}_{\bar{\mu}^+,\bar{\mu}^-}(L\mathfrak{sl}_n) \hookrightarrow U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n), \tag{3.17}$$

*uniquely determined by*

$$\begin{gathered} e_i(z) \mapsto \frac{E_i(\boldsymbol{v}^i z)}{\boldsymbol{v}-\boldsymbol{v}^{-1}}, \quad f_i(z) \mapsto \frac{F_i(\boldsymbol{v}^i z)}{\boldsymbol{v}-\boldsymbol{v}^{-1}}, \\ \psi_i^\pm(z) \mapsto (\varphi_i^\pm(\boldsymbol{v}^i z))^{-1}\varphi^\pm_{i+1}(\boldsymbol{v}^i z), \quad \phi_i^\pm \mapsto (\varphi^\pm_{1,\pm d_1^\pm}\cdots\varphi^\pm_{i,\pm d_i^\pm})^{-1}\cdot(\varphi^\pm)^{i/n}. \end{gathered} \tag{3.18}$$

*Restricting to $U^{\mathrm{sc}}_{\bar{\mu}^+,\bar{\mu}^-}(L\mathfrak{sl}_n) \subset U^{\mathrm{ad}}_{\bar{\mu}^+,\bar{\mu}^-}(L\mathfrak{sl}_n)$, this gives rise to a $\mathbb{C}(\boldsymbol{v})$-algebra embedding*

$$\iota_{\mu^+,\mu^-} : U^{\mathrm{sc}}_{\bar{\mu}^+,\bar{\mu}^-}(L\mathfrak{sl}_n) \hookrightarrow U_{\mu^+,\mu^-}(L\mathfrak{gl}_n). \tag{3.19}$$

**Remark 3.20.** For $\mu^+ = \mu^- = 0$, this recovers (an extension of) the classical embedding $U_{\boldsymbol{v}}(L\mathfrak{sl}_n) \hookrightarrow U_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ of quantum loop algebras.

**Proof of Proposition 3.16.** The proof is completely analogous to that of Proposition 2.19. $\square$

Define the generating series

$$C^{\pm}(z) := \sum_{s \geq d_1^{\pm}+\ldots+d_n^{\pm}} C^{\pm}_{\pm s} z^{\mp s} = \varphi_1^{\pm}(z)\varphi_2^{\pm}(\boldsymbol{v}^2 z)\cdots\varphi_n^{\pm}(\boldsymbol{v}^{2(n-1)}z). \tag{3.21}$$

The coefficients $C^{\pm}_{\pm s}$ are central elements of both $U_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$ and $U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$, due to the defining relations (3.2), (3.4), (3.5). We also note that $C^{\pm}_{\pm(d_1^{\pm}+\ldots+d_n^{\pm})} = \varphi^{\pm}$.

The following result provides a trigonometric version of the decomposition (2.25):

**Lemma 3.22.** *There is a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism*

$$U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n) \simeq \mathbb{C}[\{C^{\pm}_{\pm s^{\pm}}, (\varphi^{\epsilon})^{\pm 1/n}\}^{\epsilon\in\{+,-\}}_{s^{\pm}>d_1^{\pm}+\ldots+d_n^{\pm}}] \otimes_{\mathbb{C}(\boldsymbol{v})} U^{\mathrm{ad}}_{\bar{\mu}^+,\bar{\mu}^-}(L\mathfrak{sl}_n). \tag{3.23}$$

*In particular, $U^{\mathrm{ad}}_{\bar{\mu}^+,\bar{\mu}^-}(L\mathfrak{sl}_n)$ may be realized both as a subalgebra of $U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$ via (3.17) as well as a quotient algebra of $U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$ by the central ideal $(C^{\pm}_{\pm s^{\pm}} - b^{\pm}_{\pm s^{\pm}}$, $(\varphi^{\epsilon})^{\pm 1/n} - (b^{\epsilon})^{\pm 1})$ with $\epsilon \in \{+,-\}, s^{\pm} > d_1^{\pm}+\ldots+d_n^{\pm}$ for any collection of $b^{\pm}_{\pm s^{\pm}} \in \mathbb{C}$ and $b^{\epsilon} \in \mathbb{C}^{\times}$.*

**Remark 3.24.** We expect that the trigonometric version of the key result of [40], see Theorem 2.80 and Conjecture 3.75, holds. Then, the arguments similar to those of Remark 2.81 would yield the triviality of centers of the shifted quantum affine algebras $U^{\mathrm{ad}}_{\nu^+,\nu^-}(L\mathfrak{g})$ for any coweights $\nu^+, \nu^-$ of a semisimple Lie algebra $\mathfrak{g}$. Combined with (3.23) this would imply that the center of $U'_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$ coincides with $\mathbb{C}[\{C^{\pm}_{\pm s^{\pm}}, (\varphi^{\epsilon})^{\pm 1/n}\}^{\epsilon\in\{+,-\}}_{s^{\pm}>d_1^{\pm}+\ldots+d_n^{\pm}}]$ for any $\mu^+, \mu^- \in \Lambda$.

### *3.2. Homomorphism $\Psi_D$*

In this section, we generalize [18, Theorem 7.1] for the type $A_{n-1}$ Dynkin diagram with arrows pointing $i \to i+1, 1 \leq i \leq n-2$, by replacing $U^{\mathrm{ad}}_{\bar{\mu}^+,\bar{\mu}^-}(L\mathfrak{sl}_n)$ of [18, Theorem 7.1] with $U_{\mu^+,\mu^-}(L\mathfrak{gl}_n)$.

**Remark 3.25.** While similar generalizations exist for all orientations of $A_{n-1}$ Dynkin diagram, for the purposes of this paper it suffices to consider only the above equioriented case, see Remarks 2.27, 2.73.

A **$\Lambda$-valued divisor $D$ on $\mathbb{P}^1$, $\Lambda^+$-valued outside** $\{0,\infty\} \in \mathbb{P}^1$, is a formal sum

$$D = \sum_{1\le s\le N} \gamma_s \varpi_{i_s}[\mathsf{x}_s] + \mu^+[\infty] + \mu^-[0] \tag{3.26}$$

with $N \in \mathbb{N}$, $0 \le i_s < n$, $\mathsf{x}_s \in \mathbb{C}^\times$, $\gamma_s = \begin{cases} 1 & \text{if } i_s \neq 0 \\ \pm 1 & \text{if } i_s = 0 \end{cases}$, and $\mu^+, \mu^- \in \Lambda$. We will write $\mu^+ = D|_\infty$ and $\mu^- = D|_0$. Note that if $\mu^+, \mu^- \in \Lambda^+$, then $D$ is a **$\Lambda^+$-valued divisor on** $\mathbb{P}^1$. It will be convenient to present

$$D = \sum_{\mathsf{x}\in\mathbb{P}^1\setminus\{0,\infty\}} \lambda_{\mathsf{x}}[\mathsf{x}] + \mu^+[\infty] + \mu^-[0] \text{ with } \lambda_{\mathsf{x}} \in \Lambda^+, \tag{3.27}$$

related to (3.26) via $\lambda_{\mathsf{x}} := \sum_{s:\mathsf{x}_s=\mathsf{x}} \gamma_s \varpi_{i_s}$. Set $\lambda := \sum_{s=1}^N \gamma_s \varpi_{i_s} \in \Lambda^+$. Following [18], we make the following

$$\textbf{Assumption:}\quad \lambda + \mu^+ + \mu^- = a_1\alpha_1 + \ldots + a_{n-1}\alpha_{n-1} \quad \text{with } a_i \in \mathbb{N}. \tag{3.28}$$

Consider the associative $\mathbb{C}[\boldsymbol{v}, \boldsymbol{v}^{-1}]$-algebra

$$\widetilde{\mathcal{A}}^{\boldsymbol{v}} = \mathbb{C}\langle D_{i,r}^{\pm 1}, \mathsf{w}_{i,r}^{\pm 1/2}, (\mathsf{w}_{i,r} - \boldsymbol{v}^m \mathsf{w}_{i,s})^{-1}, (1-\boldsymbol{v}^l)^{-1}\rangle_{1\le i<n, m\in\mathbb{Z}, l\in\mathbb{Z}\setminus\{0\}}^{1\le r\neq s\le a_i} \tag{3.29}$$

with the defining relations

$$D_{i,r}\mathsf{w}_{j,s}^{1/2} = \boldsymbol{v}^{\delta_{i,j}\delta_{r,s}} \mathsf{w}_{j,s}^{1/2} D_{i,r}, \quad [D_{i,r}, D_{j,s}] = 0 = [\mathsf{w}_{i,r}^{1/2}, \mathsf{w}_{j,s}^{1/2}],$$
$$D_{i,r}^{\pm 1} D_{i,r}^{\mp 1} = 1 = \mathsf{w}_{i,r}^{\pm 1/2}\mathsf{w}_{i,r}^{\mp 1/2}.$$

We also define its $\mathbb{C}(\boldsymbol{v})$-counterpart

$$\widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}} := \widetilde{\mathcal{A}}^{\boldsymbol{v}} \otimes_{\mathbb{C}[\boldsymbol{v},\boldsymbol{v}^{-1}]} \mathbb{C}(\boldsymbol{v}). \tag{3.30}$$

**Remark 3.31.** The algebra $\widetilde{\mathcal{A}}^{\boldsymbol{v}}$ can be represented in the algebra of $\boldsymbol{v}$-difference operators with rational coefficients on functions of $\{\widetilde{\mathsf{w}}_{i,r}\}_{1\le i<n}^{1\le r\le a_i}$ with the conventions $\widetilde{\mathsf{w}}_{i,r}^{\pm 1} = \mathsf{w}_{i,r}^{\pm 1/2}$ by taking $D_{i,r}^{\pm 1}$ to be a $\boldsymbol{v}$-difference operator $\mathsf{D}_{i,r}^{\pm 1}$ acting via $(\mathsf{D}_{i,r}^{\pm 1}\Psi)(\widetilde{\mathsf{w}}_{1,1}, \ldots, \widetilde{\mathsf{w}}_{i,r}, \ldots, \widetilde{\mathsf{w}}_{n-1,a_{n-1}}) = \Psi(\widetilde{\mathsf{w}}_{1,1}, \ldots, \boldsymbol{v}^{\pm 1}\widetilde{\mathsf{w}}_{i,r}, \ldots, \widetilde{\mathsf{w}}_{n-1,a_{n-1}})$.

For $0 \le i \le n-1$ and $1 \le j \le n-1$, we define

$$\begin{aligned} \mathsf{Z}_i(z) &:= \prod_{1\le s\le N}^{i_s=i} \left(1 - \frac{\boldsymbol{v}^{-i}\mathsf{x}_s}{z}\right)^{\gamma_s} = \prod_{\mathsf{x}\in\mathbb{P}^1\setminus\{0,\infty\}} \left(1 - \frac{\boldsymbol{v}^{-i}\mathsf{x}}{z}\right)^{\alpha_i^\vee(\lambda_{\mathsf{x}})}, \\ \mathsf{W}_j(z) &:= \prod_{r=1}^{a_j}\left(1 - \frac{\mathsf{w}_{j,r}}{z}\right), \quad \mathsf{W}_{j,r}(z) := \prod_{1\le s\le a_j}^{s\neq r}\left(1 - \frac{\mathsf{w}_{j,s}}{z}\right), \end{aligned} \tag{3.32}$$

where $\alpha_0^\vee = -\epsilon_1^\vee$ as before. We also define

$$a_0 := 0, \quad a_n := 0, \quad \mathsf{W}_0(z) := 1, \quad \mathsf{W}_n(z) := 1.$$

The following result generalizes $A_{n-1}$-case of [18, Theorem 7.1] stated for semisimple Lie algebras $\mathfrak{g}$:

**Theorem 3.33.** *Let $D$ be as above and $\mu^+ = D|_\infty, \mu^- = D|_0$. There is a unique $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism*

$$\Psi_D \colon U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \longrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}} \tag{3.34}$$

*such that*

$$\begin{gathered}
E_i(z) \mapsto z^{-\alpha_i^\vee(\mu^+)} \cdot \prod_{t=1}^{a_i} \mathsf{w}_{i,t} \prod_{t=1}^{a_{i-1}} \mathsf{w}_{i-1,t}^{-1/2} \cdot \sum_{r=1}^{a_i} \delta\left(\frac{\boldsymbol{v}^i \mathsf{w}_{i,r}}{z}\right) \frac{\mathsf{Z}_i(\mathsf{w}_{i,r})}{\mathsf{W}_{i,r}(\mathsf{w}_{i,r})} \mathsf{W}_{i-1}(\boldsymbol{v}^{-1}\mathsf{w}_{i,r}) D_{i,r}^{-1}, \\
F_i(z) \mapsto -\boldsymbol{v}^{-1} \prod_{t=1}^{a_{i+1}} \mathsf{w}_{i+1,t}^{-1/2} \cdot \sum_{r=1}^{a_i} \delta\left(\frac{\boldsymbol{v}^{i+2} \mathsf{w}_{i,r}}{z}\right) \frac{1}{\mathsf{W}_{i,r}(\mathsf{w}_{i,r})} \mathsf{W}_{i+1}(\boldsymbol{v}\mathsf{w}_{i,r}) D_{i,r}, \\
\varphi_i^\pm(z) \mapsto \prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{-1/2} \prod_{t=1}^{a_{i-1}} \mathsf{w}_{i-1,t}^{1/2} \cdot \left( z^{d_i^+} \cdot \frac{\mathsf{W}_i(\boldsymbol{v}^{-i}z)}{\mathsf{W}_{i-1}(\boldsymbol{v}^{-i-1}z)} \prod_{0\le k\le i-1} \mathsf{Z}_k(\boldsymbol{v}^{-k}z) \right)^{\pm} = \\
\prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{-1/2} \prod_{t=1}^{a_{i-1}} \mathsf{w}_{i-1,t}^{1/2} \cdot \left( z^{\epsilon_i^\vee(\mu^+)} \cdot \frac{\mathsf{W}_i(\boldsymbol{v}^{-i}z)}{\mathsf{W}_{i-1}(\boldsymbol{v}^{-i-1}z)} \prod_{\mathsf{x}\in\mathbb{P}^1\backslash\{0,\infty\}} (1-\mathsf{x}/z)^{-\epsilon_i^\vee(\lambda_{\mathsf{x}})} \right)^{\pm}.
\end{gathered} \tag{3.35}$$

*We write $\gamma(z)^\pm$ for the expansion of a rational function $\gamma(z)$ in $z^{\mp 1}$, respectively.*

**Remark 3.36.** Let $\widetilde{\mathcal{A}}^{\boldsymbol{v},\mathrm{ext}}_{\mathrm{frac}}$ be the associative $\mathbb{C}(\boldsymbol{v})$-algebra obtained from $\widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}$ by formally adjoining $n$-th roots of $\boldsymbol{v}, \mathsf{x}_s$, and $\Psi_D \colon U'_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \to \widetilde{\mathcal{A}}^{\boldsymbol{v},\mathrm{ext}}_{\mathrm{frac}}$ be the extended homomorphism. Then, the (restriction) composition

$$U^{\mathrm{ad}}_{-\bar{\mu}^+,-\bar{\mu}^-}(L\mathfrak{sl}_n) \xrightarrow{\iota_{-\mu^+,-\mu^-}} U'_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \xrightarrow{\Psi_D} \widetilde{\mathcal{A}}^{\boldsymbol{v},\mathrm{ext}}_{\mathrm{frac}}$$

coincides with the composition of the natural isomorphism

$$U^{\mathrm{ad}}_{-\bar{\mu}^+,-\bar{\mu}^-}(L\mathfrak{sl}_n) \xrightarrow{\sim} U^{\mathrm{ad}}_{0,-\bar{\mu}^+-\bar{\mu}^-}(L\mathfrak{sl}_n)$$

and the homomorphism

$$\widetilde{\Phi}^{\bar{\lambda}}_{-\bar{\mu}^+-\bar{\mu}^-} \colon U^{\mathrm{ad}}_{0,-\bar{\mu}^+-\bar{\mu}^-}(L\mathfrak{sl}_n) \longrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v},\mathrm{ext}}_{\mathrm{frac}}$$

of [18, Theorem 7.1].

**Proof of Theorem 3.33.** First, we need to verify that under the above assignment (3.35), the images of $\varphi_i^+(z)$ (resp. $\varphi_i^-(z)$) contain only powers of $z$ which are $\leq d_i^+$ (resp. $\geq -d_i^-$), and the corresponding coefficients of $z^{d_i^+}$ (respectively of $z^{-d_i^-}$) are invertible. The claim is clear for $\varphi_i^+(z)$, while its validity for $\varphi_i^-(z)$ follows from the equality

$$-a_i + a_{i-1} + \epsilon_i^\vee(\mu^+) + \epsilon_i^\vee(\lambda) = -\epsilon_i^\vee(\mu^-),$$

due to (3.28).

Evoking the decomposition (3.23), it suffices to prove that the restrictions of the assignment (3.35) to the subalgebras $U^{\mathrm{ad}}_{-\bar{\mu}^+,-\bar{\mu}^-}(L\mathfrak{sl}_n)$ and $\mathbb{C}[\{C^\pm_{\pm s}\}_{s>d_1^\pm+\ldots+d_n^\pm}]$ determine algebra homomorphisms, whose images commute. The former is clear for the restriction to $U^{\mathrm{ad}}_{-\bar{\mu}^+,-\bar{\mu}^-}(L\mathfrak{sl}_n)$, due to Theorem 7.1 of [18] combined with Remark 3.36 above. On the other hand, we have

$$\Psi_D(C^\pm(z)) = \mathsf{A}\cdot\prod_{i=1}^{n}\prod_{\mathsf{x}\in\mathbb{P}^1\backslash\{0,\infty\}}\left(1-\boldsymbol{v}^{-2(i-1)}\frac{\mathsf{x}}{z}\right)^{-\epsilon_i^\vee(\lambda_{\mathsf{x}})} = \mathsf{A}\cdot\prod_{s=1}^{N}\prod_{k=i_s}^{n-1}\left(1-\boldsymbol{v}^{-2k}\frac{\mathsf{x}_s}{z}\right)^{\gamma_s}, \tag{3.37}$$

where $\mathsf{A} := \prod_{i=1}^n (\boldsymbol{v}^{2(i-1)}z)^{\epsilon_i^\vee(\mu^+)}$.

Thus, the restriction of $\Psi_D$ to the subalgebra $\mathbb{C}[\{C^\pm_{\pm s}\}_{s>d_1^\pm+\ldots+d_n^\pm}]$ defines an algebra homomorphism, whose image is central in $\widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}$. This completes our proof of Theorem 3.33. □

### 3.3. *Antidominantly shifted RTT quantum affine algebras of $\mathfrak{gl}_n$*

Consider the *trigonometric* $R$-matrix $R_{\mathrm{trig}}(z,w) = R^{\boldsymbol{v}}_{\mathrm{trig}}(z,w)$ given by

$$\begin{aligned} R_{\mathrm{trig}}(z,w) := (\boldsymbol{v}z-\boldsymbol{v}^{-1}w)\sum_{i=1}^n E_{ii}\otimes E_{ii} + (z-w)\sum_{i\neq j} E_{ii}\otimes E_{jj} + \\ (\boldsymbol{v}-\boldsymbol{v}^{-1})z\sum_{i<j} E_{ij}\otimes E_{ji} + (\boldsymbol{v}-\boldsymbol{v}^{-1})w\sum_{i>j} E_{ij}\otimes E_{ji}, \end{aligned} \tag{3.38}$$

cf. [9, (3.7)]. It satisfies the Yang-Baxter equation with a spectral parameter:

$$R_{\mathrm{trig};12}(u,v)R_{\mathrm{trig};13}(u,w)R_{\mathrm{trig};23}(v,w) = R_{\mathrm{trig};23}(v,w)R_{\mathrm{trig};13}(u,w)R_{\mathrm{trig};12}(u,v). \tag{3.39}$$

Fix $\mu^+,\mu^-\in\Lambda^+$. Define the *(antidominantly) shifted RTT quantum affine algebra of* $\mathfrak{gl}_n$, denoted by $U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$, to be the associative $\mathbb{C}(\boldsymbol{v})$-algebra generated by

$$\{t^\pm_{ij}[\pm r]\}^{r\in\mathbb{Z}}_{1\leq i,j\leq n}\cup\{(g^\pm_{i,\mp d_i^\pm})^{-1}\}_{i=1}^n$$

subject to the following three families of relations:

- The first family of relations may be encoded by a single RTT relation

$$R_{\mathrm{trig}}(z,w)T_1^{\epsilon}(z)T_2^{\epsilon'}(w) = T_2^{\epsilon'}(w)T_1^{\epsilon}(z)R_{\mathrm{trig}}(z,w) \tag{3.40}$$

for any $\epsilon,\epsilon' \in \{+,-\}$, where $T^{\pm}(z) \in U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)[[z,z^{-1}]] \otimes_{\mathbb{C}} \operatorname{End}\mathbb{C}^n$ are defined via

$$T^{\pm}(z) = \sum_{i,j} t^{\pm}_{ij}(z) \otimes E_{ij} \quad \text{with} \quad t^{\pm}_{ij}(z) := \sum_{r\in\mathbb{Z}} t^{\pm}_{ij}[\pm r] z^{\mp r}. \tag{3.41}$$

Thus, (3.40) is an equality in $U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)[[z,z^{-1},w,w^{-1}]] \otimes_{\mathbb{C}} (\operatorname{End}\mathbb{C}^n)^{\otimes 2}$ for any $\epsilon,\epsilon'$.
- The second family of relations encodes the fact that $T^{\pm}(z)$ admits the Gauss decomposition:

$$T^{\pm}(z) = F^{\pm}(z)\cdot G^{\pm}(z)\cdot E^{\pm}(z), \tag{3.42}$$

where $F^{\pm}(z), G^{\pm}(z), E^{\pm}(z) \in U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)((z^{\mp 1})) \otimes_{\mathbb{C}} \operatorname{End}\mathbb{C}^n$ are of the form

$$\begin{aligned} F^{\pm}(z) &= \sum_i E_{ii} + \sum_{i<j} f^{\pm}_{ji}(z) \otimes E_{ji}, \quad G^{\pm}(z) = \sum_i g^{\pm}_i(z) \otimes E_{ii}, \\ E^{\pm}(z) &= \sum_i E_{ii} + \sum_{i<j} e^{\pm}_{ij}(z) \otimes E_{ij}, \end{aligned}$$

with the matrix coefficients having the following expansions in $z$:

$$\begin{aligned} e^{+}_{ij}(z) &= \sum_{r\geq 0} e^{(r)}_{ij} z^{-r}, \quad e^{-}_{ij}(z) = \sum_{r<0} e^{(r)}_{ij} z^{-r}, \\ f^{+}_{ij}(z) &= \sum_{r>0} f^{(r)}_{ij} z^{-r}, \quad f^{-}_{ij}(z) = \sum_{r\leq 0} f^{(r)}_{ij} z^{-r}, \\ g^{+}_{i}(z) &= \sum_{r\geq -d^+_i} g^{+}_{i,r} z^{-r}, \quad g^{-}_{i}(z) = \sum_{r\geq -d^-_i} g^{-}_{i,-r} z^{r}, \end{aligned} \tag{3.43}$$

where $\{e^{(r)}_{ij}, f^{(r)}_{ji}\}^{r\in\mathbb{Z}}_{1\leq i<j\leq n} \cup \{g^{\pm}_{i,\pm s^{\pm}_i}\}^{s^{\pm}_i \geq -d^{\pm}_i}_{1\leq i\leq n} \subset U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$.
- The third family of relations is just:

$$g^{\pm}_{i,\mp d^{\pm}_i} \cdot (g^{\pm}_{i,\mp d^{\pm}_i})^{-1} = (g^{\pm}_{i,\mp d^{\pm}_i})^{-1} \cdot g^{\pm}_{i,\mp d^{\pm}_i} = 1. \tag{3.44}$$

**Remark 3.45.** (a) For $\mu^+ = \mu^- = 0$, the second family of relations (3.42), (3.43) is equivalent to the relations $t^{+}_{ij}[r] = t^{-}_{ij}[-r] = 0$ for all $i,j$ and $r<0$ as well as $t^{+}_{ji}[0] = t^{-}_{ij}[0] = 0$ for $1\leq i<j\leq n$. In this case, adjoining the inverses of $g^{\pm}_{i,0}$, cf. (3.44), is equivalent to adjoining the inverses of $t^{\pm}_{ii}[0]$. Thus, $U^{\mathrm{rtt}}_{0,0}(L\mathfrak{gl}_n)$ is the RTT quantum loop algebra of $\mathfrak{gl}_n$ of [17], or more precisely, its extended version $U^{\mathrm{rtt,ext}}_{\boldsymbol{v}}(L\mathfrak{gl}_n)$ of [23, (2.15)].

(b) Likewise, (3.43) is equivalent to a certain family of algebraic relations on $t^{\pm}_{ij}[r]$. In particular, $T^{\pm}(z) \in U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}((z^{\mp 1})) \otimes_{\mathbb{C}} \operatorname{End} \mathbb{C}^n$. For example, (3.43) for $i = 1$ are equivalent to:

$$\begin{aligned}
&t^+_{11}[r] = 0 \text{ for } r < -d^+_1, \quad t^-_{11}[-r] = 0 \text{ for } r < -d^-_1,\\
&t^+_{1j}[r] = 0 \ \text{ for } r < -d^+_1, j > 1, \quad t^-_{1j}[-r] = 0 \ \text{ for } r \le -d^-_1, j > 1,\\
&t^+_{j1}[r] = 0 \ \text{ for } r \le -d^+_1, j > 1, \quad t^-_{j1}[-r] = 0 \ \text{ for } r < -d^-_1, j > 1.
\end{aligned}$$

(c) If $\mu^+_1, \mu^-_1, \mu^+_2, \mu^-_2 \in \Lambda^+$ satisfy $\bar{\mu}^+_1 = \bar{\mu}^+_2$ and $\bar{\mu}^-_1 = \bar{\mu}^-_2$ in $\bar{\Lambda}$, that is, $\mu^+_2 = \mu^+_1 + c^+\varpi_0$ and $\mu^-_2 = \mu^-_1 + c^-\varpi_0$ with $c^+, c^- \in \mathbb{Z}$, then the assignment $T^{\pm}(z) \mapsto z^{\pm c^{\pm}} T^{\pm}(z)$ gives rise to a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism $U^{\mathrm{rtt}}_{-\mu^+_1,-\mu^-_1}(L\mathfrak{gl}_n) \xrightarrow{\sim} U^{\mathrm{rtt}}_{-\mu^+_2,-\mu^-_2}(L\mathfrak{gl}_n)$, cf. Lemma 3.13.

**Lemma 3.46.** *For any $1 \le i < j \le n$ and $r \in \mathbb{Z}$, we have the following identities:*

$$\begin{aligned}
e^{(r)}_{ij} &= (\boldsymbol{v} - \boldsymbol{v}^{-1})^{i-j+1} [e^{(0)}_{j-1,j}, [e^{(0)}_{j-2,j-1}, \cdots, [e^{(0)}_{i+1,i+2}, e^{(r)}_{i,i+1}]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}}]_{\boldsymbol{v}^{-1}},\\
f^{(r)}_{ji} &= (\boldsymbol{v}^{-1} - \boldsymbol{v})^{i-j+1} [[[\cdots [f^{(r)}_{i+1,i}, f^{(0)}_{i+2,i+1}]_{\boldsymbol{v}}, \cdots, f^{(0)}_{j-1,j-2}]_{\boldsymbol{v}}, f^{(0)}_{j,j-1}]_{\boldsymbol{v}}.
\end{aligned} \tag{3.47}$$

**Proof.** The proof is analogous to that of [19, Corollary 3.22]. □

**Corollary 3.48.** *The algebra $U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$ is generated by*

$$\{e^{(r)}_{i,i+1}, f^{(r)}_{i+1,i}, g^{\pm}_{j,\pm s^{\pm}_j}, (g^{\pm}_{j,\mp d^{\pm}_j})^{-1}\}^{r \in \mathbb{Z}, s^{\pm}_j \ge -d^{\pm}_j}_{1 \le i < n, 1 \le j \le n}.$$

The following result is a shifted version of [9, Main Theorem] and a trigonometric version of our Theorem 2.52:

**Theorem 3.49.** *For any $\mu^+, \mu^- \in \Lambda^+$, there is a unique $\mathbb{C}(\boldsymbol{v})$-algebra epimorphism*

$$\Upsilon_{-\mu^+,-\mu^-} : U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \twoheadrightarrow U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$$

*defined by*

$$E^{\pm}_i(z) \mapsto e^{\pm}_{i,i+1}(z), \quad F^{\pm}_i(z) \mapsto f^{\pm}_{i+1,i}(z), \quad \varphi^{\pm}_j(z) \mapsto g^{\pm}_j(z). \tag{3.50}$$

Modulo a trigonometric counterpart of [40], see Conjecture 3.75, the following result is proved in Section 3.4.3:

**Theorem 3.51.** *$\Upsilon_{-\mu^+,-\mu^-} : U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \xrightarrow{\sim} U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$ is a $\mathbb{C}(\boldsymbol{v})$-algebra isomorphism for any $\mu^+, \mu^- \in \Lambda^+$.*

**Remark 3.52.** (a) For $\mu^+ = \mu^- = 0$ and any $n$, the isomorphism $\Upsilon_{0,0}$ of Theorem 3.51 was established in [9, Main Theorem] (more precisely, $\Upsilon_{0,0}$ is an isomorphism between the extended versions of both algebras featuring in the Main Theorem of [9]).
(b) For $n = 2$ and $\mu^+, \mu^- \in \Lambda^+$, a long straightforward verification shows that the assignment

$$t^{\pm}_{11}(z) \mapsto \varphi^{\pm}_1(z), \quad t^{\pm}_{22}(z) \mapsto F^{\pm}_1(z)\varphi^{\pm}_1(z)E^{\pm}_1(z) + \varphi^{\pm}_2(z),$$
$$t^{\pm}_{12}(z) \mapsto \varphi^{\pm}_1(z)E^{\pm}_1(z), \quad t^{\pm}_{21}(z) \mapsto F^{\pm}_1(z)\varphi^{\pm}_1(z),$$

gives rise to a $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism $U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_2) \to U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_2)$ (the $\mathfrak{sl}_2$-counterpart of which is due to [18, Theorem 11.11]), which is clearly the inverse of $\Upsilon_{-\mu^+,-\mu^-}$. Thus, Theorem 3.51 for $n = 2$ is essentially due to [18].

### 3.4. Trigonometric Lax matrices via antidominantly shifted quantum affine algebras of $\mathfrak{gl}_n$

In this section, we construct $n \times n$ *trigonometric Lax* matrices $T_D(z)$ (with coefficients in $\widetilde{\mathcal{A}}^{\boldsymbol{v}}(z)$) for each $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying (3.28). They are explicitly defined via (3.64), (3.65) combined with (3.56), (3.58), (3.60). We note that these formulas arise naturally by considering the images of $T^{\pm}(z) \in U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)((z^{\mp 1}))\otimes_{\mathbb{C}} \mathrm{End}\,\mathbb{C}^n$ under the composition $\Psi_D \circ \Upsilon^{-1}_{-\mu^+,-\mu^-} : U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}$, assuming Theorem 3.51 has been established, see (3.53), (3.54) and Proposition 3.63. As the name indicates, $(T_D(z))^{\pm}$ satisfy the RTT relation (3.40), which is derived in Proposition 3.74. Combining the latter with the conjectured generalization of [40], see Conjecture 3.75, we finally prove Theorem 3.51 in Section 3.4.3.

We also establish the regularity (up to a rational factor (3.67)) of $T_D(z)$ in Theorem 3.68, and find simplified explicit formulas for those $T_D(z)$ which are linear in $z$ in Theorem 3.77. Finally, we show how to degenerate these trigonometric Lax matrices into the rational Lax matrices of Section 2.4.1, see Proposition 3.94.

#### 3.4.1. Construction of $T_D(z)$ and their regularity

Consider a $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$, see (3.26), satisfying the assumption (3.28). Note that $\mu^+ := D|_\infty \in \Lambda^+$ and $\mu^- := D|_0 \in \Lambda^+$. Composing $\Psi_D \colon U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}$ of (3.34) with the isomorphism $\Upsilon^{-1}_{-\mu^+,-\mu^-} : U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \xrightarrow{\sim} U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$ (assuming the validity of Theorem 3.51), we obtain an algebra homomorphism

$$\Theta_D = \Psi_D \circ \Upsilon^{-1}_{-\mu^+,-\mu^-} : U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \longrightarrow \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}. \tag{3.53}$$

Such a homomorphism is uniquely determined by two matrices

$$T^{\pm}_D(z) \in \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}((z^{\mp 1})) \otimes_{\mathbb{C}} \mathrm{End}\,\mathbb{C}^n$$

defined via

$$T_D^{\pm}(z) := \Theta_D(T^{\pm}(z)) = \Theta_D(F^{\pm}(z)) \cdot \Theta_D(G^{\pm}(z)) \cdot \Theta_D(E^{\pm}(z)). \tag{3.54}$$

**Remark 3.55.** Actually $T_D^{\pm}(z) \in \widetilde{\mathcal{A}}^{\boldsymbol{v}}((z^{\mp 1})) \otimes_{\mathbb{C}} \operatorname{End}\mathbb{C}^n$, due to the formulas (3.56), (3.58), (3.60).

Let us compute explicitly the images of the matrices $F^{\pm}(z), G^{\pm}(z), E^{\pm}(z)$ under $\Theta_D$, which shall provide an explicit formula for the matrices $T_D^{\pm}(z)$ via (3.54).

Combining $\Upsilon^{-1}_{-\mu^+,-\mu^-}(g_i^{\pm}(z)) = \varphi_i^{\pm}(z)$ with the formula for $\Psi_D(\varphi_i^{\pm}(z))$, we obtain:

$$\Theta_D(g_i^{\pm}(z)) =$$

$$\prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{-1/2} \prod_{t=1}^{a_{i-1}} \mathsf{w}_{i-1,t}^{1/2} \cdot \left( z^{\epsilon_i^{\vee}(\mu^+)} \frac{\mathsf{W}_i(\boldsymbol{v}^{-i}z)}{\mathsf{W}_{i-1}(\boldsymbol{v}^{-i-1}z)} \prod_{\mathsf{x}\in\mathbb{P}^1\backslash\{0,\infty\}} (1-\mathsf{x}/z)^{-\epsilon_i^{\vee}(\lambda_{\mathsf{x}})} \right)^{\pm}. \tag{3.56}$$

Combining $\Upsilon^{-1}_{-\mu^+,-\mu^-}(e_{i,i+1}^{\pm}(z)) = E_i^{\pm}(z)$ with the formula for $\Psi_D(E_i^{\pm}(z))$, we obtain:

$$\Theta_D(e_{i,i+1}^{\pm}(z)) = \prod_{t=1}^{a_i} \mathsf{w}_{i,t} \prod_{t=1}^{a_{i-1}} \mathsf{w}_{i-1,t}^{-1/2} \cdot \sum_{r=1}^{a_i} \left( \frac{(\boldsymbol{v}^i \mathsf{w}_{i,r})^{-\alpha_i^{\vee}(\mu^+)}}{1-\boldsymbol{v}^i\mathsf{w}_{i,r}/z} \right)^{\pm} \frac{\mathsf{Z}_i(\mathsf{w}_{i,r})\mathsf{W}_{i-1}(\boldsymbol{v}^{-1}\mathsf{w}_{i,r})}{\mathsf{W}_{i,r}(\mathsf{w}_{i,r})} D_{i,r}^{-1}. \tag{3.57}$$

As $e_{ij}^{\pm}(z) = (\boldsymbol{v}-\boldsymbol{v}^{-1})^{i-j+1}[e_{j-1,j}^{(0)}, \cdots, [e_{i+1,i+2}^{(0)}, e_{i,i+1}^{\pm}(z)]_{\boldsymbol{v}^{-1}} \cdots]_{\boldsymbol{v}^{-1}}$ due to (3.47), we thus get (cf. [19, (4.6)]):

$$\Theta_D(e_{ij}^{\pm}(z)) = (-1)^{i-j+1} \cdot \prod_{t=1}^{a_{j-1}} \mathsf{w}_{j-1,t} \prod_{k=i}^{j-2} \prod_{t=1}^{a_k} \mathsf{w}_{k,t}^{1/2} \prod_{t=1}^{a_{i-1}} \mathsf{w}_{i-1,t}^{-1/2} \times$$

$$\sum_{\substack{1\le r_i\le a_i \\ \cdots \\ 1\le r_{j-1}\le a_{j-1}}} \left\{ \left( \frac{(\boldsymbol{v}^i\mathsf{w}_{i,r_i})^{-\alpha_i^{\vee}(\mu^+)} \ldots (\boldsymbol{v}^{j-1}\mathsf{w}_{j-1,r_{j-1}})^{-\alpha_{j-1}^{\vee}(\mu^+)}}{1-\boldsymbol{v}^i\mathsf{w}_{i,r_i}/z} \right)^{\pm} \times \right.$$

$$\left. \frac{\mathsf{W}_{i-1}(\boldsymbol{v}^{-1}\mathsf{w}_{i,r_i}) \prod_{k=i}^{j-2} \mathsf{W}_{k,r_k}(\boldsymbol{v}^{-1}\mathsf{w}_{k+1,r_{k+1}})}{\prod_{k=i}^{j-1} \mathsf{W}_{k,r_k}(\mathsf{w}_{k,r_k})} \prod_{k=i}^{j-1} \mathsf{Z}_k(\mathsf{w}_{k,r_k}) \cdot \frac{\mathsf{w}_{i,r_i}}{\mathsf{w}_{j-1,r_{j-1}}} \cdot \prod_{k=i}^{j-1} D_{k,r_k}^{-1} \right\}. \tag{3.58}$$

Combining $\Upsilon^{-1}_{-\mu^+,-\mu^-}(f_{i+1,i}^{\pm}(z)) = F_i^{\pm}(z)$ with the formula for $\Psi_D(F_i^{\pm}(z))$, we obtain:

$$\Theta_D(f_{i+1,i}^{\pm}(z)) = \boldsymbol{v}^{-1} \prod_{t=1}^{a_{i+1}} \mathsf{w}_{i+1,t}^{-1/2} \cdot \sum_{r=1}^{a_i} \left( \frac{1}{1-z/\boldsymbol{v}^{i+2}\mathsf{w}_{i,r}} \right)^{\pm} \frac{\mathsf{W}_{i+1}(\boldsymbol{v}\mathsf{w}_{i,r})}{\mathsf{W}_{i,r}(\mathsf{w}_{i,r})} D_{i,r}. \tag{3.59}$$

As $f_{ji}^{\pm}(z) = (\boldsymbol{v}^{-1}-\boldsymbol{v})^{i-j+1}[[[\cdots[f_{i+1,i}^{\pm}(z), f_{i+2,i+1}^{(0)}]_{\boldsymbol{v}}, \cdots, f_{j,j-1}^{(0)}]_{\boldsymbol{v}}$ due to (3.47), we thus get (cf. [19, (4.7)]):

$$\Theta_D(f_{ji}^{\pm}(z)) = (-1)^{i-j+1}\boldsymbol{v}^{i-j} \cdot \prod_{k=i+1}^{j} \prod_{t=1}^{a_k} \mathsf{w}_{k,t}^{-1/2} \times$$

$$\sum_{\substack{1\le r_i\le a_i\\ \cdots\\ 1\le r_{j-1}\le a_{j-1}}} \left\{ \left(\frac{1}{1-z/\boldsymbol{v}^{i+2}\mathsf{w}_{i,r_i}}\right)^{\pm} \frac{\mathsf{W}_j(\boldsymbol{v}\mathsf{w}_{j-1,r_{j-1}})\prod_{k=i+1}^{j-1}\mathsf{W}_{k,r_k}(\boldsymbol{v}\mathsf{w}_{k-1,r_{k-1}})}{\prod_{k=i}^{j-1}\mathsf{W}_{k,r_k}(\mathsf{w}_{k,r_k})} \times \right.$$

$$\left. \frac{\mathsf{w}_{j-1,r_{j-1}}}{\mathsf{w}_{i,r_i}} \cdot \prod_{k=i}^{j-1} D_{k,r_k} \right\}. \qquad (3.60)$$

While the above derivation of the formulas (3.56), (3.58), (3.60) is based on yet unproved Theorem 3.51, we shall use their explicit right-hand sides from now on, without any direct referral to Theorem 3.51. More precisely, let us define $\widetilde{\mathcal{A}}^{\boldsymbol{v}}((z^{\mp 1}))$-valued $n\times n$ diagonal matrix $G^{\pm}_D(z)$, an upper-triangular matrix $E^{\pm}_D(z)$, and a lower-triangular matrix $F^{\pm}_D(z)$, whose matrix coefficients $g^{\pm;D}_i(z), e^{\pm;D}_{ij}(z), f^{\pm;D}_{ji}(z)$ are given by the right-hand sides of (3.56), (3.58), (3.60) expanded in $z^{\mp 1}$, respectively. Thus, we amend (3.54) and define

$$T^{\pm}_D(z) := F^{\pm}_D(z)G^{\pm}_D(z)E^{\pm}_D(z), \qquad (3.61)$$

so that the matrix coefficients of $T^{\pm}_D(z)$ are given by

$$T^{\pm}_D(z)_{\alpha,\beta} = \sum_{i=1}^{\min\{\alpha,\beta\}} f^{\pm;D}_{\alpha,i}(z)\cdot g^{\pm;D}_i(z)\cdot e^{\pm;D}_{i,\beta}(z) \qquad (3.62)$$

for any $1\le\alpha,\beta\le n$, where the three factors in the right-hand side of (3.62) are determined via (3.60), (3.56), (3.58), respectively, with the conventions $f^{\pm;D}_{\alpha,\alpha}(z)=1=e^{\pm;D}_{\beta,\beta}(z)$.

**Proposition 3.63.** *The matrix coefficients of the matrices $T^{+}_D(z)$ and $T^{-}_D(z)$ are the expansions of the same rational functions in $z^{-1}$ and $z$, respectively.*

**Proof.** This result follows immediately from the defining formula (3.62), since $f^{+;D}_{\alpha,i}(z)$ and $f^{-;D}_{\alpha,i}(z)$ (as well as $e^{+;D}_{i,\beta}(z)$ and $e^{-;D}_{i,\beta}(z)$, or $g^{+;D}_i(z)$ and $g^{-;D}_i(z)$, respectively) are expansions of the same rational functions in $z^{-1}$ and $z$. □

Thus, $T^{\pm}_D(z)=(T_D(z))^{\pm}$ for an $\widetilde{\mathcal{A}}^{\boldsymbol{v}}(z)$-valued $n\times n$ matrix $T_D(z)$. Explicitly, $T_D(z)$ is defined via its Gauss decomposition

$$T_D(z) := F_D(z)G_D(z)E_D(z), \qquad (3.64)$$

so that the matrix coefficients of $T_D(z)$ are given by

$$T_D(z)_{\alpha,\beta} = \sum_{i=1}^{\min\{\alpha,\beta\}} f^{D}_{\alpha,i}(z)\cdot g^{D}_i(z)\cdot e^{D}_{i,\beta}(z) \qquad (3.65)$$

for any $1 \leq \alpha, \beta \leq n$, where the three factors in the right-hand side of (3.65) are the rational functions of (3.60), (3.56), (3.58), respectively, with the conventions $f^D_{\alpha,\alpha}(z) = 1 = e^D_{\beta,\beta}(z)$.

**Remark 3.66.** We note that $T_D(z)$ is singular at $\mathsf{x} \in \mathbb{C}^\times$ if and only if $\lambda_\mathsf{x} \neq 0$. As $F_D(z)$ and $E_D(z)$ are regular in the neighborhood of $\mathsf{x}$, while $G_D(z) = (\text{regular part})\cdot(z-\mathsf{x})^{-\lambda_\mathsf{x}}$, we see that in the classical limit $T_D(z)$ represents a $GL_n$-multiplicative Higgs field on $\mathbb{P}^1$ with partial (Borel) framing at $0, \infty \in \mathbb{P}^1$ (trigonometric type) and with prescribed singularities on $D$.

We shall also need the following normalized trigonometric Lax matrices:

$$\mathsf{T}_D(z) := \frac{z^{\epsilon_1^\vee(\lambda+\mu^-)}}{\mathsf{Z}_0(z)} T_D(z), \tag{3.67}$$

with the normalization factor determined via (3.32):

$$\frac{z^{\epsilon_1^\vee(\lambda+\mu^-)}}{\mathsf{Z}_0(z)} = z^{\epsilon_1^\vee(\mu^-)} \prod_{1\leq s\leq N}^{i_s=0} (z-\mathsf{x}_s)^{-\gamma_s} = z^{\epsilon_1^\vee(\mu^-)} \prod_{\mathsf{x}\in\mathbb{P}^1\backslash\{0,\infty\}} (z-\mathsf{x})^{-\alpha_0^\vee(\lambda_\mathsf{x})}.$$

The first main result of this section establishes the regularity of these matrices:

**Theorem 3.68.** *We have $\mathsf{T}_D(z) \in \widetilde{\mathcal{A}}^{\boldsymbol{v}}[z] \otimes_\mathbb{C} \mathrm{End}\,\mathbb{C}^n$.*

**Proof.** First, we claim that $\mathsf{T}_D(z)$ is regular at $z=0$. Since $f^D_{*,*}(z), e^D_{*,*}(z)$ are clearly regular at $z=0$, it remains to show that $\frac{z^{\epsilon_1^\vee(\lambda+\mu^-)}}{\mathsf{Z}_0(z)} g^D_i(z)$ is regular at $z=0$ for any $1 \leq i \leq n$. However, the minimal power of $z$ in $\left(\frac{z^{\epsilon_1^\vee(\lambda+\mu^-)}}{\mathsf{Z}_0(z)} g^D_i(z)\right)^-$ equals

$$-a_i + a_{i-1} + \epsilon_i^\vee(\mu^+) + \epsilon_i^\vee(\lambda) + \epsilon_1^\vee(\mu^-) = \epsilon_1^\vee(\mu^-) - \epsilon_i^\vee(\mu^-) = (\alpha_1^\vee + \ldots + \alpha_{i-1}^\vee)(\mu^-) \geq 0.$$

Hence, the rational function $\frac{z^{\epsilon_1^\vee(\lambda+\mu^-)}}{\mathsf{Z}_0(z)} g^D_i(z)$ is indeed regular at $z=0$ for any $1 \leq i \leq n$.

The rest of the proof is completely analogous to our proof of Theorem 2.67. □

*3.4.2. Normalized limit description and the RTT relation for $T_D(z)$*

Consider a $\Lambda^+$-valued divisor $D = \sum_{s=1}^N \gamma_s \varpi_{i_s}[\mathsf{x}_s] + \mu^+[\infty] + \mu^-[0]$. As $\mathsf{x}_N \to \infty$, we obtain another $\Lambda^+$-valued divisor $D' = \sum_{s=1}^{N-1} \gamma_s \varpi_{i_s}[\mathsf{x}_s] + (\mu^+ + \gamma_N \varpi_{i_N})[\infty] + \mu^-[0]$, while as $\mathsf{x}_N \to 0$, we obtain yet another $\Lambda^+$-valued divisor $D'' = \sum_{s=1}^{N-1} \gamma_s \varpi_{i_s}[\mathsf{x}_s] + \mu^+[\infty] + (\mu^- + \gamma_N \varpi_{i_N})[0]$. We will now relate the corresponding matrices $T_{D'}(z), T_{D''}(z)$ to $T_D(z)$, defined via (3.64), (3.65).

If $i_N = 0$, then

$$T_{D'}(z) = (z-\mathsf{x}_N)^{-\gamma_N} T_D(z), \quad T_{D''}(z) = (1-\mathsf{x}_N/z)^{-\gamma_N} T_D(z), \tag{3.69}$$

due to the defining formula (3.64) and the equalities $F_D(z) = F_{D'}(z) = F_{D''}(z)$, $E_D(z) = E_{D'}(z) = E_{D''}(z)$, $G_D(z) = (z - \mathsf{x}_N)^{\gamma_N} G_{D'}(z) = (1 - \mathsf{x}_N/z)^{\gamma_N} G_{D''}(z)$.

Let us now consider the case $1 \leq i_N \leq n-1$ (note that $\gamma_N = 1$).

**Proposition 3.70.** *The $\mathsf{x}_N \to 0$ limit of $T_D(z)$ equals $T_{D''}(z)$.*

**Proof.** We note that $F_D(z) = F_{D''}(z)$ by (3.60), the $\mathsf{x}_N \to 0$ limit of $G_D(z)$ equals $G_{D''}(z)$ by (3.56), and the $\mathsf{x}_N \to 0$ limit of $E_D(z)$ equals $E_{D''}(z)$ by (3.58). This implies the result, due to the defining formulas (3.64), (3.65). □

To treat the case $\mathsf{x}_N \to \infty$, let us recall the notation $(-\mathsf{x}_N)^{\varpi_{i_N}} = \mathrm{diag}(1^{i_N}, (-\mathsf{x}_N^{-1})^{n-i_N})$.

**Proposition 3.71.** *The $\mathsf{x}_N \to \infty$ limit of $T_D(z) \cdot (-\mathsf{x}_N)^{\varpi_{i_N}}$ equals $T_{D'}(z)$.*

**Proof.** The proof is completely analogous to our proof of Proposition 2.75. □

**Corollary 3.72.** *(a) $T_{D''}(z)$ is a limit of $T_D(z)$.*
*(b) $T_{D'}(z)$ is a* normalized limit *of $T_D(z)$.*

For $D$ as above, we can pick a $\Lambda^+$-valued divisor $\bar{D} = \sum_{s=1}^{N+M} \gamma_s \varpi_{i_s} [\mathsf{x}_s]$, so that $\{\mathsf{x}_s\}_{s=N+1}^{N+M}$ are some points on $\mathbb{P}^1 \backslash \{0, \infty\}$ while $\sum_{s=N+1}^{N+M} \gamma_s \varpi_{i_s} = \mu^+ + \mu^-$. Note that $0, \infty \notin \mathrm{supp}(\bar{D})$, that is, $\bar{D}|_\infty = 0$ and $\bar{D}|_0 = 0$.

**Corollary 3.73.** *For any $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying (3.28), the matrix $T_D(z)$ is a normalized limit of $T_{\bar{D}}(z)$ with a $\Lambda^+$-valued divisor $\bar{D}$ satisfying $\bar{D}|_\infty = 0 = \bar{D}|_0$.*

Evoking Remark 3.52(a), we see that the original definition of $T^\pm_{\bar{D}}(z)$ via (3.53), (3.54) is valid. Hence, $T^\pm_{\bar{D}}(z)$ defined via (3.61), (3.62) indeed satisfies the RTT relation (3.40), and so is $T_{\bar{D}}(z)$. As a multiplication by diagonal $z$-independent matrices preserves (3.40), we obtain the main result of this section:

**Proposition 3.74.** *For any $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying the assumption (3.28), the matrix $T_D(z)$ defined via* (3.64), (3.65) *is Lax, i.e. it satisfies the RTT relation (3.40).*

*3.4.3. Proof of Theorem 3.51*

Due to Proposition 3.74 and the Gauss decomposition (3.64), (3.65) of $T_D(z)$ with the factors defined via (3.56), (3.58), (3.60), we see that $T_D(z)$ indeed gives rise to the algebra homomorphism $\Theta_D \colon U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \to \widetilde{\mathcal{A}}^{\boldsymbol{v}}_{\mathrm{frac}}$, given by $T^\pm(z) \mapsto (T_D(z))^\pm$, whose composition with the epimorphism $\Upsilon_{-\mu^+,-\mu^-} \colon U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \twoheadrightarrow U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$ of Theorem 3.49 coincides with the homomorphism $\Psi_D$ of (3.34). Thus, for $\mu^+, \mu^- \in \Lambda^+$ and any $\Lambda^+$-valued divisor $D$ on $\mathbb{P}^1$ satisfying (3.28) and $D|_\infty = \mu^+, D|_0 = \mu^-$, the homomorphism $\Psi_D$ factors through $\Upsilon_{-\mu^+,-\mu^-}$.

The latter observation immediately implies the injectivity of $\Upsilon_{-\mu^+,-\mu^-}$ once the following trigonometric counterpart of Theorem 2.80 is established:

**Conjecture 3.75.** *For any coweights $\mu^+, \mu^- \in \Lambda$, the intersection of kernels of the homomorphisms $\Psi_D$ of (3.34) is zero: $\bigcap_D \mathrm{Ker}(\Psi_D) = 0$, where $D$ ranges through all $\Lambda$-valued divisors on $\mathbb{P}^1$, $\Lambda^+$-valued outside $\{0, \infty\} \in \mathbb{P}^1$, satisfying (3.28) and such that $D|_\infty = \mu^+, D|_0 = \mu^-$.*

This completes our proof of Theorem 3.51 modulo Conjecture 3.75, left to a future work.

*3.4.4. Linear trigonometric Lax matrices*

In this section, we will obtain simplified explicit formulas for all $\mathsf{T}_D(z)$ that are linear in $z$.

Following Section 2.4.4, let us fix a triple of *pseudo Young diagrams* $\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-$. They give rise to $\lambda, \mu^+, \mu^- \in \Lambda^+$ via (2.87). Then, $\lambda + \mu^+ + \mu^-$ is of the form $\lambda + \mu^+ + \mu^- = \sum_{i=1}^{n-1} a_i \alpha_i$ for some $a_i \in \mathbb{C}$ iff $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}^+| + |\boldsymbol{\mu}^-| = 0$. Moreover, due to Lemma 2.88, we have:

**Lemma 3.76.** *(a) $a_i = -\sum_{j=n-i+1}^{n} (\boldsymbol{\lambda}_j + \boldsymbol{\mu}_j^+ + \boldsymbol{\mu}_j^-)$ for any $1 \leq i \leq n-1$.*
*(b) $a_i \in \mathbb{N}$ for any $1 \leq i \leq n-1$.*
*(c) $a_j - a_{j-1} = -\boldsymbol{\lambda}_{n-j+1} - \boldsymbol{\mu}_{n-j+1}^+ - \boldsymbol{\mu}_{n-j+1}^-$ for any $1 \leq j \leq n$, where we set $a_0 := 0$, $a_n := 0$.*

Thus, $\Lambda^+$-valued divisors on $\mathbb{P}^1$ satisfying (3.28) and without summands $\{-\varpi_0[\mathsf{x}]\}_{\mathsf{x} \in \mathbb{C}^\times}$ may be encoded by triples $(\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-)$ of a Young diagram $\boldsymbol{\lambda}$ of length $\leq n$ and a pair of pseudo Young diagrams $\boldsymbol{\mu}^+, \boldsymbol{\mu}^-$ with $n$ rows and of total size $|\boldsymbol{\lambda}| + |\boldsymbol{\mu}^+| + |\boldsymbol{\mu}^-| = 0$, together with a collection of points $\underline{\mathsf{x}} = \{\mathsf{x}_i\}_{i=1}^{\boldsymbol{\lambda}_1}$ of $\mathbb{C}^\times$ (so that $\mathsf{x}_i$ is assigned to the $i$-th column of $\boldsymbol{\lambda}$). Explicitly, given $\boldsymbol{\lambda}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-, \underline{\mathsf{x}}$ as above, we set

$$D = D(\boldsymbol{\lambda}, \underline{\mathsf{x}}, \boldsymbol{\mu}^+, \boldsymbol{\mu}^-) := \sum_{i=1}^{\boldsymbol{\lambda}_1} \varpi_{n-\boldsymbol{\lambda}_i^t}[\mathsf{x}_i] + \mu^+[\infty] + \mu^-[0].$$

Due to (3.69), we can actually assume that $D$ does not contain summands $\{\pm\varpi_0[\mathsf{x}]\}_{\mathsf{x} \in \mathbb{C}}$. Thus, $\boldsymbol{\lambda}_n = 0 = \boldsymbol{\mu}_n^-$, so that $\mathsf{Z}_0(z) = 1, \epsilon_1^\vee(\lambda + \mu^-) = -\boldsymbol{\lambda}_n - \boldsymbol{\mu}_n^- = 0$, and $T_D(z) = \mathsf{T}_D(z)$ is polynomial in $z$ by Theorem 2.67. Moreover, $T_D(z)_{11} = g_1^D(z)$ is a polynomial in $z$ of degree $\epsilon_1^\vee(\mu^+) = -\boldsymbol{\mu}_n^+ \geq 0$. Thus, we have $-\boldsymbol{\mu}_n^+ \leq 1$ for linear Lax matrices $T_D(z)$. If $\boldsymbol{\mu}_n^+ = 0$, then $\boldsymbol{\lambda}_i = \boldsymbol{\mu}_i^+ = \boldsymbol{\mu}_i^- = 0$ for all $i$, and so $T_D(z) = \mathsf{T}_D(z) = I_n$. Therefore, it remains to treat the case when $\boldsymbol{\lambda}_n = 0, \boldsymbol{\mu}_n^- = 0, \boldsymbol{\mu}_n^+ = -1$, which constitutes the key result of this section.

**Theorem 3.77.** *Following the above notations, assume further that $\boldsymbol{\lambda}_n = 0, \boldsymbol{\mu}_n^- = 0, \boldsymbol{\mu}_n^+ = -1$.*
*(a) The trigonometric Lax matrix $\mathsf{T}_D(z)$ is explicitly determined as follows:*

*(I) The matrix coefficients on the main diagonal are:*

$$\begin{aligned}\mathsf{T}_D(z)_{ii} = z\cdot\delta_{\boldsymbol{\mu}^+_{n-i+1},-1}\cdot\prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{-1/2}\prod_{t=1}^{a_{i-1}}\mathsf{w}_{i-1,t}^{1/2}+\\ \delta_{\boldsymbol{\mu}^-_{n-i+1},0}\cdot\prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{1/2}\prod_{t=1}^{a_{i-1}}\mathsf{w}_{i-1,t}^{-1/2}\frac{(-\boldsymbol{v}^i)^{a_i}}{(-\boldsymbol{v}^{i+1})^{a_{i-1}}}\prod_{1\le s\le\boldsymbol{\lambda}_1}^{i_s\le i-1}(-\mathsf{x}_s),\end{aligned}\tag{3.78}$$

*where* $i_s := n-\boldsymbol{\lambda}^t_s$.

*(II) The matrix coefficients above the main diagonal are:*

$$\begin{aligned}\mathsf{T}_D(z)_{ij} = z\cdot\delta_{\boldsymbol{\mu}^+_{n-i+1},-1}(-1)^{i-j+1}\cdot\prod_{t=1}^{a_{j-1}}\mathsf{w}_{j-1,t}\prod_{k=i}^{j-2}\prod_{t=1}^{a_k}\mathsf{w}_{k,t}^{1/2}\prod_{t=1}^{a_i}\mathsf{w}_{i,t}^{-1/2}\times\\ \sum_{\substack{1\le r_i\le a_i\\ \cdots\\ 1\le r_{j-1}\le a_{j-1}}}\frac{(\boldsymbol{v}^i\mathsf{w}_{i,r_i})^{-b_i^+}\cdots(\boldsymbol{v}^{j-1}\mathsf{w}_{j-1,r_{j-1}})^{-b^+_{j-1}}}{\prod_{k=i}^{j-1}\mathsf{W}_{k,r_k}(\mathsf{w}_{k,r_k})}\times\\ \frac{\mathsf{W}_{i-1}(\boldsymbol{v}^{-1}\mathsf{w}_{i,r_i})\prod_{k=i}^{j-2}\mathsf{W}_{k,r_k}(\boldsymbol{v}^{-1}\mathsf{w}_{k+1,r_{k+1}})}{\prod_{k=i}^{j-1}\mathsf{W}_{k,r_k}(\mathsf{w}_{k,r_k})}\prod_{k=i}^{j-1}\mathsf{Z}_k(\mathsf{w}_{k,r_k})\cdot\frac{\mathsf{w}_{i,r_i}}{\mathsf{w}_{j-1,r_{j-1}}}\cdot\prod_{k=i}^{j-1}D^{-1}_{k,r_k},\end{aligned}\tag{3.79}$$

*for* $i<j$*, where the constants* $b^+_r$ *are defined via* $b^+_r := \boldsymbol{\mu}^+_{n-r}-\boldsymbol{\mu}^+_{n-r+1}$.

*(III) The matrix coefficients below the main diagonal are:*

$$\begin{aligned}\mathsf{T}_D(z)_{ji} = \delta_{\boldsymbol{\mu}^-_{n-i+1},0}(-1)^{i-j+1}\boldsymbol{v}^{i-j+1}\cdot\prod_{k=i-1}^{j}\prod_{t=1}^{a_k}\mathsf{w}_{k,t}^{-1/2+\delta_{k,i}}\frac{(-\boldsymbol{v}^i)^{a_i}}{(-\boldsymbol{v}^{i+1})^{a_{i-1}}}\prod_{1\le s\le\boldsymbol{\lambda}_1}^{i_s\le i-1}(-\mathsf{x}_s)\times\\ \sum_{\substack{1\le r_i\le a_i\\ \cdots\\ 1\le r_{j-1}\le a_{j-1}}}\frac{\prod_{k=i+1}^{j-1}\mathsf{W}_{k,r_k}(\boldsymbol{v}\mathsf{w}_{k-1,r_{k-1}})\mathsf{W}_j(\boldsymbol{v}\mathsf{w}_{j-1,r_{j-1}})}{\prod_{k=i}^{j-1}\mathsf{W}_{k,r_k}(\mathsf{w}_{k,r_k})}\cdot\frac{\mathsf{w}_{j-1,r_{j-1}}}{\mathsf{w}_{i,r_i}}\cdot\prod_{k=i}^{j-1}D_{k,r_k}\end{aligned}\tag{3.80}$$

*for* $i<j$.

*(b)* $\mathsf{T}_D(z)=T_D(z)$ *is polynomial of degree* 1 *in* $z$.

**Proof.** (a) Combining the explicit formulas (3.65), (3.67) for the matrix coefficients $\mathsf{T}_D(z)_{\alpha,\beta}$ with their polynomiality of Theorem 3.68, we may immediately determine all of them explicitly. As $e^D_{*,*}(z), f^D_{*,*}(z), \frac{g^D_i(z)}{z}$ are regular at $z=\infty$ (for the latter, note that $\epsilon^\vee_i(\mu^+)-1=-\boldsymbol{\mu}^+_{n-i+1}-1\le 0$), each matrix coefficient $\mathsf{T}_D(z)_{\alpha,\beta}$ is a linear polynomial in $z$, due to Theorem 3.68.

The computation of the coefficients of $z^1$ is based on the following observations:

- The $z\to\infty$ limit of $e^D_{ij}(z)$ equals the right-hand side of (3.58) with $\frac{1}{1-\boldsymbol{v}^i\mathsf{w}_{i,r_i}/z}$ disregarded.

- The $z \to \infty$ limit of $f^D_{ji}(z)$ equals 0.
- The $z \to \infty$ limit of $\frac{g^D_i(z)}{z}$ equals $\delta_{\boldsymbol{\mu}^+_{n-i+1},-1} \cdot \prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{-1/2} \prod_{t=1}^{a_{i-1}} \mathsf{w}_{i-1,t}^{1/2}$.

The computation of the coefficients of $z^0$ is based on the following observations:

- The $z \to 0$ limit of $e^D_{ij}(z)$ equals 0.
- The $z \to 0$ limit of $f^D_{ji}(z)$ equals the right-hand side of (3.60) with $\frac{1}{1-z/\boldsymbol{v}^{i+2}\mathsf{w}_{i,r_i}}$ disregarded.
- The $z \to 0$ limit of $g^D_i(z)$ equals $\delta_{\boldsymbol{\mu}^-_{n-i+1},0} \cdot \prod_{t=1}^{a_i} \mathsf{w}_{i,t}^{1/2} \prod_{t=1}^{a_{i-1}} \mathsf{w}_{i-1,t}^{-1/2} \frac{(-\boldsymbol{v}^i)^{a_i}}{(-\boldsymbol{v}^{i+1})^{a_{i-1}}} \times \prod_{1\leq s\leq \boldsymbol{\lambda}_1}^{i_s\leq i-1} (-\mathsf{x}_s)$.

Part (b) follows immediately from part (a). $\square$

**Remark 3.81.** In the particular case when $\boldsymbol{\mu}^- = (0^n), \boldsymbol{\mu}^+ = ((-1)^n)$, and $\boldsymbol{\lambda}$ is a Young diagram of size $n$ and length $< n$, the Lax matrices $T_D(z)$ of Theorem 3.77 are closely related to the $\boldsymbol{v}$-deformed parabolic Gelfand-Tsetlin formulas (cf. [18, Proposition 12.8]), thus providing a $\boldsymbol{v}$-deformed version of Section 2.7.

We note that the trigonometric Lax matrices of Theorem 3.77 have the form $z \cdot T^+ - T^-$. Here, $T^+$ is an upper-triangular and $T^-$ is a lower-triangular $z$-independent $n \times n$ matrices, with some of their diagonal entries being zero as prescribed by the pseudo Young diagrams $\boldsymbol{\mu}^\pm$.

We conclude this section by deriving the conditions on a pair of $n \times n$ matrices $T^+, T^-$ (with values in an associative algebra $\mathcal{D}$) which are equivalent to $T(z) := z \cdot T^+ - T^-$ satisfying the trigonometric RTT relation

$$R_{\mathrm{trig}}(z,w)T_1(z)T_2(w) = T_2(w)T_1(z)R_{\mathrm{trig}}(z,w). \tag{3.82}$$

To this end, let us recall the (finite) *trigonometric* $R$-matrix $R = R^{\boldsymbol{v}}$ given by

$$R = \boldsymbol{v}^{-1} \sum_{1\leq i\leq n} E_{ii} \otimes E_{ii} + \sum_{i\neq j} E_{ii} \otimes E_{jj} + (\boldsymbol{v}^{-1} - \boldsymbol{v}) \sum_{i>j} E_{ij} \otimes E_{ji}. \tag{3.83}$$

It satisfies the Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{3.84}$$

The final result of this section is:

**Proposition 3.85.** *Matrix $T(z) = zT^+ - T^-$ satisfies the trigonometric RTT relation (3.82) if and only if $(T^+, T^-)$ satisfy the following three finite trigonometric RTT relations:*

$$RT_1^+T_2^+ = T_2^+T_1^+R, \quad RT_1^-T_2^- = T_2^-T_1^-R, \quad RT_1^-T_2^+ = T_2^+T_1^-R. \tag{3.86}$$

**Proof.** Recall the following relation between the trigonometric $R$-matrices (3.38) and (3.83):

$$R_{\mathrm{trig}}(z,w) = (z-w)R + (\boldsymbol{v}-\boldsymbol{v}^{-1})zP,$$

where $P = \sum_{i,j=1}^n E_{ij}\otimes E_{ji}$ as before. Thus, the relation (3.82) on $T(z)$ may be written as

$$\begin{aligned}&\left((z-w)R + (\boldsymbol{v}-\boldsymbol{v}^{-1})zP\right)(zT_1^+ - T_1^-)(wT_2^+ - T_2^-) = \\ &(wT_2^+ - T_2^-)(zT_1^+ - T_1^-)\left((z-w)R + (\boldsymbol{v}-\boldsymbol{v}^{-1})zP\right).\end{aligned} \tag{3.87}$$

To prove the "only if" part, we compare the coefficients of $z^1w^2, z^0w^1, z^0w^2$, and $z^2w^0$ in (3.87) to recover the equalities $RT_1^+T_2^+ = T_2^+T_1^+R, RT_1^-T_2^- = T_2^-T_1^-R, RT_1^-T_2^+ = T_2^+T_1^-R$, and $\widetilde{R}T_1^+T_2^- = T_2^-T_1^+\widetilde{R}$, respectively, where $\widetilde{R} := R + (\boldsymbol{v}-\boldsymbol{v}^{-1})P$.

To prove the "if" part, we note that multiplying the last equality of (3.86) by $R^{-1}$ both on the left and on the right, and conjugating further by the permutation operator $P$, we get $(PR^{-1}P^{-1})T_1^+T_2^- = T_2^-T_1^+(PR^{-1}P^{-1})$, which together with $PR^{-1}P^{-1} = \widetilde{R}$ finally implies

$$\widetilde{R}T_1^+T_2^- = T_2^-T_1^+\widetilde{R}. \tag{3.88}$$

Combining this with (3.86) and $\widetilde{R} = R + (\boldsymbol{v}-\boldsymbol{v}^{-1})P$, the equality (3.87) is equivalent to

$$\begin{aligned}&(\boldsymbol{v}-\boldsymbol{v}^{-1})z^2w(PT_1^+T_2^+ - T_2^+T_1^+P) + (\boldsymbol{v}-\boldsymbol{v}^{-1})z(PT_1^-T_2^- - T_2^-T_1^-P) - \\ &\qquad (\boldsymbol{v}-\boldsymbol{v}^{-1})zw(PT_1^-T_2^+ - T_2^+T_1^-P) + zw(RT_1^+T_2^- - T_2^-T_1^+R) = 0.\end{aligned}$$

In the above left-hand side, the first two summands are clearly zero as $PT_1^+T_2^+ = T_2^+T_1^+P$ and $PT_1^-T_2^- = T_2^-T_1^-P$, while the sum of the latter two equals

$$\begin{aligned}&zw\left((R + (\boldsymbol{v}-\boldsymbol{v}^{-1})P)T_1^+T_2^- - T_2^-T_1^+(R + (\boldsymbol{v}-\boldsymbol{v}^{-1})P)\right) \\ &= zw\left(\widetilde{R}T_1^+T_2^- - T_2^-T_1^+\widetilde{R}\right) = 0,\end{aligned}$$

due to (3.88).

This completes our proof of Proposition 3.85. □

**Remark 3.89.** The above proof is identical to the verification of the fact that the assignment $T^+(z) \mapsto T^+ - z^{-1}T^-, T^-(z) \mapsto T^- - zT^+$ gives rise to the (*evaluation*) homomorphism $U_{\boldsymbol{v}}^{\mathrm{rtt}}(L\mathfrak{gl}_n) \to U_{\boldsymbol{v}}^{\mathrm{rtt}}(\mathfrak{gl}_n)$. In particular, if it was not for (3.42), (3.43), we would get homomorphisms from shifted quantum affine algebras to the corresponding *contracted algebras* of [37].

*3.4.5. From trigonometric Lax matrices to rational Lax matrices*

In this section, we explain how the trigonometric Lax matrices $T^{\mathrm{trig}}_*(z)$ of Section 3.4.1 may be degenerated into the rational Lax matrices $T^{\mathrm{rat}}_*(z)$ of Section 2.4.1 (here, the superscripts trig, rat are used to distinguish between the trigonometric and the rational setups). Given a $\Lambda^+$-valued divisor $D = \sum_{s=1}^N \gamma_s \varpi_{i_s}[\mathsf{x}_s] + \mu^+[\infty] + \mu^-[0]$ on $\mathbb{P}^1$ (with $\mathsf{x}_s \in \mathbb{C}^\times$), we consider another $\Lambda^+$-valued divisor $\hat{D} = \sum_{s=1}^N \gamma_s \varpi_{i_s}[\mathsf{x}_s] + (\mu^+ + \mu^-)[\infty]$ on $\mathbb{P}^1$.

Let us make the following change of variables:

$$\boldsymbol{v} \rightsquigarrow e^{\epsilon/2}, \quad z \rightsquigarrow e^{\epsilon x}, \quad \mathsf{x}_s \rightsquigarrow e^{\epsilon x_s}; \tag{3.90}$$

$$\mathsf{w}_{i,r} \rightsquigarrow e^{\epsilon(p_{i,r}-\frac{i}{2})} = e^{\epsilon w_{i,r}}, \quad \text{where } w_{i,r} := p_{i,r} - i/2 \text{ as in Remark 2.38}; \tag{3.91}$$

$$D_{i,r} \rightsquigarrow -e^{-q_{i,r}} \epsilon^{s_i}, \quad \text{where } s_i := a_i - a_{i+1} = -\epsilon^\vee_{i+1}(\lambda + \mu^+ + \mu^-). \tag{3.92}$$

We also consider the diagonal $z$-independent matrix

$$\epsilon^{-\mu^+-\mu^-} := \mathrm{diag}(\epsilon^{-d_1}, \epsilon^{-d_2}, \cdots, \epsilon^{-d_n}) \text{ with } d_i := \epsilon^\vee_i(\mu^+ + \mu^-) = d^+_i + d^-_i. \tag{3.93}$$

The main result of this section is:

**Proposition 3.94.** $\lim\limits_{\epsilon\to 0} \left(T^{\mathrm{trig}}_D(z) \cdot \epsilon^{-\mu^+-\mu^-}\right) = T^{\mathrm{rat}}_{\hat{D}}(x)$.

**Proof.** Recall the Gauss decomposition $T^{\mathrm{trig}}_D(z) = F^{\mathrm{trig}}_D(z) G^{\mathrm{trig}}_D(z) E^{\mathrm{trig}}_D(z)$ of (3.64) with all three factors determined explicitly via (3.56), (3.58), (3.60). Then, $T^{\mathrm{trig}}_D(z) \cdot \epsilon^{-\mu^+-\mu^-}$ has the following Gauss decomposition:

$$T^{\mathrm{trig}}_D(z) \cdot \epsilon^{-\mu^+-\mu^-} = F^{\mathrm{trig}}_D(z) \cdot \left(G^{\mathrm{trig}}_D(z)\epsilon^{-\mu^+-\mu^-}\right) \cdot \left(\epsilon^{\mu^++\mu^-} E^{\mathrm{trig}}_D(z)\epsilon^{-\mu^+-\mu^-}\right). \tag{3.95}$$

On the other hand, we also have the Gauss decomposition

$$T^{\mathrm{rat}}_{\hat{D}}(x) = F^{\mathrm{rat}}_{\hat{D}}(x) \cdot G^{\mathrm{rat}}_{\hat{D}}(x) \cdot E^{\mathrm{rat}}_{\hat{D}}(x) \tag{3.96}$$

of (2.63) with all three factors determined explicitly via (2.58), (2.60), (2.62).

It remains to note that upon the above change of variables (3.90)–(3.92), the $\epsilon \to 0$ limit of each of the three factors in (3.95) exactly coincides with the corresponding factor in (3.96):

- For the diagonal factors, this immediately follows from

$$\epsilon^{-a_i}\mathsf{W}_i(\boldsymbol{v}^{-i}z) \to P_i(x), \quad \epsilon^{-a_{i-1}}\mathsf{W}_{i-1}(\boldsymbol{v}^{-i-1}z) \to P_{i-1}(x-1),$$
$$\epsilon^{-\alpha^\vee_k(\lambda)}\mathsf{Z}_k(\boldsymbol{v}^{-k}z) \to Z_k(x)$$

  as $\epsilon \to 0$, combined with the equality

$$a_i - a_{i-1} + \sum_{k=0}^{i-1} \alpha_k^\vee(\lambda) - d_i = a_i - a_{i-1} - \epsilon_i^\vee(\lambda) - \epsilon_i^\vee(\mu^+ + \mu^-)$$
$$= a_i - a_{i-1} - \epsilon_i^\vee(\lambda + \mu^+ + \mu^-) = 0;$$

- For the upper triangular factors, this follows from

$$\epsilon^{-a_k+1}\mathsf{W}_{k,r_k}(\boldsymbol{v}^{-1}\mathsf{w}_{k+1,r_{k+1}}) \to P_{k,r_k}(p_{k+1,r_{k+1}} - 1),$$
$$\epsilon^{-a_k+1}\mathsf{W}_{k,r_k}(\mathsf{w}_{k,r_k}) \to P_{k,r_k}(p_{k,r_k}),$$
$$\epsilon^{-a_{i-1}}\mathsf{W}_{i-1}(\boldsymbol{v}^{-1}\mathsf{w}_{i,r_i}) \to P_{i-1}(p_{i,r_i} - 1), \quad \frac{\epsilon}{1 - \boldsymbol{v}^i\mathsf{w}_{i,r_i}/z} \to \frac{1}{x - p_{i,r_i}}$$

as $\epsilon \to 0$, combined with the equality

$$a_{i-1} - a_{j-1} + \sum_{k=i}^{j-1} \alpha_k^\vee(\lambda) - \sum_{k=i}^{j-1} s_k + d_i - d_j =$$
$$a_{i-1} - a_i - a_{j-1} + a_j + (\epsilon_i^\vee - \epsilon_j^\vee)(\lambda + \mu^+ + \mu^-) = 0;$$

- For the lower triangular factors, this follows from

$$\epsilon^{-a_k+1}\mathsf{W}_{k,r_k}(\boldsymbol{v}\mathsf{w}_{k-1,r_{k-1}}) \to P_{k,r_k}(p_{k-1,r_{k-1}} + 1),$$
$$\epsilon^{-a_k+1}\mathsf{W}_{k,r_k}(\mathsf{w}_{k,r_k}) \to P_{k,r_k}(p_{k,r_k}),$$
$$\epsilon^{-a_j}\mathsf{W}_j(\boldsymbol{v}\mathsf{w}_{j-1,r_{j-1}}) \to P_j(p_{j-1,r_{j-1}} + 1), \quad \frac{\epsilon}{1 - z/\boldsymbol{v}^{i+2}\mathsf{w}_{i,r_i}} \to \frac{-1}{x - p_{i,r_i} - 1}$$

as $\epsilon \to 0$, combined with the equality $a_j - a_i + \sum_{k=i}^{j-1} s_k = 0$.

This completes our proof of Proposition 3.94. □

### *3.5. Six explicit linear trigonometric Lax matrices for $n = 2$*

In this section, we apply Theorem 3.77 to obtain explicitly all linear trigonometric Lax matrices $\mathsf{T}_D(z)$ for the smallest rank $n = 2$, corresponding to a triple of pseudo Young diagrams

$$\boldsymbol{\lambda} = (\boldsymbol{\lambda}_1, 0),\ \boldsymbol{\mu}^+ = (\boldsymbol{\mu}_1^+, -1),\ \boldsymbol{\mu}^- = (\boldsymbol{\mu}_1^-, 0)$$
$$\text{with } \boldsymbol{\lambda}_1 \geq 0, \boldsymbol{\mu}_1^+ \geq -1, \boldsymbol{\mu}_1^- \geq 0 \text{ and } \boldsymbol{\lambda}_1 + \boldsymbol{\mu}_1^+ + \boldsymbol{\mu}_1^- = 1.$$

We shall also compute their **quantum determinant** $\operatorname{qdet} \mathsf{T}_D(z)$, defined via

$$\operatorname{qdet} \mathsf{T}_D(z) := \mathsf{T}_D(\boldsymbol{v}^2 z)_{11}\mathsf{T}_D(z)_{22} - \boldsymbol{v}^{-1}\mathsf{T}_D(\boldsymbol{v}^2 z)_{12}\mathsf{T}_D(z)_{21}. \tag{3.97}$$

Note that $a_1 = -(\boldsymbol{\lambda}_2 + \boldsymbol{\mu}_2^+ + \boldsymbol{\mu}_2^-) = 1$ manifestly, due to Lemma 3.76(a). To simplify the formulas below, we relabel $D_1^{\pm 1}, \mathsf{w}_1^{\pm 1/2}$ by $D^{\pm 1}, \widetilde{\mathsf{w}}^{\pm 1}$, respectively.

- *Case* $\boldsymbol{\lambda}_1 = 0, \boldsymbol{\mu}_1^+ = -1, \boldsymbol{\mu}_1^- = 2$.
  We have

$$\mathsf{T}_D(z) = \begin{pmatrix} z \cdot \widetilde{\mathsf{w}}^{-1} - \boldsymbol{v}\widetilde{\mathsf{w}} & z \cdot \widetilde{\mathsf{w}} D^{-1} \\ -\boldsymbol{v}\widetilde{\mathsf{w}} D & z \cdot \widetilde{\mathsf{w}} \end{pmatrix} \tag{3.98}$$

and its quantum determinant is $\operatorname{qdet} \mathsf{T}_D(z) = \boldsymbol{v}^2 z^2$.

- *Case* $\boldsymbol{\lambda}_1 = 0, \boldsymbol{\mu}_1^+ = 0, \boldsymbol{\mu}_1^- = 1$.
  We have

$$\mathsf{T}_D(z) = \begin{pmatrix} z \cdot \widetilde{\mathsf{w}}^{-1} - \boldsymbol{v}\widetilde{\mathsf{w}} & z \cdot \boldsymbol{v}^{-1} \widetilde{\mathsf{w}}^{-1} D^{-1} \\ -\boldsymbol{v}\widetilde{\mathsf{w}} D & 0 \end{pmatrix} \tag{3.99}$$

and its quantum determinant is $\operatorname{qdet} \mathsf{T}_D(z) = z$.

- *Case* $\boldsymbol{\lambda}_1 = 0, \boldsymbol{\mu}_1^+ = 1, \boldsymbol{\mu}_1^- = 0$.
  We have

$$\mathsf{T}_D(z) = \begin{pmatrix} z \cdot \widetilde{\mathsf{w}}^{-1} - \boldsymbol{v}\widetilde{\mathsf{w}} & z \cdot \boldsymbol{v}^{-2} \widetilde{\mathsf{w}}^{-3} D^{-1} \\ -\boldsymbol{v}\widetilde{\mathsf{w}} D & -\boldsymbol{v}^{-3} \widetilde{\mathsf{w}}^{-1} \end{pmatrix} \tag{3.100}$$

and its quantum determinant is $\operatorname{qdet} \mathsf{T}_D(z) = \boldsymbol{v}^{-2}$.

- *Case* $\boldsymbol{\lambda}_1 = 1, \boldsymbol{\mu}_1^+ = -1, \boldsymbol{\mu}_1^- = 1$.
  We have

$$\mathsf{T}_D(z) = \begin{pmatrix} z \cdot \widetilde{\mathsf{w}}^{-1} - \boldsymbol{v}\widetilde{\mathsf{w}} & z \cdot \widetilde{\mathsf{w}}(1 - \boldsymbol{v}^{-1}\mathsf{x}_1/\widetilde{\mathsf{w}}^2) D^{-1} \\ -\boldsymbol{v}\widetilde{\mathsf{w}} D & z \cdot \widetilde{\mathsf{w}} \end{pmatrix} \tag{3.101}$$

and its quantum determinant is $\operatorname{qdet} \mathsf{T}_D(z) = \boldsymbol{v}^2 z(z - \boldsymbol{v}^{-2}\mathsf{x}_1)$.

- *Case* $\boldsymbol{\lambda}_1 = 1, \boldsymbol{\mu}_1^+ = 0, \boldsymbol{\mu}_1^- = 0$.
  We have

$$\mathsf{T}_D(z) = \begin{pmatrix} z \cdot \widetilde{\mathsf{w}}^{-1} - \boldsymbol{v}\widetilde{\mathsf{w}} & z \cdot \boldsymbol{v}^{-1} \widetilde{\mathsf{w}}^{-1}(1 - \boldsymbol{v}^{-1}\mathsf{x}_1/\widetilde{\mathsf{w}}^2) D^{-1} \\ -\boldsymbol{v}\widetilde{\mathsf{w}} D & \boldsymbol{v}^{-3} \widetilde{\mathsf{w}}^{-1} \mathsf{x}_1 \end{pmatrix} \tag{3.102}$$

and its quantum determinant is $\operatorname{qdet} \mathsf{T}_D(z) = z - \boldsymbol{v}^{-2}\mathsf{x}_1$.

- *Case* $\boldsymbol{\lambda}_1 = 2, \boldsymbol{\mu}_1^+ = -1, \boldsymbol{\mu}_1^- = 0$.

We have

$$\mathsf{T}_D(z) = \begin{pmatrix} z \cdot \widetilde{\mathsf{w}}^{-1} - \boldsymbol{v}\widetilde{\mathsf{w}} & z \cdot \widetilde{\mathsf{w}}(1 - \boldsymbol{v}^{-1}\mathsf{x}_1/\widetilde{\mathsf{w}}^2)(1 - \boldsymbol{v}^{-1}\mathsf{x}_2/\widetilde{\mathsf{w}}^2)D^{-1} \\ -\boldsymbol{v}\widetilde{\mathsf{w}}D & z \cdot \widetilde{\mathsf{w}} - \boldsymbol{v}^{-3}\widetilde{\mathsf{w}}^{-1}\mathsf{x}_1\mathsf{x}_2 \end{pmatrix} \tag{3.103}$$

and its quantum determinant is $\mathrm{qdet}\, \mathsf{T}_D(z) = \boldsymbol{v}^2(z - \boldsymbol{v}^{-2}\mathsf{x}_1)(z - \boldsymbol{v}^{-2}\mathsf{x}_2)$.

**Remark 3.104.** The first three Lax matrices (3.98), (3.99), (3.100) first appeared in [18] (up to a normalization factor, they coincide with those of [18, (11.4, 11.6, 11.7)] having $\mathrm{qdet} = 1$).

*3.6. Coproduct homomorphisms for shifted quantum affine algebras*

A crucial benefit of the RTT realization is that it immediately endows the quantum affine algebra of $\mathfrak{gl}_n$ with the Hopf algebra structure, in particular, the coproduct homomorphism

$$\Delta^{\mathrm{rtt}} \colon U^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \longrightarrow U^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n) \otimes U^{\mathrm{rtt}}_{\boldsymbol{v}}(L\mathfrak{gl}_n), \qquad T^{\pm}(z) \mapsto T^{\pm}(z) \otimes T^{\pm}(z). \tag{3.105}$$

The main observation of this section is that (3.105) naturally admits a shifted version:

**Proposition 3.106.** *For any $\mu_1^{\pm}, \mu_2^{\pm} \in \Lambda^+$, there is a unique $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism*

$$\Delta^{\mathrm{rtt}}_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-} \colon U^{\mathrm{rtt}}_{-\mu_1^+-\mu_2^+,-\mu_1^--\mu_2^-}(L\mathfrak{gl}_n) \longrightarrow U^{\mathrm{rtt}}_{-\mu_1^+,-\mu_1^-}(L\mathfrak{gl}_n) \otimes U^{\mathrm{rtt}}_{-\mu_2^+,-\mu_2^-}(L\mathfrak{gl}_n)$$

*defined by*

$$\Delta^{\mathrm{rtt}}_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-}(T^{\pm}(z)) = T^{\pm}(z) \otimes T^{\pm}(z). \tag{3.107}$$

**Proof.** The proof is completely analogous to our proof of Proposition 2.136 with the only minor update of the general observation we used in *loc.cit.* To be more precise, we either need to add the generators $\mathsf{e}_{ij}^{(0)}$ so that $\mathsf{e}_{ij}(z) = \sum_{r\geq 0} \mathsf{e}_{ij}^{(r)} z^{-r}$ or to add the generators $\mathsf{f}_{ji}^{(0)}$ so that $\mathsf{f}_{ji}(z) = \sum_{r\geq 0} \mathsf{f}_{ji}^{(r)} z^{-r}$. In both cases, the product $\mathsf{E}(z)\cdot\mathsf{F}(z)$ still admits the Gauss decomposition (2.138) with either $\bar{\mathsf{e}}_{ij}(z) = \sum_{r\geq 0} \bar{\mathsf{e}}_{ij}^{(r)} z^{-r}$ and $\bar{\mathsf{f}}_{ji}(z) = \sum_{r\geq 1} \bar{\mathsf{f}}_{ji}^{(r)} z^{-r}$, or $\bar{\mathsf{e}}_{ij}(z) = \sum_{r\geq 1} \bar{\mathsf{e}}_{ij}^{(r)} z^{-r}$ and $\bar{\mathsf{f}}_{ji}(z) = \sum_{r\geq 0} \bar{\mathsf{f}}_{ji}^{(r)} z^{-r}$, respectively. □

The following basic property of $\Delta^{\mathrm{rtt}}_{*,*,*,*}$ is straightforward:

**Corollary 3.108.** *For any $\mu_1^+, \mu_1^-, \mu_2^+, \mu_2^-, \mu_3^+, \mu_3^- \in \Lambda^+$, the following equality holds:*

$$(\mathrm{Id} \otimes \Delta^{\mathrm{rtt}}_{-\mu_2^+,-\mu_2^-,-\mu_3^+,-\mu_3^-}) \circ \Delta^{\mathrm{rtt}}_{-\mu_1^+,-\mu_1^-,-\mu_2^+-\mu_3^+,-\mu_2^--\mu_3^-} = (\Delta^{\mathrm{rtt}}_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-} \otimes \mathrm{Id}) \circ \Delta^{\mathrm{rtt}}_{-\mu_1^+-\mu_2^+,-\mu_1^--\mu_2^-,-\mu_3^+,-\mu_3^-}.$$

Evoking the key isomorphisms $\Upsilon_{-\mu^+,-\mu^-}\colon U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n) \xrightarrow{\sim} U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$ of Theorem 3.51 for $(\mu^+,\mu^-)$ being either of the three pairs $(\mu_1^+,\mu_1^-)$, $(\mu_2^+,\mu_2^-)$ and $(\mu_1^++\mu_2^+,\mu_1^-+\mu_2^-)$, we conclude that $\Delta^{\mathrm{rtt}}_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-}$ of (3.107) gives rise to the $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism

$$\Delta_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-}\colon U_{-\mu_1^+-\mu_2^+,-\mu_1^--\mu_2^-}(L\mathfrak{gl}_n) \longrightarrow U_{-\mu_1^+,-\mu_1^-}(L\mathfrak{gl}_n)\otimes U_{-\mu_2^+,-\mu_2^-}(L\mathfrak{gl}_n). \tag{3.109}$$

Since the algebra $U_{-\mu_1^+-\mu_2^+,-\mu_1^--\mu_2^-}(L\mathfrak{gl}_n)$ is generated by

$$\left\{E_{i,0},F_{i,0},\varphi^{\pm}_{j,\mp\epsilon_j^\vee(\mu_1^\pm+\mu_2^\pm)},(\varphi^{\pm}_{j,\mp\epsilon_j^\vee(\mu_1^\pm+\mu_2^\pm)})^{-1},\varphi^{\pm}_{j,\mp\epsilon_j^\vee(\mu_1^\pm+\mu_2^\pm)\pm 1}\right\}_{1\le i<n}^{1\le j\le n} \tag{3.110}$$

and the coefficients of the central series $C^\pm(z)$ of (3.21), as follows from Lemma 3.22, the homomorphism $\Delta_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-}$ is uniquely determined by the images of these elements:

- the images of the finite set of the generators (3.110) under $\Delta_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-}$ were computed explicitly in [18, Appendices G, H], cf. [18, Theorems G.10, G.13];
- in a complete analogy to (2.144), the images of the central series $C^\pm(z)$ are given by

$$\Delta_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-}(C^\pm(z)) = C^\pm(z)\otimes C^\pm(z). \tag{3.111}$$

The proof of (3.111) follows from the standard formulas

$$\Delta^{\mathrm{rtt}}_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-}(\mathrm{qdet}\,T^\pm(z)) = \mathrm{qdet}\,T^\pm(z)\otimes \mathrm{qdet}\,T^\pm(z)$$

combined with the trigonometric version of Proposition 2.83:

$$C^\pm(z) = \Upsilon^{-1}_{-\mu^+,-\mu^-}(\mathrm{qdet}\,T^\pm(z)).$$

Here, the *quantum determinant* $\mathrm{qdet}\,T^\pm(z)$ of $U^{\mathrm{rtt}}_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$ is defined via (cf. (3.97) for the smallest rank $n=2$ case):

$$\mathrm{qdet}\,T^\pm(z) := \sum_{\sigma\in S_n}(-\boldsymbol{v})^{-\ell(\sigma)} t^\pm_{1,\sigma(1)}(\boldsymbol{v}^{2(n-1)}z)t^\pm_{2,\sigma(2)}(\boldsymbol{v}^{2(n-2)}z)\cdots t^\pm_{n-1,\sigma(n-1)}(\boldsymbol{v}^2z)t^\pm_{n,\sigma(n)}(z). \tag{3.112}$$

Moreover, the homomorphisms (3.109) have natural $\mathfrak{sl}_n$–counterparts:

**Proposition 3.113.** *For any $\nu_1^\pm,\nu_2^\pm\in\bar{\Lambda}^+$, there is a unique $\mathbb{C}(\boldsymbol{v})$-algebra homomorphism*

$$\Delta_{-\nu_1^+,-\nu_1^-,-\nu_2^+,-\nu_2^-}\colon U^{\mathrm{sc}}_{-\nu_1^+-\nu_2^+,-\nu_1^--\nu_2^-}(L\mathfrak{sl}_n) \longrightarrow U^{\mathrm{sc}}_{-\nu_1^+,-\nu_1^-}(L\mathfrak{sl}_n)\otimes U^{\mathrm{sc}}_{-\nu_2^+,-\nu_2^-}(L\mathfrak{sl}_n)$$

*such that the following diagram is commutative*

$$\begin{tikzcd} U^{\mathrm{sc}}_{-\bar{\mu}_1^+-\bar{\mu}_2^+,-\bar{\mu}_1^--\bar{\mu}_2^-}(L\mathfrak{sl}_n) \arrow[r, "\Delta_{-\bar{\mu}_1^+,-\bar{\mu}_1^-,-\bar{\mu}_2^+,-\bar{\mu}_2^-}"] \arrow[d, "\iota_{-\mu_1^+-\mu_2^+,-\mu_1^--\mu_2^-}"'] & U^{\mathrm{sc}}_{-\bar{\mu}_1^+,-\bar{\mu}_1^-}(L\mathfrak{sl}_n) \otimes U^{\mathrm{sc}}_{-\bar{\mu}_2^+,-\bar{\mu}_2^-}(L\mathfrak{sl}_n) \arrow[d, "\iota_{-\mu_1^+,-\mu_1^-}\otimes\iota_{-\mu_2^+,-\mu_2^-}"] \\ U_{-\mu_1^+-\mu_2^+,-\mu_1^--\mu_2^-}(L\mathfrak{gl}_n) \arrow[r, "\Delta_{-\mu_1^+,-\mu_1^-,-\mu_2^+,-\mu_2^-}"'] & U_{-\mu_1^+,-\mu_1^-}(L\mathfrak{gl}_n) \otimes U_{-\mu_2^+,-\mu_2^-}(L\mathfrak{gl}_n) \end{tikzcd} \tag{3.114}$$

*for any* $\mu_1^+, \mu_1^-, \mu_2^+, \mu_2^- \in \Lambda^+$.

Evoking the defining formulas (3.18) for the embedding $\iota_{-\mu^+,-\mu^-}: U^{\mathrm{sc}}_{-\bar{\mu}^+,-\bar{\mu}^-}(L\mathfrak{sl}_n) \hookrightarrow U_{-\mu^+,-\mu^-}(L\mathfrak{gl}_n)$ of Proposition 3.16, one obtains explicit formulas for the $\Delta_{-\bar{\mu}_1^+,-\bar{\mu}_1^-,-\bar{\mu}_2^+,-\bar{\mu}_2^-}$-images of the finite generating set, following the proof of [18, Theorem 10.13] presented in [18, Appendix G]. The resulting formulas coincide with the explicit long formulas of [18, Theorem 10.16], thus providing a simpler and more conceptual proof of [18, Theorem 10.16].

**Remark 3.115.** Due to [18, Theorem 10.20], $\Delta_{-\nu_1^+,-\nu_1^-,-\nu_2^+,-\nu_2^-}$ with $\nu_1^\pm, \nu_2^\pm \in \bar{\Lambda}^+$ give rise to algebra homomorphisms

$$\Delta_{\nu_1^+,\nu_1^-,\nu_2^+,\nu_2^-}: U^{\mathrm{sc}}_{\nu_1^++\nu_2^+,\nu_1^-+\nu_2^-}(L\mathfrak{sl}_n) \longrightarrow U^{\mathrm{sc}}_{\nu_1^+,\nu_1^-}(L\mathfrak{sl}_n) \otimes U^{\mathrm{sc}}_{\nu_2^+,\nu_2^-}(L\mathfrak{sl}_n)$$

for any $\mathfrak{sl}_n$–coweights $\nu_1^\pm, \nu_2^\pm \in \bar{\Lambda}$. However, we note that $\Delta_{\nu_1^+,\nu_1^-,\nu_2^+,\nu_2^-}$ ($\nu_1^\pm, \nu_2^\pm \in \bar{\Lambda}$) are not coassociative, in contrast to Corollary 3.108.

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