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Rational Lax Matrices from Antidominantly Shifted Extended Yangians: BCD Types

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Abstract: Generalizing Frassek et al. (Adv. Math. **401**, 108283 (2022). https://doi.org/10.1016/j.aim.2022.108283), we construct a family of $SO(2r)$, $Sp(2r)$, $SO(2r+1)$ rational Lax matrices $T_D(z)$, polynomial in the spectral parameter z, parametrized by Λ^+-valued divisors D on $\mathbb{P}^1$. To this end, we provide the RTT realization of the antidominantly shifted extended Drinfeld Yangians of $\mathfrak{g} = \mathfrak{so}_{2r}, \mathfrak{sp}_{2r}, \mathfrak{so}_{2r+1}$, and of their coproduct homomorphisms.

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1. Introduction

1.1. Summary. The main results of the present paper are:

- The RTT realization of the antidominantly shifted (extended) Yangians associated to the simple Lie algebras $\mathfrak{g}$ of the classical types $\mathfrak{so}_{2r}, \mathfrak{sp}_{2r}, \mathfrak{so}_{2r+1}$, generalizing the recent isomorphisms of [JLM1] in the non-shifted case. This naturally equips those algebras with the coproduct homomorphisms, which as we show do coincide with those of [FKPRW] (obtained by rather lengthy computations in the Drinfeld realization).
- The construction of a family of (rational) Lax matrices, regular in the spectral parameter, of the corresponding type, parametrized by the divisors on the projective line $\mathbb{P}^1$ with coefficients in Λ^+, the dominant integral cone of the coweight lattice of $\mathfrak{g}$. In the simplest cases, this recovers recent constructions in the physics literature [IKK,F,KK].

Our exposition follows closely that of our previous joint work with V. Pestun [FPT], where both above constructions were carried out for $\mathfrak{g} = \mathfrak{sl}_n$ (extended version corresponding to $\mathfrak{gl}_n$).

The original definition of Yangians $Y(\mathfrak{g})$ associated to any simple Lie algebra $\mathfrak{g}$ is due to [D1], where these algebras are realized as Hopf algebras with a finite set of generators. The representation theory of such algebras is best developed using their alternative *Drinfeld realization* of these algebras proposed in [D2], though the Hopf algebra structure is much more involved in this presentation (e.g. the coproduct formula has been known since the 90s, see [KT, (2.8)–(2.11)], but its proof has never appeared in the literature until the very recent paper [GNW]).

For $\mathfrak{g} = \mathfrak{gl}_n$, a closely related algebra was studied earlier in the work of L. Faddeev's school (see e.g. [FRT]), where the algebra generators were encoded into an $n \times n$ square matrix $T(z)$ subject to a single *RTT relation*

$$R_{12}(z-w)T_1(z)T_2(w) = T_2(w)T_1(z)R_{12}(z-w) \tag{1.1}$$

involving the rational R-matrix $R(z)$ satisfying the *Yang-Baxter equation*

$$R_{12}(z)R_{13}(z+w)R_{23}(w) = R_{23}(w)R_{13}(z+w)R_{12}(z) \tag{1.2}$$

(note that the $\mathfrak{sl}_n$-version is recovered by imposing an extra relation $\mathrm{qdet}\, T(z) = 1$). The Hopf algebra structure is extremely simple in this RTT realization, which is suitable both for the development of the representation theory and study of the corresponding integrable systems.

An explicit isomorphism from the new Drinfeld to the RTT realizations of type A Yangians is constructed using the Gauss decomposition of $T(z)$, a complete proof provided in [BK1] (the trigonometric version of this result established earlier in [DF]). A similar explicit isomorphism for the remaining classical types B, C, D was only recently provided in [JLM1], where it was again constructed using the Gauss decomposition of the generating matrices $T(z)$ which are subject to the RTT relations (1.1) with the rational solutions of (1.2) first discovered in [ZZ]. However, let us emphasize that the formulas recovering the matrix $T(z)$ through the Drinfeld currents in B, C, D types are significantly harder than their counterparts in type A, see our Lemmas 2.77, 2.79, 2.80, 2.96, 2.97, 3.11, 3.12, 4.10, 4.11, generalizing partial results of [JLM1]. We note that a non-constructive existence of such an isomorphism for any $\mathfrak{g}$ was noted by V. Drinfeld back in 80s, while a detailed proof of this result was only recently provided in [Wen].

In the present paper, we are mostly interested with the shifted versions of the algebras above. Historically, the shifted Yangians $Y_\nu(\mathfrak{g})$ were first introduced for $\mathfrak{g} = \mathfrak{gl}_n$ and dominant shifts ν in [BK2], where their certain quotients were identified with type A finite W-algebras, the latter being natural quantizations of type A Slodowy slices. This construction was further generalized to any semisimple $\mathfrak{g}$ still with dominant shifts $\nu \in \Lambda^+$ in [KWWY], where it was shown that their "GKLO-type" quotients (called *truncated shifted Yangians*) quantize slices in the affine Grassmannians. To this end, the authors constructed a family of algebra homomorphisms

$$\Phi_\nu^{\lambda,\underline{x}}\colon\ Y_\nu(\mathfrak{g}) \longrightarrow \mathcal{A} \tag{1.3}$$

to the (localized) oscillator algebra $\mathcal{A}$ (generalizing the construction of [GKLO] for $\nu = 0$) parametrized by $\lambda \in \Lambda^+$ and an associated collection of points $\underline{x} \in \mathbb{C}^N$. The generalization to arbitrary shifts $\nu \in \Lambda$ was finally carried out in [BFNb, Appendix B] for simply-laced $\mathfrak{g}$ and later in [NW, §5] for non-simply-laced types, where it was also shown (using earlier arguments of A. Weekes) that their images quantize *generalized slices in the affine Grassmannians*.

In contrast to [BK2, KWWY], we consider the opposite case of antidominantly shifted Yangians (note that any shifted Yangian $Y_\nu(\mathfrak{g})$ may be embedded into the antidominantly shifted one $Y_{-\mu}(\mathfrak{g}),\ \mu \in \Lambda^+$, via the *shift homomorphisms* of [FKPRW]). For

$\mathfrak{g} = \mathfrak{so}_{2r}, \mathfrak{sp}_{2r}, \mathfrak{so}_{2r+1}$, we introduce the shifted extended Drinfeld Yangians $X_\mu(\mathfrak{g})$ related to $Y_\nu(\mathfrak{g})$ via isomorphisms

$$X_\mu(\mathfrak{g}) \simeq Y_{\bar{\mu}}(\mathfrak{g}) \otimes ZX_\mu(\mathfrak{g}) \tag{1.4}$$

where the center $ZX_\mu(\mathfrak{g})$ of $X_\mu(\mathfrak{g})$ can be explicitly described via a central Cartan current. For $\mu \in \Lambda^+$ and $\mathfrak{g}$ as above, we also introduce the shifted extended RTT Yangians $X^{\mathrm{rtt}}_{-\mu}(\mathfrak{g})$, whose generators are encoded in a single matrix $T(z)$ (the shift is reflected in the powers of z) subject to the relation (1.1). Based on and generalizing [JLM1], we construct isomorphisms

$$\Upsilon_{-\mu}\colon\ X_{-\mu}(\mathfrak{g}) \xrightarrow{\sim} X^{\mathrm{rtt}}_{-\mu}(\mathfrak{g}) \qquad \text{for any} \quad \mu \in \Lambda^+. \tag{1.5}$$

The construction of $\Upsilon_{-\mu}\colon X_{-\mu}(\mathfrak{g}) \twoheadrightarrow X^{\mathrm{rtt}}_{-\mu}(\mathfrak{g})$ is exactly the same as in [JLM1], but the proof of its injectivity is different, since the arguments of *loc.cit.* do not apply in the shifted setup.

To this end, we construct a family of $\mathcal{A}((z^{-1}))$-valued Lax matrices $T_D(z)$, parametrized by Λ^+-valued divisors D on the projective line $\mathbb{P}^1$, which can be equivalently thought of as algebra homomorphisms $\Theta_D\colon X^{\mathrm{rtt}}_{-\mu}(\mathfrak{g}) \to \mathcal{A}$ with $\mu = D|_\infty$, the coefficient of $[\infty]$. The compositions

$$\Psi_D = \Theta_D \circ \Upsilon_{-\mu}\colon\ X_{-\mu}(\mathfrak{g}) \longrightarrow \mathcal{A} \tag{1.6}$$

coincide with extended versions of (1.3). Combining this with the recent result of [W], asserting that the intersection of kernels of (1.3) as λ varies is trivial, implies the injectivity of $\Upsilon_{-\mu}$.

The aforementioned Lax matrices $T_D(z)$ are defined explicitly by providing the lower-triangular, diagonal, and upper-triangular factors in their Gauss decomposition. The exact defining formulas are exactly engineered (utilizing the new explicit formulas for the inverse of the isomorphism Υ_0 constructed in [JLM1]) to allow us match the resulting homomorphisms Ψ_D of (1.6) with extended versions of (1.3). Meanwhile, the fact that thus constructed matrices are Lax, i.e. satisfy (1.1), follows from a simple *renormalized limit* argument as we shall explain now (expected from the physics of $\mathcal{N} = 2$ ADE quiver gauge theories as explained in [FPT, p. 3]). To this end, we show that if the divisor D contains a summand $\omega_i[x]$ (with $x \in \mathbb{P}^1$ and ω_i being the i-th fundamental coweight of $\mathfrak{g}$) and D' is defined as $D' = D - \omega_i[x] + \omega_i[\infty]$, then

$$T_{D'}(z) = \lim_{x\to\infty} \left\{(-x)^{\omega_i} \cdot T_D(z)\right\} \tag{1.7}$$

realizing $T_{D'}(z)$ as an $x \to \infty$ limit of $T_D(z)$ multiplied on the left by a z-independent diagonal factor $(-x)^{\omega_i}$, the latter preserving the RTT relation (1.1). Therefore, it suffices to prove that $T_D(z)$ satisfies the RTT relation for the divisors D whose support does not contain $\infty \in \mathbb{P}^1$. However, the latter follows from the fact that Υ_0 is indeed an isomorphism as proved in [JLM1].

Similar to the type A case treated in [FPT], the Lax matrices $T_D(z)$ are actually regular in the spectral parameter z (up to a rational factor). This provides a shortcut to the explicit formulas of all linear (in z) Lax matrices $T_D(z)$, which we classify explicitly for each of the B, C, D types. We also show that some of our simplest linear and quadratic Lax matrices, after nontrivial canonical transformations, recover the recent constructions in the physics literature [IKK,F,KK] (see also [R2]). The latter results were obtained by making an ansatz for the Lax matrices and subsequently solving the conditions that arise from the RTT relation. We would like to point out that our formalism provides a recipe to write down Lax matrices of any degree in the spectral parameter (with the leading term not necessarily proportional to the identity matrix) without making such an ansatz.

The algebras $X^{\mathrm{rtt}}_{-\mu}(\mathfrak{g})$ are naturally equipped with coassociative coproduct homomorphisms

$$\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}\colon\ X^{\mathrm{rtt}}_{-\mu_1-\mu_2}(\mathfrak{g}) \longrightarrow X^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{g}) \otimes X^{\mathrm{rtt}}_{-\mu_2}(\mathfrak{g}), \qquad T(z) \mapsto T(z) \otimes T(z). \quad (1.8)$$

Evoking the isomorphisms (1.5) and the embeddings $Y_{-\bar{\mu}}(\mathfrak{g}) \hookrightarrow X_{-\mu}(\mathfrak{g})$, cf. (1.4), we obtain

$$\Delta_{-\nu_1,-\nu_2}\colon\ Y_{-\nu_1-\nu_2}(\mathfrak{g}) \longrightarrow Y_{-\nu_1}(\mathfrak{g}) \otimes Y_{-\nu_2}(\mathfrak{g}). \quad (1.9)$$

We show that the homomorphisms (1.9) precisely coincide with the coproduct homomorphisms of [FKPRW, Theorem 4.8] provided in *loc.cit.* via lengthy formulas (but suitable for any $\mathfrak{g}$).

We note that both the isomorphism (1.5) and the identifications of (1.8, 1.9) with [FKPRW] were conjectured recently (for a general $\mathfrak{g}$) in the physics literature [CGY, §7–8] (see also [DG]).

1.2. Outline of the paper. The structure of the present paper is the following:

- In Section 2, we present our results relevant to the classical type D_r ($\mathfrak{g} = \mathfrak{so}_{2r}$) in full details.
- In Section 3, we provide our results relevant to the classical type C_r (that is, for $\mathfrak{g} = \mathfrak{sp}_{2r}$). Since this is very similar to the type D_r, we only highlight the few technical differences.
- In Section 4, we provide our results relevant to the classical type B_r (that is, for $\mathfrak{g} = \mathfrak{so}_{2r+1}$). Since this is very similar to the type D_r, we only highlight the few technical differences.
- In Section 5, we briefly discuss the further directions.
- In Appendix A, we provide explicit formulas for the Lax matrices in type D_r.
- In Appendix B, we provide the shuffle algebra realization of the key homomorphisms (1.3), which allows us to derive the explicit formulas for the Lax matrices in types C_r and B_r.

2. Type D

Consider the lattice $\bar{\Lambda}^\vee = \bigoplus_{j=1}^r \mathbb{Z}\epsilon_j^\vee$, endowed with the bilinear form with $(\epsilon_i^\vee, \epsilon_j^\vee) = \delta_{i,j}$. We realize the simple positive roots $\{\alpha_i^\vee\}_{i=1}^r$ of the Lie algebra $\mathfrak{so}_{2r}$ via:

$$\alpha_1^\vee = \epsilon_1^\vee - \epsilon_2^\vee,\ \alpha_2^\vee = \epsilon_2^\vee - \epsilon_3^\vee,\ \ldots,\ \alpha_{r-1}^\vee = \epsilon_{r-1}^\vee - \epsilon_r^\vee,\ \alpha_r^\vee = \epsilon_{r-1}^\vee + \epsilon_r^\vee, \quad (2.1)$$

so that the Cartan matrix $A = (a_{ij})_{i,j=1}^r$ is symmetric and is given by $a_{ij} = (\alpha_i^\vee, \alpha_j^\vee)$.

2.1. Classical (unshifted) story. To motivate our constructions in the shifted setting, as well as to carry out the explicit computation of the corresponding Lax matrices, we start by recalling the unshifted setup.

2.1.1. Drinfeld Yangian $Y(\mathfrak{so}_{2r})$ and its extended version $X(\mathfrak{so}_{2r})$ The *Drinfeld Yangian* of $\mathfrak{so}_{2r}$, denoted by $Y(\mathfrak{so}_{2r})$, is the associative $\mathbb{C}$-algebra generated by $\{\mathsf{E}_i^{(k)}, \mathsf{F}_i^{(k)}, \mathsf{H}_i^{(k)}\}_{1\le i\le r}^{k\ge 1}$ with the following defining relations:[1]

$$[\mathsf{H}_i^{(k)}, \mathsf{H}_j^{(\ell)}] = 0, \tag{2.2}$$

$$[\mathsf{E}_i^{(k)}, \mathsf{F}_j^{(\ell)}] = \delta_{i,j}\, \mathsf{H}_i^{(k+\ell-1)}, \tag{2.3}$$

$$[\mathsf{H}_i^{(k'+1)}, \mathsf{E}_j^{(\ell)}] - [\mathsf{H}_i^{(k')}, \mathsf{E}_j^{(\ell+1)}] = \frac{(\alpha_i^\vee, \alpha_j^\vee)}{2} \{\mathsf{H}_i^{(k')}, \mathsf{E}_j^{(\ell)}\}, \tag{2.4}$$

$$[\mathsf{H}_i^{(k'+1)}, \mathsf{F}_j^{(\ell)}] - [\mathsf{H}_i^{(k')}, \mathsf{F}_j^{(\ell+1)}] = -\frac{(\alpha_i^\vee, \alpha_j^\vee)}{2} \{\mathsf{H}_i^{(k')}, \mathsf{F}_j^{(\ell)}\}, \tag{2.5}$$

$$[\mathsf{E}_i^{(k+1)}, \mathsf{E}_j^{(\ell)}] - [\mathsf{E}_i^{(k)}, \mathsf{E}_j^{(\ell+1)}] = \frac{(\alpha_i^\vee, \alpha_j^\vee)}{2} \{\mathsf{E}_i^{(k)}, \mathsf{E}_j^{(\ell)}\}, \tag{2.6}$$

$$[\mathsf{F}_i^{(k+1)}, \mathsf{F}_j^{(\ell)}] - [\mathsf{F}_i^{(k)}, \mathsf{F}_j^{(\ell+1)}] = -\frac{(\alpha_i^\vee, \alpha_j^\vee)}{2} \{\mathsf{F}_i^{(k)}, \mathsf{F}_j^{(\ell)}\}, \tag{2.7}$$

$$\sum_{\sigma\in S(1-a_{ij})} [\mathsf{E}_i^{(k_{\sigma(1)})}, [\mathsf{E}_i^{(k_{\sigma(2)})}, \cdots, [\mathsf{E}_i^{(k_{\sigma(1-a_{ij})})}, \mathsf{E}_j^{(\ell)}]\cdots]] = 0 \quad \text{for } i\neq j, \tag{2.8}$$

$$\sum_{\sigma\in S(1-a_{ij})} [\mathsf{F}_i^{(k_{\sigma(1)})}, [\mathsf{F}_i^{(k_{\sigma(2)})}, \cdots, [\mathsf{F}_i^{(k_{\sigma(1-a_{ij})})}, \mathsf{F}_j^{(\ell)}]\cdots]] = 0 \quad \text{for } i\neq j, \tag{2.9}$$

for $i, j \in \{1, \ldots, r\}$, $k, \ell, k_s \in \mathbb{Z}_{>0}$, and $k' \in \mathbb{Z}_{\ge 0}$, where we set:

$$\mathsf{H}_i^{(0)} = 1 \qquad \text{and} \qquad \{a, b\} = ab + ba. \tag{2.10}$$

Considering the generating series:

$$\mathsf{E}_i(z) := \sum_{k\ge 1} \mathsf{E}_i^{(k)} z^{-k}, \quad \mathsf{F}_i(z) := \sum_{k\ge 1} \mathsf{F}_i^{(k)} z^{-k},$$
$$\mathsf{H}_i(z) := \sum_{k\ge 0} \mathsf{H}_i^{(k)} z^{-k} = 1 + \sum_{k\ge 1} \mathsf{H}_i^{(k)} z^{-k}, \tag{2.11}$$

the defining relations (2.2)–(2.9) are easily seen to be equivalent to (cf. [JLM1, (6.1)–(6.5)]):

$$[\mathsf{H}_i(z), \mathsf{H}_j(w)] = 0, \tag{2.12}$$

$$[\mathsf{E}_i(z), \mathsf{F}_j(w)] = -\delta_{i,j} \frac{\mathsf{H}_i(z) - \mathsf{H}_i(w)}{z - w}, \tag{2.13}$$

$$[\mathsf{H}_i(z), \mathsf{E}_j(w)] = -\frac{(\alpha_i^\vee, \alpha_j^\vee)}{2} \frac{\{\mathsf{H}_i(z), \mathsf{E}_j(z) - \mathsf{E}_j(w)\}}{z - w}, \tag{2.14}$$

[1] We note that our conventions $k \ge 1$ instead of $k \ge 0$ are in charge of perceiving the Yangian as a *QFSHA* (quantum formal series Hopf algebra) which is related to a more standard viewpoint of it as a *QUEA* (quantum universal enveloping algebra) via the so-called Drinfeld-Gavarini quantum duality principle, see [D3] and [G].

$$[\mathsf{H}_i(z), \mathsf{F}_j(w)] = \frac{(\alpha_i^\vee, \alpha_j^\vee)}{2} \frac{\{\mathsf{H}_i(z), \mathsf{F}_j(z) - \mathsf{F}_j(w)\}}{z-w}, \tag{2.15}$$

$$[\mathsf{E}_i(z), \mathsf{E}_j(w)] + [\mathsf{E}_j(z), \mathsf{E}_i(w)] = -\frac{(\alpha_i^\vee, \alpha_j^\vee)}{2} \frac{\{\mathsf{E}_i(z) - \mathsf{E}_i(w), \mathsf{E}_j(z) - \mathsf{E}_j(w)\}}{z-w}, \tag{2.16}$$

$$[\mathsf{F}_i(z), \mathsf{F}_j(w)] + [\mathsf{F}_j(z), \mathsf{F}_i(w)] = \frac{(\alpha_i^\vee, \alpha_j^\vee)}{2} \frac{\{\mathsf{F}_i(z) - \mathsf{F}_i(w), \mathsf{F}_j(z) - \mathsf{F}_j(w)\}}{z-w}, \tag{2.17}$$

$$\sum_{\sigma \in S(1-a_{ij})} [\mathsf{E}_i(z_{\sigma(1)}), [\mathsf{E}_i(z_{\sigma(2)}), \cdots, [\mathsf{E}_i(z_{\sigma(1-a_{ij})}), \mathsf{E}_j(w)] \cdots]] = 0 \quad \text{for } i \neq j, \tag{2.18}$$

$$\sum_{\sigma \in S(1-a_{ij})} [\mathsf{F}_i(z_{\sigma(1)}), [\mathsf{F}_i(z_{\sigma(2)}), \cdots, [\mathsf{F}_i(z_{\sigma(1-a_{ij})}), \mathsf{F}_j(w)] \cdots]] = 0 \quad \text{for } i \neq j, \tag{2.19}$$

Likewise, following [JLM1, Theorem 5.14], the *extended Drinfeld Yangian* of $\mathfrak{so}_{2r}$, denoted by $X(\mathfrak{so}_{2r})$, is defined as the associative $\mathbb{C}$-algebra generated by $\{E_i^{(k)}, F_i^{(k)}\}_{1\le i\le r}^{k\ge 1} \cup \{D_i^{(k)}\}_{1\le i\le r+1}^{k\ge 1}$ with the following defining relations:

$$[D_i(z), D_j(w)] = 0, \tag{2.20}$$

$$[E_i(z), F_j(w)] = -\delta_{i,j} \frac{K_i(z) - K_i(w)}{z-w}, \tag{2.21}$$

$$[D_i(z), E_j(w)] = (\epsilon_i^\vee, \alpha_j^\vee) \frac{D_i(z)(E_j(z) - E_j(w))}{z-w} \quad \text{if } i \le r, \tag{2.22}$$

$$[D_{r+1}(z), E_j(w)] = \begin{cases} -(\epsilon_r^\vee, \alpha_r^\vee) \frac{D_{r+1}(z)(E_r(z)-E_r(w))}{z-w} & \text{if } j = r \\ \frac{D_{r+1}(z)(E_{r-1}(z)-E_{r-1}(w))}{z-w} & \text{if } j = r-1 \\ 0 & \text{if } j < r-1 \end{cases}, \tag{2.23}$$

$$[D_i(z), F_j(w)] = -(\epsilon_i^\vee, \alpha_j^\vee) \frac{(F_j(z) - F_j(w))D_i(z)}{z-w} \quad \text{if } i \le r, \tag{2.24}$$

$$[D_{r+1}(z), F_j(w)] = \begin{cases} (\epsilon_r^\vee, \alpha_r^\vee) \frac{(F_r(z)-F_r(w))D_{r+1}(z)}{z-w} & \text{if } j = r \\ -\frac{(F_{r-1}(z)-F_{r-1}(w))D_{r+1}(z)}{z-w} & \text{if } j = r-1 \\ 0 & \text{if } j < r-1 \end{cases}, \tag{2.25}$$

$$[E_i(z), E_i(w)] = -\frac{(\alpha_i^\vee, \alpha_i^\vee)}{2} \frac{(E_i(z) - E_i(w))^2}{z-w}, \tag{2.26}$$

$$z[E_i^\circ(z), E_j(w)] - w[E_i(z), E_j^\circ(w)] = (\alpha_i^\vee, \alpha_j^\vee)\, E_i(z)E_j(w) \quad \text{for } i \neq j, \tag{2.27}$$

$$[F_i(z), F_i(w)] = \frac{(\alpha_i^\vee, \alpha_i^\vee)}{2} \frac{(F_i(z) - F_i(w))^2}{z-w}, \tag{2.28}$$

$$z[F_i^\circ(z), F_j(w)] - w[F_i(z), F_j^\circ(w)] = -(\alpha_i^\vee, \alpha_j^\vee)\, F_j(w)F_i(z) \quad \text{for } i \neq j, \tag{2.29}$$

$$\sum_{\sigma \in S(1-a_{ij})} [E_i(z_{\sigma(1)}), [E_i(z_{\sigma(2)}), \cdots, [E_i(z_{\sigma(1-a_{ij})}), E_j(w)] \cdots]] = 0 \quad \text{for } i \neq j, \tag{2.30}$$

$$\sum_{\sigma\in S(1-a_{ij})} [F_i(z_{\sigma(1)}), [F_i(z_{\sigma(2)}), \cdots, [F_i(z_{\sigma(1-a_{ij})}), F_j(w)]\cdots]] = 0 \quad \text{for } i\neq j, \tag{2.31}$$

where the generating series are defined via:

$$\begin{aligned} E_i(z) &:= \sum_{k\geq 1} E_i^{(k)} z^{-k}, \qquad E_i^{\circ}(z) := \sum_{k\geq 2} E_i^{(k)} z^{-k},\\ F_i(z) &:= \sum_{k\geq 1} F_i^{(k)} z^{-k}, \qquad F_i^{\circ}(z) := \sum_{k\geq 2} F_i^{(k)} z^{-k}, \end{aligned} \tag{2.32}$$

as well as:

$$\begin{aligned} D_i(z) &:= \sum_{k\geq 0} D_i^{(k)} z^{-k} = 1 + \sum_{k\geq 1} D_i^{(k)} z^{-k},\\ K_i(z) &:= \begin{cases} D_i(z)^{-1} D_{i+1}(z) & \text{if } i<r\\ D_{r-1}(z)^{-1} D_{r+1}(z) & \text{if } i=r \end{cases}. \end{aligned} \tag{2.33}$$

Let us define the elements $\{C_r^{(k)}\}_{k\geq 1}$ of $X(\mathfrak{so}_{2r})$ via:

$$C_r(z) = 1 + \sum_{k\geq 1} C_r^{(k)} z^{-k} := \prod_{i=1}^{r-1} \frac{D_i(z+i-r)}{D_i(z+i-r+1)} \cdot D_r(z) D_{r+1}(z). \tag{2.34}$$

The following result follows from [JLM1, Main Theorem, Theorem 5.8]:

Lemma 2.35. *The elements $\{C_r^{(k)}\}_{k\geq 1}$ are in the center of $X(\mathfrak{so}_{2r})$.*

This result is actually an immediate corollary of the defining relations (2.20, 2.22–2.25), as the proof below shows. This will allow us to generalize it to the shifted setup in Subsection 2.2.1.

Proof. $C_r(z)$ obviously commutes with all $\{D_i(w)\}_{i=1}^{r+1}$, due to (2.20). We shall now verify that it also commutes with all $\{E_i(w)\}_{i=1}^{r}$ (cf. [BK1, Theorem 7.2] for the type A counterpart); the commutativity with $\{F_i(w)\}_{i=1}^{r}$ is completely analogous and is left to the interested reader.

- For $i\leq r-2$, the relations (2.22, 2.23) imply:

$$(z-w+1)D_i(z)E_i(w) - D_i(z)E_i(z) = (z-w)E_i(w)D_i(z), \tag{2.36}$$
$$(z-w-1)D_{i+1}(z)E_i(w) + D_{i+1}(z)E_i(z) = (z-w)E_i(w)D_{i+1}(z). \tag{2.37}$$

  Setting $w=z-1$ in (2.37), we find:

$$E_i(z-1)D_{i+1}(z) = D_{i+1}(z)E_i(z). \tag{2.38}$$

  Now, calculating $(z-w)E_i(w)D_i(z)D_{i+1}(z+1)$ using (2.36)–(2.38), we find that it equals $(z-w)D_i(z)D_{i+1}(z+1)E_i(w)$. Hence, $E_i(w)$ commutes with $D_i(z)D_{i+1}(z+1)$. But it also commutes with $D_j(z)$ for $j\neq i, i+1$, due to (2.22, 2.23). Thus, $[C_r(z), E_i(w)] = 0$ for $i\leq r-2$.

- For $i = r-1$, applying the same arguments we see that $E_{r-1}(w)$ commutes both with $D_{r-1}(z)D_r(z+1)$ and $D_{r+1}(z)D_r(z+1)$, hence, it also commutes with

$$\frac{D_{r-1}(z-1)}{D_{r-1}(z)}D_r(z)D_{r+1}(z) = \frac{(D_{r-1}(z-1)D_r(z))\cdot(D_r(z+1)D_{r+1}(z))}{D_{r-1}(z)D_r(z+1)}.$$

As $[E_{r-1}(w), D_j(w)] = 0$ for $j < r-1$ by (2.22), we thus get the equality $[C_r(z), E_{r-1}(w)] = 0$.

- For $i = r$, applying the same arguments we see that $E_r(w)$ commutes with $D_r(z)D_{r+1}(z+1)$ as well as with $D_{r-1}(z)D_{r+1}(z+1)$, hence, it also commutes with

$$\frac{D_{r-1}(z-1)}{D_{r-1}(z)}D_r(z)D_{r+1}(z) = \frac{(D_{r-1}(z-1)D_{r+1}(z))\cdot(D_r(z)D_{r+1}(z+1))}{D_{r-1}(z)D_{r+1}(z+1)}.$$

As $[E_r(w), D_j(w)] = 0$ for $j < r-1$ by (2.22), we thus get the equality $[C_r(z), E_r(w)] = 0$. □

On the other hand, comparing the defining relations of $Y(\mathfrak{so}_{2r})$ and $X(\mathfrak{so}_{2r})$, it is easy to check (see [JLM1, Proposition 6.2]) that there is a natural homomorphism

$$\iota_0 \colon Y(\mathfrak{so}_{2r}) \longrightarrow X(\mathfrak{so}_{2r}), \tag{2.39}$$

determined by:

$$\mathsf{E}_i(z) \mapsto \begin{cases} E_i(z+\frac{i-1}{2}) & \text{if } i<r \\ E_r(z+\frac{r-2}{2}) & \text{if } i=r \end{cases}, \qquad \mathsf{F}_i(z) \mapsto \begin{cases} F_i(z+\frac{i-1}{2}) & \text{if } i<r \\ F_r(z+\frac{r-2}{2}) & \text{if } i=r \end{cases},$$
$$\mathsf{H}_i(z) \mapsto \begin{cases} D_i(z+\frac{i-1}{2})^{-1}D_{i+1}(z+\frac{i-1}{2}) & \text{if } i<r \\ D_{r-1}(z+\frac{r-2}{2})^{-1}D_{r+1}(z+\frac{r-2}{2}) & \text{if } i=r \end{cases}. \tag{2.40}$$

Lemma 2.41. ι_0 *of* (2.39) *is an embedding and we have a tensor product algebra decomposition:*

$$X(\mathfrak{so}_{2r}) \simeq Y(\mathfrak{so}_{2r}) \otimes_{\mathbb{C}} \mathbb{C}[\{C_r^{(k)}\}_{k\geq 1}]. \tag{2.42}$$

Proof. Given an abstract polynomial algebra $\mathcal{B} = \mathbb{C}[\{D_i^{(k)}\}_{1\leq i\leq r+1}^{k\geq 1}]$, define the elements $\{\bar{D}_i^{(k)}\}_{1\leq i\leq r}^{k\geq 1}$ and $\{C_r^{(k)}\}_{k\geq 1}$ of $\mathcal{B}$ via

$$\bar{D}_i(z) := 1 + \sum_{k\geq 1}\bar{D}_i^{(k)}z^{-k} = D_i(z)^{-1}D_{i+1}(z), \qquad 1\leq i<r,$$

$$\bar{D}_r(z) := 1 + \sum_{k\geq 1}\bar{D}_r^{(k)}z^{-k} = D_{r-1}(z)^{-1}D_{r+1}(z),$$

$$C_r(z) := 1 + \sum_{k\geq 1}C_r^{(k)}z^{-k} = \prod_{i=1}^{r-1}\frac{D_i(z+i-r)}{D_i(z+i-r+1)}\cdot D_r(z)D_{r+1}(z),$$

where $D_i(z) := 1 + \sum_{k\geq 1}D_i^{(k)}z^{-k}$. It is clear that $\{\bar{D}_i^{(k)}\}_{1\leq i\leq r}^{k\geq 1} \cup \{C_r^{(k)}\}_{k\geq 1}$ provide an alternative collection of generators of the polynomial algebra $\mathcal{B}$, so that we have:

$$\mathcal{B} \simeq \mathbb{C}[\{C_r^{(k)}\}_{k\geq 1}] \otimes_{\mathbb{C}} \mathbb{C}[\{\bar{D}_i^{(k)}\}_{1\leq i\leq r}^{k\geq 1}].$$

Applying this in our setup, we get a tensor product decomposition of vector spaces:

$$X(\mathfrak{so}_{2r}) \simeq Z \otimes_{\mathbb{C}} X'(\mathfrak{so}_{2r}), \tag{2.43}$$

where Z is a $\mathbb{C}$-subalgebra generated by $\{C_r^{(k)}\}_{k\geq 1}$ and $X'(\mathfrak{so}_{2r})$ is the $\mathbb{C}$-subalgebra generated by $\{E_i^{(k)}, F_i^{(k)}, \bar{D}_i^{(k)}\}_{1\leq i\leq r}^{k\geq 1}$. Moreover, the defining relations (2.20)–(2.25) are equivalent to Z being central (as explained above) and the commutation relations between $\bar{D}_i^{(k)}$ and $E_i^{(k)}, F_i^{(k)}$ exactly matching those of $Y(\mathfrak{so}_{2r})$ through (2.40). Thus, ι_0 of (2.39, 2.40) is indeed injective, and furthermore (2.43) precisely recovers the tensor product algebra decomposition (2.42). □

2.1.2. RTT Yangian $Y^{\mathrm{rtt}}(\mathfrak{so}_{2r})$ and its extended version $X^{\mathrm{rtt}}(\mathfrak{so}_{2r})$ It will be convenient to use the following notations:

$$N = 2r, \qquad \kappa = r - 1, \\ i' = N + 1 - i \quad \text{for} \quad 1 \leq i \leq N. \tag{2.44}$$

Following [ZZ], we consider the rational *R-matrix* $R(z)$ given by:

$$R(z) = \mathrm{Id} + \frac{\mathrm{P}}{z} - \frac{\mathrm{Q}}{z + \kappa} \tag{2.45}$$

with $\mathrm{P}, \mathrm{Q} \in \operatorname{End} \mathbb{C}^N \otimes \operatorname{End} \mathbb{C}^N$ defined via:

$$\mathrm{P} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{ji}, \qquad \mathrm{Q} = \sum_{i,j=1}^{N} E_{ij} \otimes E_{i'j'}. \tag{2.46}$$

We note the following relations:

$$\mathrm{P}^2 = \mathrm{Id}, \qquad \mathrm{Q}^2 = N\mathrm{Q}, \qquad \mathrm{PQ} = \mathrm{QP} = \mathrm{Q},$$

which imply that $R(z)$ of (2.45) satisfies the Yang-Baxter equation with a spectral parameter:

$$R_{12}(z)R_{13}(z + w)R_{23}(w) = R_{23}(w)R_{13}(z + w)R_{12}(z). \tag{2.47}$$

The *extended RTT Yangian* of $\mathfrak{so}_{2r}$, denoted by $X^{\mathrm{rtt}}(\mathfrak{so}_{2r})$, is the associative $\mathbb{C}$-algebra generated by $\{t_{ij}^{(k)}\}_{1\leq i,j\leq N}^{k\geq 1}$ with the following defining relation (the so-called *RTT relation*):

$$R_{12}(z - w)T_1(z)T_2(w) = T_2(w)T_1(z)R_{12}(z - w), \tag{2.48}$$

where $T(z) \in X^{\mathrm{rtt}}(\mathfrak{so}_{2r})[[z^{-1}]] \otimes_{\mathbb{C}} \operatorname{End} \mathbb{C}^N$ is defined via:

$$T(z) = \sum_{i,j=1}^{N} t_{ij}(z) \otimes E_{ij} \qquad \text{with} \qquad t_{ij}(z) := \sum_{k\geq 0} t_{ij}^{(k)} z^{-k} = \delta_{i,j} + \sum_{k\geq 1} t_{ij}^{(k)} z^{-k}, \tag{2.49}$$

where we set $t_{ij}^{(0)} = \delta_{i,j}$. Thus, (2.48) is an equality in $X^{\mathrm{rtt}}(\mathfrak{so}_{2r})[[z^{-1}, w^{-1}]] \otimes_{\mathbb{C}} (\operatorname{End} \mathbb{C}^N)^{\otimes 2}$, which can be explicitly written as:

$$[t_{ij}(z), t_{k\ell}(w)] = \frac{1}{z-w}\Big(t_{kj}(w)t_{i\ell}(z) - t_{kj}(z)t_{i\ell}(w)\Big) + \frac{1}{z-w+\kappa}\left(\delta_{k,i'}\sum_{p=1}^{N} t_{pj}(z)t_{p'\ell}(w) - \delta_{\ell,j'}\sum_{p=1}^{N} t_{kp'}(w)t_{ip}(z)\right). \tag{2.50}$$

These formulas immediately imply the following simple result, which will be needed later:

Corollary 2.51. *If* $\mathsf{T}^\circ(z)$ *satisfies* (2.48) *and* $\mathsf{T} = \mathrm{diag}(\mathsf{t}_1, \ldots, \mathsf{t}_{2r})$ *is a diagonal* z*-independent matrix such that* $\mathsf{t}_1\mathsf{t}_{2r} = \mathsf{t}_2\mathsf{t}_{2r-1} = \ldots = \mathsf{t}_r\mathsf{t}_{r+1}$*, then* $\bar{\mathsf{T}}^\circ(z) := \mathsf{T}\cdot\mathsf{T}^\circ(z)$ *also satisfies* (2.48).

The *RTT Yangian* of $\mathfrak{so}_{2r}$, denoted by $Y^{\mathrm{rtt}}(\mathfrak{so}_{2r})$, is the subalgebra of $X^{\mathrm{rtt}}(\mathfrak{so}_{2r})$ which consists of the elements stable under the automorphisms:

$$\mu_f\colon T(z) \mapsto f(z)T(z), \qquad \forall\, f(z) = 1 + f_1 z^{-1} + f_2 z^{-2} + \ldots \in \mathbb{C}[[z^{-1}]]. \tag{2.52}$$

At the same time, $Y^{\mathrm{rtt}}(\mathfrak{so}_{2r})$ may also be viewed as a quotient of $X^{\mathrm{rtt}}(\mathfrak{so}_{2r})$. To this end, we recall the following tensor product decomposition (see [AMR, Theorem 3.1, Corollary 3.9]):

$$X^{\mathrm{rtt}}(\mathfrak{so}_{2r}) \simeq ZX^{\mathrm{rtt}}(\mathfrak{so}_{2r}) \otimes_{\mathbb{C}} Y^{\mathrm{rtt}}(\mathfrak{so}_{2r}), \tag{2.53}$$

where $ZX^{\mathrm{rtt}}(\mathfrak{so}_{2r})$ is the center of $X^{\mathrm{rtt}}(\mathfrak{so}_{2r})$. Explicitly, $ZX^{\mathrm{rtt}}(\mathfrak{so}_{2r})$ is a polynomial algebra in the coefficients $\{\mathsf{z}_N^{(k)}\}_{k\geq 1}$ of the series

$$\mathsf{z}_N(z) = 1 + \sum_{k\geq 1} \mathsf{z}_N^{(k)} z^{-k}, \tag{2.54}$$

determined from (with I_N denoting the $N \times N$ identity matrix):

$$T'(z-\kappa)T(z) = T(z)T'(z-\kappa) = \mathsf{z}_N(z)\mathrm{I}_N, \tag{2.55}$$

where the prime denotes the matrix transposition along the antidiagonal, that is:

$$(X')_{ij} = X_{j'i'} \quad \text{for any} \quad N \times N \text{ matrix } X. \tag{2.56}$$

Therefore, the *RTT Yangian* $Y^{\mathrm{rtt}}(\mathfrak{so}_{2r})$ may also be realized as a quotient of $X^{\mathrm{rtt}}(\mathfrak{so}_{2r})$ by:

$$\mathsf{z}_N(z) = 1 + \sum_{k\geq 1} b_k z^{-k} \qquad \text{for any collection of } b_k \in \mathbb{C}, \tag{2.57}$$

though it is common ([AMR, Corollary 3.2]) to choose $b_{\geq 1} = 0$, so that (2.57) reads $\mathsf{z}_N(z) = 1$.

2.1.3. From RTT to Drinfeld realization Consider the Gauss decomposition of the matrix $T(z)$ of (2.49):

$$T(z) = F(z) \cdot H(z) \cdot E(z), \tag{2.58}$$

where $H(z)$ is diagonal:

$$H(z) = \begin{pmatrix} h_1(z) & 0 & \cdots & 0 \\ 0 & h_2(z) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_N(z) \end{pmatrix}, \tag{2.59}$$

$F(z)$ is lower-triangular:

$$F(z) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ f_{2,1}(z) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ f_{N,1}(z) & \cdots & f_{N,N-1}(z) & 1 \end{pmatrix}, \tag{2.60}$$

and $E(z)$ is upper-triangular:

$$E(z) = \begin{pmatrix} 1 & e_{1,2}(z) & \cdots & e_{1,N}(z) \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & e_{N-1,N}(z) \\ 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{2.61}$$

The following explicit identification of the Drinfeld and RTT extended Yangians of $\mathfrak{so}_{2r}$ constitutes the key result of [JLM1]:

Theorem 2.62 ([JLM1, Theorem 5.14]). *There is a $\mathbb{C}$-algebra isomorphism:*

$$\Upsilon_0 \colon X(\mathfrak{so}_{2r}) \xrightarrow{\sim} X^{\mathrm{rtt}}(\mathfrak{so}_{2r}), \tag{2.63}$$

defined by:

$$E_i(z) \mapsto \begin{cases} e_{i,i+1}(z) & \text{if } i < r \\ e_{r-1,r+1}(z) & \text{if } i = r \end{cases}, \qquad F_i(z) \mapsto \begin{cases} f_{i+1,i}(z) & \text{if } i < r \\ f_{r+1,r-1}(z) & \text{if } i = r \end{cases} \tag{2.64}$$

and

$$D_j(z) \mapsto h_j(z) \quad \text{for } 1 \le j \le r+1. \tag{2.65}$$

Combining the Theorem above with Lemma 2.41, we obtain the following explicit identification of the Drinfeld and RTT Yangians of $\mathfrak{so}_{2r}$:

Theorem 2.66 ([JLM1, Main Theorem]). *The composition of the algebra embedding ι_0 (2.39) and the algebra isomorphism Υ_0 (2.63) gives rise to a $\mathbb{C}$-algebra isomorphism:*

$$\Upsilon_0 \circ \iota_0 \colon Y(\mathfrak{so}_{2r}) \xrightarrow{\sim} Y^{\mathrm{rtt}}(\mathfrak{so}_{2r}).$$

Explicitly, it is given by:

$$\mathsf{E}_i(z) \mapsto \begin{cases} e_{i,i+1}(z+\frac{i-1}{2}) & \text{if } i<r \\ e_{r-1,r+1}(z+\frac{r-2}{2}) & \text{if } i=r \end{cases},$$
$$\mathsf{F}_i(z) \mapsto \begin{cases} f_{i+1,i}(z+\frac{i-1}{2}) & \text{if } i<r \\ f_{r+1,r-1}(z+\frac{r-2}{2}) & \text{if } i=r \end{cases}, \tag{2.67}$$
$$\mathsf{H}_i(z) \mapsto \begin{cases} h_i(z+\frac{i-1}{2})^{-1}h_{i+1}(z+\frac{i-1}{2}) & \text{if } i<r \\ h_{r-1}(z+\frac{r-2}{2})^{-1}h_{r+1}(z+\frac{r-2}{2}) & \text{if } i=r \end{cases}.$$

Remark 2.68. (a) We note that our R-matrix $R(z)$ of (2.45) is related to the one of [JLM1, (2.6)], to be denoted by $R^{\mathrm{JLM}}(z)$, via $R(z) = R^{\mathrm{JLM}}(-z)$. Therefore, our matrix $T(z)$ of (2.49) is related to the one of [JLM1, (2.10)], to be denoted by $T^{\mathrm{JLM}}(z)$, via $T(z) = T^{\mathrm{JLM}}(-z)$. This explains the sign difference between our relations (2.21)–(2.29) and those of [JLM1, Theorem 5.14].
(b) Accordingly, our formulas (2.67) agree with those of [JLM1], once we identify the generating series $\mathsf{E}_i(z), \mathsf{F}_i(z), \mathsf{H}_i(z)$ of $Y(\mathfrak{so}_{2r})$ with $\xi_i^-(-z), \xi_i^+(-z), \kappa_i(-z)$ of [JLM1, (1.5)], respectively.

Remark 2.69. Evoking the series $C_r(z)$ of (2.34) and $\mathsf{z}_N(z)$ of (2.54), we note that:

$$\mathsf{z}_N(z) = \prod_{i=1}^{r-1} \frac{h_i(z+i-r)}{h_i(z+i-r+1)} \cdot h_r(z)h_{r+1}(z) = \Upsilon_0(C_r(z)) \tag{2.70}$$

with the first equality due to [JLM1, Theorem 5.8]. Combining (2.70) with Theorems 2.62, 2.66, Lemma 2.41, and the isomorphism $ZX^{\mathrm{rtt}}(\mathfrak{so}_{2r}) \simeq \mathbb{C}[\{\mathsf{z}_N^{(k)}\}_{k\geq 1}]$, we see that the center of $Y(\mathfrak{so}_{2r})$ is trivial, while the center of $X(\mathfrak{so}_{2r})$ is a polynomial algebra in $\{C_r^{(k)}\}_{k\geq 1}$.

2.1.4. From Drinfeld to RTT realization To simplify some of the upcoming formulas, let us introduce the following notations:

$$e_i(z) = \sum_{k\geq 1} e_i^{(k)} z^{-k} := \begin{cases} e_{i,i+1}(z) & \text{if } i<r \\ e_{r-1,r+1}(z) & \text{if } i=r \end{cases},$$
$$f_i(z) = \sum_{k\geq 1} f_i^{(k)} z^{-k} := \begin{cases} f_{i+1,i}(z) & \text{if } i<r \\ f_{r+1,r-1}(z) & \text{if } i=r \end{cases}. \tag{2.71}$$

According to Theorem 2.62, the coefficients of $\{e_i(z), f_i(z)\}_{i=1}^r \cup \{h_j(z)\}_{j=1}^{r+1}$ generate the algebra $X^{\mathrm{rtt}}(\mathfrak{so}_{2r})$. In this Subsection, we record the explicit formulas (those of [JLM1] as well as some new ones) for all other entries of the matrices $F(z), H(z), E(z)$ in (2.58)–(2.61).

But first let us recall the key ingredient of [JLM1]: the algebra embeddings $X^{\mathrm{rtt}}(\mathfrak{so}_{2(r-s)}) \hookrightarrow X^{\mathrm{rtt}}(\mathfrak{so}_{2r})$ for any $0 \leq s < r$. To this end, consider the following $(2r-2s)\times(2r-2s)$ submatrices:

$$H^{[s]}(z) = \begin{pmatrix} h_{s+1}(z) & 0 & \cdots & 0 \\ 0 & h_{s+2}(z) & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_{(s+1)'}(z) \end{pmatrix}, \tag{2.72}$$

$$F^{[s]}(z) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ f_{s+2,s+1}(z) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ f_{(s+1)',s+1}(z) & \cdots & f_{(s+1)',(s+2)'}(z) & 1 \end{pmatrix}, \tag{2.73}$$

$$E^{[s]}(z) = \begin{pmatrix} 1 & e_{s+1,s+2}(z) & \cdots & e_{s+1,(s+1)'}(z) \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & e_{(s+2)',(s+1)'}(z) \\ 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{2.74}$$

Then, according to [JLM1, Proposition 4.1], the matrix

$$T^{[s]}(z) := F^{[s]}(z) \cdot H^{[s]}(z) \cdot E^{[s]}(z) \tag{2.75}$$

coincides with the image of the corresponding T-matrix of $X^{\mathrm{rtt}}(\mathfrak{so}_{2(r-s)})$ under the embedding $X^{\mathrm{rtt}}(\mathfrak{so}_{2(r-s)}) \hookrightarrow X^{\mathrm{rtt}}(\mathfrak{so}_{2r})$ of [JLM1, Theorem 3.7] constructed using the quasideterminants. While we omit the details of the latter construction, but let us record an important corollary that provides a powerful "rank-reduction" tool that will be used through the rest of this Subsection:

Corollary 2.76 ([JLM1, Corollary 4.2]). *The subalgebra of $X^{\mathrm{rtt}}(\mathfrak{so}_{2r})$ generated by the coefficients of all matrix coefficients of the matrix $T^{[s]}(z)$ (2.75) is isomorphic to $X^{\mathrm{rtt}}(\mathfrak{so}_{2(r-s)})$.*

- *Matrix H(z) explicitly*.

Lemma 2.77. *For $1 \le i \le r-1$, we have:*

$$h_{i'}(z) = \frac{1}{h_i(z+i-r+1)} \cdot \prod_{j=i+1}^{r-1} \frac{h_j(z+j-r)}{h_j(z+j-r+1)} \cdot h_r(z) h_{r+1}(z). \tag{2.78}$$

Proof. For $i=1$, this follows from (2.70) combined with the equality $h_N(z) = \frac{\mathsf{z}_N(z)}{h_1(z-r+1)}$ of [JLM1, (5.14)] (obtained by comparing the (N,N) matrix coefficients of both sides of the equality $T'(z-\kappa) = \mathsf{z}_N(z) T(z)^{-1}$). The general case follows now from Corollary 2.76. □

- *Matrix E(z) explicitly.*

The following result is essentially due to [JLM1]:[2]

Lemma 2.79. (a) $e_{r,r+1}(z) = 0$.
(b) $e_{r,(r-1)'}(z) = -e_r(z)$.
(c) $e_{(i+1)',i'}(z) = -e_i(z+i-r+1)$ *for* $1 \le i \le r-1$.
(d) $e_{i,j+1}(z) = -[e_{i,j}(z), e_j^{(1)}]$ *for* $1 \le i < j \le r-1$.
(e) $e_{i,j'}(z) = [e_{i,(j+1)'}(z), e_j^{(1)}]$ *for* $1 \le i < j \le r-1$.
(f) $e_{i,r'}(z) = -[e_{i,r-1}(z), e_r^{(1)}]$ *for* $1 \le i \le r-2$.
(g) $e_{i',j'}(z) = [e_{i',(j+1)'}(z), e_j^{(1)}]$ *for* $1 \le j \le i-2 \le r-2$.

[2] Note a sign and index errors in the equality from part (f) as stated in [JLM1].

Proof. (a, b) follow from Corollary 2.76 and their validity for $r = 2$ (the latter follow from the results of [AMR, §4] in a straightforward way, see [JLM1, Lemma 5.3]).

(c) is [JLM1, Proposition 5.7] (due to Corollary 2.76, it suffices to prove it for $i = 1$ case, in which case it follows by comparing the $(N-1, N)$ matrix coefficients of both sides of the equality $T'(z-\kappa) = T(z)^{-1}\mathsf{z}_N(z)$ and using the equality $h_1(z)e_1(z) = e_1(z+1)h_1(z)$, a result of applying (2.50) to the computation of $[t_{11}(z), t_{12}(z+1)] = [h_1(z), h_1(z+1)e_{1,2}(z+1)]$).

(d, e, f) are [JLM1, Lemma 5.15] (due to Corollary 2.76, it suffices to prove them for $i = 1$, in which case they follow by evaluating the w^{-1}-coefficients in the expressions $[t_{1j}(z), t_{j,j+1}(w)]$, $[t_{1,(j+1)'}(z), t_{(j+1)',j'}(w)]$, $[t_{1,r-1}(z), t_{r-1,r+1}(w)]$, respectively, using (2.50), combined with the equalities $t_{1k}(z) = h_1(z)e_{1,k}(z)$ and $e^{(1)}_{(i+1)',i'} = -e^{(1)}_i$, the latter due to part (c)).

(g) follows immediately from [JLM1, Proposition 5.6] (based on the observation that multiplying the bottom-right $r\times r$ submatrices of $F(z)$, $H(z)$, $E(z)$ provides an $r\times r$ matrix satisfying the RTT relation of type A) and the equality $e^{(1)}_{(j+1)',j'} = -e^{(1)}_j$ due to part (c). □

The remaining matrix coefficients of $E(z)$ from (2.61) are recovered via:

Lemma 2.80. (a) $e_{i,i'}(z) = [e_{i,(i+1)'}(z), e^{(1)}_i] - e_i(z)e_{i,(i+1)'}(z)$ *for* $1 \le i \le r-1$.
(b) $e_{i+1,i'}(z) = [e_{i+1,(i+1)'}(z), e^{(1)}_i] + e_i(z)e_{i+1,(i+1)'}(z) - e_{i,(i+1)'}(z)$ *for* $1 \le i \le r-2$.
(c) $e_{i,j'}(z) = [e_{i,(j+1)'}(z), e^{(1)}_j]$ *for* $1 \le j \le i-2 \le r-2$.

Proof. (a) Due to Corollary 2.76, it suffices to establish this equality for $i = 1$. Comparing the w^{-1}-coefficients in the equality $[t_{1,2r-1}(z), t_{2r-1,2r}(w)] = \frac{t_{2r-1,2r-1}(z)t_{1,2r}(w)-t_{2r-1,2r-1}(w)t_{1,2r}(z)}{w-z}$ of (2.50), we get: $[t_{1,2r-1}(z), t^{(1)}_{2r-1,2r}] = -t_{1,2r}(z)$. Note that $t^{(1)}_{2r-1,2r} = e^{(1)}_{2r-1,2r} = -e^{(1)}_{12}$, due to Lemma 2.79(c). Combining this with the identities $t_{1k}(z) = h_1(z)e_{1,k}(z)$, we find:

$$[h_1(z), e^{(1)}_1]e_{1,2r-1}(z) + h_1(z)[e_{1,2r-1}(z), e^{(1)}_1] = h_1(z)e_{1,2r}(z). \tag{2.81}$$

On the other hand, we have $[t_{11}(z), t_{12}(w)] = \frac{t_{11}(w)t_{12}(z)-t_{11}(z)t_{12}(w)}{z-w}$, so that $[h_1(z), e_1(w)] = \frac{h_1(z)(e_1(z)-e_1(w))}{z-w}$. Comparing the w^{-1}-coefficients of both sides of the latter equality, we get:

$$[h_1(z), e^{(1)}_1] = -h_1(z)e_1(z). \tag{2.82}$$

Combining the formulas (2.81, 2.82), we immediately obtain the desired equality:

$$e_{1,2r}(z) = [e_{1,2r-1}(z), e^{(1)}_1] - e_1(z)e_{1,2r-1}(z). \tag{2.83}$$

(b) Due to Corollary 2.76, it suffices to establish this equality for $i = 1$. To this end, let us compare the w^{-1}-coefficients in the equality

$$[t_{2,2r-1}(z), t_{2r-1,2r}(w)] = \frac{t_{2r-1,2r-1}(z)t_{2,2r}(w) - t_{2r-1,2r-1}(w)t_{2,2r}(z)}{w-z} + \frac{\sum_p t_{p,2r-1}(z)t_{p',2r}(w)}{z-w+r-1}$$

of (2.50), which together with the aforementioned equality $t^{(1)}_{2r-1,2r} = -e^{(1)}_1$ implies:

$$[t_{2,2r-1}(z), e^{(1)}_1] = t_{2,2r}(z) + t_{1,2r-1}(z). \tag{2.84}$$

Note that

$$t_{2,2r-1}(z) = h_2(z)e_{2,2r-1}(z) + f_1(z)h_1(z)e_{1,2r-1}(z). \tag{2.85}$$

Comparing the w^{-1}-coefficients of both sides of $[t_{21}(z), t_{12}(w)] = \frac{t_{11}(z)t_{22}(w)-t_{11}(w)t_{22}(z)}{w-z}$, we get $[f_1(z)h_1(z), e^{(1)}_1] = t_{11}(z) - t_{22}(z) = h_1(z) - t_{22}(z)$, so that:

$$[f_1(z)h_1(z)e_{1,2r-1}(z), e^{(1)}_1] = (h_1(z) - t_{22}(z))\, e_{1,2r-1}(z) + t_{21}(z)[e_{1,2r-1}(z), e^{(1)}_1]. \tag{2.86}$$

We also have $[h_2(z), e^{(1)}_1] = h_2(z)e_1(z)$, so that:

$$[h_2(z)e_{2,2r-1}(z), e^{(1)}_1] = h_2(z)\left(e_1(z)e_{2,2r-1}(z) + [e_{2,2r-1}(z), e^{(1)}_1]\right). \tag{2.87}$$

Combining the formulas (2.84)–(2.87) with (2.83), we immediately obtain the desired equality:

$$e_{2,2r}(z) = [e_{2,2r-1}(z), e^{(1)}_1] + e_1(z)e_{2,2r-1}(z) - e_{1,2r-1}(z). \tag{2.88}$$

(c) Due to Corollary 2.76, it suffices to establish this equality for $j = 1$. We shall proceed by induction on i. Comparing the w^{-1}-coefficients in both parts of $[t_{i,2r-1}(z), t_{2r-1,2r}(w)] = \frac{t_{2r-1,2r-1}(z)t_{i,2r}(w)-t_{2r-1,2r-1}(w)t_{i,2r}(z)}{w-z}$ of (2.50), and evoking $t^{(1)}_{2r-1,2r} = -e^{(1)}_1$, we obtain:

$$[t_{i,2r-1}(z), e^{(1)}_1] = t_{i,2r}(z). \tag{2.89}$$

Note that the series featuring in (2.89) are explicitly given by:

$$t_{i,2r}(z) = h_i(z)e_{i,2r}(z) + \sum_{j=1}^{i-1} f_{i,j}(z)h_j(z)e_{j,2r}(z),$$
$$t_{i,2r-1}(z) = h_i(z)e_{i,2r-1}(z) + \sum_{j=1}^{i-1} f_{i,j}(z)h_j(z)e_{j,2r-1}(z). \tag{2.90}$$

Comparing the w^{-1}-coefficients in both sides of $[t_{i1}(z), t_{12}(w)] = \frac{t_{11}(z)t_{i2}(w)-t_{11}(w)t_{i2}(z)}{w-z}$, we obtain $[t_{i1}(z), e^{(1)}_1] = -t_{i2}(z) = -f_{i,2}(z)h_2(z) - f_{i,1}(z)h_1(z)e_1(z)$, so that:

$$[f_{i,1}(z)h_1(z)e_{1,2r-1}(z), e^{(1)}_1] = f_{i,1}(z)h_1(z)\left([e_{1,2r-1}(z), e^{(1)}_1] - e_1(z)e_{1,2r-1}(z)\right) - f_{i,2}(z)h_2(z)e_{1,2r-1}(z). \tag{2.91}$$

For $j=2$, we have $[f_{i,2}(z), e_1^{(1)}]=0$ and $[h_2(z), e_1^{(1)}]=h_2(z)e_1(z)$, so that:

$$[f_{i,2}(z)h_2(z)e_{2,2r-1}(z), e_1^{(1)}] = f_{i,2}(z)h_2(z)\left(e_1(z)e_{2,2r-1}(z)+[e_{2,2r-1}(z), e_1^{(1)}]\right). \tag{2.92}$$

Finally, for $2<j\leq i-1$, we clearly have $[f_{i,j}(z), e_1^{(1)}]=0=[h_j(z), e_1^{(1)}]$, so that:

$$[f_{i,j}(z)h_j(z)e_{j,2r-1}(z), e_1^{(1)}] = f_{i,j}(z)h_j(z)[e_{j,2r-1}(z), e_1^{(1)}] = f_{i,j}(z)h_j(z)e_{j,2r}(z) \tag{2.93}$$

with the last equality due to the induction assumption.

Combining the formulas (2.83, 2.88, 2.89–2.93), we immediately obtain the desired equality:

$$e_{i,2r}(z) = [e_{i,2r-1}(z), e_1^{(1)}] \quad \text{for} \quad 3\leq i\leq r. \tag{2.94}$$

This completes our proof of Lemma 2.80. □

Let us record the recursive relations that follow from the above two Lemmas:

$$\begin{aligned}
&e_{i,j+1}(z) = [e_j^{(1)}, [e_{j-1}^{(1)}, \cdots, [e_{i+2}^{(1)}, [e_{i+1}^{(1)}, e_i(z)]]\cdots]], \qquad 1\leq i<j\leq r-1,\\
&e_{i,r'}(z) = [e_r^{(1)}, [e_{r-2}^{(1)}, \cdots, [e_{i+2}^{(1)}, [e_{i+1}^{(1)}, e_i(z)]]\cdots]], \qquad 1\leq i\leq r-2,\\
&e_{i,j'}(z) = [[\cdots[[e_{i,r'}(z), e_{r-1}^{(1)}], e_{r-2}^{(1)}], \cdots, e_{j+1}^{(1)}], e_j^{(1)}], \qquad 1\leq i<j\leq r-1,\\
&e_{i,j'}(z) = [[\cdots[[e_{i,(i-1)'}(z), e_{i-2}^{(1)}], e_{i-3}^{(1)}], \cdots, e_{j+1}^{(1)}], e_j^{(1)}], \qquad 1\leq j\leq i-2\leq r-2,\\
&e_{i',j'}(z) = [[\cdots[[e_{i',(i-1)'}(z), e_{i-2}^{(1)}], e_{i-3}^{(1)}], \cdots, e_{j+1}^{(1)}], e_j^{(1)}], \qquad 1\leq j\leq i-2\leq r-2.
\end{aligned} \tag{2.95}$$

• *Matrix F(z) explicitly.*

The following result is essentially due to [JLM1][3] and is proved exactly as Lemma 2.79:

Lemma 2.96. (a) $f_{r+1,r}(z)=0$.
(b) $f_{(r-1)',r}(z) = -f_r(z)$.
(c) $f_{i',(i+1)'}(z) = -f_i(z+i-r+1)$ *for* $1\leq i\leq r-1$.
(d) $f_{j+1,i}(z) = -[f_j^{(1)}, f_{j,i}(z)]$ *for* $1\leq i<j\leq r-1$.
(e) $f_{j',i}(z) = [f_j^{(1)}, f_{(j+1)',i}(z)]$ *for* $1\leq i<j\leq r-1$.
(f) $f_{r',i}(z) = -[f_r^{(1)}, f_{r-1,i}(z)]$ *for* $1\leq i\leq r-2$.
(g) $f_{j',i'}(z) = [f_j^{(1)}, f_{(j+1)',i'}(z)]$ *for* $1\leq j\leq i-2\leq r-2$.

The remaining matrix coefficients of $F(z)$ (2.60) are recovered via the analogue of Lemma 2.80:

Lemma 2.97. (a) $f_{i',i}(z) = [f_i^{(1)}, f_{(i+1)',i}(z)] - f_{(i+1)',i}(z)f_i(z)$ *for* $1\leq i\leq r-1$.
(b) $f_{i',i+1}(z) = [f_i^{(1)}, f_{(i+1)',i+1}(z)] + f_{(i+1)',i+1}(z)f_i(z) - f_{(i+1)',i}(z)$ *for* $1\leq i\leq r-2$.
(c) $f_{j',i}(z) = [f_j^{(1)}, f_{(j+1)',i}(z)]$ *for* $1\leq j\leq i-2\leq r-2$.

[3] Note a sign and index errors in the equality from part (f) as stated in [JLM1].

Let us record the recursive relations that follow from the above two Lemmas:

$$\begin{aligned}
&f_{j+1,i}(z)=[[\cdots[[f_i(z),f_{i+1}^{(1)}],f_{i+2}^{(1)}],\cdots,f_{j-1}^{(1)}],f_j^{(1)}], \qquad 1\le i<j\le r-1,\\
&f_{r',i}(z)=[[\cdots[[f_i(z),f_{i+1}^{(1)}],f_{i+2}^{(1)}],\cdots,f_{r-2}^{(1)}],f_r^{(1)}], \qquad 1\le i\le r-2,\\
&f_{j',i}(z)=[f_j^{(1)},[f_{j+1}^{(1)},\cdots,[f_{r-2}^{(1)},[f_{r-1}^{(1)},f_{r',i}(z)]]\cdots]], \qquad 1\le i<j\le r-1,\\
&f_{j',i}(z)=[f_j^{(1)},[f_{j+1}^{(1)},\cdots,[f_{i-3}^{(1)},[f_{i-2}^{(1)},f_{(i-1)',i}(z)]]\cdots]], \qquad 1\le j\le i-2\le r-2,\\
&f_{j',i'}(z)=[f_j^{(1)},[f_{j+1}^{(1)},\cdots,[f_{i-3}^{(1)},[f_{i-2}^{(1)},f_{(i-1)',i'}(z)]]\cdots]], \qquad 1\le j\le i-2\le r-2.
\end{aligned} \tag{2.98}$$

2.2. Shifted story.

2.2.1. Shifted extended Drinfeld Yangians of $\mathfrak{so}_{2r}$ Consider the *extended* lattice $\Lambda^\vee=\bigoplus_{j=1}^{r+1}\mathbb{Z}\epsilon_j^\vee=\bar{\Lambda}^\vee\oplus\mathbb{Z}\epsilon_{r+1}^\vee$, endowed with the bilinear form via $(\epsilon_i^\vee,\epsilon_j^\vee)=\delta_{i,j}$. We shall need the following family of elements $\{\hat{\alpha}_i^\vee\}_{i=1}^r$ of $\Lambda^\vee$:

$$\hat{\alpha}_1^\vee=\epsilon_1^\vee-\epsilon_2^\vee,\ \hat{\alpha}_2^\vee=\epsilon_2^\vee-\epsilon_3^\vee,\ \ldots,\ \hat{\alpha}_{r-1}^\vee=\epsilon_{r-1}^\vee-\epsilon_r^\vee,\ \hat{\alpha}_r^\vee=\epsilon_{r-1}^\vee-\epsilon_{r+1}^\vee. \tag{2.99}$$

Let $\Lambda=\bigoplus_{j=1}^{r+1}\mathbb{Z}\epsilon_j$ be the dual lattice with $\epsilon_i^\vee(\epsilon_j)=\delta_{i,j}$. Identifying the dual space $(\Lambda^\vee\otimes_\mathbb{Z}\mathbb{C})^*$ with $\Lambda^\vee\otimes_\mathbb{Z}\mathbb{C}$ via the form $(\cdot,\cdot)$, the lattice Λ gets naturally identified with $\Lambda^\vee$ via $\epsilon_i\leftrightarrow\epsilon_i^\vee$. We will also need another $\mathbb{Z}$-basis: $\Lambda=\bigoplus_{i=0}^r\mathbb{Z}\varpi_i$ with

$$\varpi_{r-1}:=-\epsilon_r,\ \varpi_r:=-\epsilon_{r+1},\ \varpi_i=-\epsilon_{i+1}-\epsilon_{i+2}-\ldots-\epsilon_{r+1}\quad \text{for } 0\le i<r-1. \tag{2.100}$$

For $\mu\in\Lambda$, define $\underline{d}=\{d_j\}_{j=1}^{r+1}\in\mathbb{Z}^{r+1}$ and $\underline{b}=\{b_i\}_{i=1}^r\in\mathbb{Z}^r$ via:

$$d_j:=\epsilon_j^\vee(\mu), \tag{2.101}$$

and $\underline{b}=\{b_i\}_{i=1}^r\in\mathbb{Z}^r$ via:

$$b_i:=\hat{\alpha}_i^\vee(\mu), \tag{2.102}$$

so that:

$$b_1=d_1-d_2,\ b_2=d_2-d_3,\ \ldots,\ b_{r-1}=d_{r-1}-d_r,\ b_r=d_{r-1}-d_{r+1}. \tag{2.103}$$

For $\mu\in\Lambda$, define the *shifted extended Drinfeld Yangian of* $\mathfrak{so}_{2r}$, denoted by $X_\mu(\mathfrak{so}_{2r})$, to be the associative $\mathbb{C}$-algebra generated by $\{E_i^{(k)},F_i^{(k)}\}_{1\le i\le r}^{k\ge1}\cup\{D_i^{(k_i)}\}_{1\le i\le r+1}^{k_i\ge d_i+1}$ with the defining relations (2.20, 2.22–2.31) and the following replacement of (2.21):

$$[E_i(z),F_j(w)]=-\delta_{i,j}\,\frac{\underline{\underline{K_i(z)}}-\underline{\underline{K_i(w)}}}{z-w}, \tag{2.104}$$

where $E_i(z)$, $E_i^\circ(z)$, $F_i(z)$, $F_i^\circ(z)$ are defined via (2.32), $D_i(z)$, $K_i(z)$ are defined via:

$$D_i(z) := \sum_{k\geq d_i} D_i^{(k)} z^{-k} = z^{-d_i} + \sum_{k\geq d_i+1} D_i^{(k)} z^{-k},$$

$$K_i(z) = \sum_{k\geq -b_i} K_i^{(k)} z^{-k} := \begin{cases} D_i(z)^{-1} D_{i+1}(z) & \text{if } i < r \\ D_{r-1}(z)^{-1} D_{r+1}(z) & \text{if } i = r \end{cases}, \tag{2.105}$$

with the conventions

$$D_i^{(d_i)} = 1 = K_i^{(-b_i)},$$

and finally $\underline{K_i(z)}$ denotes the principal part of $K_i(z)$:

$$\underline{K_i(z)} := \sum_{k\geq \max\{1,-b_i\}} K_i^{(k)} z^{-k}. \tag{2.106}$$

Remark 2.107. For $\mu = 0$, we obviously get $X_0(\mathfrak{so}_{2r}) \simeq X(\mathfrak{so}_{2r})$.

Similar to our proof of Lemma 2.35, we note that the coefficients $\{C_r^{(k)}\}_{k\geq d_r+d_{r+1}+1}$ of the series

$$C_r(z) = z^{-d_r-d_{r+1}} + \sum_{k>d_r+d_{r+1}} C_r^{(k)} z^{-k} := \prod_{i=1}^{r-1} \frac{D_i(z+i-r)}{D_i(z+i-r+1)} \cdot D_r(z) D_{r+1}(z) \tag{2.108}$$

are central elements of $X_\mu(\mathfrak{so}_{2r})$, which is an immediate corollary of the relations (2.22)–(2.25).

Let $\bar{\Lambda} = \bigoplus_{i=1}^r \mathbb{Z}\omega_i$ be the coweight lattice of $\mathfrak{so}_{2r}$, where $\{\omega_i\}_{i=1}^r$ are the standard fundamental coweights of $\mathfrak{so}_{2r}$, i.e. $\alpha_i^\vee(\omega_j) = \delta_{i,j}$ for $1 \leq i, j \leq r$. There is a natural $\mathbb{Z}$-linear projection:

$$\Lambda \longrightarrow \bar{\Lambda}, \quad \mu \mapsto \bar{\mu} \quad \text{definedvia} \quad \alpha_i^\vee(\bar{\mu}) = \hat{\alpha}_i^\vee(\mu) \quad \text{for } 1 \leq i \leq r. \tag{2.109}$$

Explicitly, we have:

$$\Lambda \ni \mu \mapsto \bar{\mu} = \sum_{i=1}^r b_i \omega_i \in \bar{\Lambda}$$

with $b_i = \hat{\alpha}_i^\vee(\mu)$, cf. (2.102), so that:

$$\bar{\varpi}_0 = 0, \quad \bar{\varpi}_i = \omega_i \quad \text{for} \quad 1 \leq i \leq r.$$

The algebra $X_\mu(\mathfrak{so}_{2r})$ depends only on the associated $\mathfrak{so}_{2r}$–coweight $\bar{\mu}$, up to an isomorphism:

Lemma 2.110. *If $\mu_1, \mu_2 \in \Lambda$ satisfy $\bar{\mu}_1 = \bar{\mu}_2 \in \bar{\Lambda}$, then the assignment*

$$E_i^{(k)} \mapsto E_i^{(k)}, \quad F_i^{(k)} \mapsto F_i^{(k)}, \quad D_i^{(k_i)} \mapsto D_i^{(k_i - \epsilon_i^\vee(\mu_1-\mu_2))} \tag{2.111}$$

gives rise to a $\mathbb{C}$-algebra isomorphism

$$X_{\mu_1}(\mathfrak{so}_{2r}) \xrightarrow{\sim} X_{\mu_2}(\mathfrak{so}_{2r}).$$

Proof. The assignment (2.111) is clearly compatible with the defining relations (2.20, 2.22–2.31, 2.104), thus giving rise to a $\mathbb{C}$-algebra homomorphism $X_{\mu_1}(\mathfrak{so}_{2r}) \to X_{\mu_2}(\mathfrak{so}_{2r})$. Switching μ_1 and μ_2, we obtain the inverse homomorphism $X_{\mu_2}(\mathfrak{so}_{2r}) \to X_{\mu_1}(\mathfrak{so}_{2r})$. Hence, the result. □

Let us also recall the *shifted Drinfeld Yangians of* $\mathfrak{so}_{2r}$ introduced in [BFNb, Definition B.2]. To this end, fix a coweight $\nu \in \bar{\Lambda}$ and set $b_i := \alpha_i^\vee(\nu)$ for $1 \leq i \leq r$. The *shifted Drinfeld Yangian of* $\mathfrak{so}_{2r}$, denoted by $Y_\nu(\mathfrak{so}_{2r})$, is the associative $\mathbb{C}$-algebra generated by $\{\mathsf{E}_i^{(k)}, \mathsf{F}_i^{(k)}, \mathsf{H}_i^{(\ell_i)}\}_{1\leq i\leq r}^{k\geq 1, \ell_i > -b_i}$ with the defining relations (2.12, 2.14–2.19) and the following replacement of (2.13):

$$[\mathsf{E}_i(z), \mathsf{F}_j(w)] = -\delta_{i,j} \frac{\underline{\mathsf{H}_i(z)} - \underline{\mathsf{H}_i(w)}}{z-w}, \tag{2.112}$$

where $\mathsf{E}_i(z), \mathsf{F}_i(z)$ are defined via (2.11), $\mathsf{H}_i(z)$ are defined via:

$$\mathsf{H}_i(z) := \sum_{k\geq -b_i} \mathsf{H}_i^{(k)} z^{-k} = z^{b_i} + \sum_{k\geq 1-b_i} \mathsf{H}_i^{(k)} z^{-k},$$

with the conventions $\mathsf{H}_i^{(-b_i)} = 1$, and finally $\underline{\mathsf{H}_i(z)}$ denotes the principal part of $\mathsf{H}_i(z)$:

$$\underline{\mathsf{H}_i(z)} := \sum_{k\geq \max\{1,-b_i\}} \mathsf{H}_i^{(k)} z^{-k}. \tag{2.113}$$

The explicit relation between the shifted Yangians $X_\mu(\mathfrak{so}_{2r})$ and $Y_\nu(\mathfrak{so}_{2r})$ is as follows:

Proposition 2.114. *For any* $\mu \in \Lambda$, *the assignment* (2.40) *gives rise to a* $\mathbb{C}$*-algebra embedding*

$$\iota_\mu \colon Y_{\bar{\mu}}(\mathfrak{so}_{2r}) \hookrightarrow X_\mu(\mathfrak{so}_{2r}). \tag{2.115}$$

Furthermore, we have a tensor product algebra decomposition:

$$X_\mu(\mathfrak{so}_{2r}) \simeq Y_{\bar{\mu}}(\mathfrak{so}_{2r}) \otimes_{\mathbb{C}} \mathbb{C}[\{C_r^{(k)}\}_{k\geq d_r+d_{r+1}+1}]. \tag{2.116}$$

Remark 2.117. For $\mu = 0$, this exactly recovers (2.39) and Lemma 2.41.

Proof. The proof is completely analogous to that of Lemma 2.41 treating the special case $\mu = 0$ (while Lemma 2.41 follows from the results of [JLM1] combined with the isomorphism (2.63), let us stress right away that our proof was only using the defining relations (2.20)–(2.31)).

The compatibility of the assignment (2.40) with the defining relations of $Y_{\bar{\mu}}(\mathfrak{so}_{2r})$ is straightforward, giving rise to a $\mathbb{C}$-algebra homomorphism $\iota_\mu \colon Y_{\bar{\mu}}(\mathfrak{so}_{2r}) \to X_\mu(\mathfrak{so}_{2r})$. The injectivity of ι_μ as well as the tensor product algebra decomposition (2.116) are immediate after switching from the coefficients of the generating Cartan series $\{D_i(z)\}_{i=1}^{r+1}$ to the coefficients of the central Cartan series $\{C_r(z)\}$ of (2.108) and the series $\left\{D_i(z)^{-1}D_{i+1}(z)\right\}_{i=1}^{r-1} \cup \{D_{r-1}(z)^{-1}D_{r+1}(z)\}$, as in our proof of Lemma 2.41. □

Corollary 2.118. $Y_{\bar{\mu}}(\mathfrak{so}_{2r})$ *may be realized both as a subalgebra of* $X_\mu(\mathfrak{so}_{2r})$ *via (2.115) as well as a quotient of* $X_\mu(\mathfrak{so}_{2r})$ *by the central ideal* $(C_r^{(k)} - c_k)_{k>d_r+d_{r+1}}$ *for any* $c_k \in \mathbb{C}$.

Similar to Remark 2.69 and [FPT, Lemma 2.26], we have:

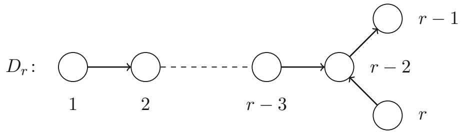

Fig. 1. Oriented Dynkin diagram of type D_r

Lemma 2.119. *(a) The center of the shifted Yangian $Y_\nu(\mathfrak{so}_{2r})$ is trivial for any $\nu \in \bar{\Lambda}$.*
(b) The center of the shifted extended Yangian $X_\mu(\mathfrak{so}_{2r})$ is $\mathbb{C}[\{C_r^{(k)}\}_{k>d_r+d_{r+1}}]$ for any $\mu \in \Lambda$.

Proof. Part (a) is a general result which follows from [W] as explained in [FPT, Remark 2.81]. Part (b) follows from (a), the decomposition (2.116), and the series $C_r(z)$ being central. □

2.2.2. Homomorphisms Ψ_D In this Subsection, we generalize [BFNb, Theorem B.15] for the type D_r Dynkin diagram with arrows pointing $i \to i+1$ for $1 \le i \le r-2$ and $r \to r-2$ (see Fig. 1), by replacing $Y_{\bar{\mu}}(\mathfrak{so}_{2r})$ of *loc.cit.* with $X_\mu(\mathfrak{so}_{2r})$. We closely follow the presentation of [FPT, §2.2] for type A.

Remark 2.120. While similar generalizations exist for all orientations of D_r Dynkin diagram, it suffices to consider only the above one for the purposes of this paper, see Remark 2.191.

An element $\lambda \in \Lambda$ will be called *dominant*, denoted by $\lambda \in \Lambda^+$, if the corresponding $\mathfrak{so}_{2r}$–coweight $\bar{\lambda}$ (2.109) is dominant: $\bar{\lambda} \in \bar{\Lambda}^+$. Thus, $\sum_{i=0}^r c_i \varpi_i$ is dominant iff $c_i \in \mathbb{N}$ for $1 \le i \le r$.

A Λ-*valued divisor D on* $\mathbb{P}^1$, Λ^+-*valued outside* $\{\infty\} \in \mathbb{P}^1$, is a formal sum:

$$D = \sum_{1\le s\le N} \gamma_s \varpi_{i_s}[x_s] + \mu[\infty] \tag{2.121}$$

with $N \in \mathbb{N}$, $0 \le i_s \le r$, $x_s \in \mathbb{C}$, $\gamma_s = \begin{cases} 1 & \text{if } i_s \ne 0 \\ \pm 1 & \text{if } i_s = 0 \end{cases}$, and $\mu \in \Lambda$. We will write

$$\mu = D|_\infty. \tag{2.122}$$

If $\mu \in \Lambda^+$, we call D a Λ^+-*valued divisor on* $\mathbb{P}^1$. It will be convenient to present D also as:

$$D = \sum_{x\in\mathbb{P}^1\backslash\{\infty\}} \lambda_x[x] + \mu[\infty] \quad \text{with } \lambda_x \in \Lambda^+, \tag{2.123}$$

related to (2.121) via $\lambda_x = D|_x := \sum_{1\le s\le N}^{x_s=x} \gamma_s \varpi_{i_s}$. Define $\lambda \in \Lambda^+$ via:

$$\lambda := \sum_{1\le s\le N} \gamma_s \varpi_{i_s} = \sum_{x\in\mathbb{P}^1\backslash\{\infty\}} D|_x. \tag{2.124}$$

Let $\{\alpha_i\}_{i=1}^r \subset \bar{\Lambda}$ denote the simple coroots of $\mathfrak{so}_{2r}$, explicitly given by:

$$\alpha_1 = \epsilon_1 - \epsilon_2, \ \ldots, \ \alpha_{r-2} = \epsilon_{r-2} - \epsilon_{r-1}, \ \alpha_{r-1} = \epsilon_{r-1} - \epsilon_r, \ \alpha_r = \epsilon_{r-1} + \epsilon_r. \tag{2.125}$$

We also consider the following family of elements $\{\hat{\alpha}_i\}_{i=1}^r \subset \Lambda$ given by:

$$\begin{aligned} \hat{\alpha}_1 &= \epsilon_1 - \epsilon_2,\ \ldots,\ \hat{\alpha}_{r-2} = \epsilon_{r-2} - \epsilon_{r-1}, \\ \hat{\alpha}_{r-1} &= \epsilon_{r-1} - \epsilon_r + \epsilon_{r+1},\ \hat{\alpha}_r = \epsilon_{r-1} + \epsilon_r - \epsilon_{r+1}, \end{aligned} \tag{2.126}$$

which are the "lifts" of α_i from (2.125) in the sense of (2.109), that is:

$$\bar{\hat{\alpha}}_i = \alpha_i \qquad \text{for } 1 \leq i \leq r. \tag{2.127}$$

Following [BFNb], we make the following

$$\textbf{Assumption}: \qquad \lambda + \mu = a_1\hat{\alpha}_1 + \ldots + a_r\hat{\alpha}_r \quad \text{with } a_i \in \mathbb{N}. \tag{2.128}$$

Let us record the explicit formulas for the coefficients a_i of (2.128):

$$\begin{aligned} a_k &= (\epsilon_1^\vee + \ldots + \epsilon_k^\vee)(\lambda+\mu) \qquad \text{for} \quad 1 \leq k \leq r-2, \\ a_{r-1} &= \frac{(\epsilon_1^\vee + \ldots + \epsilon_{r-1}^\vee - \epsilon_r^\vee)(\lambda+\mu)}{2}, \\ a_r &= \frac{(\epsilon_1^\vee + \ldots + \epsilon_{r-1}^\vee + \epsilon_r^\vee)(\lambda+\mu)}{2}. \end{aligned} \tag{2.129}$$

Remark 2.130. Note that D of (2.121) satisfies the assumption (2.128) iff all quantities in the right-hand sides of (2.129) are non-negative integers and $(\epsilon_r^\vee + \epsilon_{r+1}^\vee)(\lambda+\mu) = 0$.

Consider the associative $\mathbb{C}$-algebra

$$\mathcal{A} = \mathbb{C}\left\langle p_{i,k}, e^{\pm q_{i,k}}, (p_{i,k} - p_{i,\ell} + m)^{-1} \right\rangle_{1\leq i\leq r, m\in\mathbb{Z}}^{1\leq k\neq\ell\leq a_i} \tag{2.131}$$

with the defining relations:

$$[e^{\pm q_{i,k}}, p_{j,\ell}] = \mp\delta_{i,j}\delta_{k,\ell}e^{\pm q_{i,k}}, \quad [p_{i,k}, p_{j,\ell}] = 0 = [e^{q_{i,k}}, e^{q_{j,\ell}}], \quad e^{\pm q_{i,k}}e^{\mp q_{i,k}} = 1.$$

Remark 2.132. (a) This algebra $\mathcal{A}$ can be represented in the algebra of difference operators with rational coefficients on functions of $\{p_{i,k}\}_{1\leq i\leq r}^{1\leq k\leq a_i}$ by taking $e^{\mp q_{i,k}}$ to be a difference operator $\mathsf{D}_{i,k}^{\pm 1}$ that acts as

$$(\mathsf{D}_{i,k}^{\pm 1}\Psi)(p_{1,1}, \ldots, p_{i,k}, \ldots, p_{r,a_r}) = \Psi(p_{1,1}, \ldots, p_{i,k} \pm 1, \ldots, p_{r,a_r}).$$

(b) The *total number of pairs of* (p,q)*-oscillators in the algebra* $\mathcal{A}$ will refer to the sum $\sum_{i=1}^r a_i$.

For $0 \leq i \leq r$ and $1 \leq j \leq r$, we define:

$$\begin{aligned} P_j(z) &:= \prod_{k=1}^{a_j}(z - p_{j,k}), \quad P_{j,\ell}(z) := \prod_{1\leq k\leq a_j}^{k\neq\ell}(z - p_{j,k}), \\ Z_i(z) &:= \prod_{1\leq s\leq N}^{i_s=i}(z - x_s)^{\gamma_s} = \prod_{x\in\mathbb{P}^1\backslash\{\infty\}}(z-x)^{\widetilde{\alpha}_i^\vee(\lambda_x)}, \end{aligned} \tag{2.133}$$

where $\{\widetilde{\alpha}_i^\vee\}_{i=0}^r$ is a $\mathbb{Z}$-basis of $\Lambda^\vee$ dual to the $\mathbb{Z}$-basis $\{\varpi_i\}_{i=0}^r$ of Λ. Explicitly, we have:

$$\begin{gathered}\widetilde{\alpha}_0^\vee = -\epsilon_1^\vee,\ \widetilde{\alpha}_1^\vee = \epsilon_1^\vee - \epsilon_2^\vee,\ \widetilde{\alpha}_2^\vee = \epsilon_2^\vee - \epsilon_3^\vee,\ \ldots, \\ \widetilde{\alpha}_{r-1}^\vee = \epsilon_{r-1}^\vee - \epsilon_r^\vee,\ \widetilde{\alpha}_r^\vee = \epsilon_{r-1}^\vee - \epsilon_{r+1}^\vee. \end{gathered} \tag{2.134}$$

We also set:

$$a_0 := 0, \quad a_{r+1} := 0, \quad P_0(z) := 1, \quad P_{r+1}(z) := 1. \tag{2.135}$$

The following result generalizes the D_r-case of [BFNb, Theorem B.15] stated for semisimple Lie algebras $\mathfrak{g}$ (preceded by [GKLO] for the trivial shift and by [KWWY] for dominant shifts):

Theorem 2.136. *Let D be as in (2.121), satisfying the assumption (2.128), and set $\mu = D|_\infty$. There is a unique $\mathbb{C}$-algebra homomorphism*

$$\Psi_D \colon X_{-\mu}(\mathfrak{so}_{2r}) \longrightarrow \mathcal{A}, \tag{2.137}$$

determined by the following assignment:

$$E_i(z) \mapsto \begin{cases} \sum_{k=1}^{a_i} \frac{P_{i-1}(p_{i,k}-1)}{(z-p_{i,k})P_{i,k}(p_{i,k})} e^{q_{i,k}} & \text{if } i \le r-3 \\ \sum_{k=1}^{a_{r-2}} \frac{P_{r-3}(p_{r-2,k}-1)P_r(p_{r-2,k})}{(z-p_{r-2,k})P_{r-2,k}(p_{r-2,k})} e^{q_{r-2,k}} & \text{if } i = r-2 \\ \sum_{k=1}^{a_{r-1}} \frac{P_{r-2}(p_{r-1,k}-1)}{(z-p_{r-1,k})P_{r-1,k}(p_{r-1,k})} e^{q_{r-1,k}} & \text{if } i = r-1 \\ \sum_{k=1}^{a_r} \frac{1}{(z-p_{r,k})P_{r,k}(p_{r,k})} e^{q_{r,k}} & \text{if } i = r \end{cases},$$

$$F_i(z) \mapsto \begin{cases} -\sum_{k=1}^{a_i} \frac{Z_i(p_{i,k}+1)P_{i+1}(p_{i,k}+1)}{(z-p_{i,k}-1)P_{i,k}(p_{i,k})} e^{-q_{i,k}} & \text{if } i \le r-2 \\ -\sum_{k=1}^{a_{r-1}} \frac{Z_{r-1}(p_{r-1,k}+1)}{(z-p_{r-1,k}-1)P_{r-1,k}(p_{r-1,k})} e^{-q_{r-1,k}} & \text{if } i = r-1 \\ -\sum_{k=1}^{a_r} \frac{Z_r(p_{r,k}+1)P_{r-2}(p_{r,k})}{(z-p_{r,k}-1)P_{r,k}(p_{r,k})} e^{-q_{r,k}} & \text{if } i = r \end{cases},$$

$$D_i(z) \mapsto \begin{cases} \frac{P_i(z)}{P_{i-1}(z-1)} \cdot \prod_{k=0}^{i-1} Z_k(z) & \text{if } i \le r-2 \\ \frac{P_{r-1}(z)P_r(z)}{P_{r-2}(z-1)} \cdot \prod_{k=0}^{r-2} Z_k(z) & \text{if } i = r-1 \\ \frac{P_r(z)}{P_{r-1}(z-1)} \cdot \prod_{k=0}^{r-1} Z_k(z) & \text{if } i = r \\ \frac{P_{r-1}(z)}{P_r(z-1)} \cdot \prod_{k=0}^{r-2} Z_k(z) \cdot Z_r(z) & \text{if } i = r+1 \end{cases}$$

$$= \prod_{x \in \mathbb{P}^1 \setminus \{\infty\}} (z-x)^{-\epsilon_i^\vee(\lambda_x)} \cdot \begin{cases} \frac{P_i(z)}{P_{i-1}(z-1)} & \text{if } i \le r-2 \\ \frac{P_{r-1}(z)P_r(z)}{P_{r-2}(z-1)} & \text{if } i = r-1 \\ \frac{P_r(z)}{P_{r-1}(z-1)} & \text{if } i = r \\ \frac{P_{r-1}(z)}{P_r(z-1)} & \text{if } i = r+1 \end{cases}. \tag{2.138}$$

Remark 2.139. To compare this with [BFNb, §B(ii)], let us identify $\mathcal{A}$ with $\tilde{\mathcal{A}}$ of *loc.cit.* and the points x_s with the parameters z_s of *loc.cit.* (assigned respectively to the summands of $\bar{\lambda} = \sum_{1 \le s \le N}^{i_s \neq 0} \omega_{i_s}$) via:

$$p_{i,k} \leftrightarrow \begin{cases} w_{i,k} + \frac{i-1}{2} & \text{if } i < r \\ w_{r,k} + \frac{r-2}{2} & \text{if } i = r \end{cases},$$
$$e^{\pm q_{i,k}} \leftrightarrow \mathsf{u}_{i,k}^{\mp 1}, \tag{2.140}$$
$$x_s \leftrightarrow \begin{cases} z_s + \frac{i_s}{2} & \text{if } 1 \le i_s < r \\ z_s + \frac{r-1}{2} & \text{if } i_s = r \end{cases}.$$

Then, the (restriction) composition

$$Y_{-\bar{\mu}}(\mathfrak{so}_{2r}) \xrightarrow{\iota_{-\mu}} X_{-\mu}(\mathfrak{so}_{2r}) \xrightarrow{\Psi_D} \mathcal{A} \tag{2.141}$$

is explicitly given by:

$$\mathsf{E}_i(z) \mapsto \sum_{k=1}^{a_i} \frac{\prod_{h\in Q:\mathrm{i}(h)=i} W_{\mathrm{o}(h)}(w_{i,k}-\frac{1}{2})}{(z-w_{i,k})W_{i,k}(w_{i,k})}\mathsf{u}_{i,k}^{-1},$$
$$\mathsf{F}_i(z) \mapsto -\sum_{k=1}^{a_i} \frac{\mathsf{Z}_i(w_{i,k}+1)\prod_{h\in Q:\mathrm{o}(h)=i} W_{\mathrm{i}(h)}(w_{i,k}+\frac{1}{2})}{(z-w_{i,k}-1)W_{i,k}(w_{i,k})}\mathsf{u}_{i,k}, \tag{2.142}$$
$$\mathsf{H}_i(z) \mapsto \frac{\mathsf{Z}_i(z)\prod_{h\in Q\cup\bar{Q}:\mathrm{o}(h)=i} W_{\mathrm{i}(h)}(z-\frac{1}{2})}{W_i(z)W_i(z-1)},$$

where Q (resp. $\bar{Q}$) denotes the set of oriented (resp. oppositely oriented) edges of the Dynkin diagram from Fig. 1, the notation $\mathrm{i}(h) = i$ (resp. $\mathrm{o}(h) = i$) for an edge $h \in Q$ (or $h \in Q \cup \bar{Q}$) is to indicate that h points towards (resp. away from) the i-th node, and the generating series in (2.142) are defined via:

$$W_i(z) = \prod_{k=1}^{a_i}(z-w_{i,k}),\ W_{i,\ell}(z) = \prod_{1\le k\le a_i}^{k\neq\ell}(z-w_{i,k}),\ \mathsf{Z}_i(z) = \prod_{1\le s\le N}^{i_s=i}\left(z-z_s-\tfrac{1}{2}\right).$$

Thus, the composition

$$\Psi_D \circ \iota_{-\mu} \colon Y_{-\bar{\mu}}(\mathfrak{so}_{2r}) \longrightarrow \mathcal{A}$$

essentially coincides with the version of the homomorphism $\Phi^{\bar{\lambda}}_{-\bar{\mu}}$ of [BFNb, Theorem B.15], where the signs of all $E_i(z)$ and $F_i(z)$ are reversed, and the $\mathsf{Z}_i(w_{i,k})$-factors in $E_i(z)$-currents are now replaced with the $\mathsf{Z}_i(w_{i,k}+1)$-factors in $F_i(z)$-currents, cf. [FT1, Remark C.3].

Proof of Theorem 2.136. First, let us verify that under the above assignment (2.138), the image of $D_i(z)$ is of the form z^{d_i} + (lower order terms in z) for all $1 \le i \le r+1$. Let $\deg_i$ denote the leading power of z in the image of $D_i(z)$ (clearly the coefficient of $z^{\deg_i}$ equals 1). Then, we have:

$$\deg_i = -\sum_{x\in\mathbb{P}^1\backslash\{\infty\}} \epsilon_i^\vee(\lambda_x) + \begin{cases} a_i - a_{i-1} & \text{if } i \neq r\pm 1 \\ a_{r-1}+a_r-a_{r-2} & \text{if } i = r-1 \\ a_{r-1}-a_r & \text{if } i = r+1 \end{cases}. \tag{2.143}$$

Note that $\sum_{x\in\mathbb{P}^1\backslash\{\infty\}} \lambda_x + \mu = \lambda + \mu = a_1\hat{\alpha}_1 + \ldots + a_r\hat{\alpha}_r$ (2.128), so that:

$$\sum_{x\in\mathbb{P}^1\backslash\{\infty\}} \epsilon_i^\vee(\lambda_x) + \epsilon_i^\vee(\mu) = \epsilon_i^\vee(a_1\hat{\alpha}_1 + \ldots + a_r\hat{\alpha}_r)$$

$$= \begin{cases} a_i - a_{i-1} & \text{if } i \neq r \pm 1 \\ a_{r-1} + a_r - a_{r-2} & \text{if } i = r-1 \\ a_{r-1} - a_r & \text{if } i = r+1 \end{cases}. \tag{2.144}$$

Combining (2.143, 2.144), we thus obtain the desired equality:

$$\deg_i = \epsilon_i^\vee(\mu) = d_i. \tag{2.145}$$

Evoking the algebra decomposition (2.116)

$$X_{-\mu}(\mathfrak{so}_{2r}) \simeq Y_{-\bar{\mu}}(\mathfrak{so}_{2r}) \otimes_{\mathbb{C}} \mathbb{C}[\{C_r^{(k)}\}_{k>-d_r-d_{r+1}}],$$

it suffices to prove that the restrictions of the assignment (2.138) to the subalgebras $Y_{-\bar{\mu}}(\mathfrak{so}_{2r})$ and $\mathbb{C}[\{C_r^{(k)}\}_{k>-d_r-d_{r+1}}]$ determine algebra homomorphisms, whose images commute. The former is clear for the restriction to $Y_{-\bar{\mu}}(\mathfrak{so}_{2r})$, due to Theorem B.15 of [BFNb] combined with Remark 2.139 above. On the other hand, we have:

$$\Psi_D(C_r(z)) = \prod_{i=0}^{r-2} \Big(Z_i(z) Z_i(z+i-r+1) \Big) \cdot Z_{r-1}(z) Z_r(z). \tag{2.146}$$

Thus, the restriction of Ψ_D to the polynomial algebra $\mathbb{C}[\{C_r^{(k)}\}_{k>-d_r-d_{r+1}}]$ defines an algebra homomorphism, whose image is central in $\mathcal{A}$. This completes our proof of Theorem 2.136. □

Remark 2.147. Our choice of $\hat{\alpha}_i \in \Lambda$ in (2.126) "lifting" $\alpha_i \in \bar{\Lambda}$ of (2.125) in the sense of (2.127) is exactly to guarantee the equality (2.144); moreover, the latter determines $\hat{\alpha}_i$ uniquely.

2.2.3. Antidominantly shifted extended RTT Yangians of $\mathfrak{so}_{2r}$ Fix $\mu \in \Lambda^+$. Define the *antidominantly shifted extended RTT Yangian of* $\mathfrak{so}_{2r}$, denoted by $X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r})$, to be the associative $\mathbb{C}$-algebra generated by $\{t_{ij}^{(k)}\}_{1\leq i,j\leq 2r}^{k\in\mathbb{Z}}$ subject to the following two families of relations:

- The RTT relation (2.48) with $T(z) \in X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r})[[z, z^{-1}]] \otimes_{\mathbb{C}} \mathrm{End}\, \mathbb{C}^{2r}$ defined via:

$$T(z) = \sum_{i,j} t_{ij}(z) \otimes E_{ij} \qquad \text{with} \qquad t_{ij}(z) := \sum_{k\in\mathbb{Z}} t_{ij}^{(k)} z^{-k}. \tag{2.148}$$

- The second family of relations encodes the fact that $T(z)$ admits the Gauss decomposition:

$$T(z) = F(z) \cdot H(z) \cdot E(z), \tag{2.149}$$

where $F(z), H(z), E(z) \in X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r})((z^{-1})) \otimes_{\mathbb{C}} \mathrm{End}\, \mathbb{C}^{2r}$ are of the form

$$F(z) = \sum_i E_{ii} + \sum_{i<j} f_{j,i}(z) \otimes E_{ji}, \ H(z) = \sum_i h_i(z) \otimes E_{ii},$$

$$E(z) = \sum_i E_{ii} + \sum_{i<j} e_{i,j}(z) \otimes E_{ij}$$

with the matrix coefficients having the following expansions in z:

$$e_{i,j}(z) = \sum_{k\geq 1} e^{(k)}_{i,j} z^{-k}, \quad f_{j,i}(z) = \sum_{k\geq 1} f^{(k)}_{j,i} z^{-k} \quad \text{for } 1 \leq i < j \leq 2r,$$
$$h_i(z) = z^{d_i} + \sum_{k\geq 1-d_i} h^{(k)}_i z^{-k}, \quad h_{i'}(z) = z^{d'_i} + \sum_{k\geq 1-d'_i} h^{(k)}_{i'} z^{-k} \quad \text{for } 1 \leq i \leq r, \tag{2.150}$$

with $i' = 2r+1-i$ as in (2.44) and $d'_i \in \mathbb{Z}$ defined via:

$$d'_i := d_r + d_{r+1} - d_i \qquad \text{for } 1 \leq i \leq r. \tag{2.151}$$

Note that $d'_r = d_{r+1}$. We also note that $\mu \in \Lambda^+$ implies the following inequalities:

$$d_1 \geq d_2 \geq \cdots \geq d_{r-1} \geq \max\{d_r, d'_r\} \geq \min\{d_r, d'_r\} \geq d'_{r-1} \geq \cdots \geq d'_1. \tag{2.152}$$

Remark 2.153. (a) For $\mu = 0$, the second family of relations (2.149, 2.150) is equivalent to the relations $t^{(k)}_{ij} = 0$ for $k < 0$ and $t^{(0)}_{ij} = \delta_{i,j}$, so that $X^{\mathrm{rtt}}_0(\mathfrak{so}_{2r}) \simeq X^{\mathrm{rtt}}(\mathfrak{so}_{2r})$.

(b) If $\mu_1, \mu_2 \in \Lambda^+$ satisfy $\bar{\mu}_1 = \bar{\mu}_2 \in \bar{\Lambda}$, that is, $\mu_2 = \mu_1 + c\varpi_0$ with $c \in \mathbb{Z}$, then the assignment

$$T(z) \mapsto z^c T(z)$$

gives rise to a $\mathbb{C}$-algebra isomorphism $X^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{so}_{2r}) \xrightarrow{\sim} X^{\mathrm{rtt}}_{-\mu_2}(\mathfrak{so}_{2r})$, cf. Lemma 2.110.

Similar to the $\mu = 0$ case, $X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r})$ is generated by

$$e^{(k)}_{i,i+1},\ e^{(k)}_{r-1,r+1},\ f^{(k)}_{i+1,i},\ f^{(k)}_{r+1,r-1},\ h^{(s_j)}_j \tag{2.154}$$

for all $1 \leq i \leq r-1$, $1 \leq j \leq r+1$, $k \geq 1$, $s_j \geq 1-d_j$. Furthermore, all the other generators $e^{(k)}_{i,j}, f^{(k)}_{j,i}, h^{(k)}_i$ of (2.150) are expressed via (2.154) by exactly the same formulas as in the $\mu = 0$ case, treated in details in Subsection 2.1.4. This immediately implies the following result:

Proposition 2.155. *For any $\mu \in \Lambda^+$, there is a unique $\mathbb{C}$-algebra epimorphism*

$$\Upsilon_{-\mu}\colon X_{-\mu}(\mathfrak{so}_{2r}) \twoheadrightarrow X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r})$$

defined by the formulas (2.64, 2.65).

One of our key results (the proof of which is deferred to Subsection 2.3.2) is the following generalization of Theorem 2.62 (corresponding to the case $\mu = 0$):

Theorem 2.156. $\Upsilon_{-\mu}\colon X_{-\mu}(\mathfrak{so}_{2r}) \xrightarrow{\sim} X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r})$ *is a $\mathbb{C}$-algebra isomorphism for any $\mu \in \Lambda^+$.*

2.2.4. Coproduct homomorphisms One of the key benefits of the RTT realization is that it immediately endows the (extended) Yangian of $\mathfrak{so}_{2r}$ with the Hopf algebra structure, in particular, the coproduct homomorphism:

$$\Delta^{\mathrm{rtt}}\colon X^{\mathrm{rtt}}(\mathfrak{so}_{2r}) \longrightarrow X^{\mathrm{rtt}}(\mathfrak{so}_{2r}) \otimes X^{\mathrm{rtt}}(\mathfrak{so}_{2r}), \qquad T(z) \mapsto T(z) \otimes T(z). \tag{2.157}$$

The main observation of this Subsection is that (2.157) naturally admits a shifted version:

Proposition 2.158. *For any $\mu_1, \mu_2 \in \Lambda^+$, there is a unique $\mathbb{C}$-algebra homomorphism*

$$\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}\colon X^{\mathrm{rtt}}_{-\mu_1-\mu_2}(\mathfrak{so}_{2r}) \longrightarrow X^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{so}_{2r}) \otimes X^{\mathrm{rtt}}_{-\mu_2}(\mathfrak{so}_{2r}) \tag{2.159}$$

defined by:

$$\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}(T(z)) = T(z) \otimes T(z). \tag{2.160}$$

Proof. The proof is completely analogous to its type A counterpart established in [FPT, Proposition 2.136]: the arguments of *loc.cit.* apply on the nose, due to (2.152) as well as $e_{r,r+1}(z) = 0 = f_{r+1,r}(z)$ (to treat the possible case $d_r < d'_r$), cf. Lemmas 2.79(a), 2.96(a). □

Similar to [FPT, Corollary 2.141], we note that $\Delta^{\mathrm{rtt}}_{*,*}$ (2.159) satisfy the natural coassociativity:

Corollary 2.161. *For any $\mu_1, \mu_2, \mu_3 \in \Lambda^+$, the following diagram is commutative:*

$$\begin{array}{ccc}
X^{\mathrm{rtt}}_{-\mu_1-\mu_2-\mu_3}(\mathfrak{so}_{2r}) & \xrightarrow{\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2-\mu_3}} & X^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{so}_{2r}) \otimes X^{\mathrm{rtt}}_{-\mu_2-\mu_3}(\mathfrak{so}_{2r}) \\
\Big\downarrow {\scriptstyle \Delta^{\mathrm{rtt}}_{-\mu_1-\mu_2,-\mu_3}} & & \Big\downarrow {\scriptstyle \mathrm{Id}\otimes\Delta^{\mathrm{rtt}}_{-\mu_2,-\mu_3}} \\
X^{\mathrm{rtt}}_{-\mu_1-\mu_2}(\mathfrak{so}_{2r}) \otimes X^{\mathrm{rtt}}_{-\mu_3}(\mathfrak{so}_{2r}) & \xrightarrow[\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}\otimes\mathrm{Id}]{} & X^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{so}_{2r}) \otimes X^{\mathrm{rtt}}_{-\mu_2}(\mathfrak{so}_{2r}) \otimes X^{\mathrm{rtt}}_{-\mu_3}(\mathfrak{so}_{2r})
\end{array}$$

Evoking the key isomorphism $\Upsilon_{-\mu}\colon X_{-\mu}(\mathfrak{so}_{2r}) \xrightarrow{\sim} X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r})$ of Theorem 2.156 for $\mu = \mu_1, \mu_2, \mu_1+\mu_2$, we conclude that $\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}$ of (2.159) gives rise to the $\mathbb{C}$-algebra homomorphism

$$\Delta_{-\mu_1,-\mu_2}\colon X_{-\mu_1-\mu_2}(\mathfrak{so}_{2r}) \longrightarrow X_{-\mu_1}(\mathfrak{so}_{2r}) \otimes X_{-\mu_2}(\mathfrak{so}_{2r}). \tag{2.162}$$

Proposition 2.163. *For any $\mu_1, \mu_2 \in \Lambda^+$, the above $\mathbb{C}$-algebra homomorphism* (2.162)

$$\Delta_{-\mu_1,-\mu_2}\colon X_{-\mu_1-\mu_2}(\mathfrak{so}_{2r}) \longrightarrow X_{-\mu_1}(\mathfrak{so}_{2r}) \otimes X_{-\mu_2}(\mathfrak{so}_{2r})$$

is uniquely determined by specifying the image of the central series $C_r(z)$ of (2.108) *via:*

$$C_r(z) \mapsto C_r(z) \otimes C_r(z), \tag{2.164}$$

and the following formulas (for any $1 \leq i \leq r$ and $1 \leq j \leq r+1$):

$$\begin{aligned}
F_i^{(k)} &\mapsto F_i^{(k)} \otimes 1 \quad \text{for} \quad 1 \leq k \leq \hat{\alpha}_i^\vee(\mu_1), \\
F_i^{(\hat{\alpha}_i^\vee(\mu_1)+1)} &\mapsto F_i^{(\hat{\alpha}_i^\vee(\mu_1)+1)} \otimes 1 + 1 \otimes F_i^{(1)}, \\
E_i^{(k)} &\mapsto 1 \otimes E_i^{(k)} \quad \text{for} \quad 1 \leq k \leq \hat{\alpha}_i^\vee(\mu_2),
\end{aligned}$$

$$
\begin{aligned}
E_i^{(\hat{\alpha}_i^\vee(\mu_2)+1)} &\mapsto 1\otimes E_i^{(\hat{\alpha}_i^\vee(\mu_2)+1)} + E_i^{(1)}\otimes 1,\\
D_j^{(-\epsilon_j^\vee(\mu_1+\mu_2)+1)} &\mapsto D_j^{(-\epsilon_j^\vee(\mu_1)+1)}\otimes 1 + 1\otimes D_j^{(-\epsilon_j^\vee(\mu_2)+1)},\\
D_j^{(-\epsilon_j^\vee(\mu_1+\mu_2)+2)} &\mapsto D_j^{(-\epsilon_j^\vee(\mu_1)+2)}\otimes 1 + 1\otimes D_j^{(-\epsilon_j^\vee(\mu_2)+2)}\\
&+ D_j^{(-\epsilon_j^\vee(\mu_1)+1)}\otimes D_j^{(-\epsilon_j^\vee(\mu_2)+1)} + \sum_{\gamma^\vee\in\Delta^+}(\tilde{\epsilon}_j^\vee,\gamma^\vee)E_{\gamma^\vee}^{(1)}\otimes F_{\gamma^\vee}^{(1)},
\end{aligned}
\tag{2.165}
$$

with

$$
\tilde{\epsilon}_j^\vee=\epsilon_j^\vee \ \text{ for } j\le r,\qquad \tilde{\epsilon}_{r+1}^\vee=-\epsilon_r^\vee, \tag{2.166}
$$

and the root generators $\{E_{\gamma^\vee}^{(1)},F_{\gamma^\vee}^{(1)}\}_{\gamma\in\Delta^+}$ *defined via (cf. (2.95, 2.98)):*

$$
\begin{aligned}
E_{\epsilon_i^\vee-\epsilon_j^\vee}^{(1)} &= [E_{j-1}^{(1)},[E_{j-2}^{(1)},[E_{j-3}^{(1)},\cdots,[E_{i+1}^{(1)},E_i^{(1)}]\cdots]]],\\
F_{\epsilon_i^\vee-\epsilon_j^\vee}^{(1)} &= [[[\cdots[F_i^{(1)},F_{i+1}^{(1)}],\cdots,F_{j-3}^{(1)}],F_{j-2}^{(1)}],F_{j-1}^{(1)}],\\
E_{\epsilon_i^\vee+\epsilon_j^\vee}^{(1)} &= [\cdots[[E_r^{(1)},[E_{r-2}^{(1)},[E_{r-3}^{(1)},\cdots,[E_{i+1}^{(1)},E_i^{(1)}]\cdots]]],E_{r-1}^{(1)}],\cdots,E_j^{(1)}],\\
F_{\epsilon_i^\vee+\epsilon_j^\vee}^{(1)} &= [F_j^{(1)},\cdots,[F_{r-1}^{(1)},[[[\cdots[F_i^{(1)},F_{i+1}^{(1)}],\cdots,F_{r-3}^{(1)}],F_{r-2}^{(1)}],F_{r-1}^{(1)}]]\cdots]
\end{aligned}
\tag{2.167}
$$

for $1\le i<j\le r$*, where* $\Delta^+=\left\{\epsilon_i^\vee\pm\epsilon_j^\vee\right\}_{1\le i<j\le r}$ *is the set of positive roots of* $\mathfrak{so}_{2r}$*.*

The proof of this result is completely analogous to that of [FPT, Proposition 2.143] with the only non-trivial computation of $\Delta_{-\mu_1,-\mu_2}(D_j^{(-\epsilon_j^\vee(\mu_1+\mu_2)+2)})$ based on the identifications:

$$
\begin{aligned}
\Upsilon_{-\mu}\colon\ & E_{\epsilon_i^\vee-\epsilon_j^\vee}^{(1)}\mapsto e_{i,j}^{(1)},\ E_{\epsilon_i^\vee+\epsilon_j^\vee}^{(1)}\mapsto e_{i,j'}^{(1)},\\
& F_{\epsilon_i^\vee-\epsilon_j^\vee}^{(1)}\mapsto f_{j,i}^{(1)},\ F_{\epsilon_i^\vee+\epsilon_j^\vee}^{(1)}\mapsto f_{j',i}^{(1)}\quad \forall\, 1\le i<j\le r
\end{aligned}
$$

and the equalities $e_{i,j}^{(1)}=-e_{j',i'}^{(1)},\ f_{j,i}^{(1)}=-f_{i',j'}^{(1)}$ for $1\le i<j\le 2r$, cf. (A.4, A.6, B.29, B.30). Let us note that the formula (2.164) is a direct corollary of the formulas $\mathsf{z}_N(z)=\Upsilon_{-\mu}(C_r(z))$ and $T(z)T'(z-\kappa)=\mathsf{z}_N(z)\mathrm{I}_N$ established in Lemma 2.184 and Proposition 2.186 below.

The above result provides a conceptual and elementary proof of [FKPRW, Theorem 4.8]:

Proposition 2.168. *(a) For any* $\nu_1,\nu_2\in\bar{\Lambda}^+$*, there is a unique* $\mathbb{C}$*-algebra homomorphism*

$$
\Delta_{-\nu_1,-\nu_2}\colon Y_{-\nu_1-\nu_2}(\mathfrak{so}_{2r})\longrightarrow Y_{-\nu_1}(\mathfrak{so}_{2r})\otimes Y_{-\nu_2}(\mathfrak{so}_{2r}) \tag{2.169}
$$

such that the following diagram is commutative for any $\mu_1,\mu_2\in\Lambda^+$*:*

$$
\begin{array}{ccc}
Y_{-\bar{\mu}_1-\bar{\mu}_2}(\mathfrak{so}_{2r}) & \xrightarrow{\Delta_{-\bar{\mu}_1,-\bar{\mu}_2}} & Y_{-\bar{\mu}_1}(\mathfrak{so}_{2r})\otimes Y_{-\bar{\mu}_2}(\mathfrak{so}_{2r})\\
{\scriptstyle \iota_{-\mu_1-\mu_2}}\big\downarrow & & \big\downarrow{\scriptstyle \iota_{-\mu_1}\otimes\iota_{-\mu_2}}\\
X_{-\mu_1-\mu_2}(\mathfrak{so}_{2r}) & \xrightarrow{\Delta_{-\mu_1,-\mu_2}} & X_{-\mu_1}(\mathfrak{so}_{2r})\otimes X_{-\mu_2}(\mathfrak{so}_{2r})
\end{array}
\tag{2.170}
$$

for any $\mu_1, \mu_2 \in \Lambda^+$.
(b) The homomorphism $\Delta_{-\nu_1,-\nu_2}$ *is uniquely determined by the following formulas:*

$$\begin{aligned}
&\mathsf{F}_i^{(k)} \mapsto \mathsf{F}_i^{(k)} \otimes 1 \quad \text{for} \quad 1 \le k \le \alpha_i^\vee(\nu_1),\\
&\mathsf{F}_i^{(\alpha_i^\vee(\nu_1)+1)} \mapsto \mathsf{F}_i^{(\alpha_i^\vee(\nu_1)+1)} \otimes 1 + 1 \otimes \mathsf{F}_i^{(1)},\\
&\mathsf{E}_i^{(k)} \mapsto 1 \otimes \mathsf{E}_i^{(k)} \quad \text{for} \quad 1 \le k \le \alpha_i^\vee(\nu_2),\\
&\mathsf{E}_i^{(\alpha_i^\vee(\nu_2)+1)} \mapsto 1 \otimes \mathsf{E}_i^{(\alpha_i^\vee(\nu_2)+1)} + \mathsf{E}_i^{(1)} \otimes 1,\\
&\mathsf{H}_i^{(\alpha_i^\vee(\nu_1+\nu_2)+1)} \mapsto \mathsf{H}_i^{(\alpha_i^\vee(\nu_1)+1)} \otimes 1 + 1 \otimes \mathsf{H}_i^{(\alpha_i^\vee(\nu_2)+1)},\\
&\mathsf{H}_i^{(\alpha_i^\vee(\nu_1+\nu_2)+2)} \mapsto \mathsf{H}_i^{(\alpha_i^\vee(\nu_1)+2)} \otimes 1 + 1 \otimes \mathsf{H}_i^{(\alpha_i^\vee(\nu_2)+2)}\\
&\quad + \mathsf{H}_i^{(\alpha_i^\vee(\nu_1)+1)} \otimes \mathsf{H}_i^{(\alpha_i^\vee(\nu_2)+1)} - \sum_{\gamma^\vee \in \Delta^+} (\alpha_i^\vee, \gamma^\vee) \mathsf{E}_{\gamma^\vee}^{(1)} \otimes \mathsf{F}_{\gamma^\vee}^{(1)},
\end{aligned} \tag{2.171}$$

with the root generators $\{\mathsf{E}_{\gamma^\vee}^{(1)}, \mathsf{F}_{\gamma^\vee}^{(1)}\}_{\gamma \in \Delta^+}$ *defined exactly as in* (2.167)*, but using* $\mathsf{E}_i^{(1)}$ *and* $\mathsf{F}_i^{(1)}$ *instead of* $E_i^{(1)}$ *and* $F_i^{(1)}$*, respectively.*

Proof. This follows immediately from the formulas (2.165) of Proposition 2.163 combined with the formulas (2.40) for the embedding $\iota_{-\mu}\colon Y_{-\bar{\mu}}(\mathfrak{so}_{2r}) \hookrightarrow X_{-\mu}(\mathfrak{so}_{2r})$ of Proposition 2.114. In particular, the proof of the last formula in (2.171) uses the equality $\alpha_i^\vee = \begin{cases} \tilde{\epsilon}_i^\vee - \tilde{\epsilon}_{i+1}^\vee & \text{if } i < r \\ \tilde{\epsilon}_{r-1}^\vee - \tilde{\epsilon}_{r+1}^\vee & \text{if } i = r \end{cases}$ with $\tilde{\epsilon}_j^\vee$ defined in (2.166). □

Remark 2.172. (a) As our formulas (2.171) coincide with those of [FKPRW, Theorem 4.8], this provides a confirmative answer to the question raised in the end of [CGY, §8], in type D.
(b) A simple argument (see [FKPRW, Theorem 4.12]) shows that the coproduct homomorphisms $\Delta_{-\nu_1,-\nu_2}$ of (2.169) with $\nu_1, \nu_2 \in \bar{\Lambda}^+$ give rise to a family of coproduct homomorphisms $\Delta_{\nu_1,\nu_2}\colon Y_{\nu_1+\nu_2}(\mathfrak{so}_{2r}) \to Y_{\nu_1}(\mathfrak{so}_{2r}) \otimes Y_{\nu_2}(\mathfrak{so}_{2r})$ for any pair of $\mathfrak{so}_{2r}$–coweights $\nu_1, \nu_2 \in \bar{\Lambda}$. However, let us note that Δ_{ν_1,ν_2} ($\nu_1, \nu_2 \in \bar{\Lambda}$) are not coassociative, in contrast to Corollary 2.161.

2.3. Lax matrices.

2.3.1. Motivation, explicit construction, and the normalized limit description Consider a Λ^+-valued divisor D on $\mathbb{P}^1$, see (2.121), satisfying the assumption (2.128). Note that $\mu := D|_\infty \in \Lambda^+$. Assuming the validity of Theorem 2.156, let us compose $\Psi_D\colon X_{-\mu}(\mathfrak{so}_{2r}) \to \mathcal{A}$ of (2.137) with $\Upsilon_{-\mu}^{-1}\colon X_{-\mu}^{\mathrm{rtt}}(\mathfrak{so}_{2r}) \xrightarrow{\sim} X_{-\mu}(\mathfrak{so}_{2r})$ to get a homomorphism:

$$\Theta_D = \Psi_D \circ \Upsilon_{-\mu}^{-1}\colon X_{-\mu}^{\mathrm{rtt}}(\mathfrak{so}_{2r}) \longrightarrow \mathcal{A}. \tag{2.173}$$

Such a homomorphism is uniquely determined by $T_D(z) \in \mathcal{A}((z^{-1})) \otimes_{\mathbb{C}} \mathrm{End}\, \mathbb{C}^{2r}$ defined via:

$$T_D(z) := \Theta_D(T(z)) = \Theta_D(F(z)) \cdot \Theta_D(H(z)) \cdot \Theta_D(E(z)). \tag{2.174}$$

While the above definition (2.174) of $T_D(z)$ is based on yet unproved Theorem 2.156, we can combine the formulas (2.138) for Ψ_D with those of Subsection 2.1.4 to recover the explicit sought-after images $f^D_{j,i}(z), e^D_{i,j}(z), h^D_i(z) \in \mathcal{A}((z^{-1}))$ of the generating series $f_{j,i}(z), e_{i,j}(z), h_i(z)$, the matrix coefficients of $F(z), H(z), E(z)$ in (2.149). Thus, we amend (2.174) and define:

$$T_D(z) := F^D(z) \cdot H^D(z) \cdot E^D(z) \tag{2.175}$$

with $F^D(z), E^D(z), H^D(z)$ being the lower-triangular, upper-triangular, and diagonal matrices with matrix coefficients $f^D_{j,i}(z), e^D_{i,j}(z), h^D_i(z)$ obtained from the explicit formulas (2.138) for the images of $\{e_i(z), f_i(z)\}_{i=1}^r \cup \{h_j(z)\}_{j=1}^{r+1}$ combined with Lemmas 2.77, 2.79, 2.80, 2.96, 2.97. The explicit formulas for $f^D_{j,i}(z), e^D_{i,j}(z), h^D_i(z)$ are presented in Appendix A, cf. [FPT, §2.4.1]. Therefore, the matrix coefficients of $T_D(z)$ are given by:

$$T_D(z)_{\alpha,\beta} = \sum_{i=1}^{\min\{\alpha,\beta\}} f^D_{\alpha,i}(z) \cdot h^D_i(z) \cdot e^D_{i,\beta}(z) \tag{2.176}$$

for any $1 \leq \alpha, \beta \leq 2r$, with the conventions $f^D_{\alpha,\alpha}(z) = 1 = e^D_{\beta,\beta}(z)$.

Definition 2.177. For an associative algebra $\mathcal{B}$, a $\mathcal{B}((z^{-1}))$-valued $2r \times 2r$ matrix $\mathsf{T}(z)$ is called **Lax** (of type D_r) if it satisfies the RTT relation (2.48) with the R-matrix $R(z)$ of (2.45).

Following the arguments of [FPT, §2.4.2], let us show that $T_D(z) \in \mathcal{A}((z^{-1}))$ (2.176) are Lax. To this end, consider a Λ^+-valued divisor $D = \sum_{s=1}^N \gamma_s \varpi_{i_s}[x_s] + \mu[\infty]$. As the point x_N tends to ∞ (denoted $x_N \to \infty$), we obtain another Λ^+-valued divisor $D' = \sum_{s=1}^{N-1} \gamma_s \varpi_{i_s}[x_s] + (\mu + \gamma_N \varpi_{i_N})[\infty]$. Similar to [FPT, Proposition 2.75], the matrix $T_{D'}(z)$ of (2.176) is related to $T_D(z)$ via:

$$T_{D'}(z) = \lim_{x_N \to \infty} \Big\{ (-x_N)^{\gamma_N \varpi_{i_N}} \cdot T_D(z) \Big\}, \tag{2.178}$$

where x^ν (with $x \in \mathbb{C}^\times, \nu \in \Lambda$) denotes the following $2r \times 2r$ diagonal z-independent matrix:

$$x^\nu = \mathrm{diag}\left(x^{\epsilon^\vee_1(\nu)}, \ldots, x^{\epsilon^\vee_{r-1}(\nu)}, x^{\epsilon^\vee_r(\nu)}, x^{\epsilon'^\vee_r(\nu)}, x^{\epsilon'^\vee_{r-1}(\nu)}, \ldots, x^{\epsilon'^\vee_1(\nu)} \right) \tag{2.179}$$

with

$$\epsilon'^\vee_i := \epsilon^\vee_r + \epsilon^\vee_{r+1} - \epsilon^\vee_i \qquad \text{for } 1 \leq i \leq r.$$

Remark 2.180. In contrast to [FPT], we note that the normalization factor $(-x_N)^{\gamma_N \varpi_{i_N}}$ appears on the left of $T_D(z)$ in (2.178), due to our present choice (2.138) of using $Z_k(z)$-factors in the Ψ_D-images of $F_k(z)$-currents rather than $E_k(z)$-currents, cf. Remark 2.139.

In view of (2.178), $T_{D'}(z)$ can be constructed as a *normalized limit* of $T_D(z)$, hence we get:

Corollary 2.181. *For any Λ^+-valued divisor D on $\mathbb{P}^1$ satisfying* (2.128)*, the matrix $T_D(z)$ of* (2.176) *is a normalized limit of $T_{\bar{D}}(z)$ with a Λ^+-valued divisor $\bar{D}$ satisfying $\bar{D}|_\infty = 0$.*

Note that the condition $\bar{D}|_\infty = 0$ corresponds to the unshifted case ($\mu = 0$), in which case Theorem 2.156 holds due to Theorem 2.62. Therefore, $T_{\bar{D}}(z)$ defined via (2.176) can also be recovered via (2.173, 2.174), hence, $T_{\bar{D}}(z)$ is Lax. Since multiplication by x^ν preserves (2.48) (due to Corollary 2.51 and $\epsilon_1^\vee + \epsilon'^\vee_1 = \epsilon_2^\vee + \epsilon'^\vee_2 = \ldots = \epsilon_r^\vee + \epsilon'^\vee_r$), we finally obtain:

Proposition 2.182. *For any Λ^+-valued divisor D on $\mathbb{P}^1$ satisfying the assumption* (2.128)*, the matrix $T_D(z)$ defined via* (2.176) *is Lax, i.e. it satisfies the RTT relation* (2.48)*.*

2.3.2. Proof of the key isomorphism Reversing the argument from the previous Subsection, we note that the Lax matrix $T_D(z)$ (Proposition 2.182) gives rise to the algebra homomorphism $\Theta_D\colon X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r}) \to \mathcal{A}$, whose composition with the epimorphism $\Upsilon_{-\mu}\colon X_{-\mu}(\mathfrak{so}_{2r}) \twoheadrightarrow X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r})$ of Proposition 2.155 coincides with the homomorphism Ψ_D (2.137). Thus, for $\mu \in \Lambda^+$ and any Λ^+-valued divisor D on $\mathbb{P}^1$, see (2.121), satisfying (2.128) and $D|_\infty = \mu$, the homomorphism Ψ_D does factor through $\Upsilon_{-\mu}$. This observation immediately implies the injectivity of $\Upsilon_{-\mu}$, due to the recent result of [W]:[4]

Theorem 2.183 ([W]). *For any coweight ν of a semisimple Lie algebra $\mathfrak{g}$, the intersection of kernels of the homomorphisms $\Phi^*_{-\nu}$ of* [BFNb, Theorem B.15] *is zero: $\bigcap_\lambda \mathrm{Ker}(\Phi^\lambda_{-\nu}) = 0$, where λ ranges through all dominant coweights of $\mathfrak{g}$ such that $\lambda + \nu = \sum a_i\alpha_i$ with $a_i \in \mathbb{N}$, α_i being simple coroots of $\mathfrak{g}$, and points $\{z_i\}$ of* loc.cit. *specialized to arbitrary complex parameters.*

This completes our proof of Theorem 2.156. Combining this with Lemma 2.119(b), we obtain:

Lemma 2.184. *For any $\mu \in \Lambda^+$, the center of $X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r})$ is a polynomial algebra in the coefficients $\{\mathsf{z}_N^{(k)}\}_{k>d_r+d_{r+1}}$ of the series:*

$$\mathsf{z}_N(z) = \sum_{k\geq d_r+d_{r+1}} \mathsf{z}_N^{(k)} z^{-k} = \Upsilon_{-\mu}(C_r(z)) = \prod_{i=1}^{r-1} \frac{h_i(z+i-r)}{h_i(z+i-r+1)} \cdot h_r(z)h_{r+1}(z). \tag{2.185}$$

The above argument can be also used to establish the *crossing relation* for $X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r})$:

Proposition 2.186. *For any $\mu \in \Lambda^+$, the matrix $T(z)$ of* (2.148) *satisfies:*

$$T(z)T'(z-\kappa) = T'(z-\kappa)T(z) = \mathsf{z}_N(z)\mathrm{I}_N. \tag{2.187}$$

Proof. According to (the extended version of) Theorem 2.183, it suffices to verify:

$$T_D(z)T'_D(z-\kappa) = T'_D(z-\kappa)T_D(z) = \Theta_D(\mathsf{z}_N(z))\mathrm{I}_N \tag{2.188}$$

for any Λ^+-valued divisor D on $\mathbb{P}^1$ satisfying (2.128) and $D|_\infty = \mu$. According to (2.55), the equality (2.188) obviously holds for D such that $D|_\infty = 0$. Therefore, the validity

[4] Actually, we need the extended version of Theorem 2.183 (now with the points Z_i, cf. (2.140), ranging over all $\mathbb{C}$), which nevertheless follows immediately from the algebra decomposition (2.116) and the formula (2.146).

of (2.188) for any D follows now from the "normalized limit" construction (2.178). To this end, using the notations of *loc.cit.*, the validity of (2.188) for D implies the one for D' as follows from:

$$T_{D'}(z)T'_{D'}(z-\kappa) = \lim_{x_N\to\infty}\Big((-x_N)^{\gamma_N\varpi_{i_N}}\cdot T_D(z)T'_D(z-\kappa)\cdot\big((-x_N)^{\gamma_N\varpi_{i_N}}\big)'\Big) =$$
$$\lim_{x_N\to\infty}\Big((-x_N)^{\gamma_N\varpi_{i_N}}\cdot\big((-x_N)^{\gamma_N\varpi_{i_N}}\big)'\cdot\Theta_D(\mathsf{z}_N(z))\mathrm{I}_N\Big) = \Theta_{D'}(\mathsf{z}_N(z))\mathrm{I}_N,$$

where the last equality follows from

$$x^{\varpi_i}\cdot\big(x^{\varpi_i}\big)' = x^{-2+\delta_{i,r-1}+\delta_{i,r}}\cdot\mathrm{I}_N$$

and the explicit formulas (2.146, 2.185). □

2.3.3. Regularity of Lax matrices Consider the following normalized version of $T_D(z)$:

$$\mathsf{T}_D(z) := T_D(z)/Z_0(z), \tag{2.189}$$

with the normalization factor $Z_0(z)$ defined in (2.133). The key property of these matrices is their regularity in (the spectral parameter) z:

Theorem 2.190. *We have* $\mathsf{T}_D(z)\in\mathcal{A}[z]\otimes_{\mathbb{C}}\mathrm{End}\,\mathbb{C}^{2r}$.

This straightforward verification, based on the explicit formulas of Appendix A, is completely analogous (though is more tedious) to its type A counterpart of [FPT, Theorem 2.67].

Remark 2.191. Similar to [BFNb, Theorem B.15], Theorem 2.136 can be generalized by constructing the homomorphisms $\Psi_D\colon X_{-\mu}(\mathfrak{so}_{2r})\to\mathcal{A}$ for any orientation of D_r Dynkin diagram, so that $\Psi_D\circ\iota_{-\mu}$ is to $\Phi^{\bar\lambda}_{-\bar\mu}$ of [BFNb] as in Remark 2.139 (note that the images of $D_i(z)$ are independent of the orientation, hence, so is the image of $C_r(z)$, see (2.108, 2.146)). However, extending $\mathcal{A}$ to its localization $\mathcal{A}_{\mathrm{loc}}$ by the multiplicative set generated by $\{p_{i,k}-p_{j,\ell}+m\}_{k\le a_i,\ell\le a_j}^{m\in\mathbb{Z}}$ with (i,j) connected by an edge, these homomorphisms are compositions of (2.137) with algebra automorphisms of $\mathcal{A}_{\mathrm{loc}}$. Thus, similar to [FPT, Remark 2.73], the resulting Lax matrices are equivalent, up to algebra automorphisms of $\mathcal{A}_{\mathrm{loc}}$, to the above $T_D(z)$, cf. Remark 2.120.

2.3.4. Linear Lax matrices The regularity of Theorem 2.190 provides a shortcut to the computation of the Lax matrices $T_D(z)$ defined, in general, as a product of three complicated matrices $F^D(z)$, $H^D(z)$, $E^D(z)$ in (2.175). Let us illustrate this in the case of the linear ones, i.e. those of degree 1 in the spectral parameter z. We shall use the following notations:

$$e^D_{i,j}(z)=\sum_{k\ge1}e^{(D)k}_{i,j}z^{-k},\quad f^D_{j,i}(z)=\sum_{k\ge1}f^{(D)k}_{j,i}z^{-k},\quad h^D_i(z)=\sum_{k\in\mathbb{Z}}h^{(D)k}_i z^{-k}. \tag{2.192}$$

Let us also recall the coefficients $a_i\in\mathbb{N}$ from (2.128, 2.129).

Proposition 2.193. *(a) The normalized Lax matrix* $\mathsf{T}_D(z)$ *of* (2.189) *is linear iff* $a_1=1$.
(b) Any linear normalized Lax matrix $\mathsf{T}_D(z)$ *is explicitly determined as follows:*

- *The diagonal entries are:*

$$\mathsf{T}_D(z)_{i,i} = z \cdot h_i^{(D),-1} + h_i^{(D)0}, \qquad 1 \le i \le 2r. \tag{2.194}$$

- *The entries above the main diagonal are:*

$$\mathsf{T}_D(z)_{i,j} = \begin{cases} e_{i,j}^{(D)1} & \text{if } h_i^{(D),-1} \neq 0 \\ 0 & \text{otherwise} \end{cases}, \qquad 1 \le i < j \le 2r. \tag{2.195}$$

- *The entries below the main diagonal are:*

$$\mathsf{T}_D(z)_{j,i} = \begin{cases} f_{j,i}^{(D)1} & \text{if } h_i^{(D),-1} \neq 0 \\ 0 & \text{otherwise} \end{cases}, \qquad 1 \le i < j \le 2r. \tag{2.196}$$

Proof. (a) As $\mathsf{T}_D(z)_{1,1} = \frac{h_1^D(z)}{Z_0(z)} = P_1(z)$ is a polynomial in z of degree a_1, the condition $a_1 \le 1$ is necessary for $\mathsf{T}_D(z)$ to be of degree ≤ 1 in z. On the other hand, $\deg_z h_1^D(z) \ge \deg_z h_i^D(z)$ for any $1 < i \le 2r$, due to (2.152). Combining this with $\deg_z e_{i,j}^D(z), \deg_z f_{j,i}^D(z) < 0$, we conclude that $a_1 \le 1$ is also a sufficient condition for $\mathsf{T}_D(z)$ to be of degree ≤ 1 in z. Moreover, $\mathsf{T}_D(z)$ is actually of degree < 1 in z if and only if $a_1 < 1$. This concludes our proof of part (a).

(b) This follows immediately from the regularity result of Theorem 2.190 combined with the formula (2.176) and the observation that $\deg_z e_{i,j}^D(z), \deg_z f_{j,i}^D(z) < 0$ for any $i < j$. □

Let us now describe all Λ^+-valued divisors D on $\mathbb{P}^1$ satisfying (2.128) such that $\deg_z \mathsf{T}_D(z) = 1$. Define $\lambda, \mu \in \Lambda^+$ via (2.122, 2.124), so that $\lambda + \mu = \sum_{j=0}^r b_j \varpi_j$ with $b_0 \in \mathbb{Z},\ b_1, \ldots, b_r \in \mathbb{N}$. Then, the assumption (2.128) implies:

$$\sum_{j=0}^{r} b_j \varpi_j = \sum_{i=1}^{r} a_i \hat{\alpha}_i \qquad \text{with} \quad a_i \in \mathbb{N}. \tag{2.197}$$

Decomposing both sides of (2.197) in the basis $\{\epsilon_i\}_{i=1}^{r+1}$, we can express b_j's via a_i's:

$$\begin{gathered} b_0 = -a_1,\ b_1 = 2a_1 - a_2,\ b_i = -a_{i-1} + 2a_i - a_{i+1} \quad \text{for } 2 \le i \le r-3, \\ b_{r-2} = -a_{r-3} + 2a_{r-2} - a_{r-1} - a_r,\ b_{r-1} = -a_{r-2} + 2a_{r-1},\ b_r = -a_{r-2} + 2a_r. \end{gathered} \tag{2.198}$$

Likewise, let us also express a_i's via b_j's:

$$\begin{aligned} a_i &= \sum_{k=0}^{i} (k-i) b_k = \sum_{k=1}^{i-1} k b_k + i \sum_{k=i}^{r-2} b_k + \frac{i}{2}(b_{r-1} + b_r) \quad \text{for } 1 \le i \le r-2, \\ a_{r-1} &= \frac{1}{2} \sum_{k=0}^{r-1} (k+2-r) b_k = \frac{1}{2} \left(\sum_{k=1}^{r-2} k b_k + \frac{r}{2} b_{r-1} + \frac{r-2}{2} b_r \right), \\ a_r &= \frac{1}{2} \sum_{k=0}^{r} (k-r) b_k = \frac{1}{2} \left(\sum_{k=1}^{r-2} k b_k + \frac{r-2}{2} b_{r-1} + \frac{r}{2} b_r \right), \end{aligned} \tag{2.199}$$

where we used the equality (arising from the comparison of the coefficients of ϵ_r and ϵ_{r+1}):

$$b_0 = -b_1 - \ldots - b_{r-2} - \frac{1}{2}(b_{r-1} + b_r). \tag{2.200}$$

Note that (2.200) uniquely recovers b_0 in terms of $b_1, \ldots, b_r$ and forces $b_{r-1} + b_r$ to be even. We also note that the total number of pairs of (p, q)-oscillators in the algebra $\mathcal{A}$ (2.131) equals:

$$\sum_{i=1}^{r} a_i = -\sum_{k=0}^{r} \frac{(r-k)(r-k-1)}{2} b_k = \sum_{k=1}^{r-2} k\left(r - \frac{k+1}{2}\right) b_k + \frac{r(r-1)}{4}(b_{r-1} + b_r). \tag{2.201}$$

Combining the above formulas (2.199, 2.200) with Proposition 2.193(a), we thus conclude that the normalized Lax matrix $\mathsf{T}_D(z)$ is linear only for the following configurations of b_i's:

(1) $b_0 = -1,\ b_j = 1,\ b_1 = \ldots = b_{j-1} = b_{j+1} = \ldots = b_r = 0$ for an even $1 \leq j \leq r-2$,

(2) $\begin{cases} b_0 = -1,\ b_1 = \ldots = b_{r-2} = 0,\ b_{r-1} = b_r = 1 & \text{if } r \text{ is odd} \\ b_0 = -1,\ b_1 = \ldots = b_{r-2} = 0,\ \{b_{r-1}, b_r\} = \{0, 2\} & \text{if } r \text{ is even} \end{cases}$.

As b_0 is uniquely determined via (2.200) and does not affect the normalized Lax matrix $\mathsf{T}_D(z)$, we shall rather focus on the corresponding values of the dominant $\mathfrak{so}_{2r}$-coweights $\bar{\lambda}, \bar{\mu} \in \bar{\Lambda}^+$.

- <u>Case (1)</u> : $\bar{\lambda} + \bar{\mu} = \omega_j$ for even $1 \leq j \leq r-2$.

In this case, we have $a_1 = 1, \ldots, a_{j-1} = j-1, a_j = \ldots = a_{r-2} = j, a_{r-1} = a_r = j/2$, and the total number of pairs of (p, q)-oscillators is $\frac{j(2r-j-1)}{2}$, see (2.199, 2.201). For $\bar{\lambda} = \omega_j, \bar{\mu} = 0$ we get a *non-degenerate* Lax matrix with z appearing on the entire diagonal:

$$\mathsf{T}_{\varpi_j[x]-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{2r,2r}) + O(1), \tag{2.202}$$

depending on the additional parameter $x \in \mathbb{C}$ (note that it is independent of the point $y \in \mathbb{P}^1$).

The normalized limit (2.178) of (2.202) as $x \to \infty$ recovers the Lax matrix corresponding to $\bar{\lambda} = 0, \bar{\mu} = \omega_j$, which is *degenerate* as it contains z only in the first j diagonal entries:

$$\mathsf{T}_{\varpi_j[\infty]-\varpi_0[y]} = z(E_{11} + \ldots + E_{jj}) + O(1) \tag{2.203}$$

and also satisfies:

$$\mathsf{T}_{\varpi_j[\infty]-\varpi_0[y]}(z)_{k,k} = \begin{cases} 1 & \text{if } j+1 \leq k \leq (j+1)' \\ 0 & \text{if } j' \leq k \leq 1' \end{cases}. \tag{2.204}$$

- <u>Case (2) for odd r</u> : $\bar{\lambda} + \bar{\mu} = \omega_{r-1} + \omega_r$.

In this case, we have $a_1 = 1, \ldots, a_{r-2} = r-2, a_{r-1} = a_r = \frac{r-1}{2}$, and the total number of pairs of (p,q)-oscillators is $\frac{r(r-1)}{2}$, see (2.199, 2.201).
For $\bar{\lambda} = \omega_{r-1} + \omega_r$, $\bar{\mu} = 0$ we get a *non-degenerate* Lax matrix with z on the entire diagonal:

$$\mathsf{T}_{\varpi_{r-1}[x_1]+\varpi_r[x_2]-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{2r,2r}) + O(1), \tag{2.205}$$

which depends on two additional parameters $x_1, x_2 \in \mathbb{C}$ (but is independent of $y \in \mathbb{P}^1$). The normalized limit (2.178) of (2.205) as $x_2 \to \infty$ recovers the Lax matrix corresponding to $\bar{\lambda} = \omega_{r-1}$, $\bar{\mu} = \omega_r$, which is *degenerate* as it contains z only in the first r diagonal entries:

$$\mathsf{T}_{\varpi_{r-1}[x_1]+\varpi_r[\infty]-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{rr}) + O(1) \tag{2.206}$$

and also satisfies:

$$\mathsf{T}_{\varpi_{r-1}[x_1]+\varpi_r[\infty]-\varpi_0[y]}(z)_{k,k} = 1 \qquad \text{for} \quad r' \leq k \leq 1'. \tag{2.207}$$

Likewise, the normalized limit of (2.205) as $x_1 \to \infty$ recovers the Lax matrix corresponding to $\bar{\lambda} = \omega_r$, $\bar{\mu} = \omega_{r-1}$, which is *degenerate* as it contains z only in r of its diagonal entries:

$$\mathsf{T}_{\varpi_r[x_2]+\varpi_{r-1}[\infty]-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{r-1,r-1} + E_{r+1,r+1}) + O(1) \tag{2.208}$$

and also satisfies:

$$\mathsf{T}_{\varpi_r[x_2]+\varpi_{r-1}[\infty]-\varpi_0[y]}(z)_{k,k} = 1 \qquad \text{for} \quad k \in \{r, r+2, r+3, \ldots, 2r\}. \tag{2.209}$$

Finally, the normalized limit of (2.206) as $x_1 \to \infty$, or equivalently of (2.208) as $x_2 \to \infty$, recovers the Lax matrix corresponding to $\bar{\lambda} = 0$, $\bar{\mu} = \omega_{r-1} + \omega_r$, which is even more *degenerate*:

$$\mathsf{T}_{\varpi_{r-1}[\infty]+\varpi_r[\infty]-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{r-1,r-1}) + O(1) \tag{2.210}$$

and also satisfying:

$$\mathsf{T}_{\varpi_{r-1}[\infty]+\varpi_r[\infty]-\varpi_0[y]}(z)_{k,k} = \begin{cases} 1 & \text{if } k = r, r' \\ 0 & \text{if } r' < k \leq 1' \end{cases}. \tag{2.211}$$

- Case (2) for even r : $\bar{\lambda} + \bar{\mu} = 2\omega_{r-1}$ or $2\omega_r$.

In this case, we have $a_1 = 1, \ldots, a_{r-2} = r-2$ and $\{a_{r-1}, a_r\} = \{\frac{r}{2}, \frac{r}{2} - 1\}$, and the total number of pairs of (p,q)-oscillators is again $\frac{r(r-1)}{2}$, see (2.199, 2.201).
For $\bar{\lambda} = 2\omega_{r-1}$, $\bar{\mu} = 0$ we get a *non-degenerate* Lax matrix with z on the entire diagonal:

$$\mathsf{T}_{\varpi_{r-1}([x_1]+[x_2])-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{2r,2r}) + O(1), \tag{2.212}$$

which depends, in a symmetric way, on additional parameters $x_1, x_2 \in \mathbb{C}$, but not on $y \in \mathbb{P}^1$. The normalized limit (2.178) of (2.212) as $x_2 \to \infty$ recovers the Lax matrix corresponding to $\bar{\lambda} = \omega_{r-1}$, $\bar{\mu} = \omega_{r-1}$, which is *degenerate* as it contains z only in half of its diagonal entries:

$$\mathsf{T}_{\varpi_{r-1}([x_1]+[\infty])-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{r-1,r-1} + E_{r+1,r+1}) + O(1) \tag{2.213}$$

and also satisfies:

$$\mathsf{T}_{\varpi_{r-1}([x_1]+[\infty])-\varpi_0[y]}(z)_{k,k} = 1 \qquad \text{for} \quad k \in \{r, r+2, r+3, \ldots, 2r\}. \tag{2.214}$$

Finally, the normalized limit (2.178) of (2.213) as $x_1 \to \infty$ recovers the Lax matrix corresponding to $\bar{\lambda} = 0$, $\bar{\mu} = 2\omega_{r-1}$, which also contains z in half of its diagonal entries:

$$\mathsf{T}_{2\varpi_{r-1}[\infty]-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{r-1,r-1} + E_{r+1,r+1}) + O(1), \tag{2.215}$$

but is more degenerate in the other diagonal entries:

$$\mathsf{T}_{2\varpi_{r-1}[\infty]-\varpi_0[y]}(z)_{k,k} = 0 \qquad \text{for} \quad k \in \{r, r+2, r+3, \ldots, 2r\}. \tag{2.216}$$

For $\bar{\lambda} = 2\omega_r$, $\bar{\mu} = 0$ we get a *non-degenerate* Lax matrix with z on the entire diagonal:

$$\mathsf{T}_{\varpi_r([x_1]+[x_2])-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{2r,2r}) + O(1), \tag{2.217}$$

which depends, in a symmetric way, on additional parameters $x_1, x_2 \in \mathbb{C}$, but not on $y \in \mathbb{P}^1$. The normalized limit (2.178) of (2.217) as $x_2 \to \infty$ recovers the Lax matrix corresponding to $\bar{\lambda} = \omega_r$, $\bar{\mu} = \omega_r$, which is *degenerate* as it contains z only in half of its diagonal entries:

$$\mathsf{T}_{\varpi_r([x_1]+[\infty])-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{rr}) + O(1) \tag{2.218}$$

and also satisfies:

$$\mathsf{T}_{\varpi_r([x_1]+[\infty])-\varpi_0[y]}(z)_{k,k} = 1 \qquad \text{for} \quad r' \leq k \leq 1'. \tag{2.219}$$

Finally, the normalized limit (2.178) of (2.218) as $x_1 \to \infty$ recovers the Lax matrix corresponding to $\bar{\lambda} = 0$, $\bar{\mu} = 2\omega_r$, which also contains z in half of its diagonal entries:

$$\mathsf{T}_{2\varpi_r[\infty]-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{rr}) + O(1) \tag{2.220}$$

but is more degenerate in the other diagonal entries:

$$\mathsf{T}_{2\varpi_r[\infty]-\varpi_0[y]}(z)_{k,k} = 0 \qquad \text{for} \quad r' \leq k \leq 1'. \tag{2.221}$$

We conclude this Subsection with the following important *unitarity* property of the above non-degenerate linear Lax matrices (recall the parameter $\kappa = r - 1$, see (2.44)), cf. [R2, (3.8)]:

Proposition 2.222. *(a) For any even $1 \leq \jmath \leq r-2$, the corresponding non-degenerate linear Lax matrix $L_\jmath(z) := \mathsf{T}_{\varpi_\jmath[x]-\varpi_0[y]}\left(z + x + \frac{\kappa-\jmath}{2}\right)$ is unitary:*

$$L_\jmath(z)L_\jmath(-z) = \left[\left(\frac{\kappa-\jmath}{2}\right)^2 - z^2\right] \mathrm{I}_N.$$

(b) Consider $D = \varpi_\imath[x_1] + \varpi_\jmath[x_2] - \varpi_0[y]$ with $\imath = \jmath \in \{r-1, r\}$ if r is even or $\{\imath, \jmath\} = \{r-1, r\}$ if r is odd. Then, the non-degenerate linear Lax matrix $L(z) := \mathsf{T}_D\left(z + \frac{x_1+x_2}{2}\right)$ is unitary:

$$L(z)L(-z) = \left[\left(\frac{x_1-x_2}{2}\right)^2 - z^2\right] \mathrm{I}_N.$$

Remark 2.223. We note that such unitarity property can be regarded as a consequence of the natural constraints that arise for a linear solution when inserted into the RTT relation (2.48), see [K, (18)].

Proof. (a) Combining Theorem 2.190 and Proposition 2.193 with the equalities

$$h_i^{(D)0} = -h_{i'}^{(D)0} - 2x + \jmath, \quad e_{i,j}^{(D)1} = -e_{j',i'}^{(D)1}, \quad f_{j,i}^{(D)1} = -f_{i',j'}^{(D)1},$$

due to (A.1, A.4, A.6), we obtain:

$$\mathsf{T}'_{\varpi_\jmath[x]-\varpi_0[y]}(z) = -\mathsf{T}_{\varpi_\jmath[x]-\varpi_0[y]}(-z + 2x - \jmath). \tag{2.224}$$

According to the crossing relation of Proposition 2.186, see formula (2.188), we also have:

$$\mathsf{T}_{\varpi_\jmath[x]-\varpi_0[y]}(z)\mathsf{T}'_{\varpi_\jmath[x]-\varpi_0[y]}(z - r + 1) = (z - x)(z - x - r + \jmath + 1)\mathrm{I}_N. \tag{2.225}$$

The result now follows by combining (2.224, 2.225).

(b) The proof is completely analogous to that of part (a) and is left to the interested reader (in particular, $h_i^{(D)0} = -h_{i'}^{(D)0} + r - 1 - x_1 - x_2$). □

2.3.5. Examples In this Subsection, we explain how the type D_r linear and quadratic Lax matrices recently constructed by the first author in [F] arise as particular examples of our general construction.

• Example 1 : Consider the following Λ^+-valued divisor on $\mathbb{P}^1$:

$$D = \begin{cases} \varpi_{r-1}[x] + \varpi_r[\infty] - \varpi_0[y] & \text{if } r \text{ is odd} \\ \varpi_r[x] + \varpi_r[\infty] - \varpi_0[y] & \text{if } r \text{ is even} \end{cases} \tag{2.226}$$

depending on $x \in \mathbb{C}$ (note that $\mathsf{T}_D(z)$ is independent of $y \in \mathbb{P}^1$), so that the total number of (p, q)-oscillators in the algebra $\mathcal{A}$ equals $a_1 + \ldots + a_r = \frac{r(r-1)}{2}$.

According to (2.206–2.207, 2.218–2.219), the corresponding normalized Lax matrix $\mathsf{T}_D(z)$ has the block form:

$$\mathsf{T}_D(z) = \begin{pmatrix} z\mathrm{I}_r + \mathsf{F} & \mathsf{B} \\ \mathsf{C} & \mathrm{I}_r \end{pmatrix}, \tag{2.227}$$

where $\mathsf{B}, \mathsf{C}, \mathsf{F}$ are z-independent $r \times r$ matrices. We have the following properties of B, C:

Lemma 2.228. (a) *The matrices* B, C *are skew-symmetric with respect to their antidiagonals:*

$$\mathsf{B}_{ij} = -\mathsf{B}_{r+1-j,r+1-i}, \qquad \mathsf{C}_{ij} = -\mathsf{C}_{r+1-j,r+1-i}.$$

In particular, $\mathsf{B}_{i,r+1-i} = \mathsf{C}_{i,r+1-i} = 0$ *for any* $1 \le i \le r$.

(b) *The matrix coefficients* $\{\mathsf{B}_{ij}\}_{i,j=1}^r$ *of the matrix* B *pairwise commute.*

(c) *The matrix coefficients* $\{\mathsf{C}_{ij}\}_{i,j=1}^r$ *of the matrix* C *pairwise commute.*

(d) *The commutation among the matrix coefficients of* B *and* C *is given by:*

$$[\mathsf{B}_{ij}, \mathsf{C}_{k\ell}] = \delta_{i,\ell}\delta_{j,k} - \delta_{i,r+1-k}\delta_{j,r+1-\ell}.$$

Proof. (a) According to Proposition 2.193, we have $\mathsf{B}_{ij} = e^{(D)1}_{i,r+j}$, $\mathsf{B}_{r+1-j,r+1-i} = e^{(D)1}_{r+1-j,2r+1-i}$. Combining this with $e^{(D)1}_{i,r+j} = -e^{(D)1}_{r+1-j,2r+1-i}$, due to (A.4), we obtain the desired equality $\mathsf{B}_{ij} = -\mathsf{B}_{r+1-j,r+1-i}$. The proof of $\mathsf{C}_{ij} = -\mathsf{C}_{r+1-j,r+1-i}$ is completely analogous.

(b, c) Those follow immediately from the ansatz (2.227) and the RTT relation (2.50) applied to the evaluation of $[\mathsf{B}_{ij}, \mathsf{B}_{k\ell}] = [t_{i,r+j}(z), t_{k,r+\ell}(w)]$ or $[\mathsf{C}_{ij}, \mathsf{C}_{k\ell}] = [t_{r+i,j}(z), t_{r+k,\ell}(w)]$.

(d) This also follows from the ansatz (2.227) and the RTT relation (2.50). Indeed, evaluating $[\mathsf{B}_{ij}, \mathsf{C}_{k\ell}] = [t_{i,r+j}(z), t_{r+k,\ell}(w)]$ via (2.50), the first summand is easily seen to equal $\delta_{i,\ell}\delta_{j,k}$, while computing the leading term of the second summand, we get $-\delta_{i,r+1-k}\delta_{j,r+1-\ell}$. □

It will be convenient to relabel the matrices B, C as $\bar{\mathbf{A}}, -\mathbf{A}$, respectively:

$$\mathsf{B} = \bar{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{a}}_{1,r'} & \cdots & \bar{\mathbf{a}}_{1,2'} & 0 \\ \vdots & \iddots & \iddots & -\bar{\mathbf{a}}_{1,2'} \\ \bar{\mathbf{a}}_{r-1,r'} & \iddots & \iddots & \vdots \\ 0 & -\bar{\mathbf{a}}_{r-1,r'} & \cdots & -\bar{\mathbf{a}}_{1,r'} \end{pmatrix},$$

$$-\mathsf{C} = \mathbf{A} = \begin{pmatrix} \mathbf{a}_{r',1} & \cdots & \mathbf{a}_{r',r-1} & 0 \\ \vdots & \iddots & \iddots & -\mathbf{a}_{r',r-1} \\ \mathbf{a}_{2',1} & \iddots & \iddots & \vdots \\ 0 & -\mathbf{a}_{2',1} & \cdots & -\mathbf{a}_{r',1} \end{pmatrix}, \tag{2.229}$$

with the matrix coefficients satisfying the following relations:

$$[\mathbf{a}_{i',j}, \bar{\mathbf{a}}_{k,\ell'}] = \delta_{i,\ell}\delta_{j,k}, \quad [\mathbf{a}_{i',j}, \mathbf{a}_{k',\ell}] = 0, \quad [\bar{\mathbf{a}}_{i,j'}, \bar{\mathbf{a}}_{k,\ell'}] = 0, \tag{2.230}$$

due to Lemma 2.228. Then, a tedious straightforward calculation (cf. [FPT, Theorem 2.133]) yields:

$$\mathsf{T}_D(z) = \left(\begin{array}{c|c} (z+x)\mathrm{I}_r - \bar{\mathbf{A}}\mathbf{A} & \bar{\mathbf{A}} \\ \hline -\mathbf{A} & \mathrm{I}_r \end{array}\right) \tag{2.231}$$

which can also be written in the following factorized form:

$$\mathsf{T}_D(z) = \left(\begin{array}{c|c} \mathrm{I}_r & \bar{\mathbf{A}} \\ \hline 0 & \mathrm{I}_r \end{array}\right) \left(\begin{array}{c|c} (z+x)\mathrm{I}_r & 0 \\ \hline 0 & \mathrm{I}_r \end{array}\right) \left(\begin{array}{c|c} \mathrm{I}_r & 0 \\ \hline -\mathbf{A} & \mathrm{I}_r \end{array}\right). \tag{2.232}$$

The type D_r Lax matrix of the form (2.231, 2.232) was recently discovered in [F, (4.3)].

• Example 2 : Consider the following Λ^+-valued divisor on $\mathbb{P}^1$:

$$D = \begin{cases} \varpi_{r-1}[x_1] + \varpi_r[x_2] - \varpi_0[y] & \text{if } r \text{ is odd} \\ \varpi_r[x_1] + \varpi_r[x_2] - \varpi_0[y] & \text{if } r \text{ is even} \end{cases} \tag{2.233}$$

depending on $x_1, x_2 \in \mathbb{C}$ (while $\mathsf{T}_D(z)$ does not depend of $y \in \mathbb{P}^1$), so that the total number of (p,q)-oscillators in the algebra $\mathcal{A}$ equals $a_1 + \ldots + a_r = \frac{r(r-1)}{2}$.

We expect that the normalized non-degenerate linear Lax matrix $\mathsf{T}_D(z)$, see (2.205, 2.217), is equivalent, up to a (nontrivial) canonical transformation, to the Lax matrix $\mathcal{L}(z)$ of [F, (5.4)] (cf. [R2, (3.6)]). The latter was defined via:

$$\mathcal{L}(z) = \left(\begin{array}{c|c} (z+x_1)\mathrm{I}_r - \bar{\mathbf{A}}\mathbf{A} & \bar{\mathbf{A}}(x_2 - x_1 + \mathbf{A}\bar{\mathbf{A}}) \\ \hline -\mathbf{A} & (z+x_2)\mathrm{I}_r + \mathbf{A}\bar{\mathbf{A}} \end{array}\right) \tag{2.234}$$

with the matrices $\bar{\mathbf{A}}, \mathbf{A}$ as in (2.229) encoding $\frac{r(r-1)}{2}$ pairs of oscillators (2.230), which can also be written in the following factorized form:

$$\mathcal{L}(z) = \left(\begin{array}{c|c} \mathrm{I}_r & \bar{\mathbf{A}} \\ \hline 0 & \mathrm{I}_r \end{array}\right) \left(\begin{array}{c|c} (z+x_1)\mathrm{I}_r & 0 \\ \hline -\mathbf{A} & (z+x_2)\mathrm{I}_r \end{array}\right) \left(\begin{array}{c|c} \mathrm{I}_r & -\bar{\mathbf{A}} \\ \hline 0 & \mathrm{I}_r \end{array}\right). \tag{2.235}$$

• Example 3 : Consider the following Λ^+-valued divisor on $\mathbb{P}^1$:

$$D = \varpi_1([x] + [\infty]) - \varpi_0([y_1] + [y_2]) \tag{2.236}$$

depending on $x \in \mathbb{C}$ (while $\mathsf{T}_D(z)$ does not depend of $y_1, y_2 \in \mathbb{P}^1$), so that the total number of (p,q)-oscillators in the algebra $\mathcal{A}$ equals $a_1 + \ldots + a_r = 2(r-1)$.

We expect that the normalized quadratic Lax matrix $\mathsf{T}_D(z)$ is equivalent, up to a (nontrivial) canonical transformation, to the Lax matrix $\mathsf{L}(z+x)$ of [F, (4.12)]. The latter was defined via:

$$\mathsf{L}(z) = \left(\begin{array}{c|c|c} z^2 + z(2 - \frac{N}{2} - \bar{\mathbf{w}}\mathbf{w}) + \frac{1}{4}\bar{\mathbf{w}}\mathbf{J}\bar{\mathbf{w}}^t\mathbf{w}^t\mathbf{J}\mathbf{w} & z\bar{\mathbf{w}} - \frac{1}{2}\bar{\mathbf{w}}\mathbf{J}\bar{\mathbf{w}}^t\mathbf{w}^t\mathbf{J} & -\frac{1}{2}\bar{\mathbf{w}}\mathbf{J}\bar{\mathbf{w}}^t \\ \hline -z\mathbf{w} + \frac{1}{2}\mathbf{J}\bar{\mathbf{w}}^t\mathbf{w}^t\mathbf{J}\mathbf{w} & z\mathrm{I} - \mathbf{J}\bar{\mathbf{w}}^t\mathbf{w}^t\mathbf{J} & -\mathbf{J}\bar{\mathbf{w}}^t \\ \hline -\frac{1}{2}\mathbf{w}^t\mathbf{J}\mathbf{w} & \mathbf{w}^t\mathbf{J} & 1 \end{array}\right) \tag{2.237}$$

with

$$\mathrm{I} = \mathrm{I}_{N-2} = \begin{pmatrix} 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \qquad \mathrm{J} = \mathrm{J}_{N-2} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & \iddots & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \tag{2.238}$$

and $\mathbf{w}, \bar{\mathbf{w}}$ encoding $N-2 = 2(r-1)$ pairs of oscillators:

$$\bar{\mathbf{w}} = (\bar{\mathbf{a}}_2, \ldots, \bar{\mathbf{a}}_r, \bar{\mathbf{a}}_{r'}, \ldots, \bar{\mathbf{a}}_{2'}), \qquad \mathbf{w} = (\mathbf{a}_2, \ldots, \mathbf{a}_r, \mathbf{a}_{r'}, \ldots, \mathbf{a}_{2'})^t, \tag{2.239}$$

so that

$$[\mathbf{a}_i, \bar{\mathbf{a}}_j] = \delta_{i,j}, \qquad [\mathbf{a}_i, \mathbf{a}_j] = 0, \qquad [\bar{\mathbf{a}}_i, \bar{\mathbf{a}}_j] = 0. \tag{2.240}$$

The matrix $\mathsf{L}(z)$ of (2.237) can also be written in the following factorized form, see [F, (4.15)]:

$$\mathsf{L}(z) = \left(\begin{array}{c|c|c} 1 & \bar{\mathbf{w}} & -\frac{1}{2}\bar{\mathbf{w}}\mathrm{J}\bar{\mathbf{w}}^t \\ \hline 0 & \mathrm{I} & -\mathrm{J}\bar{\mathbf{w}}^t \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|c|c} z(z-\frac{N}{2}+2) & 0 & 0 \\ \hline 0 & z\mathrm{I} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|c|c} 1 & 0 & 0 \\ \hline -\mathbf{w} & \mathrm{I} & 0 \\ \hline -\frac{1}{2}\mathbf{w}^t\mathrm{J}\mathbf{w} & \mathbf{w}^t\mathrm{J} & 1 \end{array}\right). \tag{2.241}$$

• Example 4 : Consider the following Λ^+-valued divisor on $\mathbb{P}^1$:

$$D = \varpi_1([x_1]+[x_2]) - \varpi_0([y_1]+[y_2]) \tag{2.242}$$

depending on $x_1, x_2 \in \mathbb{C}$ (while $\mathsf{T}_D(z)$ does not depend of $y_1, y_2 \in \mathbb{P}^1$), so that the total number of (p,q)-oscillators in the algebra $\mathcal{A}$ equals $a_1 + \ldots + a_r = 2(r-1)$.

We expect that the normalized quadratic Lax matrix $\mathsf{T}_D(z)$ is equivalent, up to a (non-trivial) canonical transformation, to the Lax matrix $\mathfrak{L}_{x_1,-x_2}(z+x_1)$ of [F, (5.36, 5.38)] (see Remark 4.37 where a relation to [R2, (3.11)] is discussed). The latter was defined via:

$$\mathfrak{L}_{x_1,x_2}(z) = \left(\begin{array}{c|c|c} 1 & \bar{\mathbf{w}} & -\frac{1}{2}\bar{\mathbf{w}}\mathrm{J}\bar{\mathbf{w}}^t \\ \hline 0 & \mathrm{I} & -\mathrm{J}\bar{\mathbf{w}}^t \\ \hline 0 & 0 & 1 \end{array}\right) \cdot D_{x_1,x_2}(z) \cdot \left(\begin{array}{c|c|c} 1 & -\bar{\mathbf{w}} & -\frac{1}{2}\bar{\mathbf{w}}\mathrm{J}\bar{\mathbf{w}}^t \\ \hline 0 & \mathrm{I} & \mathrm{J}\bar{\mathbf{w}}^t \\ \hline 0 & 0 & 1 \end{array}\right) \tag{2.243}$$

with I, J, $\mathbf{w}$, $\bar{\mathbf{w}}$ as in (2.238)–(2.240) and the middle factor explicitly given by:

$$D_{x_1,x_2}(z) = \left(\begin{array}{c|c|c} (z-x_1)(z-x_1-\frac{N}{2}+2) & 0 & 0 \\ \hline -\mathbf{w}(z-x_1) & (z-x_1)(z-x_2)\mathrm{I} & 0 \\ \hline -\frac{1}{2}\mathbf{w}^t\mathrm{J}\mathbf{w} & \mathbf{w}^t\mathrm{J}(z-x_2) & (z-x_2)(z-x_2-\frac{N}{2}+2) \end{array}\right).$$

We conclude this Subsection with the following observation:

Remark 2.244. We note that the degeneration phenomena observed in [F, (5.11, 5.42)]:

(1) degeneration of the Lax matrix (2.234) into the one of (2.231)
(2) degeneration of the Lax matrix (2.243) into the one of (2.237)

exactly agree with our general normalized limit construction (2.178).

3. Type C

The type C_r is completely similar to the type D_r, which we considered in details above. Thus, we'll be brief, only stating the key results and highlighting the few technical differences.

3.1. Classical (unshifted) story. We shall realize the simple positive roots $\{\alpha_i^\vee\}_{i=1}^r$ of the Lie algebra $\mathfrak{sp}_{2r}$ in $\bar{\Lambda}^\vee$ via:

$$\alpha_1^\vee = \epsilon_1^\vee - \epsilon_2^\vee,\ \alpha_2^\vee = \epsilon_2^\vee - \epsilon_3^\vee,\ \ldots,\ \alpha_{r-1}^\vee = \epsilon_{r-1}^\vee - \epsilon_r^\vee,\ \alpha_r^\vee = 2\epsilon_r^\vee. \tag{3.1}$$

The *Drinfeld Yangian* of $\mathfrak{sp}_{2r}$, denoted by $Y(\mathfrak{sp}_{2r})$, is defined similarly to $Y(\mathfrak{so}_{2r})$: it is generated by $\{\mathsf{E}_i^{(k)}, \mathsf{F}_i^{(k)}, \mathsf{H}_i^{(k)}\}_{1\le i\le r}^{k\ge 1}$ subject to the defining relations (2.2)–(2.9), with $\alpha_i^\vee$ of (3.1). The *extended Drinfeld Yangian* of $\mathfrak{sp}_{2r}$, denoted by $X(\mathfrak{sp}_{2r})$, is defined alike $X(\mathfrak{so}_{2r})$: it is generated by $\{E_i^{(k)}, F_i^{(k)}\}_{1\le i\le r}^{k\ge 1} \cup \{D_i^{(k)}\}_{1\le i\le r+1}^{k\ge 1}$ subject to (2.20)–(2.31) with the modification:

$$\begin{aligned}
[D_{r+1}(z), E_{r-1}(w)] &= \frac{D_{r+1}(z)(E_{r-1}(z-2) - E_{r-1}(w))}{z-w-2},\\
[D_{r+1}(z), F_{r-1}(w)] &= -\frac{(F_{r-1}(z-2) - F_{r-1}(w))D_{r+1}(z)}{z-w-2}.
\end{aligned} \tag{3.2}$$

The central elements $\{C_r^{(k)}\}_{k\ge 1}$ of $X(\mathfrak{sp}_{2r})$ are now defined via (cf. (2.34)):

$$C_r(z) = 1 + \sum_{k\ge 1} C_r^{(k)} z^{-k} := \prod_{i=1}^{r-1} \frac{D_i(z+i-r-2)}{D_i(z+i-r-1)} \cdot D_r(z-2) D_{r+1}(z). \tag{3.3}$$

Furthermore, a natural analogue of Lemma 2.41 holds with $\iota_0 \colon Y(\mathfrak{sp}_{2r}) \hookrightarrow X(\mathfrak{sp}_{2r})$ given by:

$$\begin{aligned}
&\mathsf{E}_i(z) \mapsto \begin{cases} E_i(z+\frac{i-1}{2}) & \text{if } i<r \\ E_r(z+\frac{r}{2}) & \text{if } i=r \end{cases}, \qquad \mathsf{F}_i(z) \mapsto \begin{cases} F_i(z+\frac{i-1}{2}) & \text{if } i<r \\ F_r(z+\frac{r}{2}) & \text{if } i=r \end{cases},\\
&\mathsf{H}_i(z) \mapsto \begin{cases} D_i(z+\frac{i-1}{2})^{-1} D_{i+1}(z+\frac{i-1}{2}) & \text{if } i<r \\ D_r(z+\frac{r}{2})^{-1} D_{r+1}(z+\frac{r}{2}) & \text{if } i=r \end{cases}.
\end{aligned} \tag{3.4}$$

The *extended RTT Yangian* of $\mathfrak{sp}_{2r}$, denoted by $X^{\mathrm{rtt}}(\mathfrak{sp}_{2r})$, is defined similarly to $X^{\mathrm{rtt}}(\mathfrak{so}_{2r})$: it is generated by $\{t_{ij}^{(k)}\}_{1\le i,j\le N}^{k\ge 1}$ $(N=2r)$ subject to the RTT relation (2.48) with the R-matrix $R(z)$ given by (2.45), but with the following modifications of $\kappa \in \mathbb{C}$ and $\mathrm{Q} \in \operatorname{End}\mathbb{C}^N \otimes \operatorname{End}\mathbb{C}^N$:

$$\kappa = r+1, \qquad \mathrm{Q} = \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j\, E_{ij} \otimes E_{i'j'} \quad \text{with} \quad \varepsilon_i = \begin{cases} 1 & \text{if } 1\le i\le r \\ -1 & \text{if } r' \le i \le 1' \end{cases}. \tag{3.5}$$

The *RTT Yangian* of $\mathfrak{sp}_{2r}$, denoted by $Y^{\mathrm{rtt}}(\mathfrak{sp}_{2r})$, is defined alike $Y^{\mathrm{rtt}}(\mathfrak{so}_{2r})$: it is the subalgebra of $X^{\mathrm{rtt}}(\mathfrak{sp}_{2r})$ consisting of the elements stable under the automorphisms (2.52). However, it can be also realized as a quotient of $X^{\mathrm{rtt}}(\mathfrak{sp}_{2r})$ as in (2.57), due to the natural analogue of (2.53), where the center $ZX^{\mathrm{rtt}}(\mathfrak{sp}_{2r})$ of $X^{\mathrm{rtt}}(\mathfrak{sp}_{2r})$ is explicitly described as a polynomial algebra in the coefficients $\{\mathsf{z}_N^{(k)}\}_{k\ge 1}$ of the series $\mathsf{z}_N(z) = 1 + \sum_{k\ge 1} \mathsf{z}_N^{(k)} z^{-k}$ determined from:

$$T'(z-\kappa)T(z) = T(z)T'(z-\kappa) = \mathsf{z}_N(z)\mathrm{I}_N, \tag{3.6}$$

where in the present setup the matrix transposition (2.56) should be redefined via:

$$(X')_{ij} = \varepsilon_i \varepsilon_j X_{j'i'} \quad \text{for any} \quad N\times N \text{ matrix } X. \tag{3.7}$$

In the notations of Subsection 2.1.3, the analogue of Theorem 2.62 still holds, explicitly:

$$\Upsilon_0\colon \quad E_k(z) \mapsto \begin{cases} e_{k,k+1}(z) & \text{if } k<r \\ \frac{e_{r,r+1}(z)}{2} & \text{if } k=r \end{cases}, \quad F_i(z) \mapsto f_{i+1,i}(z), \quad D_j(z) \mapsto h_j(z) \tag{3.8}$$

for all $i \leq r$, $j \leq r+1$. Hence, a natural analogue of Theorem 2.66 holds with $\Upsilon_0 \circ \iota_0$ given by:

$$\begin{gathered} \mathsf{E}_i(z) \mapsto \begin{cases} e_{i,i+1}(z+\frac{i-1}{2}) & \text{if } i<r \\ \frac{1}{2}e_{r,r+1}(z+\frac{r}{2}) & \text{if } i=r \end{cases}, \qquad \mathsf{F}_i(z) \mapsto \begin{cases} f_{i+1,i}(z+\frac{i-1}{2}) & \text{if } i<r \\ f_{r+1,r}(z+\frac{r}{2}) & \text{if } i=r \end{cases}, \\ \mathsf{H}_i(z) \mapsto \begin{cases} h_i(z+\frac{i-1}{2})^{-1}h_{i+1}(z+\frac{i-1}{2}) & \text{if } i<r \\ h_r(z+\frac{r}{2})^{-1}h_{r+1}(z+\frac{r}{2}) & \text{if } i=r \end{cases}. \end{gathered} \tag{3.9}$$

We note that our conventions are to those of [JLM1] as in type D_r, see Remark 2.68 for details.

Accordingly, the algebra $X^{\mathrm{rtt}}(\mathfrak{sp}_{2r})$ is generated by the coefficients of $\{h_j(z)\}_{j=1}^{r+1}$ as well as of:

$$e_i(z) = \sum_{k\geq 1} e_i^{(k)} z^{-k} := e_{i,i+1}(z), \quad f_i(z) = \sum_{k\geq 1} f_i^{(k)} z^{-k} := f_{i+1,i}(z), \quad 1 \leq i \leq r. \tag{3.10}$$

We shall now record the explicit formulas for all other entries of the matrices $F(z)$, $H(z)$, $E(z)$ from (2.58)–(2.61). The following result is essentially due to [JLM1][5]:

Lemma 3.11. (a) $h_{i'}(z) = \frac{1}{h_i(z+i-r-1)} \cdot \prod_{j=i+1}^{r-1} \frac{h_j(z+j-r-2)}{h_j(z+j-r-1)} \cdot h_r(z-2)h_{r+1}(z)$ *for* $1 \leq i \leq r-1$.
(b) $e_{(i+1)',i'}(z) = -e_i(z+i-r-1)$ *for* $1 \leq i \leq r-1$.
(c) $e_{i,j+1}(z) = -[e_{i,j}(z), e_j^{(1)}]$ *for* $1 \leq i < j \leq r-1$.
(d) $e_{i,j'}(z) = [e_{i,(j+1)'}(z), e_j^{(1)}]$ *for* $1 \leq i < j \leq r-1$.
(e) $e_{i,r'}(z) = -\frac{1}{2}[e_{i,r}(z), e_r^{(1)}]$ *for* $1 \leq i \leq r-1$.
(f) $e_{i',j'}(z) = [e_{i',(j+1)'}(z), e_j^{(1)}]$ *for* $1 \leq j \leq i-2 \leq r-2$.
(g) $e_{i,i'}(z) = [e_{i,(i+1)'}(z), e_i^{(1)}] - e_i(z)e_{i,(i+1)'}(z)$ *for* $1 \leq i \leq r-1$.
(h) $f_{i',(i+1)'}(z) = -f_i(z+i-r-1)$ *for* $1 \leq i \leq r-1$.
(i) $f_{j+1,i}(z) = -[f_j^{(1)}, f_{j,i}(z)]$ *for* $1 \leq i < j \leq r-1$.
(j) $f_{j',i}(z) = [f_j^{(1)}, f_{(j+1)',i}(z)]$ *for* $1 \leq i < j \leq r-1$.
(k) $f_{r',i}(z) = -\frac{1}{2}[f_r^{(1)}, f_{r,i}(z)]$ *for* $1 \leq i \leq r-1$.
(l) $f_{j',i'}(z) = [f_j^{(1)}, f_{(j+1)',i'}(z)]$ *for* $1 \leq j \leq i-2 \leq r-2$.
(m) $f_{i',i}(z) = [f_i^{(1)}, f_{(i+1)',i}(z)] - f_{(i+1)',i}(z)f_i(z)$ *for* $1 \leq i \leq r-1$.

The remaining matrix coefficients of $E(z)$ and $F(z)$ are recovered via the following analogues of Lemmas 2.80 and 2.97:

Lemma 3.12. (a) $e_{i,j'}(z) = [e_{i,(j+1)'}(z), e_j^{(1)}]$ *for* $1 \leq j \leq i-2 \leq r-2$.

[5] Note the missing summands in the equalities from parts (g, m) as stated in [JLM1].

(b) $e_{i+1,i'}(z) = [e_{i+1,(i+1)'}(z), e_i^{(1)}] + e_i(z)e_{i+1,(i+1)'}(z) - e_{i,(i+1)'}(z)$ *for* $1 \le i \le r-1$.
(c) $f_{j',i}(z) = [f_j^{(1)}, f_{(j+1)',i}(z)]$ *for* $1 \le j \le i-2 \le r-2$.
(d) $f_{i',i+1}(z) = [f_i^{(1)}, f_{(i+1)',i+1}(z)] + f_{(i+1)',i+1}(z)f_i(z) - f_{(i+1)',i}(z)$ *for* $1 \le i \le r-1$.

3.2. Shifted story. We shall use the same *extended* lattice $\Lambda^\vee$, but $\{\hat{\alpha}_i^\vee\}_{i=1}^r$ of $\Lambda^\vee$ are now defined via:

$$\hat{\alpha}_i^\vee = \epsilon_i^\vee - \epsilon_{i+1}^\vee \qquad \text{for} \quad 1 \le i \le r. \tag{3.13}$$

We shall also use the same notation for the dual lattice $\Lambda = \bigoplus_{j=1}^{r+1} \mathbb{Z}\epsilon_j = \bigoplus_{i=0}^{r} \mathbb{Z}\varpi_i$ with

$$\varpi_i = -\epsilon_{i+1} - \epsilon_{i+2} - \ldots - \epsilon_{r+1} \qquad \text{for} \quad 0 \le i \le r. \tag{3.14}$$

For $\mu \in \Lambda$, define $\underline{d} = \{d_j\}_{j=1}^{r+1} \in \mathbb{Z}^{r+1}$, $\underline{b} = \{b_i\}_{i=1}^r \in \mathbb{Z}^r$ via (2.101, 2.102); so that $b_i = d_i - d_{i+1}$ for all i.

The *shifted extended Drinfeld Yangian of* $\mathfrak{sp}_{2r}$, denoted by $X_\mu(\mathfrak{sp}_{2r})$, is defined similarly: it is generated by $\{E_i^{(k)}, F_i^{(k)}\}_{1\le i\le r}^{k\ge 1} \cup \{D_i^{(k_i)}\}_{1\le i\le r+1}^{k_i\ge d_i+1}$ subject to (2.20, 2.22–2.31, 2.104, 3.2). Like in Lemma 2.110, $X_\mu(\mathfrak{sp}_{2r})$ depends only on the image of μ under (2.109), up to an isomorphism.

For $\nu \in \bar{\Lambda}$, the *shifted Drinfeld Yangian of* $\mathfrak{sp}_{2r}$, denoted by $Y_\nu(\mathfrak{sp}_{2r})$, is defined likewise. It is related to $X_\mu(\mathfrak{sp}_{2r})$ via a natural analogue of Proposition 2.114 with $\iota_\mu \colon Y_{\bar{\mu}}(\mathfrak{sp}_{2r}) \hookrightarrow X_\mu(\mathfrak{sp}_{2r})$ determined by (3.4) and the central elements $\{C_r^{(k)}\}_{k\ge d_r+d_{r+1}+1}$ of $X_\mu(\mathfrak{sp}_{2r})$ defined via:

$$C_r(z) = z^{-d_r-d_{r+1}} + \sum_{k>d_r+d_{r+1}} C_r^{(k)} z^{-k} := \prod_{i=1}^{r-1} \frac{D_i(z+i-r-2)}{D_i(z+i-r-1)} \cdot D_r(z-2)D_{r+1}(z). \tag{3.15}$$

The natural analogues of Corollary 2.118 and Lemma 2.119 still hold in the present setup.

We shall use the same notations (2.121)–(2.124) for Λ-*valued divisors* D *on* $\mathbb{P}^1$, Λ^+-*valued outside* $\{\infty\} \in \mathbb{P}^1$. The simple coroots $\{\alpha_i\}_{i=1}^r \subset \bar{\Lambda}$ of $\mathfrak{sp}_{2r}$ are explicitly given by:

$$\alpha_1 = \epsilon_1 - \epsilon_2, \ \ldots, \ \alpha_{r-2} = \epsilon_{r-2} - \epsilon_{r-1}, \ \alpha_{r-1} = \epsilon_{r-1} - \epsilon_r, \ \alpha_r = \epsilon_r. \tag{3.16}$$

We also consider $\{\hat{\alpha}_i\}_{i=1}^r \subset \Lambda$, which are the "lifts" of $\{\alpha_i\}$ from (3.16) in the sense of (2.127):

$$\hat{\alpha}_1 = \epsilon_1 - \epsilon_2, \ \ldots, \ \hat{\alpha}_{r-2} = \epsilon_{r-2} - \epsilon_{r-1}, \ \hat{\alpha}_{r-1} = \epsilon_{r-1} - \epsilon_r + \epsilon_{r+1}, \ \hat{\alpha}_r = \epsilon_r - \epsilon_{r+1}. \tag{3.17}$$

From now on, we shall impose the following assumption on D (cf. (2.128)):

$$\textbf{Assumption}: \qquad \lambda + \mu = a_1\hat{\alpha}_1 + \ldots + a_r\hat{\alpha}_r \quad \text{with} \quad a_i \in \mathbb{N}. \tag{3.18}$$

The above coefficients a_i in (3.18) are explicitly given by:

$$a_i = (\epsilon_1^\vee + \ldots + \epsilon_i^\vee)(\lambda + \mu) \quad \text{ for } \quad 1 \le i \le r. \tag{3.19}$$

Thus, (3.18) is equivalent to $(\epsilon_r^\vee + \epsilon_{r+1}^\vee)(\lambda + \mu) = 0$ and $\sum_{k=1}^{i} \epsilon_k^\vee(\lambda + \mu) \in \mathbb{N}$ for all $1 \le i \le r$.

Consider the algebra $\mathcal{A}$ defined as in (2.131) with the following important modification:

$$[e^{\pm q_{i,k}}, p_{j,\ell}] = \mp \frac{(\alpha_i^\vee, \alpha_i^\vee)}{2} \delta_{i,j} \delta_{k,\ell} e^{\pm q_{i,k}}, \tag{3.20}$$

so that $[e^{\pm q_{r,k}}, p_{r,k}] = \mp 2 e^{\pm q_{r,k}}$. Then, as in Theorem 2.136, we have an algebra homomorphism

$$\Psi_D \colon X_{-\mu}(\mathfrak{sp}_{2r}) \longrightarrow \mathcal{A}, \tag{3.21}$$

determined by the following assignment (keeping the notations (2.133, 2.135)):

$$\begin{gathered}
E_i(z) \mapsto \begin{cases} \sum_{k=1}^{a_i} \frac{P_{i-1}(p_{i,k}-1)}{(z-p_{i,k}) P_{i,k}(p_{i,k})} e^{q_{i,k}} & \text{if } i < r \\ \sum_{k=1}^{a_r} \frac{P_{r-1}(p_{r,k}-1) P_{r-1}(p_{r,k}-2)}{2(z-p_{r,k}) P_{r,k}(p_{r,k})} e^{q_{r,k}} & \text{if } i = r \end{cases}, \\
F_i(z) \mapsto \begin{cases} -\sum_{k=1}^{a_i} \frac{Z_i(p_{i,k}+1) P_{i+1}(p_{i,k}+1)}{(z-p_{i,k}-1) P_{i,k}(p_{i,k})} e^{-q_{i,k}} & \text{if } i < r \\ -\sum_{k=1}^{a_r} \frac{Z_r(p_{r,k}+2)}{(z-p_{r,k}-2) P_{r,k}(p_{r,k})} e^{-q_{r,k}} & \text{if } i = r \end{cases}, \\
D_i(z) \mapsto \begin{cases} \frac{P_i(z)}{P_{i-1}(z-1)} \cdot \prod_{k=0}^{i-1} Z_k(z) & \text{if } i \le r \\ \frac{P_{r-1}(z-2)}{P_r(z-2)} \cdot \prod_{k=0}^{r} Z_k(z) & \text{if } i = r+1 \end{cases}.
\end{gathered} \tag{3.22}$$

The proof is analogous to that of Theorem 2.136 and is based on the explicit formula

$$\Psi_D(C_r(z)) = \prod_{i=0}^{r-1} \Big(Z_i(z) Z_i(z + i - r - 1) \Big) \cdot Z_r(z) \tag{3.23}$$

as well as the comparison to the homomorphisms of [NW]. Precisely, identifying $\mathcal{A}$ with $\tilde{\mathcal{A}}$ of *loc.cit.* and the points x_s with the parameters z_s of *loc.cit.* via:

$$p_{i,k} \leftrightarrow \begin{cases} w_{i,k} + \frac{i-1}{2} & \text{if } i < r \\ w_{r,k} + \frac{r}{2} & \text{if } i = r \end{cases}, \qquad e^{\pm q_{i,k}} \leftrightarrow \mathsf{u}_{i,k}^{\mp 1}, \qquad x_s \leftrightarrow \begin{cases} z_s + \frac{i_s}{2} & \text{if } 1 \le i_s < r \\ z_s + \frac{r+1}{2} & \text{if } i_s = r \end{cases},$$

the (restriction) composition $Y_{-\bar{\mu}}(\mathfrak{sp}_{2r}) \xrightarrow{\iota_{-\mu}} X_{-\mu}(\mathfrak{sp}_{2r}) \xrightarrow{\Psi_D} \mathcal{A}$ is given by the formulas (B.4) of Appendix B (applied to the type C_r Dynkin diagram with the arrows pointing $i \to i+1$ for $1 \le i < r$), which essentially coincide with the homomorphisms $\Phi_{-\bar{\mu}}^{\bar{\lambda}}$ of [NW, Theorem 5.4].

For $\mu \in \Lambda^+$, the *antidominantly shifted extended RTT Yangian of* $\mathfrak{sp}_{2r}$, denoted by $X_{-\mu}^{\mathrm{rtt}}(\mathfrak{sp}_{2r})$, is defined similarly to $X_{-\mu}^{\mathrm{rtt}}(\mathfrak{so}_{2r})$: it is generated by $\{t_{ij}^{(k)}\}_{1 \le i,j \le 2r}^{k \in \mathbb{Z}}$ subject to the RTT relation (2.48) and the restriction (2.150) on the matrix coefficients of the matrices $F(z)$, $H(z)$, $E(z)$ with $d_i' \in \mathbb{Z}$ defined as in (2.151). We note that $\mu \in \Lambda^+$ implies now the following inequalities:

$$d_1 \ge d_2 \ge \cdots \ge d_{r-1} \ge d_r \ge d_r' \ge d_{r-1}' \ge \cdots \ge d_1'. \tag{3.24}$$

One of our key results in the type C_r is the natural analogue of Theorem 2.156:

Theorem 3.25. *For any $\mu \in \Lambda^+$, the assignment* (3.8) *gives rise to the algebra isomorphism* $\Upsilon_{-\mu}\colon X_{-\mu}(\mathfrak{sp}_{2r}) \xrightarrow{\sim} X^{\mathrm{rtt}}_{-\mu}(\mathfrak{sp}_{2r})$.

Similarly to the type D_r, the assignment (2.160) gives rise to the coproduct homomorphisms

$$\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}\colon X^{\mathrm{rtt}}_{-\mu_1-\mu_2}(\mathfrak{sp}_{2r}) \longrightarrow X^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{sp}_{2r}) \otimes X^{\mathrm{rtt}}_{-\mu_2}(\mathfrak{sp}_{2r}) \qquad \forall\, \mu_1,\mu_2 \in \Lambda^+,$$

for any $\mu_1,\mu_2 \in \Lambda^+$, coassociative in the sense of Corollary 2.161. Evoking the isomorphism of Theorem 3.25 and the algebra embedding $\iota_\mu\colon Y_{\bar{\mu}}(\mathfrak{sp}_{2r}) \hookrightarrow X_\mu(\mathfrak{sp}_{2r})$, we obtain the coproduct homomorphisms

$$\Delta_{-\nu_1,-\nu_2}\colon Y_{-\nu_1-\nu_2}(\mathfrak{sp}_{2r}) \longrightarrow Y_{-\nu_1}(\mathfrak{sp}_{2r}) \otimes Y_{-\nu_2}(\mathfrak{sp}_{2r}) \tag{3.26}$$

for any $\nu_1,\nu_2 \in \bar{\Lambda}^+$. Explicitly, the homomorphism (3.26) is uniquely determined by the formulas (2.171) with the root generators $\{\mathsf{E}^{(1)}_{\gamma^\vee}, \mathsf{F}^{(1)}_{\gamma^\vee}\}_{\gamma^\vee \in \Delta^+}$ defined via:

$$\begin{aligned}
\mathsf{E}^{(1)}_{\epsilon^\vee_i-\epsilon^\vee_j} &= [\mathsf{E}^{(1)}_{j-1}, [\mathsf{E}^{(1)}_{j-2}, [\mathsf{E}^{(1)}_{j-3}, \cdots, [\mathsf{E}^{(1)}_{i+1}, \mathsf{E}^{(1)}_{i}]\cdots]]],\\
\mathsf{F}^{(1)}_{\epsilon^\vee_i-\epsilon^\vee_j} &= [[[\cdots[\mathsf{F}^{(1)}_{i}, \mathsf{F}^{(1)}_{i+1}], \cdots, \mathsf{F}^{(1)}_{j-3}], \mathsf{F}^{(1)}_{j-2}], \mathsf{F}^{(1)}_{j-1}],\\
&\qquad\qquad 1 \le i < j \le r
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
\mathsf{E}^{(1)}_{\epsilon^\vee_i+\epsilon^\vee_j} &= -[\cdots[[\mathsf{E}^{(1)}_{r-1}, [\mathsf{E}^{(1)}_{r-2}, [\mathsf{E}^{(1)}_{r-3}, \cdots, [\mathsf{E}^{(1)}_{i+1}, \mathsf{E}^{(1)}_{i}]\cdots]]], \mathsf{E}^{(1)}_{r}], \cdots, \mathsf{E}^{(1)}_{j}],\\
\mathsf{F}^{(1)}_{\epsilon^\vee_i+\epsilon^\vee_j} &= -2^{-\delta_{i,j}}[\mathsf{F}^{(1)}_{j}, \cdots, [\mathsf{F}^{(1)}_{r}, [[[\cdots[\mathsf{F}^{(1)}_{i}, \mathsf{F}^{(1)}_{i+1}], \cdots, \mathsf{F}^{(1)}_{r-3}], \mathsf{F}^{(1)}_{r-2}], \mathsf{F}^{(1)}_{r-1}]]\cdots],\\
&\qquad\qquad 1 \le i \le j \le r,
\end{aligned} \tag{3.28}$$

where $\Delta^+ = \left\{\epsilon^\vee_i - \epsilon^\vee_j\right\}_{1\le i<j\le r} \cup \left\{\epsilon^\vee_i + \epsilon^\vee_j\right\}_{1\le i\le j\le r}$ is the set of positive roots of $\mathfrak{sp}_{2r}$.

Remark 3.29. As our formulas (2.171) coincide with those of [FKPRW, Theorem 4.8], this provides a confirmative answer to the question raised in the end of [CGY, §8], in the type C_r.

3.3. Lax matrices. Similar to the type D_r, the proof of Theorem 3.25 goes through the faithfulness result of [W], see Theorem 2.183, and the construction of the Lax matrices $T_D(z)$. To this end, for any Λ^+-valued divisor D on $\mathbb{P}^1$ satisfying (3.18), we construct the matrix $T_D(z)$ via (2.175, 2.176) with the matrix coefficients $f^D_{j,i}(z), e^D_{i,j}(z), h^D_i(z)$ obtained from the explicit formulas (3.22) combined with Lemmas 3.11, 3.12. Using the "normalized limit" procedure (2.178), we conclude that Corollary 2.181 applies in the present setup. Combining this with Υ_0 being an isomorphism [JLM1], we conclude (as in Proposition 2.182) that $T_D(z)$ are Lax (of type C_r).

Similarly to Proposition 2.186, the matrix $T(z)$ (encoding all generators of $X^{\mathrm{rtt}}_{-\mu}(\mathfrak{sp}_{2r})$) still satisfies the *crossing* relation (2.187) with the central series $\mathsf{z}_N(z)$ defined via (cf. (2.185)):

$$\mathsf{z}_N(z) = \Upsilon_{-\mu}(C_r(z)) = \prod_{i=1}^{r-1} \frac{h_i(z+i-r-2)}{h_i(z+i-r-1)} \cdot h_r(z-2)h_{r+1}(z). \tag{3.30}$$

In Appendix B (see Theorem B.17, Lemma B.25), we use the *shuffle algebra* approach to derive the uniform formulas for the matrix coefficients $e^D_{i,j}(z)$, $f^D_{j,i}(z)$, which are rather inaccessible if derived iteratively via Lemmas 3.11, 3.12. This allows to prove the analogue of Theorem 2.190:

Theorem 3.31. *The Lax matrix* $\mathsf{T}_D(z) = \frac{T_D(z)}{Z_0(z)}$ *is regular in* z*, i.e.* $\mathsf{T}_D(z) \in \mathcal{A}[z] \otimes_{\mathbb{C}} \mathrm{End}\, \mathbb{C}^{2r}$.

Similar to type D_r, the result above provides a shortcut to the computation of the Lax matrices $T_D(z)$ defined, in general, as a product of three complicated matrices $F^D(z)$, $H^D(z)$, $E^D(z)$. In particular, the natural analogue of Proposition 2.193 holds. To this end, let us now describe all Λ^+-valued divisors D on $\mathbb{P}^1$ satisfying (3.18) such that $\deg_z \mathsf{T}_D(z) = 1$. Define $\lambda, \mu \in \Lambda^+$ via (2.122, 2.124), so that $\lambda + \mu = \sum_{j=0}^r b_j \varpi_j$ with $b_0 \in \mathbb{Z}$, $b_1, \ldots, b_r \in \mathbb{N}$. Then, the assumption (3.18) implies that the corresponding coefficients $a_i \in \mathbb{N}$ are related to b_j's via:

$$a_i = b_1 + 2b_2 + \ldots + (i-1)b_{i-1} + i(b_i + \ldots + b_{r-1}) + \frac{i}{2} b_r, \qquad 1 \le i \le r, \quad (3.32)$$

and $b_0 = -b_1 - \ldots - b_{r-1} - \frac{b_r}{2}$, which uniquely recovers b_0 in terms of $b_1, \ldots, b_r$ and forces b_r to be even. We also note that the total number of pairs of (p, q)-oscillators in $\mathcal{A}$ equals:

$$\sum_{i=1}^r a_i = \sum_{k=1}^{r-1} \frac{k(2r-k+1)}{2} b_k + \frac{r(r+1)}{4} b_r. \quad (3.33)$$

Combining the above formulas (3.32) with Proposition 2.193(a) in type C_r, we thus conclude that the normalized Lax matrix $\mathsf{T}_D(z)$ is linear only for the following configurations of b_i's:

(1) $b_0 = -1,\ b_j = 1,\ b_1 = \ldots = b_{j-1} = b_{j+1} = \ldots = b_r = 0$ for some $1 \le j \le r-1$,
(2) $b_0 = -1,\ b_1 = \ldots = b_{r-1} = 0,\ b_r = 2$.

As b_0 is uniquely determined via $b_1, \ldots, b_r$ and does not affect the Lax matrix $\mathsf{T}_D(z)$, we shall rather focus on the corresponding values of the dominant $\mathfrak{sp}_{2r}$-coweights $\bar{\lambda}, \bar{\mu} \in \bar{\Lambda}^+$.

• Case (1) : $\bar{\lambda} + \bar{\mu} = \omega_j$ for $1 \le j \le r-1$.
In this case, $a_1 = 1, \ldots, a_{j-1} = j-1, a_j = \ldots = a_r = j$, so that the total number of pairs of (p, q)-oscillators is $\frac{j(2r-j+1)}{2}$, see (3.32, 3.33). We obtain two Lax matrices: the *non-degenerate* one, depending on the additional parameter $x \in \mathbb{C}$ (but independent of the parameter $y \in \mathbb{P}^1$):

$$\mathsf{T}_{\varpi_j[x]-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{2r,2r}) + O(1), \quad (3.34)$$

and its normalized limit as $x \to \infty$, which is *degenerate* with z in the first j diagonal entries:

$$\mathsf{T}_{\varpi_j[\infty]-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{jj}) + O(1) \quad (3.35)$$

and also satisfying:

$$\mathsf{T}_{\varpi_j[\infty]-\varpi_0[y]}(z)_{k,k} = \begin{cases} 1 & \text{if } j+1 \le k \le (j+1)' \\ 0 & \text{if } j' \le k \le 1' \end{cases}.$$

• Case (2) : $\bar{\lambda} + \bar{\mu} = 2\omega_r$.
In this case, we have $a_1 = 1, \ldots, a_r = r$, and the total number of pairs of (p,q)-oscillators is $\frac{r(r+1)}{2}$, see (3.32, 3.33). We thus obtain three Lax matrices: the *non-degenerate* one, depending in a symmetric way on additional parameters $x_1, x_2 \in \mathbb{C}$ (but independent of $y \in \mathbb{P}^1$):

$$\mathsf{T}_{\varpi_r([x_1]+[x_2])-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{2r,2r}) + O(1), \tag{3.36}$$

its normalized limit as $x_2 \to \infty$, which is *degenerate* with z only in the first r diagonal entries:

$$\mathsf{T}_{\varpi_r[x_1]+\varpi_r[\infty]-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{rr}) + O(1), \tag{3.37}$$

and its further $x_1 \to \infty$ normalized limit, which also contains z only in r of its diagonal entries:

$$\mathsf{T}_{2\varpi_r[\infty]-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{rr}) + O(1). \tag{3.38}$$

The diagonal z-independent entries of these degenerate Lax matrices are explicitly given by:

$$\mathsf{T}_{\varpi_r[x_1]+\varpi_r[\infty]-\varpi_0[y]}(z)_{k,k} = 1, \quad \mathsf{T}_{2\varpi_r[\infty]-\varpi_0[y]}(z)_{k,k} = 0 \qquad \text{for} \quad r' \le k \le 1'. \tag{3.39}$$

Completely analogously to Proposition 2.222, we have the following *unitarity* property of the corresponding non-degenerate linear Lax matrices (recall the parameter $\kappa = r+1$, see (3.5)):

Proposition 3.40. *The non-degenerate Lax matrices* $L_\jmath(z) := \mathsf{T}_{\varpi_\jmath[x]-\varpi_0[y]}\left(z + x + \frac{\kappa-\jmath}{2}\right)$ *for* $1 \le \jmath < r$*, as well as* $L_r(z) := \mathsf{T}_{\varpi_r([x_1]+[x_2])-\varpi_0[\infty]}\left(z + \frac{x_1+x_2}{2}\right)$*, are unitary:*

$$L_\jmath(z)L_\jmath(-z) = \left[\left(\frac{\kappa-\jmath}{2}\right)^2 - z^2\right]\mathrm{I}_N, \qquad L_r(z)L_r(-z) = \left[\left(\frac{x_1-x_2}{2}\right)^2 - z^2\right]\mathrm{I}_N.$$

We conclude this Section by presenting a few interesting examples of the Lax matrices $\mathsf{T}_D(z)$.

• Example 1 : $D = \varpi_1[\infty] - \varpi_0[y]$ (note that $\mathsf{T}_D(z)$ is independent of $y \in \mathbb{P}^1$, as before).

In this case, $a_1 = \ldots = a_r = 1$. To simplify our notations, let us relabel $\{p_{i,1}, e^{\pm q_{i,1}}\}_{i=1}^r$ by $\{p_i, e^{\pm q_i}\}_{i=1}^r$, so that $[p_i, e^{q_j}] = \delta_{i,j} \cdot \begin{cases} e^{q_i} & \text{if } i < r \\ 2e^{q_r} & \text{if } i = r \end{cases}$. Then, we immediately find:

$$\mathsf{T}_D(z) = \begin{pmatrix} z - p_1 & * & * & \cdots & * & * \\ * & 1 & 0 & \cdots & 0 & 0 \\ * & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & 0 & 0 & \cdots & 1 & 0 \\ * & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \tag{3.41}$$

with the nontrivial entries, marked by $*$ above, explicitly given by:

$$\begin{aligned}
\mathsf{T}_D(z)_{1,j} &= (-1)^j e^{\sum_{k=1}^{j-1} q_k}, \\
\mathsf{T}_D(z)_{1,j'} &= (-1)^{j+1}(p_j - p_{j-1} - 1)e^{\sum_{k=1}^{r} q_k + \sum_{k=j}^{r-1} q_k}, \ 1 < j \le r, \\
\mathsf{T}_D(z)_{j,1} &= (-1)^j (p_j - p_{j-1} - 1)e^{-\sum_{k=1}^{j-1} q_k}, \\
\mathsf{T}_D(z)_{j',1} &= (-1)^j e^{-\sum_{k=1}^{r} q_k - \sum_{k=j}^{r-1} q_k}, \ 1 < j \le r, \\
\mathsf{T}_D(z)_{1,1'} &= e^{2\sum_{k=1}^{r-1} q_k + q_r}, \ \mathsf{T}_D(z)_{1',1} = -e^{-2\sum_{k=1}^{r-1} q_k - q_r}.
\end{aligned}$$

These entries may be written more invariantly as:

$$\begin{aligned}
&\mathsf{T}_D(z)_{1,j} = -\bar{\mathbf{a}}_1 \mathbf{a}_j, \quad \mathsf{T}_D(z)_{1,j'} = \bar{\mathbf{a}}_1 \bar{\mathbf{a}}_j, \quad 1 < j \le r, \\
&\mathsf{T}_D(z)_{j,1} = -\bar{\mathbf{a}}_1^{-1} \bar{\mathbf{a}}_j, \quad \mathsf{T}_D(z)_{j',1} = -\bar{\mathbf{a}}_1^{-1} \mathbf{a}_j, \quad 1 < j \le r, \\
&\mathsf{T}_D(z)_{1,1'} = \bar{\mathbf{a}}_1^2, \quad \mathsf{T}_D(z)_{1',1} = -\bar{\mathbf{a}}_1^{-2}, \quad \mathsf{T}_D(z)_{1,1} = z - 1 - \bar{\mathbf{a}}_1 \mathbf{a}_1,
\end{aligned} \tag{3.42}$$

where we used the following canonical transformation with ($[\mathbf{a}_i, \bar{\mathbf{a}}_j] = \delta_{i,j}$ and $[\mathbf{a}_i, \mathbf{a}_j] = 0 = [\bar{\mathbf{a}}_i, \bar{\mathbf{a}}_j]$):

$$\begin{aligned}
&\bar{\mathbf{a}}_1 = -e^{\sum_{k=1}^{r-1} q_k + \frac{1}{2} q_r}, \quad \bar{\mathbf{a}}_i = (-1)^i (p_i - p_{i-1} - 1)e^{\sum_{k=i}^{r-1} q_k + \frac{1}{2} q_r}, \quad 1 < i \le r, \\
&\mathbf{a}_1 = -p_1 e^{-\sum_{k=1}^{r-1} q_k - \frac{1}{2} q_r}, \quad \mathbf{a}_i = (-1)^i e^{-\sum_{k=i}^{r-1} q_k - \frac{1}{2} q_r}, \quad 1 < i \le r.
\end{aligned} \tag{3.43}$$

• *Example 2*: $D = \varpi_1[x] - \varpi_0[y]$ with $x \in \mathbb{C}$ ($\mathsf{T}_D(z)$ is independent of $y \in \mathbb{P}^1$, as before).

As in the previous example, we have $a_1 = \ldots = a_r = 1$, and we shall use $\{p_i, e^{\pm q_i}\}_{i=1}^r$ instead of $\{p_{i,1}, e^{\pm q_{i,1}}\}_{i=1}^r$. Then, the corresponding Lax matrix becomes:

$$\mathsf{T}_D(z) = (z - x - 1)\mathrm{I}_{2r} + (\bar{\mathbf{a}}_1, \ldots, \bar{\mathbf{a}}_r, \mathbf{a}_r, \ldots, \mathbf{a}_1)^t \cdot (-\mathbf{a}_1, \ldots, -\mathbf{a}_r, \bar{\mathbf{a}}_r, \ldots, \bar{\mathbf{a}}_1) \tag{3.44}$$

after the following canonical transformation ($[\mathbf{a}_i, \bar{\mathbf{a}}_j] = \delta_{i,j}$, $[\mathbf{a}_i, \mathbf{a}_j] = 0 = [\bar{\mathbf{a}}_i, \bar{\mathbf{a}}_j]$):

$$\begin{aligned}
&\bar{\mathbf{a}}_1 = -e^{\sum_{k=1}^{r-1} q_k + \frac{1}{2} q_r}, \quad \bar{\mathbf{a}}_i = (-1)^i (p_i - p_{i-1} - 1)e^{\sum_{k=i}^{r-1} q_k + \frac{1}{2} q_r}, \quad 1 < i \le r, \\
&\mathbf{a}_1 = -(p_1 - x) e^{-\sum_{k=1}^{r-1} q_k - \frac{1}{2} q_r}, \quad \mathbf{a}_i = (-1)^i e^{-\sum_{k=i}^{r-1} q_k - \frac{1}{2} q_r}, \quad 1 < i \le r.
\end{aligned} \tag{3.45}$$

The type C_r Lax matrix of the form (3.44) first appeared in [IKK, (4.34)].

Remark 3.46. In contrast to the natural "normalized limit" relation (2.178) in the (p, q)-oscillators, such construction in the "polynomial" $(\mathbf{a}, \bar{\mathbf{a}})$-oscillators is more involved. In particular, to recover the Lax matrix (3.42) from (3.44), one should first apply the canonical transformation $\mathbf{a}_1 \rightsquigarrow \mathbf{a}_1 - x\bar{\mathbf{a}}_1^{-1}$ (preserving all other $\mathbf{a}, \bar{\mathbf{a}}$-oscillators), and only afterwards consider the $x \to \infty$ limit of the product on the left with the diagonal factor $\mathrm{diag}(1, -x^{-1}, \ldots, -x^{-1}, x^{-2})$.

• Example 3 : $D = \varpi_r[x] + \varpi_r[\infty] - \varpi_0[y]$ with $x \in \mathbb{C}$ ($\mathsf{T}_D(z)$ is independent of $y \in \mathbb{P}^1$).

In this case, $a_1 = 1, \dots, a_r = r$. According to (3.37, 3.39), $\mathsf{T}_D(z)$ has the block form:

$$\mathsf{T}_D(z) = \begin{pmatrix} z\mathrm{I}_r + \mathsf{F} & \mathsf{B} \\ \mathsf{C} & \mathrm{I}_r \end{pmatrix}, \tag{3.47}$$

where $\mathsf{B}, \mathsf{C}, \mathsf{F}$ are z-independent $r \times r$ matrices. The following properties of B, C are established exactly as in Lemma 2.228:

Lemma 3.48. (a) *The matrices* B *and* C *are symmetric with respect to their antidiagonals:*

$$\mathsf{B}_{ij} = \mathsf{B}_{r+1-j,r+1-i}, \qquad \mathsf{C}_{ij} = \mathsf{C}_{r+1-j,r+1-i}.$$

(b) *The matrix coefficients* $\{\mathsf{B}_{ij}\}_{i,j=1}^{r}$ *of the matrix* B *pairwise commute.*
(c) *The matrix coefficients* $\{\mathsf{C}_{ij}\}_{i,j=1}^{r}$ *of the matrix* C *pairwise commute.*
(d) *The commutation among the matrix coefficients of* B *and* C *is given by:*

$$[\mathsf{B}_{ij}, \mathsf{C}_{k\ell}] = \delta_{i,\ell}\delta_{j,k} + \delta_{i,r+1-k}\delta_{j,r+1-\ell}.$$

It will be convenient to relabel the matrices B, C as $\bar{\mathbf{A}}, -\mathbf{A}$, respectively, cf. (2.229):

$$\mathsf{B} = \bar{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{a}}_{1,r'} & \cdots & \bar{\mathbf{a}}_{1,2'} & 2\bar{\mathbf{a}}_{1,1'} \\ \vdots & \iddots & \iddots & \bar{\mathbf{a}}_{1,2'} \\ \bar{\mathbf{a}}_{r-1,r'} & \iddots & \iddots & \vdots \\ 2\bar{\mathbf{a}}_{r,r'} & \bar{\mathbf{a}}_{r-1,r'} & \cdots & \bar{\mathbf{a}}_{1,r'} \end{pmatrix}, \quad -\mathsf{C} = \mathbf{A} = \begin{pmatrix} \mathbf{a}_{r',1} & \cdots & \mathbf{a}_{r',r-1} & \mathbf{a}_{r',r} \\ \vdots & \iddots & \iddots & \mathbf{a}_{r',r-1} \\ \mathbf{a}_{2',1} & \iddots & \iddots & \vdots \\ \mathbf{a}_{1',1} & \mathbf{a}_{2',1} & \cdots & \mathbf{a}_{r',1} \end{pmatrix}$$

with the matrix coefficients satisfying the following relations:

$$[\mathbf{a}_{i',j}, \bar{\mathbf{a}}_{k,\ell'}] = \delta_{i,\ell}\delta_{j,k}, \quad [\mathbf{a}_{i',j}, \mathbf{a}_{k',\ell}] = 0, \quad [\bar{\mathbf{a}}_{i,j'}, \bar{\mathbf{a}}_{k,\ell'}] = 0, \tag{3.49}$$

due to Lemma 3.48. Then, a tedious straightforward calculation yields:

$$\mathsf{T}_D(z) = \left(\begin{array}{c|c} (z+x)\mathrm{I}_r - \bar{\mathbf{A}}\mathbf{A} & \bar{\mathbf{A}} \\ \hline -\mathbf{A} & \mathrm{I}_r \end{array}\right). \tag{3.50}$$

We note that (3.50) is the exact $\mathfrak{sp}_{2r}$-analogue of the type D_r Lax matrix of (2.231, 2.232).

• <u>Example 4</u> : $D = \varpi_r([x_1] + [x_2]) - \varpi_0[y]$ with $x_1, x_2 \in \mathbb{C}$ ($\mathsf{T}_D(z)$ is independent of $y \in \mathbb{P}^1$).

Applying the arguments of [F] (see [FKT] for more details) to the Lax matrix of (3.50) and keeping the same notations for the matrices $\bar{\mathbf{A}}, \mathbf{A}$, we immediately obtain the following non-degenerate Lax matrix of type C_r (cf. [R2, (3.7)]):

$$\mathcal{L}(z) = \left(\begin{array}{c|c} (z+x_1)\mathrm{I}_r - \bar{\mathbf{A}}\mathbf{A} & \bar{\mathbf{A}}(x_2 - x_1 + \mathbf{A}\bar{\mathbf{A}}) \\ \hline -\mathbf{A} & (z+x_2)\mathrm{I}_r + \mathbf{A}\bar{\mathbf{A}} \end{array}\right) =$$
$$\left(\begin{array}{c|c} \mathrm{I}_r & \bar{\mathbf{A}} \\ \hline 0 & \mathrm{I}_r \end{array}\right) \left(\begin{array}{c|c} (z+x_1)\mathrm{I}_r & 0 \\ \hline -\mathbf{A} & (z+x_2)\mathrm{I}_r \end{array}\right) \left(\begin{array}{c|c} \mathrm{I}_r & -\bar{\mathbf{A}} \\ \hline 0 & \mathrm{I}_r \end{array}\right). \tag{3.51}$$

The type C_r Lax matrix of the form (3.51) was recently discovered in [KK, §6.2] and can be viewed as the exact $\mathfrak{sp}_{2r}$-analogue of the type D_r Lax matrix of (2.234, 2.235). We expect that $\mathcal{L}(z)$ is equivalent, up to a canonical transformation, to $\mathsf{T}_{\varpi_r([x_1]+[x_2])-\varpi_0[y]}(z)$.

4. Type B

The type B_r is also quite similar to the type D_r, which we considered in details above. Thus, we'll be brief, only stating the key results and highlighting the few technical differences.

4.1. Classical (unshifted) story. We shall realize the simple positive roots $\{\alpha_i^\vee\}_{i=1}^r$ of the Lie algebra $\mathfrak{so}_{2r+1}$ in $\bar{\Lambda}^\vee$ via:

$$\alpha_1^\vee = \epsilon_1^\vee - \epsilon_2^\vee,\ \alpha_2^\vee = \epsilon_2^\vee - \epsilon_3^\vee,\ \ldots,\ \alpha_{r-1}^\vee = \epsilon_{r-1}^\vee - \epsilon_r^\vee,\ \alpha_r^\vee = \epsilon_r^\vee. \tag{4.1}$$

The *Drinfeld Yangian* of $\mathfrak{so}_{2r+1}$, denoted by $Y(\mathfrak{so}_{2r+1})$, is defined similarly to $Y(\mathfrak{so}_{2r})$: it is generated by $\{\mathsf{E}_i^{(k)}, \mathsf{F}_i^{(k)}, \mathsf{H}_i^{(k)}\}_{1\le i\le r}^{k\ge 1}$ subject to the relations (2.2)–(2.9), with $\alpha_i^\vee$ of (4.1). The *extended Drinfeld Yangian* of $\mathfrak{so}_{2r+1}$, denoted by $X(\mathfrak{so}_{2r+1})$, is defined alike $X(\mathfrak{so}_{2r})$: it is generated by $\{E_i^{(k)}, F_i^{(k)}\}_{1\le i\le r}^{k\ge 1} \cup \{D_i^{(k)}\}_{1\le i\le r+1}^{k\ge 1}$ subject to (2.20)–(2.31) with the modification:

$$\begin{gathered}
[D_{r+1}(z), E_j(w)] = \begin{cases} -\frac{D_{r+1}(z)(E_r(z)-E_r(w))}{2(z-w)} + \frac{(E_r(z+1)-E_r(w))D_{r+1}(z)}{2(z-w+1)} & \text{if } j=r \\ 0 & \text{if } j<r \end{cases}, \\
[D_{r+1}(z), F_j(w)] = \begin{cases} \frac{D_{r+1}(z)(F_r(z)-F_r(w))}{2(z-w)} - \frac{(F_r(z+1)-F_r(w))D_{r+1}(z)}{2(z-w+1)} & \text{if } j=r \\ 0 & \text{if } j<r \end{cases}.
\end{gathered} \tag{4.2}$$

The central elements $\{C_r^{(k)}\}_{k\ge 1}$ of $X(\mathfrak{so}_{2r+1})$ are now defined via (cf. (2.34)):

$$C_r(z) = 1 + \sum_{k\ge 1} C_r^{(k)} z^{-k} := \prod_{i=1}^{r} \frac{D_i(z+i-r-\frac{1}{2})}{D_i(z+i-r+\frac{1}{2})} \cdot D_{r+1}(z) D_{r+1}(z+\tfrac{1}{2}). \tag{4.3}$$

Also, a natural analogue of Lemma 2.41 holds with $\iota_0 \colon Y(\mathfrak{so}_{2r+1}) \hookrightarrow X(\mathfrak{so}_{2r+1})$ defined via:

$$\begin{gathered}
\mathsf{E}_i(z) \mapsto E_i\left(z + \tfrac{i-1}{2}\right), \quad \mathsf{F}_i(z) \mapsto F_i\left(z + \tfrac{i-1}{2}\right), \\
\mathsf{H}_i(z) \mapsto D_i\left(z + \tfrac{i-1}{2}\right)^{-1} D_{i+1}\left(z + \tfrac{i-1}{2}\right), \quad \text{for any} \quad 1 \le i \le r.
\end{gathered} \tag{4.4}$$

Define N and κ in the present setup via:

$$N = 2r+1, \qquad \kappa = r - \tfrac{1}{2}. \tag{4.5}$$

The *extended RTT Yangian* of $\mathfrak{so}_{2r+1}$, denoted by $X^{\mathrm{rtt}}(\mathfrak{so}_{2r+1})$, is defined alike $X^{\mathrm{rtt}}(\mathfrak{so}_{2r})$: it is generated by $\{t_{ij}^{(k)}\}_{1\le i,j\le N}^{k\ge 1}$ subject to the RTT relation (2.48) with the R-matrix $R(z)$ given by (2.45). The *RTT Yangian* of $\mathfrak{so}_{2r+1}$, denoted by $Y^{\mathrm{rtt}}(\mathfrak{so}_{2r+1})$, is defined similarly to $Y^{\mathrm{rtt}}(\mathfrak{so}_{2r})$: it is the subalgebra of $X^{\mathrm{rtt}}(\mathfrak{so}_{2r+1})$ consisting of the elements stable under the automorphisms (2.52). However, it can be also realized as a quotient of $X^{\mathrm{rtt}}(\mathfrak{so}_{2r+1})$ as in (2.57), due to the natural analogue of (2.53), where the center $ZX^{\mathrm{rtt}}(\mathfrak{so}_{2r+1})$ of $X^{\mathrm{rtt}}(\mathfrak{so}_{2r+1})$ is explicitly described as a polynomial algebra in the coefficients $\{\mathsf{z}_N^{(k)}\}_{k\ge 1}$ of the series $\mathsf{z}_N(z) = 1 + \sum_{k\ge 1} \mathsf{z}_N^{(k)} z^{-k}$ determined from (keeping the notations (2.56)):

$$T'(z-\kappa)T(z) = T(z)T'(z-\kappa) = \mathsf{z}_N(z)\mathrm{I}_N. \tag{4.6}$$

In the notations of Subsection 2.1.3, the analogue of Theorem 2.62 still holds, explicitly:

$$\Upsilon_0\colon\quad E_i(z)\mapsto e_{i,i+1}(z),\quad F_i(z)\mapsto f_{i+1,i}(z),\quad D_j(z)\mapsto h_j(z) \tag{4.7}$$

for all $i \leq r$, $j \leq r+1$. Hence, a natural analogue of Theorem 2.66 holds with $\Upsilon_0\circ\iota_0$ given by:

$$\begin{aligned}&\mathsf{E}_i(z)\mapsto e_{i,i+1}(z+\tfrac{i-1}{2}),\quad \mathsf{F}_i(z)\mapsto f_{i+1,i}(z+\tfrac{i-1}{2}),\\ &\mathsf{H}_i(z)\mapsto h_i(z+\tfrac{i-1}{2})^{-1}h_{i+1}(z+\tfrac{i-1}{2}),\quad \text{for any}\quad 1\leq i\leq r.\end{aligned} \tag{4.8}$$

We note that our conventions are to those of [JLM1] as in type D_r, see Remark 2.68 for details.

Accordingly, $X^{\mathrm{rtt}}(\mathfrak{so}_{2r+1})$ is generated by the coefficients of $\{h_j(z)\}_{j=1}^{r+1}$ as well as of:

$$e_i(z)=\sum_{k\geq 1}e_i^{(k)}z^{-k}:=e_{i,i+1}(z),\quad f_i(z)=\sum_{k\geq 1}f_i^{(k)}z^{-k}:=f_{i+1,i}(z),\quad 1\leq i\leq r. \tag{4.9}$$

We shall now record the explicit formulas for all other entries of the matrices $F(z)$, $H(z)$, $E(z)$. The following result, the B type analogue of Lemmas 2.79 and 2.96, is essentially due to [JLM1]:

Lemma 4.10. (a) $h_{i'}(z)=\frac{1}{h_i(z+i-r+\frac{1}{2})}\cdot\prod_{j=i+1}^{r}\frac{h_j(z+j-r-\frac{1}{2})}{h_j(z+j-r+\frac{1}{2})}\cdot h_{r+1}(z)h_{r+1}(z+\frac{1}{2})$ *for* $1\leq i\leq r$.
(b) $e_{(i+1)',i'}(z)=-e_i(z+i-r+\frac{1}{2})$ *for* $1\leq i\leq r$.
(c) $e_{i,j+1}(z)=-[e_{i,j}(z),e_j^{(1)}]$ *for* $1\leq i<j\leq r$.
(d) $e_{i,j'}(z)=[e_{i,(j+1)'}(z),e_j^{(1)}]$ *for* $1\leq i<j\leq r$.
(e) $e_{i',j'}(z)=[e_{i',(j+1)'}(z),e_j^{(1)}]$ *for* $1\leq j\leq i-2\leq r-2$.
(f) $f_{i',(i+1)'}(z)=-f_i(z+i-r+\frac{1}{2})$ *for* $1\leq i\leq r$.
(g) $f_{j+1,i}(z)=-[f_j^{(1)},f_{j,i}(z)]$ *for* $1\leq i<j\leq r$.
(h) $f_{j',i}(z)=[f_j^{(1)},f_{(j+1)',i}(z)]$ *for* $1\leq i<j\leq r$.
(i) $f_{j',i'}(z)=[f_j^{(1)},f_{(j+1)',i'}(z)]$ *for* $1\leq j\leq i-2\leq r-2$.

The remaining matrix coefficients of $E(z)$ and $F(z)$ are recovered via the following analogues of Lemmas 2.80 and 2.97:

Lemma 4.11. (a) $e_{i,i'}(z)=[e_{i,(i+1)'}(z),e_i^{(1)}]-e_i(z)e_{i,(i+1)'}(z)$ *for* $1\leq i\leq r$.
(b) $e_{i+1,i'}(z)=[e_{i+1,(i+1)'}(z),e_i^{(1)}]+e_i(z)e_{i+1,(i+1)'}(z)-e_{i,(i+1)'}(z)$ *for* $1\leq i\leq r-1$.
(c) $e_{i,j'}(z)=[e_{i,(j+1)'}(z),e_j^{(1)}]$ *for* $1\leq j\leq i-2\leq r-1$.
(d) $f_{i',i}(z)=[f_i^{(1)},f_{(i+1)',i}(z)]-f_{(i+1)',i}(z)f_i(z)$ *for* $1\leq i\leq r$.
(e) $f_{i',i+1}(z)=[f_i^{(1)},f_{(i+1)',i+1}(z)]+f_{(i+1)',i+1}(z)f_i(z)-f_{(i+1)',i}(z)$ *for* $1\leq i\leq r-1$.
(f) $f_{j',i}(z)=[f_j^{(1)},f_{(j+1)',i}(z)]$ *for* $1\leq j\leq i-2\leq r-1$.

4.2. Shifted story. We shall use the same *extended* lattice $\Lambda^\vee$, but $\{\hat{\alpha}_i^\vee\}_{i=1}^r$ of $\Lambda^\vee$ are now defined via:

$$\hat{\alpha}_i^\vee = \epsilon_i^\vee - \epsilon_{i+1}^\vee \quad \text{for} \quad 1 \leq i \leq r. \tag{4.12}$$

We shall also use the same notation for the dual lattice $\Lambda = \bigoplus_{j=1}^{r+1} \mathbb{Z}\epsilon_j = \bigoplus_{i=0}^{r} \mathbb{Z}\varpi_i$ with

$$\varpi_i = -\epsilon_{i+1} - \epsilon_{i+2} - \ldots - \epsilon_{r+1} \quad \text{for} \quad 0 \leq i \leq r. \tag{4.13}$$

For $\mu \in \Lambda$, define $\underline{d} = \{d_j\}_{j=1}^{r+1} \in \mathbb{Z}^{r+1}$, $\underline{b} = \{b_i\}_{i=1}^r \in \mathbb{Z}^r$ via (2.101, 2.102); so that $b_i = d_i - d_{i+1}$ for all i.

The *shifted extended Drinfeld Yangian of* $\mathfrak{so}_{2r+1}$, denoted by $X_\mu(\mathfrak{so}_{2r+1})$, is defined similarly: it is generated by $\{E_i^{(k)}, F_i^{(k)}\}_{1\leq i\leq r}^{k\geq 1} \cup \{D_i^{(k_i)}\}_{1\leq i\leq r+1}^{k_i\geq d_i+1}$ subject to (2.20, 2.22–2.31, 2.104, 4.2). Up to an isomorphism, $X_\mu(\mathfrak{so}_{2r+1})$ depends only on the image of μ under (2.109), cf. Lemma 2.110.

For $\nu \in \bar{\Lambda}$, the *shifted Drinfeld Yangian of* $\mathfrak{so}_{2r+1}$, denoted by $Y_\nu(\mathfrak{so}_{2r+1})$, is defined likewise. We note that a natural analogue of Proposition 2.114 holds with the algebra embedding $\iota_\mu \colon Y_{\bar{\mu}}(\mathfrak{so}_{2r+1}) \hookrightarrow X_\mu(\mathfrak{so}_{2r+1})$ determined by (4.4) and the central elements $\{C_r^{(k)}\}_{k\geq 2d_{r+1}+1}$ of $X_\mu(\mathfrak{so}_{2r+1})$ defined via:

$$C_r(z) = z^{-2d_{r+1}} + \sum_{k>2d_{r+1}} C_r^{(k)} z^{-k} := \prod_{i=1}^{r} \frac{D_i(z+i-r-\frac{1}{2})}{D_i(z+i-r+\frac{1}{2})} \cdot D_{r+1}(z) D_{r+1}(z+\tfrac{1}{2}). \tag{4.14}$$

The natural analogues of Corollary 2.118 and Lemma 2.119 still hold in the present setup.

We shall use the same notations (2.121)–(2.124) for Λ-*valued divisors* D *on* $\mathbb{P}^1$, Λ^+-*valued outside* $\{\infty\} \in \mathbb{P}^1$. The simple coroots $\{\alpha_i\}_{i=1}^r \subset \bar{\Lambda}$ of $\mathfrak{so}_{2r+1}$ are explicitly given by:

$$\alpha_1 = \epsilon_1 - \epsilon_2, \ \ldots, \ \alpha_{r-2} = \epsilon_{r-2} - \epsilon_{r-1}, \ \alpha_{r-1} = \epsilon_{r-1} - \epsilon_r, \ \alpha_r = 2\epsilon_r. \tag{4.15}$$

We also consider $\{\hat{\alpha}_i\}_{i=1}^r \subset \Lambda$, which are the "lifts" of $\{\alpha_i\}$ from (4.15) in the sense of (2.127):

$$\hat{\alpha}_1 = \epsilon_1 - \epsilon_2, \ \ldots, \ \hat{\alpha}_{r-2} = \epsilon_{r-2} - \epsilon_{r-1}, \ \hat{\alpha}_{r-1} = \epsilon_{r-1} - \epsilon_r, \ \hat{\alpha}_r = 2\epsilon_r. \tag{4.16}$$

From now on, we shall impose the following assumption on D (cf. (2.128)):

$$\textbf{Assumption}: \quad \lambda + \mu = a_1\hat{\alpha}_1 + \ldots + a_r\hat{\alpha}_r \quad \text{with} \quad a_i \in \mathbb{N}. \tag{4.17}$$

The above coefficients a_i are explicitly given by:

$$\begin{aligned} a_i &= (\epsilon_1^\vee + \ldots + \epsilon_i^\vee)(\lambda + \mu) \quad \text{for} \quad 1 \leq i \leq r-1, \\ a_r &= \frac{(\epsilon_1^\vee + \ldots + \epsilon_r^\vee)(\lambda + \mu)}{2}. \end{aligned} \tag{4.18}$$

Thus, (4.17) is equivalent to $\epsilon_{r+1}^\vee(\lambda + \mu) = 0$ and $2^{-\delta_{i,r}} \sum_{k=1}^{i} \epsilon_k^\vee(\lambda + \mu) \in \mathbb{N}$ for all $1 \leq i \leq r$.

Consider the algebra $\mathcal{A}$ defined as in (2.131) but with the modified relation (3.20) in place, so that $[e^{\pm q_{r,k}}, p_{r,k}] = \mp\frac{1}{2}e^{\pm q_{r,k}}$. Then, as in Theorem 2.136, we have an algebra homomorphism

$$\Psi_D \colon X_{-\mu}(\mathfrak{so}_{2r+1}) \longrightarrow \mathcal{A}, \tag{4.19}$$

determined by the following assignment (keeping the notations (2.133, 2.135)):

$$E_i(z) \mapsto 2^{\delta_{i,r}} \cdot \sum_{k=1}^{a_i} \frac{P_{i-1}(p_{i,k}-1)}{(z-p_{i,k})P_{i,k}(p_{i,k})} e^{q_{i,k}},$$

$$F_i(z) \mapsto \begin{cases} -\sum_{k=1}^{a_i} \frac{Z_i(p_{i,k}+1)P_{i+1}(p_{i,k}+1)}{(z-p_{i,k}-1)P_{i,k}(p_{i,k})} e^{-q_{i,k}} & \text{if } i \le r-2 \\ -\sum_{k=1}^{a_{r-1}} \frac{Z_{r-1}(p_{r-1,k}+1)P_r(p_{r-1,k}+1)P_r(p_{r-1,k}+\frac{3}{2})}{(z-p_{r-1,k}-1)P_{r-1,k}(p_{r-1,k})} e^{-q_{r-1,k}} & \text{if } i = r-1 \\ -\sum_{k=1}^{a_r} \frac{Z_r(p_{r,k}+\frac{1}{2})}{(z-p_{r,k}-\frac{1}{2})P_{r,k}(p_{r,k})} e^{-q_{r,k}} & \text{if } i = r \end{cases}, \tag{4.20}$$

$$D_i(z) \mapsto \begin{cases} \frac{P_i(z)}{P_{i-1}(z-1)} \cdot \prod_{k=0}^{i-1} Z_k(z) & \text{if } i \le r-1 \\ \frac{P_r(z)P_r(z+\frac{1}{2})}{P_{r-1}(z-1)} \cdot \prod_{k=0}^{r-1} Z_k(z) & \text{if } i = r \\ \frac{P_r(z+\frac{1}{2})}{P_r(z-\frac{1}{2})} \cdot \prod_{k=0}^{r} Z_k(z) & \text{if } i = r+1 \end{cases}.$$

The proof is analogous to that of Theorem 2.136 and is based on the explicit formula

$$\Psi_D(C_r(z)) = \prod_{i=0}^{r} \Big(Z_i(z) Z_i(z+i-r+\tfrac{1}{2}) \Big) \tag{4.21}$$

as well as the comparison to the homomorphisms of [NW]. Precisely, identifying $\mathcal{A}$ with $\tilde{\mathcal{A}}$ of *loc.cit.* and the points x_s with the parameters z_s of *loc.cit.* via:

$$p_{i,k} \leftrightarrow w_{i,k} + \frac{i-1}{2}, \qquad e^{\pm q_{i,k}} \leftrightarrow \mathsf{u}_{i,k}^{\mp 1}, \qquad x_s \leftrightarrow z_s + \frac{i_s}{2},$$

the (restriction) composition $Y_{-\bar{\mu}}(\mathfrak{so}_{2r+1}) \xrightarrow{\iota_{-\mu}} X_{-\mu}(\mathfrak{so}_{2r+1}) \xrightarrow{\Psi_D} \mathcal{A}$ is given by the formulas (B.4) of Appendix B (applied to the type B_r Dynkin diagram with the arrows pointing $i \to i+1$ for $1 \le i < r$), which essentially coincide with the homomorphisms $\Phi_{-\bar{\mu}}^{\bar{\lambda}}$ of [NW].

The *antidominantly shifted extended RTT Yangian of* $\mathfrak{so}_{2r+1}$, denoted by $X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r+1})$ (with $\mu \in \Lambda^+$), is defined similarly to $X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r})$: it is generated by $\{t_{ij}^{(k)}\}_{1 \le i,j \le 2r+1}^{k \in \mathbb{Z}}$ subject to the RTT relation (2.48) and the restriction (2.150) on the matrix coefficients of the matrices $F(z)$, $H(z)$, $E(z)$ with $d'_i \in \mathbb{Z}$ defined in the present setup via:

$$d'_i := 2d_{r+1} - d_i \qquad \text{for} \quad 1 \le i \le r. \tag{4.22}$$

We note that $\mu \in \Lambda^+$ implies now the following inequalities:

$$d_1 \ge d_2 \ge \cdots \ge d_{r-1} \ge d_r \ge d_{r+1} \ge d'_r \ge d'_{r-1} \ge \cdots \ge d'_1. \tag{4.23}$$

One of our key results in the type B_r is the natural analogue of Theorem 2.156:

Theorem 4.24. *For any $\mu \in \Lambda^+$, the assignment (4.7) gives rise to the algebra isomorphism $\Upsilon_{-\mu}\colon X_{-\mu}(\mathfrak{so}_{2r+1}) \xrightarrow{\sim} X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r+1})$.*

Similarly to the type D_r, the assignment (2.160) gives rise to the coproduct homomorphisms

$$\Delta^{\mathrm{rtt}}_{-\mu_1,-\mu_2}\colon X^{\mathrm{rtt}}_{-\mu_1-\mu_2}(\mathfrak{so}_{2r+1}) \longrightarrow X^{\mathrm{rtt}}_{-\mu_1}(\mathfrak{so}_{2r+1}) \otimes X^{\mathrm{rtt}}_{-\mu_2}(\mathfrak{so}_{2r+1}) \qquad \forall\, \mu_1, \mu_2 \in \Lambda^+,$$

coassociative in the sense of Corollary 2.161. Evoking the isomorphism of Theorem 4.24 and the algebra embedding $\iota_\mu\colon Y_{\bar{\mu}}(\mathfrak{so}_{2r+1}) \hookrightarrow X_\mu(\mathfrak{so}_{2r+1})$, we obtain the coproduct homomorphisms

$$\Delta_{-\nu_1,-\nu_2}\colon Y_{-\nu_1-\nu_2}(\mathfrak{so}_{2r+1}) \longrightarrow Y_{-\nu_1}(\mathfrak{so}_{2r+1}) \otimes Y_{-\nu_2}(\mathfrak{so}_{2r+1}) \tag{4.25}$$

for any $\nu_1, \nu_2 \in \bar{\Lambda}^+$. Explicitly, the homomorphism (4.25) is uniquely determined by the formulas (2.171) with the root generators $\{\mathsf{E}^{(1)}_{\gamma^\vee}, \mathsf{F}^{(1)}_{\gamma^\vee}\}_{\gamma\in\Delta^+}$ defined via:

$$\begin{aligned}
&\mathsf{E}^{(1)}_{\epsilon_i^\vee-\epsilon_j^\vee} = [\mathsf{E}^{(1)}_{j-1}, [\mathsf{E}^{(1)}_{j-2}, [\mathsf{E}^{(1)}_{j-3}, \cdots, [\mathsf{E}^{(1)}_{i+1}, \mathsf{E}^{(1)}_{i}]\cdots]]],\\
&\mathsf{F}^{(1)}_{\epsilon_i^\vee-\epsilon_j^\vee} = [[[\cdots[\mathsf{F}^{(1)}_{i}, \mathsf{F}^{(1)}_{i+1}], \cdots, \mathsf{F}^{(1)}_{j-3}], \mathsf{F}^{(1)}_{j-2}], \mathsf{F}^{(1)}_{j-1}],\\
&\mathsf{E}^{(1)}_{\epsilon_i^\vee} = [\mathsf{E}^{(1)}_{r}, [\mathsf{E}^{(1)}_{r-1}, [\mathsf{E}^{(1)}_{r-2}, \cdots, [\mathsf{E}^{(1)}_{i+1}, \mathsf{E}^{(1)}_{i}]\cdots]]],\\
&\mathsf{F}^{(1)}_{\epsilon_i^\vee} = [[[\cdots[\mathsf{F}^{(1)}_{i}, \mathsf{F}^{(1)}_{i+1}], \cdots, \mathsf{F}^{(1)}_{r-2}], \mathsf{F}^{(1)}_{r-1}], \mathsf{F}^{(1)}_{r}],\\
&\mathsf{E}^{(1)}_{\epsilon_i^\vee+\epsilon_j^\vee} = [\cdots[[\mathsf{E}^{(1)}_{r}, [\mathsf{E}^{(1)}_{r-1}, [\mathsf{E}^{(1)}_{r-2}, \cdots, [\mathsf{E}^{(1)}_{i+1}, \mathsf{E}^{(1)}_{i}]\cdots]]], \mathsf{E}^{(1)}_{r}], \cdots, \mathsf{E}^{(1)}_{j}],\\
&\mathsf{F}^{(1)}_{\epsilon_i^\vee+\epsilon_j^\vee} = [\mathsf{F}^{(1)}_{j}, \cdots, [\mathsf{F}^{(1)}_{r}, [[[\cdots[\mathsf{F}^{(1)}_{i}, \mathsf{F}^{(1)}_{i+1}], \cdots, \mathsf{F}^{(1)}_{r-2}], \mathsf{F}^{(1)}_{r-1}], \mathsf{F}^{(1)}_{r}]]\cdots],\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le i < j \le r,
\end{aligned} \tag{4.26}$$

where $\Delta^+ = \left\{\epsilon_i^\vee \pm \epsilon_j^\vee\right\}_{1\le i<j\le r} \cup \left\{\epsilon_i^\vee\right\}_{1\le i\le r}$ is the set of positive roots of $\mathfrak{so}_{2r+1}$.

Remark 4.27. We note that the last formula of (2.165) holds with the following update of (2.166): following update of the formula (2.166):

$$\tilde{\epsilon}_j^\vee = \epsilon_j^\vee \quad \text{for} \quad j \le r, \qquad \tilde{\epsilon}^\vee_{r+1} = 0. \tag{4.28}$$

To this end, let us point out that the $j = r + 1$ case of the last formula of (2.165) is due to the equalities $e^{(1)}_{r+1,r+i} = -e^{(1)}_{r+2-i,r+1}$ and $f^{(1)}_{r+i,r+1} = -f^{(1)}_{r+1,r+2-i}$ which follow from (B.30).

Remark 4.29. As our formulas (2.171) coincide with those of [FKPRW, Theorem 4.8], this provides a confirmative answer to the question raised in the end of [CGY, §8], in the type B_r.

4.3. Lax matrices. Similar to type D_r, the proof of Theorem 4.24 goes through the faithfulness result of [W], see Theorem 2.183, and the construction of the Lax matrices $T_D(z)$. To this end, for any Λ^+-valued divisor D on $\mathbb{P}^1$ satisfying (4.17), we construct the matrix $T_D(z)$ via (2.175, 2.176) with the matrix coefficients $f^D_{j,i}(z), e^D_{i,j}(z), h^D_i(z)$ obtained from the explicit formulas (4.20) combined with Lemmas 4.10, 4.11. Using the same "normalized limit" procedure (2.178), we conclude that Corollary 2.181 applies in the present setup. Combining this with Υ_0 being an isomorphism [JLM1], we conclude as in Proposition 2.182 that $T_D(z)$ are Lax (of type B_r).

Similarly to Proposition 2.186, the matrix $T(z)$ (encoding all generators of $X^{\mathrm{rtt}}_{-\mu}(\mathfrak{so}_{2r+1})$) still satisfies the *crossing* relation (2.187) with the central series $\mathsf{z}_N(z)$ defined via (cf. (2.185)):

$$\mathsf{z}_N(z) = \Upsilon_{-\mu}(C_r(z)) = \prod_{i=1}^{r} \frac{h_i(z+i-r-\frac{1}{2})}{h_i(z+i-r+\frac{1}{2})} \cdot h_{r+1}(z)h_{r+1}(z+\tfrac{1}{2}). \tag{4.30}$$

In Appendix B (see Theorem B.17, Lemma B.27), we use the *shuffle algebra* approach to derive the uniform formulas for the matrix coefficients $e^D_{i,j}(z), f^D_{j,i}(z)$, which are rather inaccessible if derived iteratively via Lemmas 4.10, 4.11. This allows to prove the analogue of Theorem 2.190:

Theorem 4.31. *The Lax matrix* $\mathsf{T}_D(z) = \frac{T_D(z)}{Z_0(z)}$ *is regular, i.e.* $\mathsf{T}_D(z) \in \mathcal{A}[z] \otimes_{\mathbb{C}} \mathrm{End}\, \mathbb{C}^{2r+1}$.

Similar to type D_r, the result above provides a shortcut to the computation of the Lax matrices $T_D(z)$ defined, in general, as a product of three complicated matrices $F^D(z), H^D(z), E^D(z)$. In particular, the natural analogue of Proposition 2.193 holds. To this end, let us now describe all Λ^+-valued divisors D on $\mathbb{P}^1$ satisfying (4.17) such that $\deg_z \mathsf{T}_D(z) = 1$. Define $\lambda, \mu \in \Lambda^+$ via (2.122, 2.124), so that $\lambda + \mu = \sum_{j=0}^{r} b_j \varpi_j$ with $b_0 \in \mathbb{Z},\ b_1, \ldots, b_r \in \mathbb{N}$. Then, the assumption (4.17) implies that the corresponding coefficients $a_i \in \mathbb{N}$ are related to b_j's via:

$$\begin{aligned} a_i &= b_1 + 2b_2 + \ldots + (i-1)b_{i-1} + i(b_i + \ldots + b_r), \qquad 1 \le i \le r-1, \\ a_r &= \frac{1}{2}\Big(b_1 + 2b_2 + \ldots + (r-1)b_{r-1} + rb_r\Big), \end{aligned} \tag{4.32}$$

as well as $b_0 = -b_1 - \ldots - b_{r-1} - b_r$, which uniquely recovers b_0 in terms of $b_1, \ldots, b_r$. We also note that the total number of pairs of (p, q)-oscillators in the algebra $\mathcal{A}$ equals:

$$\sum_{i=1}^{r} a_i = \sum_{k=1}^{r} \frac{k(2r-k)}{2} b_k.$$

Combining the above formulas (4.32) with Proposition 2.193(a) in type B_r, we thus conclude that the normalized Lax matrix $\mathsf{T}_D(z)$ is linear only for the following configurations of b_i's:

- $b_0 = -1,\ b_j = 1,\ b_1 = \ldots = b_{j-1} = b_{j+1} = \ldots = b_r = 0$ for an even $1 \le j \le r$.

As b_0 is uniquely determined via $b_1, \ldots, b_r$ and does not affect the Lax matrix $\mathsf{T}_D(z)$, we shall rather focus on the corresponding values of the dominant $\mathfrak{so}_{2r+1}$-coweights $\bar{\lambda}, \bar{\mu} \in \bar{\Lambda}^+$.

In the above case of $\bar{\lambda} + \bar{\mu} = \omega_j$, we have $a_1 = 1, \ldots, a_{j-1} = j-1, a_j = \ldots = a_{r-1} = j, a_r = \frac{j}{2}$, and the total of $\frac{j(2r-j)}{2}$ pairs of (p,q)-oscillators. We obtain two Lax matrices: the *non-degenerate* one, depending on the additional parameter $x \in \mathbb{C}$ (but independent of $y \in \mathbb{P}^1$):

$$\mathsf{T}_{\varpi_j[x]-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{2r+1,2r+1}) + O(1), \tag{4.33}$$

and its normalized limit as $x \to \infty$, which is *degenerate* with z in the first j diagonal entries:

$$\mathsf{T}_{\varpi_j[\infty]-\varpi_0[y]}(z) = z(E_{11} + \ldots + E_{jj}) + O(1) \tag{4.34}$$

and also satisfying:

$$\mathsf{T}_{\varpi_j[\infty]-\varpi_0[y]}(z)_{k,k} = \begin{cases} 1 & \text{if } j+1 \le k \le (j+1)' \\ 0 & \text{if } j' \le k \le 1' \end{cases}.$$

Completely analogously to Proposition 2.222, we have the following *unitarity* property of the corresponding non-degenerate linear Lax matrices (recall the parameter $\kappa = r - \frac{1}{2}$, see (4.5)):

Proposition 4.35. *For any even* $1 \le \jmath \le r$*, the corresponding linear non-degenerate Lax matrix* $L_\jmath(z) := \mathsf{T}_{\varpi_\jmath[x]-\varpi_0[y]}\left(z + x + \frac{\kappa - \jmath}{2}\right)$ *is unitary:*

$$L_\jmath(z) L_\jmath(-z) = \left[\left(\frac{\kappa - \jmath}{2}\right)^2 - z^2\right] \mathrm{I}_N.$$

Motivated by the Examples 3 and 4 in type D_r, we expect that $\mathsf{T}_{\varpi_1([x_1]+[x_2])-\varpi_0([y_1]+[y_2])}(z)$ and its normalized limit $\mathsf{T}_{\varpi_1([x]+[\infty])-\varpi_0([y_1]+[y_2])}(z)$ are equivalent, up to canonical transformations, to the type B_r quadratic Lax matrices given by the formulas (2.243) and (2.237), respectively, with I, J of (2.238) and $\bar{\mathbf{w}}, \mathbf{w}$ encoding $N - 2 = 2r - 1$ pairs of oscillators, cf. (2.239):

$$\bar{\mathbf{w}} = (\bar{\mathbf{a}}_2, \ldots, \bar{\mathbf{a}}_r, \bar{\mathbf{a}}_{r+1}, \bar{\mathbf{a}}_{r'}, \ldots, \bar{\mathbf{a}}_{2'}), \qquad \mathbf{w} = (\mathbf{a}_2, \ldots, \mathbf{a}_r, \mathbf{a}_{r+1}, \mathbf{a}_{r'}, \ldots, \mathbf{a}_{2'})^t. \tag{4.36}$$

Remark 4.37. Let us define $\mathfrak{L}_{x_1,x_2}(z)$ via (2.243) with I, J as in (2.238) and $\bar{\mathbf{w}}, \mathbf{w}$ as in (4.36). Consider the expansion of the Lax matrix

$$L_{x_1,x_2}(z) = \mathfrak{L}_{x_1,x_2}(z + a) = z^2 + z M_{x_1,x_2} + G_{x_1,x_2} \tag{4.38}$$

with the shift a of the spectral parameter given by:

$$a = \frac{x_1 + x_2 - 1}{2}. \tag{4.39}$$

Using the equalities

$$\mathbf{w}^t \bar{\mathbf{w}}^t = \bar{\mathbf{w}}\mathbf{w} + N - 2, \qquad [\bar{\mathbf{w}} \mathrm{J} \bar{\mathbf{w}}^t, \mathbf{a}_i] = -2\bar{\mathbf{a}}_{i'},$$

cf. (4.36), it is straightforward to see from (2.243) that the linear term in (4.38) reads:

$$M_{x_1,x_2} = \left(\begin{array}{c|c|c} -x_1+x_2-\frac{N}{2}+1-\bar{\mathbf{w}}\mathbf{w} & M_{[12]} & 0 \\ \hline -\mathbf{w} & \mathbf{w}\bar{\mathbf{w}}-\mathrm{J}\bar{\mathbf{w}}^t\mathbf{w}^t\mathrm{J}-\mathrm{I} & M_{[23]} \\ \hline 0 & \mathbf{w}^t\mathrm{J} & x_1-x_2+\frac{N}{2}-1+\bar{\mathbf{w}}\mathbf{w} \end{array}\right)$$

with the row $M_{[12]}$ and the column $M_{[23]}$ explicitly given by:

$$M_{[12]} = \left(x_1 - x_2 + \frac{N}{2} - 2 + \bar{\mathbf{w}}\mathbf{w}\right)\bar{\mathbf{w}} - \frac{1}{2}\bar{\mathbf{w}}\mathrm{J}\bar{\mathbf{w}}^t\mathbf{w}^t\mathrm{J}, \tag{4.40}$$

$$M_{[23]} = -\left(x_1 - x_2 + \frac{N}{2} - 2 + \bar{\mathbf{w}}\mathbf{w}\right)\mathrm{J}\bar{\mathbf{w}}^t + \frac{1}{2}\bar{\mathbf{w}}\mathrm{J}\bar{\mathbf{w}}^t \cdot \mathbf{w}. \tag{4.41}$$

It is easily seen from the formulas above that the matrix coefficients $M_{ij} = (M_{x_1,x_2})_{i,j}$ satisfy

$$M_{ij} = -M_{j'i'} \tag{4.42}$$

as well as obey the following commutation relations:

$$[M_{ij}, M_{k\ell}] = \delta_{i,\ell}M_{kj} - \delta_{j',\ell}M_{ki'} - \delta_{i,k'}M_{\ell' j} + \delta_{j,k}M_{\ell' i'}. \tag{4.43}$$

Finally, we can show by direct computation that M_{x_1,x_2} satisfies the characteristic identity:

$$(M_{x_1,x_2}+1)(2M_{x_1,x_2}+N+2x_1-2x_2-2)(2M_{x_1,x_2}+N-2x_1+2x_2-2) = 0, \tag{4.44}$$

while the free term G_{x_1,x_2} in (4.38) is expressed via the linear term M_{x_1,x_2} as follows:

$$G_{x_1,x_2} = \frac{1}{2}M^2_{x_1,x_2} + \frac{1}{4}(N-2)M_{x_1,x_2} + \frac{1}{4}\left(N-3-(x_1-x_2)^2\right)\mathrm{I}_N. \tag{4.45}$$

Let us further introduce the parameter m via:

$$x_1 - x_2 = 1 - m - \frac{N}{2} \tag{4.46}$$

so that the characteristic identity (4.44) for $M = M_{x_1,x_2}$ becomes

$$(M-m)(M+N+m-2)(M+1) = 0, \tag{4.47}$$

thus exactly coinciding with [R2, (3.9)]. Furthermore, after an additional shift in the spectral parameter, the Lax matrix $L(z) = L_{x_1,x_2}(z)$ can be written as (taking (4.45) into an account):

$$L\left(z+\frac{N-2}{4}\right) = z\left(z+\frac{N-2}{2}\right) + zM + \frac{1}{2}\left(M^2 + (N-2)M - \frac{m(m+N-2)-N+3}{2}\right)$$

which coincides with [R2, (3.11)]. However, our oscillator realisation differs from that of [R2].

5. Further Directions

5.1. Trigonometric version. The constructions and results of the present paper admit natural trigonometric counterparts. To this end, recall the *shifted quantum affine algebras* $U_{\nu^+,\nu^-}(L\mathfrak{g})$, introduced in [FT1, §5], which are associative $\mathbb{C}(\boldsymbol{v})$-algebras depending on a pair of shifts $\nu^+,\nu^- \in \bar{\Lambda}$. Based on and generalizing the isomorphism between the new Drinfeld and the RTT realizations of extended quantum affine algebras in the classical types B_r, C_r, D_r recently established in [JLM2,JLM3], it turns out that the *shifted extended quantum affine algebras* $U^{\mathrm{ext}}_{-\mu^+,-\mu^-}(L\mathfrak{g})$ with $\mu^+,\mu^- \in \Lambda^+$ admit the RTT realization $U^{\mathrm{ext}}_{-\mu^+,-\mu^-}(L\mathfrak{g}) \xrightarrow{\sim} U^{\mathrm{rtt,ext}}_{-\mu^+,-\mu^-}(L\mathfrak{g})$ alike (1.5). This can be viewed as a natural generalization of [FPT, Theorem 3.51] for type A_r and shall be addressed elsewhere.

As an immediate corollary, we obtain the following two important structures:

- coproduct homomorphisms

$$\Delta_{\nu_1^+,\nu_1^-,\nu_2^+,\nu_2^-} : U_{\nu_1^++\nu_2^+,\nu_1^-+\nu_2^-}(L\mathfrak{g}) \longrightarrow U_{\nu_1^+,\nu_1^-}(L\mathfrak{g}) \otimes U_{\nu_2^+,\nu_2^-}(L\mathfrak{g})$$

- $\mathbb{Z}[\boldsymbol{v},\boldsymbol{v}^{-1}]$ integral forms $\mathfrak{U}_{\nu^+,\nu^-}(L\mathfrak{g}) \subset U_{\nu^+,\nu^-}(L\mathfrak{g})$ compatible with $\Delta_{\nu_1^+,\nu_1^-,\nu_2^+,\nu_2^-}$

for classical types B_r, C_r, D_r, generalizing the only known case type A_r of [FT1,FT2].

Combining the above RTT realization with [FT1, Theorem 7.1], one obtains trigonometric Lax matrices $T_D^{\mathrm{trig}}(z)$ which can be degenerated to $T_{D+D|_0([\infty]-[0])}(z)$, cf. [FPT, Proposition 3.94].

5.2. Integrable systems. As yet another important application of our key isomorphism (1.5) and its aforementioned trigonometric version, the RTT presentation provides (cf. [MM]) interesting algebraic quantum integrable systems that appear on the corresponding quantized (K-theoretic) Coulomb branches of $4d$ supersymmetric $\mathcal{N}=2$ quiver gauge theories, cf. [NP,NPS] and [BFNa,BFNb]. To this end, note that the $\mathbb{C}[\hbar]$-version of the homomorphisms (B.3) factor through the quantized Coulomb branches [NW, Theorem 5.6], cf. [FT1, Theorem 8.5] in the trigonometric case.

Let us also note that $T_D(z)T'_D(-z)$ satisfy the reflection equation [GR, (4.1)], thus giving rise to shifted versions of reflection algebras (aka twisted extended Yangians [GR, Theorem 4.2]) of types B, C, D. We expect the latter to be related to integrable systems with boundary.

5.3. Polynomial solutions and Q-operators. As mentioned in the introduction (with more details provided in Subsections 2.3.5, 3.3, 4.3), some of the simplest examples of our Lax matrices $\mathsf{T}_D(z)$ are equivalent (up to highly nontrivial canonical transformations) to the *polynomial* (as they take values in non-localized oscillator algebras) Lax matrices constructed quite recently in the physics literature. A very interesting question is to understand which of our Lax matrices $\mathsf{T}_D(z)$ can be transformed (up to canonical transformations) to the polynomial ones. We note that one of the advantages of our construction is a natural limit procedure (2.178) which becomes highly nontrivial for the polynomial Lax matrices, see e.g. Remark 3.46. However, the polynomial Lax matrices have an obvious advantage of allowing to take traces, thus leading to Q-operators as discussed below.

As outlined in [F], the polynomial solutions for D_r-type can be used to construct Q-operators following [BLZ,BFLMS]. The corresponding QQ-system for D_r-type spin chains has been recently proposed in [FFK], see also [ESV] for a different approach. We remark that only a subset of the Q-operators is constructed in [F], namely those corresponding to the end nodes of the D_r Dynkin diagram for which the evaluation map does exist. While the remaining Q-operators are determined by the QQ-system, a direct construction for those is not known. The asymptotic behaviour of the Q-operators in the spectral parameter can be extracted from the algebraic Bethe ansatz, cf. [R1], and solutions with the appropriate asymptotic behaviour are obtained in the Table 1 below, suggesting that a construction from our Lax matrices may be possible. However, the Lax matrices $\mathsf{T}_D(z)$ of the present paper are not polynomial in the oscillators and a trace prescription remains to be found.

The situation for B_r and C_r types is similar. The Lax matrices for Q-operators corresponding to the nodes of the Dynkin diagram where the evaluation map exists are presented here in the polynomial form, see [FKT] for more details. For the non-polynomial (in oscillators) solutions $\mathsf{T}_D(z)$ of the present paper we are in the same position as for D_r-type discussed above. The expected asymptotic behaviour of the corresponding Q-operators is spelled out in Tables 2 and 3 below. A study of the QQ-system for the spin chains of type B_r and C_r is outstanding.

Table 1. Solutions of D_r-type with the expected asymptotic behavior and number of oscillator pairs for Q-operator at the node i

Q_i	$\vec{a}=(a_1,\dots,a_r)$	$\vec{b}=(b_1,\dots,b_r)$	$\vec{Z}=(\deg Z_1,\dots,\deg Z_r)$	#osc.
$1\le i\le r-2$	$(\underbrace{2,4,6,\dots,2i}_{i},\underbrace{2i,\dots,2i}_{r-i-2},i,i)$	$(\underbrace{0,\dots,0,2}_{i},\underbrace{0,\dots,0}_{r-i})$	$(\underbrace{0,\dots,0,1}_{i},\underbrace{0,\dots,0}_{r-i})$	$i(2r-i-1)$
$i=r-1$				
even r	$(1,2,3,\dots,r-2,\frac{r}{2},\frac{r}{2}-1)$	$(0,\dots,0,2,0)$	$(0,\dots,0,1,0)$	$\frac{r(r-1)}{2}$
odd r	$(1,2,3,\dots,r-2,\frac{r-1}{2},\frac{r-1}{2})$	$(0,\dots,0,1,1)$		
$i=r$				
even r	$(1,2,3,\dots,r-2,\frac{r}{2}-1,\frac{r}{2})$	$(0,\dots,0,0,2)$	$(0,\dots,0,0,1)$	$\frac{r(r-1)}{2}$
odd r	$(1,2,3,\dots,r-2,\frac{r-1}{2},\frac{r-1}{2})$	$(0,\dots,0,1,1)$		

Table 2. Solutions of C_r-type with the expected asymptotic behavior and number of oscillator pairs for Q-operator at the node i

Q_i	$\vec{a}=(a_1,\dots,a_r)$	$\vec{b}=(b_1,\dots,b_r)$	$\vec{Z}=(\deg Z_1,\dots,\deg Z_r)$	#osc.
$1\le i<r$	$(\underbrace{2,4,6,\dots,2i}_{i},\underbrace{2i,\dots,2i}_{r-i})$	$(\underbrace{0,\dots,0,2}_{i},\underbrace{0,\dots,0}_{r-i})$	$(\underbrace{0,\dots,0,1}_{i},\underbrace{0,\dots,0}_{r-i})$	$i(2r-i+1)$
$i=r$	$(1,2,3,\dots,r)$	$(0,\dots,0,2)$	$(0,\dots,0,1)$	$\frac{r(r+1)}{2}$

Table 3. Solutions of B_r-type with the expected asymptotic behavior and number of oscillator pairs for Q-operator at the node i

Q_i	$\vec{a}=(a_1,\dots,a_r)$	$\vec{b}=(b_1,\dots,b_r)$	$\vec{Z}=(\deg Z_1,\dots,\deg Z_r)$	#osc.
$1\le i<r$	$(\underbrace{2,4,6,\dots,2i}_{i},\underbrace{2i,\dots,2i,i}_{r-i-1})$	$(\underbrace{0,\dots,0,2}_{i},\underbrace{0,\dots,0}_{r-i})$	$(\underbrace{0,\dots,0,1}_{i},\underbrace{0,\dots,0}_{r-i})$	$i(2r-i)$
$i=r$	$(2,4,6,\dots,2(r-1),r)$	$(0,\dots,0,2)$	$(0,\dots,0,1)$	r^2

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Appendix A. Explicit formulas in type D

In this Appendix, we record the explicit formulas for the matrices $F^D(z)$, $H^D(z)$, $E^D(z)$ the product of which recovers the Lax matrices $T_D(z)$ in type D_r, see (2.175, 2.176). All proofs are straightforward and are based on Lemmas 2.77, 2.79, 2.80, 2.96, 2.97 of Subsection 2.1.4.

• Matrix $H^D(z)$ explicitly.

$$
h_i^D(z) = \begin{cases} \frac{P_i(z)}{P_{i-1}(z-1)} \cdot \prod_{k=0}^{i-1} Z_k(z) & \text{if } i \leq r-2 \\ \frac{P_{r-1}(z)P_r(z)}{P_{r-2}(z-1)} \cdot \prod_{k=0}^{r-2} Z_k(z) & \text{if } i = r-1 \\ \frac{P_r(z)}{P_{r-1}(z-1)} \cdot \prod_{k=0}^{r-1} Z_k(z) & \text{if } i = r \end{cases}
$$
$$
h_{i'}^D(z) = \begin{cases} \frac{P_{i-1}(z+i-r)}{P_i(z+i-r)} \cdot \prod_{k=0}^{r} Z_k(z) \prod_{k=i}^{r-2} Z_k(z+k-r+1) & \text{if } i \leq r-2 \\ \frac{P_{r-2}(z-1)}{P_{r-1}(z-1)P_r(z-1)} \cdot \prod_{k=0}^{r} Z_k(z) & \text{if } i = r-1 \\ \frac{P_{r-1}(z)}{P_r(z-1)} \cdot \prod_{k=0}^{r-2} Z_k(z) \cdot Z_r(z) & \text{if } i = r \end{cases} \tag{A.1}
$$

• Matrix $E^D(z)$ explicitly.

For $1 \leq i < j \leq r-2$, we get:

$$
e_{i,j}^D(z) = (-1)^{j-i-1} \sum_{\substack{1 \leq k_i \leq a_i \\ \cdots \\ 1 \leq k_{j-1} \leq a_{j-1}}} \frac{P_{i-1}(p_{i,k_i}-1) \cdot \prod_{s=i}^{j-2} P_{s,k_s}(p_{s+1,k_{s+1}}-1)}{(z-p_{i,k_i}) \prod_{s=i}^{j-1} P_{s,k_s}(p_{s,k_s})} \cdot e^{\sum_{s=i}^{j-1} q_{s,k_s}}
$$

For $1 \leq i \leq r-2$, we get:

$$e^D_{i,r-1}(z) = (-1)^{r-i} \sum_{\substack{1 \leq k_i \leq a_i \\ \cdots \\ 1 \leq k_{r-2} \leq a_{r-2}}} \frac{P_{i-1}(p_{i,k_i}-1) \cdot \prod_{s=i}^{r-3} P_{s,k_s}(p_{s+1,k_{s+1}}-1) \cdot P_r(p_{r-2,k_{r-2}})}{(z-p_{i,k_i}) \prod_{s=i}^{r-2} P_{s,k_s}(p_{s,k_s})} \times e^{\sum_{s=i}^{r-2} q_{s,k_s}}$$

For $1 \leq i \leq r-2$, we get:

$$e^D_{i,r}(z) = (-1)^{r-i-1} \sum_{\substack{1 \leq k_i \leq a_i \\ \cdots \\ 1 \leq k_{r-1} \leq a_{r-1}}} \frac{P_{i-1}(p_{i,k_i}-1) \cdot \prod_{s=i}^{r-2} P_{s,k_s}(p_{s+1,k_{s+1}}-1) \cdot P_r(p_{r-2,k_{r-2}})}{(z-p_{i,k_i}) \prod_{s=i}^{r-1} P_{s,k_s}(p_{s,k_s})} \times e^{\sum_{s=i}^{r-1} q_{s,k_s}}$$

while $e^D_{r-1,r}(z)$ is given by the same formula (with $i = r-1$) but with $P_r(p_{r-2,k_{r-2}})$ omitted.

For $1 \leq i < j \leq r-2$, we get:

$$e^D_{j',i'}(z) = (-1)^{j-i} \sum_{\substack{1 \leq k_i \leq a_i \\ \cdots \\ 1 \leq k_{j-1} \leq a_{j-1}}} \frac{P_{i-1}(p_{i,k_i}-1) \cdot \prod_{s=i}^{j-2} P_{s,k_s}(p_{s+1,k_{s+1}}-1)}{(z-p_{j-1,k_{j-1}}+j-r) \prod_{s=i}^{j-1} P_{s,k_s}(p_{s,k_s})} \cdot e^{\sum_{s=i}^{j-1} q_{s,k_s}}$$

For $1 \leq i \leq r-2$, we get:

$$e^D_{(r-1)',i'}(z) = (-1)^{r-i+1} \sum_{\substack{1 \leq k_i \leq a_i \\ \cdots \\ 1 \leq k_{r-2} \leq a_{r-2}}} \frac{P_{i-1}(p_{i,k_i}-1) \cdot \prod_{s=i}^{r-3} P_{s,k_s}(p_{s+1,k_{s+1}}-1) \cdot P_r(p_{r-2,k_{r-2}})}{(z-p_{r-2,k_{r-2}}-1) \prod_{s=i}^{r-2} P_{s,k_s}(p_{s,k_s})} \times e^{\sum_{s=i}^{r-2} q_{s,k_s}}$$

For $1 \leq i \leq r-2$, we get:

$$e^D_{r',i'}(z) = (-1)^{r-i} \sum_{\substack{1 \leq k_i \leq a_i \\ \cdots \\ 1 \leq k_{r-1} \leq a_{r-1}}} \frac{P_{i-1}(p_{i,k_i}-1) \cdot \prod_{s=i}^{r-2} P_{s,k_s}(p_{s+1,k_{s+1}}-1) \cdot P_r(p_{r-2,k_{r-2}})}{(z-p_{r-1,k_{r-1}}) \prod_{s=i}^{r-1} P_{s,k_s}(p_{s,k_s})} \times e^{\sum_{s=i}^{r-1} q_{s,k_s}}$$

while $e^D_{r',(r-1)'}(z)$ is given by the same formula (with $i = r-1$) but with $P_r(p_{r-2,k_{r-2}})$ omitted.

For $1 \leq i \leq r-2$, we get:

$$e^D_{i,r'}(z) = (-1)^{r-i} \sum_{\substack{1\leq k_i\leq a_i \\ \cdots \\ 1\leq k_{r-2}\leq a_{r-2} \\ 1\leq k_r\leq a_r}} \frac{P_{i-1}(p_{i,k_i}-1)\cdot \prod_{s=i}^{r-3} P_{s,k_s}(p_{s+1,k_{s+1}}-1)\cdot P_{r,k_r}(p_{r-2,k_{r-2}})}{(z-p_{i,k_i})\prod_{s=i}^{r-2} P_{s,k_s}(p_{s,k_s})\cdot P_{r,k_r}(p_{r,k_r})} \times e^{\sum_{s=i}^{r-2} q_{s,k_s}+q_{r,k_r}}$$

while $e^D_{r-1,r'}(z)$ equals $\Psi_D(E_r(z))$ specified in (2.138).

For $1\leq i\leq r-2$, we get:

$$e^D_{i,(r-1)'}(z) = (-1)^{r-i} \sum_{\substack{1\leq k_i\leq a_i \\ \cdots \\ 1\leq k_r\leq a_r}} \frac{P_{i-1}(p_{i,k_i}-1)\cdot \prod_{s=i}^{r-2} P_{s,k_s}(p_{s+1,k_{s+1}}-1)\cdot P_{r,k_r}(p_{r-2,k_{r-2}})}{(z-p_{i,k_i})\prod_{s=i}^{r} P_{s,k_s}(p_{s,k_s})} \times e^{\sum_{s=i}^{r} q_{s,k_s}}$$

For $1\leq i<j\leq r-2$, we get:

$$e^D_{i,j'}(z) = (-1)^{j-i} \cdot \sum_{\substack{I_i\subset\{1,\dots,a_i\} \\ \cdots \\ I_r\subset\{1,\dots,a_r\}}}^{|I_s|=1+\delta_{s\in\{j,\dots,r-2\}}} \frac{P_{i-1}(p_{i,k_i}-1)\cdot \prod_{i\leq s\leq r-2}^{k\in I_{s+1}} P_{s,I_s}(p_{s+1,k}-1)\cdot \prod_{k\in I_{r-2}} P_{r,I_r}(p_{r-2,k})}{(z-p_{i,k_i})\prod_{i\leq s\leq r}^{k\in I_s} P_{s,I_s}(p_{s,k})} \times e^{\sum_{i\leq s\leq r}^{k\in I_s} q_{s,k}},$$

the sum taken over all subsets $I_s\subset\{1,\dots,a_s\}$, $i\leq s\leq r$, of cardinality $|I_i|=\dots=|I_{j-1}|=1$, $|I_j|=\dots=|I_{r-2}|=2$, $|I_{r-1}|=|I_r|=1$, and the natural generalization of (2.133) being used:

$$P_{s,I_s}(z) := \prod_{1\leq k\leq a_s}^{k\notin I_s} (z-p_{s,k}). \tag{A.2}$$

For $1\leq i\leq r-3$, we get:

$$e^D_{i,i'}(z) = -\sum_{\substack{I_i\subset\{1,\dots,a_i\} \\ \cdots \\ I_r\subset\{1,\dots,a_r\}}}^{|I_s|=1+\delta_{s\in\{i,\dots,r-2\}}} \left\{ \frac{\prod_{k\in I_i} P_{i-1}(p_{i,k}-1)\cdot \prod_{i\leq s\leq r-2}^{k\in I_{s+1}} P_{s,I_s}(p_{s+1,k}-1)\cdot \prod_{k\in I_{r-2}} P_{r,I_r}(p_{r-2,k})}{\prod_{k\in I_i}(z-p_{i,k})\cdot \prod_{i\leq s\leq r}^{k\in I_s} P_{s,I_s}(p_{s,k})} \cdot e^{\sum_{i\leq s\leq r}^{k\in I_s} q_{s,k}} \right\}$$

while $e^D_{r,r'}(z)=0$, due to Lemma 2.79(a), and $e^D_{r-1,(r-1)'}(z)$ is given by:

$$e^{D}_{r-1,(r-1)'}(z) \;=\; -\sum_{\substack{1\leq k_{r-1}\leq a_{r-1}\\ 1\leq k_r\leq a_r}} \frac{P_{r-2}(p_{r-1,k_{r-1}}-1)}{(z-p_{r-1,k_{r-1}})(z-p_{r,k_r})\prod_{s=r-1}^{r} P_{s,k_s}(p_{s,k_s})} \times e^{q_{r-1,k_{r-1}}+q_{r,k_r}}$$

For $2\leq i\leq r-2$, we get:

$$e^{D}_{i,(i-1)'}(z) = \sum_{\substack{I_{i-1}\subset\{1,\ldots,a_{i-1}\}\\ \cdots\\ I_r\subset\{1,\ldots,a_r\}}}^{|I_s|=1+\delta_{s\in\{i,\ldots,r-2\}}} \frac{z-p_{i-1,k_{i-1}}-1}{\prod_{k\in I_i}(z-p_{i,k})} \times \frac{P_{i-2}(p_{i-1,k_{i-1}}-1)\cdot\prod_{i-1\leq s\leq r-2}^{k\in I_{s+1}} P_{s,I_s}(p_{s+1,k}-1)\cdot\prod_{k\in I_{r-2}} P_{r,I_r}(p_{r-2,k})}{\prod_{i-1\leq s\leq r}^{k\in I_s} P_{s,I_s}(p_{s,k})} \times e^{\sum_{i-1\leq s\leq r}^{k\in I_s} q_{s,k}}$$

while the $i=r-1,r$ counterparts of this formula are as follows:

$$e^{D}_{r-1,(r-2)'}(z) \;=\; -\sum_{\substack{1\leq k_{r-2}\leq a_{r-2}\\ 1\leq k_{r-1}\leq a_{r-1}\\ 1\leq k_r\leq a_r}} \frac{z-p_{r-2,k_{r-2}}-1}{(z-p_{r-1,k_{r-1}})(z-p_{r,k_r})} \times \frac{P_{r-3}(p_{r-2,k_{r-2}}-1)P_{r-2,k_{r-2}}(p_{r-1,k_{r-1}}-1)P_{r,k_r}(p_{r-2,k_{r-2}})}{\prod_{s=r-2}^{r} P_{s,k_s}(p_{s,k_s})}\cdot e^{\sum_{s=r-2}^{r} q_{s,k_s}}$$

and

$$e^{D}_{r,(r-1)'}(z) \;=\; -\sum_{1\leq k_r\leq a_r} \frac{1}{(z-p_{r,k_r})P_{r,k_r}(p_{r,k_r})}\cdot e^{q_{r,k_r}}$$

For $1\leq j\leq i-2\leq r-4$, we get:

$$e^{D}_{i,j'}(z) \;=\; (-1)^{i-j+1} \sum_{\substack{I_j\subset\{1,\ldots,a_j\}\\ \cdots\\ I_r\subset\{1,\ldots,a_r\}}}^{|I_s|=1+\delta_{s\in\{i,\ldots,r-2\}}} \frac{z-p_{i-1,k_{i-1}}-1}{\prod_{k\in I_i}(z-p_{i,k})} \times \frac{P_{j-1}(p_{j,k_j}-1)\cdot\prod_{j\leq s\leq r-2}^{k\in I_{s+1}} P_{s,I_s}(p_{s+1,k}-1)\cdot\prod_{k\in I_{r-2}} P_{r,I_r}(p_{r-2,k})}{\prod_{j\leq s\leq r}^{k\in I_s} P_{s,I_s}(p_{s,k})}\cdot e^{\sum_{j\leq s\leq r}^{k\in I_s} q_{s,k}}$$

while the $i=r-1,r$ counterparts of this formula are as follows:

$$e^{D}_{r-1,j'}(z) \;=\; (-1)^{r-j+1} \sum_{\substack{1\leq k_j\leq a_j\\ \cdots\\ 1\leq k_r\leq a_r}} \frac{(z-p_{r-2,k_{r-2}}-1)}{(z-p_{r-1,k_{r-1}})(z-p_{r,k_r})} \times \frac{P_{j-1}(p_{j,k_j}-1)\cdot\prod_{s=j}^{r-2} P_{s,k_s}(p_{s+1,k_{s+1}}-1)\cdot P_{r,k_r}(p_{r-2,k_{r-2}})}{\prod_{s=j}^{r} P_{s,k_s}(p_{s,k_s})}\cdot e^{\sum_{s=j}^{r} q_{s,k_s}}$$

and

$$e^D_{r,j'}(z) = (-1)^{r-j+1} \sum_{\substack{1\le k_j\le a_j\\ \cdots\\ 1\le k_{r-2}\le a_{r-2}\\ 1\le k_r\le a_r}} \frac{P_{j-1}(p_{j,k_j}-1)\cdot \prod_{s=j}^{r-3} P_{s,k_s}(p_{s+1,k_{s+1}}-1)\cdot P_{r,k_r}(p_{r-2,k_{r-2}})}{(z-p_{r,k_r})\prod_{s=j}^{r-2} P_{s,k_s}(p_{s,k_s})\cdot P_{r,k_r}(p_{r,k_r})} \cdot e^{\sum_{s=j}^{r-2} q_{s,k_s}+q_{r,k_r}}$$

Remark A.3. In the notations $e^D_{i,j}(z)=\sum_{k\ge 1} e^{(D)k}_{i,j} z^{-k}$, see (2.192), the above formulas imply:

$$e^{(D)1}_{i,j} = -e^{(D)1}_{j',i'}, \qquad \forall\, 1\le i<j\le 2r. \tag{A.4}$$

• <u>Matrix $F^D(z)$ explicitly</u>.
For $1\le i<j\le r-1$, we get:

$$f^D_{j,i}(z) = (-1)^{j-i} \sum_{\substack{1\le k_i\le a_i\\ \cdots\\ 1\le k_{j-1}\le a_{j-1}}} \frac{\prod_{s=i}^{j-1} Z_s(p_{s,k_s}+1)\cdot \prod_{s=i+1}^{j-1} P_{s,k_s}(p_{s-1,k_{s-1}}+1)\cdot P_j(p_{j-1,k_{j-1}}+1)}{(z-p_{i,k_i}-1)\prod_{s=i}^{j-1} P_{s,k_s}(p_{s,k_s})} \times e^{-\sum_{s=i}^{j-1} q_{s,k_s}}$$

For $1\le i\le r-1$, we get:

$$f^D_{r,i}(z) = (-1)^{r-i} \sum_{\substack{1\le k_i\le a_i\\ \cdots\\ 1\le k_{r-1}\le a_{r-1}}} \frac{\prod_{s=i}^{r-1} Z_s(p_{s,k_s}+1)\cdot \prod_{s=i+1}^{r-1} P_{s,k_s}(p_{s-1,k_{s-1}}+1)}{(z-p_{i,k_i}-1)\prod_{s=i}^{r-1} P_{s,k_s}(p_{s,k_s})} \times e^{-\sum_{s=i}^{r-1} q_{s,k_s}}$$

For $1\le i<j\le r-1$, we get:

$$f^D_{i',j'}(z) = (-1)^{j-i-1} \sum_{\substack{1\le k_i\le a_i\\ \cdots\\ 1\le k_{j-1}\le a_{j-1}}} \frac{\prod_{s=i}^{j-1} Z_s(p_{s,k_s}+1)\cdot \prod_{s=i+1}^{j-1} P_{s,k_s}(p_{s-1,k_{s-1}}+1)\cdot P_j(p_{j-1,k_{j-1}}+1)}{(z-p_{j-1,k_{j-1}}+j-r-1)\prod_{s=i}^{j-1} P_{s,k_s}(p_{s,k_s})} \times e^{-\sum_{s=i}^{j-1} q_{s,k_s}}$$

For $1\le i\le r-1$, we get:

$$f^D_{i',r'}(z) = (-1)^{r-i+1} \sum_{\substack{1\le k_i\le a_i\\ \cdots\\ 1\le k_{r-1}\le a_{r-1}}} \frac{\prod_{s=i}^{r-1} Z_s(p_{s,k_s}+1)\cdot \prod_{s=i+1}^{r-1} P_{s,k_s}(p_{s-1,k_{s-1}}+1)}{(z-p_{r-1,k_{r-1}}-1)\prod_{s=i}^{r-1} P_{s,k_s}(p_{s,k_s})} \cdot e^{-\sum_{s=i}^{r-1} q_{s,k_s}}$$

For $1 \leq i \leq r-2$, we get:

$$f^D_{r',i}(z) = (-1)^{r-i+1} \sum_{\substack{1\leq k_i\leq a_i\\ \cdots \\ 1\leq k_{r-2}\leq a_{r-2}\\ 1\leq k_r\leq a_r}} P_{r-2,k_{r-2}}(p_{r,k_r})P_{r-1}(p_{r-2,k_{r-2}}+1)$$
$$\times \frac{\prod_{i\leq s\leq r}^{s\neq r-1} Z_s(p_{s,k_s}+1)\cdot\prod_{s=i+1}^{r-2} P_{s,k_s}(p_{s-1,k_{s-1}}+1)}{(z-p_{i,k_i}-1)\prod_{s=i}^{r-2} P_{s,k_s}(p_{s,k_s})\cdot P_{r,k_r}(p_{r,k_r})}\cdot e^{-\sum_{s=i}^{r-2} q_{s,k_s}-q_{r,k_r}}$$

while $f^D_{r',r-1}(z)$ equals $\Psi_D(F_r(z))$ specified in (2.138).

For $1 \leq i \leq r-2$, we get:

$$f^D_{(r-1)',i}(z) = (-1)^{r-i+1}$$
$$\sum_{\substack{1\leq k_i\leq a_i\\ \cdots \\ 1\leq k_r\leq a_r}} \frac{\prod_{s=i}^{r} Z_s(p_{s,k_s}+1)\cdot\prod_{s=i+1}^{r-1} P_{s,k_s}(p_{s-1,k_{s-1}}+1)\cdot P_{r-2,k_{r-2}}(p_{r,k_r})}{(z-p_{i,k_i}-1)\prod_{s=i}^{r} P_{s,k_s}(p_{s,k_s})}\cdot e^{-\sum_{s=i}^{r} q_{s,k_s}}$$

For $1 \leq i < j \leq r-2$, we get:

$$f^D_{j',i}(z) = (-1)^{j-i+1}\ \cdot$$
$$\sum_{\substack{I_i\subset\{1,\ldots,a_i\}\\ \cdots \\ I_r\subset\{1,\ldots,a_r\}}}^{|I_s|=1+\delta_{s\in\{j,\ldots,r-2\}}} \frac{\prod_{i\leq s\leq r}^{k\in I_s} Z_s(p_{s,k}+1)\cdot\prod_{i+1\leq s\leq r-1}^{k\in I_{s-1}} P_{s,I_s}(p_{s-1,k}+1)\cdot P_{r-2,I_{r-2}}(p_{r,k_r})}{(z-p_{i,k_i}-1)\prod_{i\leq s\leq r}^{k\in I_s} P_{s,I_s}(p_{s,k})}$$
$$\times\, e^{-\sum_{i\leq s\leq r}^{k\in I_s} q_{s,k}}$$

where we use the above notation (A.2) and k_r denotes the only element of I_r, i.e. $I_r = \{k_r\}$.

For $1 \leq i \leq r-2$, we get:

$$f^D_{i',i}(z) = -\sum_{\substack{I_i\subset\{1,\ldots,a_i\}\\ \cdots \\ I_r\subset\{1,\ldots,a_r\}}}^{|I_s|=1+\delta_{s\in\{i,\ldots,r-2\}}}$$
$$\left\{\frac{\prod_{i\leq s\leq r}^{k\in I_s} Z_s(p_{s,k}+1)\cdot\prod_{i+1\leq s\leq r-1}^{k\in I_{s-1}} P_{s,I_s}(p_{s-1,k}+1)\cdot P_{r-2,I_{r-2}}(p_{r,k_r})}{\prod_{k\in I_i}(z-p_{i,k}-1)\cdot\prod_{i\leq s\leq r}^{k\in I_s} P_{s,I_s}(p_{s,k})}\cdot e^{-\sum_{i\leq s\leq r}^{k\in I_s} q_{s,k}}\right\}$$

with $I_r = \{k_r\}$, while $f^D_{r',r}(z) = 0$, due to Lemma 2.96(a), and $f^D_{(r-1)',r-1}(z)$ is given by:

$$f^D_{(r-1)',r-1}(z) = -\sum_{\substack{1\leq k_{r-1}\leq a_{r-1}\\ 1\leq k_r\leq a_r}} \frac{Z_{r-1}(p_{r-1,k_{r-1}}+1)Z_r(p_{r,k_r}+1)P_{r-2}(p_{r,k_r})}{\prod_{s=r-1}^{r}(z-p_{s,k_s}-1)\cdot\prod_{s=r-1}^{r} P_{s,k_s}(p_{s,k_s})}$$
$$\times\, e^{-q_{r-1,k_{r-1}}-q_{r,k_r}}$$

For $2 \le i \le r-2$, we get:

$$f^D_{(i-1)',i}(z) = -\sum_{\substack{I_{i-1}\subset\{1,\ldots,a_{i-1}\}\\ \cdots \\ I_r\subset\{1,\ldots,a_r\}}}^{|I_s|=1+\delta_{s\in\{i,\ldots,r-2\}}} \frac{z-p_{i-1,k_{i-1}}-2}{\prod_{k\in I_i}(z-p_{i,k}-1)} \times \frac{\prod_{i-1\le s\le r}^{k\in I_s} Z_s(p_{s,k}+1)\cdot\prod_{i\le s\le r-1}^{k\in I_{s-1}} P_{s,I_s}(p_{s-1,k}+1)\cdot P_{r-2,I_{r-2}}(p_{r,k_r})}{\prod_{i-1\le s\le r}^{k\in I_s} P_{s,I_s}(p_{s,k})}\cdot e^{-\sum_{i-1\le s\le r}^{k\in I_s} q_{s,k}}$$

with $I_r = \{k_r\}$, while the $i = r-1, r$ counterparts of this formula are as follows:

$$f^D_{(r-2)',r-1}(z) = \sum_{\substack{1\le k_{r-2}\le a_{r-2}\\ 1\le k_{r-1}\le a_{r-1}\\ 1\le k_r\le a_r}} \frac{z-p_{r-2,k_{r-2}}-2}{(z-p_{r-1,k_{r-1}}-1)(z-p_{r,k_r}-1)} \times \frac{\prod_{s=r-2}^{r} Z_s(p_{s,k_s}+1)\cdot P_{r-2,k_{r-2}}(p_{r,k_r})P_{r-1,k_{r-1}}(p_{r-2,k_{r-2}}+1)}{\prod_{s=r-2}^{r} P_{s,k_s}(p_{s,k_s})}\cdot e^{-\sum_{s=r-2}^{r} q_{s,k_s}}$$

and

$$f^D_{(r-1)',r}(z) = \sum_{1\le k_r\le a_r} \frac{Z_r(p_{r,k_r}+1)P_{r-2}(p_{r,k_r})}{(z-p_{r,k_r}-1)P_{r,k_r}(p_{r,k_r})}\cdot e^{-q_{r,k_r}}$$

For $1 \le j \le i-2 \le r-4$, we get:

$$f^D_{j',i}(z) = (-1)^{i-j}\sum_{\substack{I_j\subset\{1,\ldots,a_j\}\\ \cdots \\ I_r\subset\{1,\ldots,a_r\}}}^{|I_s|=1+\delta_{s\in\{i,\ldots,r-2\}}} \frac{z-p_{i-1,k_{i-1}}-2}{\prod_{k\in I_i}(z-p_{i,k}-1)} \times \frac{\prod_{j\le s\le r}^{k\in I_s} Z_s(p_{s,k}+1)\cdot\prod_{j+1\le s\le r-1}^{k\in I_{s-1}} P_{s,I_s}(p_{s-1,k}+1)\cdot P_{r-2,I_{r-2}}(p_{r,k_r})}{\prod_{j\le s\le r}^{k\in I_s} P_{s,I_s}(p_{s,k})}\cdot e^{-\sum_{j\le s\le r}^{k\in I_s} q_{s,k}}$$

with $I_r = \{k_r\}$, while the $i = r-1, r$ counterparts of this formula are as follows:

$$f^D_{j',r-1}(z) = (-1)^{r-j}\sum_{\substack{1\le k_j\le a_j\\ \cdots \\ 1\le k_r\le a_r}} \frac{z-p_{r-2,k_{r-2}}-2}{(z-p_{r-1,k_{r-1}}-1)(z-p_{r,k_r}-1)} \times \frac{\prod_{s=j}^{r} Z_s(p_{s,k_s}+1)\cdot\prod_{s=j+1}^{r-1} P_{s,k_s}(p_{s-1,k_{s-1}}+1)\cdot P_{r-2,k_{r-2}}(p_{r,k_r})}{\prod_{s=j}^{r} P_{s,k_s}(p_{s,k_s})}\cdot e^{-\sum_{s=j}^{r} q_{s,k_s}}$$

and

$$f^D_{j',r}(z) = (-1)^{r-j}\sum_{\substack{1\le k_j\le a_j\\ \cdots \\ 1\le k_{r-2}\le a_{r-2}\\ 1\le k_r\le a_r}} P_{r-2,k_{r-2}}(p_{r,k_r})P_{r-1}(p_{r-2,k_{r-2}}+1) \times \frac{\prod_{j\le s\le r}^{s\ne r-1} Z_s(p_{s,k_s}+1)\cdot\prod_{s=j+1}^{r-2} P_{s,k_s}(p_{s-1,k_{s-1}}+1)}{(z-p_{r,k_r}-1)\prod_{s=j}^{r-2} P_{s,k_s}(p_{s,k_s})\cdot P_{r,k_r}(p_{r,k_r})}\cdot e^{-\sum_{s=j}^{r-2} q_{s,k_s}-q_{r,k_r}}$$

Remark A.5. In the notations $f^D_{j,i}(z) = \sum_{k\ge 1} f^{(D)k}_{j,i} z^{-k}$, see (2.192), the above formulas imply:

$$f^{(D)1}_{j,i} = -f^{(D)1}_{i',j'}, \qquad \forall\, 1 \le i < j \le 2r. \tag{A.6}$$

Appendix B. Explicit formulas in types B and C

In this Appendix, we provide a shuffle realization of the key homomorphisms of our paper. To simplify the exposition, we shall follow the uniform formulas of [NW] in the (w, u)-oscillators (generalizing those of [BFNb] to non-simply-laced cases in the spirit of [GKLO, FT1]).

B.1. Homomorphisms $\Phi^{\bar{\lambda}}_{\bar{\mu}}$. Let $\mathfrak{g}$ be a simple Lie algebra of rank r, and let $\{\alpha_i^\vee\}_{i=1}^r$ (resp. $\{\alpha_i\}_{i=1}^r$) be the simple roots (resp. simple coroots) of $\mathfrak{g}$. Let $(\cdot,\cdot)$ denote the corresponding pairing on the root lattice, and set $\mathsf{d}_i := \frac{(\alpha_i^\vee,\alpha_i^\vee)}{2}$. Let $(a_{ij})_{i,j=1}^r$ be the Cartan matrix of $\mathfrak{g}$, so that $\mathsf{d}_i a_{ij} = (\alpha_i^\vee, \alpha_j^\vee)$.

We also choose an orientation of the graph $\mathrm{Dyn}_\mathfrak{g}$ obtained from the Dynkin diagram of $\mathfrak{g}$ by replacing all multiple edges with simple ones. The notation $j - i$ (resp. $j \to i$ or $j \leftarrow i$) is to indicate an edge (resp. oriented edge pointing towards i or j) between the vertices $i, j \in \mathrm{Dyn}_\mathfrak{g}$.

Fix a coweight $\bar{\mu}$ of $\mathfrak{g}$, and let $Y_{\bar{\mu}}(\mathfrak{g})$ denote the corresponding shifted (Drinfeld) Yangian of $\mathfrak{g}$, cf. [BFNb, NW], whose generators are encoded into the series $\mathsf{E}_i(z), \mathsf{F}_i(z), \mathsf{H}_i(z)$ as before. We also fix a dominant coweight $\bar{\lambda} = \omega_{i_1} + \ldots + \omega_{i_N}$ (ω_k being the k-th fundamental coweight) such that $\bar{\lambda} + \bar{\mu} = a_1\alpha_1 + \ldots + a_r\alpha_r$ with $a_i \in \mathbb{N}$, and choose a collection of points $z_1, \ldots, z_N \in \mathbb{C}$.

Consider the associative $\mathbb{C}$-algebra (cf. (2.131, 3.20))

$$\tilde{\mathcal{A}} = \mathbb{C}\Big\langle w_{i,k},\ \mathsf{u}_{i,k}^{\pm 1},\ (w_{i,k} - w_{i,\ell} + m\mathsf{d}_i)^{-1}\Big\rangle_{1\le i\le r, m\in\mathbb{Z}}^{1\le k\ne \ell\le a_i}$$

with the defining relations:

$$[\mathsf{u}_{i,k}, w_{j,\ell}] = \mathsf{d}_i\delta_{i,j}\delta_{k,\ell}\mathsf{u}_{i,k}, \quad [w_{i,k}, w_{j,\ell}] = 0 = [\mathsf{u}_{i,k}, \mathsf{u}_{j,\ell}], \quad \mathsf{u}_{i,k}^{\pm 1}\mathsf{u}_{i,k}^{\mp 1} = 1.$$

Set $a_0 := 0,\ a_{r+1} := 0,\ W_0(z) := 1,\ W_{r+1}(z) = 1$. For $1 \le i \le r$, we also define:

$$W_i(z) := \prod_{k=1}^{a_i}(z - w_{i,k}), \quad W_{i,\ell}(z) := \prod_{1\le k\le a_i}^{k\ne\ell}(z - w_{i,k}), \quad \mathsf{Z}_i(z) := \prod_{1\le s\le N}^{i_s = i}\left(z - z_s - \tfrac{1}{2}\right).$$

Remark B.1. The shift by $-\frac{1}{2}$ above is purely historical [BFNb], and can be absorbed into z_s.

The following is a rational counterpart of [FT1, Theorem 7.1] (cf. [NW, Theorem 5.4]):

Theorem B.2. *There is a unique* $\mathbb{C}$*-algebra homomorphism*

$$\Phi^{\bar{\lambda}}_{\bar{\mu}} \colon Y_{\bar{\mu}}(\mathfrak{g}) \longrightarrow \tilde{\mathcal{A}}, \tag{B.3}$$

determined by the following assignment:

$$\mathsf{E}_i(z) \mapsto \frac{1}{\mathsf{d}_i}\sum_{k=1}^{a_i} \frac{\prod_{j\to i}\prod_{p=1}^{-a_{ji}} W_j(w_{i,k} - \frac{1}{2}(\alpha_i^\vee, \alpha_j^\vee) - p\mathsf{d}_j)}{(z - w_{i,k})W_{i,k}(w_{i,k})}\mathsf{u}_{i,k}^{-1},$$

$$\mathsf{F}_i(z) \mapsto -\sum_{k=1}^{a_i} \frac{\mathsf{Z}_i(w_{i,k}+\mathsf{d}_i)\prod_{j\leftarrow i}\prod_{p=1}^{-a_{ji}} W_j(w_{i,k}+\mathsf{d}_i-\frac{1}{2}(\alpha_i^\vee,\alpha_j^\vee)-p\mathsf{d}_j)}{(z-w_{i,k}-\mathsf{d}_i)W_{i,k}(w_{i,k})}\mathsf{u}_{i,k},$$
$$\mathsf{H}_i(z) \mapsto \frac{\mathsf{Z}_i(z)\prod_{j-i}\prod_{p=1}^{-a_{ji}} W_j(z-\frac{1}{2}(\alpha_i^\vee,\alpha_j^\vee)-p\mathsf{d}_j)}{W_i(z)W_i(z-\mathsf{d}_i)}. \tag{B.4}$$

B.2. Shuffle algebra realization of $\Phi^{\bar{\lambda}}_{\bar{\mu}}$. Let $Y^+(\mathfrak{g})$ and $Y^-(\mathfrak{g})$ denote the subalgebras of the Drinfeld Yangian $Y(\mathfrak{g})$ generated by $\{\mathsf{E}_i^{(k)}\}_{1\le i\le r}^{k\ge 1}$ and $\{\mathsf{F}_i^{(k)}\}_{1\le i\le r}^{k\ge 1}$, respectively. They can also be described as algebras generated by $\{\mathsf{E}_i^{(k)}\}_{1\le i\le r}^{k\ge 1}$ and $\{\mathsf{F}_i^{(k)}\}_{1\le i\le r}^{k\ge 1}$ subject to the relations (2.6, 2.8) and (2.7, 2.9), respectively.

Remark B.5. Note the algebra isomorphisms $Y^-(\mathfrak{g}) \xrightarrow{\sim} Y^+(\mathfrak{g})^{\mathrm{op}}$ determined via $\mathsf{F}_i^{(k)} \mapsto \mathsf{E}_i^{(k)}$ (given an algebra A, we use A^{op} to denote the algebra with the opposite multiplication).

For any coweight ν of $\mathfrak{g}$, we define the subalgebras $Y_\nu^\pm(\mathfrak{g})$ of the shifted Yangian $Y_\nu(\mathfrak{g})$ likewise. According to [FKPRW, Corollary 3.15], we have algebra isomorphisms for any ν:

$$Y_\nu^+(\mathfrak{g}) \xrightarrow{\sim} Y^+(\mathfrak{g}), \qquad \mathsf{E}_i^{(k)} \mapsto \mathsf{E}_i^{(k)},$$
$$Y_\nu^-(\mathfrak{g}) \xrightarrow{\sim} Y^-(\mathfrak{g}), \qquad \mathsf{F}_i^{(k)} \mapsto \mathsf{F}_i^{(k)}. \tag{B.6}$$

Consider an $\mathbb{N}^r$-graded $\mathbb{C}$-vector space

$$\mathbb{S}^{(\mathfrak{g})} = \bigoplus_{\underline{k}=(k_1,\ldots,k_r)\in\mathbb{N}^r} \mathbb{S}^{(\mathfrak{g})}_{\underline{k}}, \tag{B.7}$$

where $\mathbb{S}^{(\mathfrak{g})}_{\underline{k}}$ consists of $\prod_{i=1}^r S(k_i)$-symmetric rational functions in the variables $\{x_{i,k}\}_{1\le i\le r}^{1\le k\le k_i}$. We also fix a matrix of rational functions $(\zeta_{ij}(z))_{i,j=1}^r$ via:

$$\zeta_{ij}(z) = 1 + \frac{(\alpha_i^\vee,\alpha_j^\vee)}{2z} = 1 + \frac{\mathsf{d}_i a_{ij}}{2z}. \tag{B.8}$$

Let us define the *shuffle product* $\star$ on $\mathbb{S}^{(\mathfrak{g})}$: given $F\in\mathbb{S}^{(\mathfrak{g})}_{\underline{k}}$, $G\in\mathbb{S}^{(\mathfrak{g})}_{\underline{\ell}}$, define $F\star G\in\mathbb{S}^{(\mathfrak{g})}_{\underline{k}+\underline{\ell}}$ via

$$(F\star G)(x_{1,1},\ldots,x_{1,k_1+\ell_1};\ldots;x_{r,1},\ldots,x_{r,k_r+\ell_r}) := \frac{1}{\underline{k}!\cdot\underline{\ell}!}$$
$$\times \mathrm{Sym}\left(F\left(\{x_{i,k}\}_{1\le i\le r}^{1\le k\le k_i}\right) G\left(\{x_{i',k'}\}_{1\le i'\le r}^{k_{i'}<k'\le k_{i'}+\ell_{i'}}\right)\cdot \prod_{1\le i\le r}^{1\le i'\le r}\prod_{k\le k_i}^{k'>k_{i'}} \zeta_{ii'}(x_{i,k}-x_{i',k'})\right). \tag{B.9}$$

Here, $\underline{k}! = \prod_{i=1}^r k_i!$, while the *symmetrization* of $f\in\mathbb{C}(\{x_{i,1},\ldots,x_{i,m_i}\}_{1\le i\le r})$ is defined via:

$$\mathrm{Sym}\,(f)\left(\{x_{i,1},\ldots,x_{i,m_i}\}_{1\le i\le r}\right) := \sum_{(\sigma_1,\ldots,\sigma_r)\in S(m_1)\times\cdots\times S(m_r)} f\left(\{x_{i,\sigma_i(1)},\ldots,x_{i,\sigma_i(m_i)}\}_{1\le i\le r}\right).$$

This endows $\mathbb{S}^{(\mathfrak{g})}$ with a structure of an associative $\mathbb{C}$-algebra with the unit $\mathbf{1}\in\mathbb{S}^{(\mathfrak{g})}_{(0,\ldots,0)}$. We are interested in a certain $\mathbb{C}$-subspace of $\mathbb{S}^{(\mathfrak{g})}$ defined by the *pole* and *wheel conditions*:

- We say that $F\in\mathbb{S}^{(\mathfrak{g})}_{\underline{k}}$ satisfies the *pole conditions* if

$$F=\frac{f(x_{1,1},\ldots,x_{r,k_r})}{\prod_{i-j}^{\text{unordered}}\prod_{k\le k_i}^{k'\le k_j}(x_{i,k}-x_{j,k'})},\ \text{ where } f\in\left(\mathbb{C}[\{x_{i,k}\}_{1\le i\le r}^{1\le k\le k_i}]\right)^{S(k_1)\times\cdots\times S(k_r)}. \tag{B.10}$$

- We say that $F\in\mathbb{S}^{(\mathfrak{g})}_{\underline{k}}$ satisfies the *wheel conditions* if for any connected $i-j$, we have:

$$F\left(\{x_{i,k}\}\right)\Big|_{(x_{i1},x_{i2},x_{i3},\ldots,x_{i,1-a_{ij}})\mapsto(w,w+\mathsf{d}_i,w+2\mathsf{d}_i,\ldots,w+\mathsf{d}_ia_{ij}),\ x_{j1}\mapsto w+\frac{\mathsf{d}_ia_{ij}}{2}}=0. \tag{B.11}$$

Let $S^{(\mathfrak{g})}_{\underline{k}}\subset\mathbb{S}^{(\mathfrak{g})}_{\underline{k}}$ denote the subspace of all elements F satisfying these two conditions and set

$$S^{(\mathfrak{g})}:=\bigoplus_{\underline{k}\in\mathbb{N}^r}S^{(\mathfrak{g})}_{\underline{k}}.$$

It is straightforward to check that $S^{(\mathfrak{g})}\subset\mathbb{S}^{(\mathfrak{g})}$ is $\star$-closed. The resulting algebra $\left(S^{(\mathfrak{g})},\star\right)$ is called the *(rational) shuffle algebra of type* $\mathfrak{g}$. It is related to $Y^+(\mathfrak{g})$ via the embedding:

$$\Upsilon\colon Y^+(\mathfrak{g})\hookrightarrow S^{(\mathfrak{g})},\qquad \mathsf{E}_i^{(k)}\mapsto x_{i,1}^{k-1}\quad\text{for}\quad 1\le i\le r,\ k\ge 1. \tag{B.12}$$

In view of Remark B.5, we also get:

$$\Upsilon\colon Y^-(\mathfrak{g})\hookrightarrow S^{(\mathfrak{g}),\mathrm{op}},\qquad \mathsf{F}_i^{(k)}\mapsto x_{i,1}^{k-1}\quad\text{for}\quad 1\le i\le r,\ k\ge 1. \tag{B.13}$$

Remark B.14. The above embeddings Υ of (B.12, B.13) are expected to be actually algebra isomorphisms, similar to the trigonometric counterpart as was recently established in [NT]. This has been proved in (super version of) the type A in [T, §6-7].

The key result of this Appendix is the construction of the algebra homomorphisms

$$\widetilde{\Phi}^{\bar{\lambda}}_{\bar{\mu}}\colon\quad S^{(\mathfrak{g})}\longrightarrow\tilde{\mathcal{A}},\qquad S^{(\mathfrak{g}),\mathrm{op}}\longrightarrow\tilde{\mathcal{A}}, \tag{B.15}$$

compatible with $\Phi^{\bar{\lambda}}_{\bar{\mu}}$ (B.3) with respect to the isomorphisms (B.6) and embeddings (B.12, B.13). To this end, for $1\le i\le r$ and $1\le\ell\le a_i$, we define:

$$Y_{i,\ell}(z):=\frac{\prod_{j\to i}\prod_{p=1}^{-a_{ji}}W_j(z-\frac{1}{2}(\alpha_i^\vee,\alpha_j^\vee)-p\mathsf{d}_j)}{\mathsf{d}_i\cdot W_{i,\ell}(z)},$$
$$Y'_{i,\ell}(z):=-\frac{\mathsf{Z}_i(z+\mathsf{d}_i)\prod_{j\leftarrow i}\prod_{p=1}^{-a_{ji}}W_j(z+\mathsf{d}_i-\frac{1}{2}(\alpha_i^\vee,\alpha_j^\vee)-p\mathsf{d}_j)}{W_{i,\ell}(z)}. \tag{B.16}$$

Theorem B.17. *(a) The assignment*

$$S^{(\mathfrak{g})}_{(k_1,\ldots,k_r)} \ni E \mapsto \sum_{\substack{m^{(1)}_1+\ldots+m^{(1)}_{a_1}=k_1 \\ \cdots \\ m^{(r)}_1+\ldots+m^{(r)}_{a_r}=k_r}}^{m^{(i)}_k\in\mathbb{N}} \left\{ \prod_{i=1}^{r}\prod_{k=1}^{a_i}\prod_{p=1}^{m^{(i)}_k} Y_{i,k}\Big(w_{i,k}-(p-1)\mathsf{d}_i\Big) \right.$$

$$\times E\left(\Big\{w_{i,k}-(p-1)\mathsf{d}_i\Big\}_{\substack{1\le i\le r\\ 1\le k\le a_i\\ 1\le p\le m^{(i)}_k}}\right)$$

$$\times \prod_{i=1}^{r}\prod_{k=1}^{a_i}\prod_{1\le p_1<p_2\le m^{(i)}_k} \zeta^{-1}_{ii}\Big((w_{i,k}-(p_1-1)\mathsf{d}_i)\ -\ (w_{i,k}-(p_2-1)\mathsf{d}_i)\Big)$$

$$\times \prod_{i=1}^{r}\prod_{1\le k_1\ne k_2\le a_i}\prod_{1\le p_1\le m^{(i)}_{k_1}}^{1\le p_2\le m^{(i)}_{k_2}} \zeta^{-1}_{ii}\Big((w_{i,k_1}-(p_1-1)\mathsf{d}_i)\ -\ (w_{i,k_2}-(p_2-1)\mathsf{d}_i)\Big)$$

$$\times \prod_{j\to i}\prod_{1\le k_1\le a_i}^{1\le k_2\le a_j}\prod_{1\le p_1\le m^{(i)}_{k_1}}^{1\le p_2\le m^{(j)}_{k_2}} \zeta^{-1}_{ij}\Big((w_{i,k_1}-(p_1-1)\mathsf{d}_i)\ -\ (w_{j,k_2}-(p_2-1)\mathsf{d}_j)\Big)$$

$$\left. \times \prod_{i=1}^{r}\prod_{k=1}^{a_i} \mathsf{u}_{i,k}^{-m^{(i)}_k} \right\} \tag{B.18}$$

gives rise to the algebra homomorphism

$$\widetilde{\Phi}^{\bar{\lambda}}_{\bar{\mu}}\colon S^{(\mathfrak{g})} \longrightarrow \tilde{\mathcal{A}}. \tag{B.19}$$

Moreover, the composition

$$Y^+_{\bar{\mu}}(\mathfrak{g}) \xrightarrow[\sim]{(\text{B.6})} Y^+(\mathfrak{g}) \xrightarrow{\Upsilon} S^{(\mathfrak{g})} \xrightarrow{\widetilde{\Phi}^{\bar{\lambda}}_{\bar{\mu}}} \tilde{\mathcal{A}} \tag{B.20}$$

coincides with the restriction of the homomorphism $\Phi^{\bar{\lambda}}_{\bar{\mu}}$ (B.3) *to the subalgebra* $Y^+_{\bar{\mu}}(\mathfrak{g}) \subset Y_{\bar{\mu}}(\mathfrak{g})$.

(b) The assignment

$$S^{(\mathfrak{g}),\mathrm{op}}_{(k_1,\ldots,k_r)} \ni F \mapsto \sum_{\substack{m^{(1)}_1+\ldots+m^{(1)}_{a_1}=k_1 \\ \cdots \\ m^{(r)}_1+\ldots+m^{(r)}_{a_r}=k_r}}^{m^{(i)}_k\in\mathbb{N}} \left\{ \prod_{i=1}^{r}\prod_{k=1}^{a_i}\prod_{p=1}^{m^{(i)}_k} Y'_{i,k}\Big(w_{i,k}+(p-1)\mathsf{d}_i\Big) \right.$$

$$\times F\left(\Big\{w_{i,k}+p\mathsf{d}_i\Big\}_{\substack{1\le i\le r\\ 1\le k\le a_i\\ 1\le p\le m^{(i)}_k}}\right)$$

$$\times \prod_{i=1}^{r}\prod_{k=1}^{a_i}\prod_{1\le p_1<p_2\le m_k^{(i)}} \zeta_{ii}^{-1}\Big((w_{i,k}+p_2\mathsf{d}_i)-(w_{i,k}+p_1\mathsf{d}_i)\Big)$$

$$\times \prod_{i=1}^{r}\prod_{1\le k_1\neq k_2\le a_i}\prod_{1\le p_1\le m_{k_1}^{(i)}}^{1\le p_2\le m_{k_2}^{(i)}} \zeta_{ii}^{-1}\Big((w_{i,k_2}+p_2\mathsf{d}_i)-(w_{i,k_1}+p_1\mathsf{d}_i)\Big)$$

$$\times \prod_{j\leftarrow i}\prod_{1\le k_1\le a_i}^{1\le k_2\le a_j}\prod_{1\le p_1\le m_{k_1}^{(i)}}^{1\le p_2\le m_{k_2}^{(j)}} \zeta_{ji}^{-1}\Big((w_{j,k_2}+p_2\mathsf{d}_j)-(w_{i,k_1}+p_1\mathsf{d}_i)\Big)$$

$$\times \prod_{i=1}^{r}\prod_{k=1}^{a_i}\mathsf{u}_{i,k}^{m_k^{(i)}}\Bigg\} \tag{B.21}$$

gives rise to the algebra homomorphism

$$\widetilde{\Phi}_{\bar{\mu}}^{\bar{\lambda}}\colon S^{(\mathfrak{g}),\mathrm{op}} \longrightarrow \tilde{\mathcal{A}}. \tag{B.22}$$

Moreover, the composition

$$Y_{\bar{\mu}}^{-}(\mathfrak{g}) \xrightarrow[\sim]{(\mathrm{B.6})} Y^{-}(\mathfrak{g}) \xrightarrow{\Upsilon} S^{(\mathfrak{g}),\mathrm{op}} \xrightarrow{\widetilde{\Phi}_{\bar{\mu}}^{\bar{\lambda}}} \tilde{\mathcal{A}} \tag{B.23}$$

coincides with the restriction of the homomorphism $\Phi_{\bar{\mu}}^{\bar{\lambda}}$ (B.3) *to the subalgebra* $Y_{\bar{\mu}}^{-}(\mathfrak{g}) \subset Y_{\bar{\mu}}(\mathfrak{g})$.

The proof is straightforward and is left to the interested reader. We note that a trigonometric type A counterpart of this result played a crucial role in [FT2], see Theorem 4.11 of *loc.cit.*

B.3. Application to the Lax matrices of types B and C. The key application of Theorem B.17 to the main subject of the present paper is that it allows to obtain explicit formulas for the matrix coefficients of $E^D(z)$, $F^D(z)$ featuring in our definition of the Lax matrices $T_D(z)$ (2.175). In type D_r this recovers the formulas of Appendix A (which were rather derived using the relations of Lemmas 2.79, 2.80, 2.96, 2.97), while in types C_r and B_r this provides concise formulas (used in the proofs of Theorems 3.31, 4.31), which are quite inaccessible if derived iteratively via Lemmas 3.11, 3.12 or 4.10, 4.11, respectively.

Let $\mathfrak{g}$ be either $\mathfrak{so}_N$ ($N = 2r, 2r+1$) or $\mathfrak{sp}_N$ ($N = 2r$). Let $X^+(\mathfrak{g})$ and $X^-(\mathfrak{g})$ denote the subalgebras of the corresponding extended Drinfeld Yangian $X(\mathfrak{g})$, generated by $\{E_i^{(k)}\}_{1\le i\le r}^{k\ge 1}$ and $\{F_i^{(k)}\}_{1\le i\le r}^{k\ge 1}$, respectively. Likewise, let $X^{\mathrm{rtt},+}(\mathfrak{g})$ and $X^{\mathrm{rtt},-}(\mathfrak{g})$ denote the subalgebras of the corresponding extended RTT Yangian $X^{\mathrm{rtt}}(\mathfrak{g})$, generated by $\{e_{i,j}^{(k)}\}_{1\le i<j\le N}^{k\ge 1}$ and $\{f_{j,i}^{(k)}\}_{1\le i<j\le N}^{k\ge 1}$, respectively. Then, we have the following natural algebra isomorphisms:

$$X^{\mathrm{rtt},+}(\mathfrak{g}) \xrightarrow{\sim} X^{+}(\mathfrak{g}) \xrightarrow{\sim} Y^{+}(\mathfrak{g}), \qquad X^{\mathrm{rtt},-}(\mathfrak{g}) \xrightarrow{\sim} X^{-}(\mathfrak{g}) \xrightarrow{\sim} Y^{-}(\mathfrak{g}). \tag{B.24}$$

Let $\{\mathsf{E}_{ij}^{(k)}\}_{1\le i<j\le N}^{k\ge 1}$ and $\{\mathsf{F}_{ji}^{(k)}\}_{1\le i<j\le N}^{k\ge 1}$ denote the images of $e_{i,j}^{(k)}$ and $f_{j,i}^{(k)}$ in $Y^{+}(\mathfrak{g})$ and $Y^{-}(\mathfrak{g})$ under the composition maps of (B.24), respectively, and consider their generating series:

$$\mathsf{E}_{ij}(z) := \sum_{k\ge 1}\mathsf{E}_{ij}^{(k)}z^{-k}, \qquad \mathsf{F}_{ji}(z) := \sum_{k\ge 1}\mathsf{F}_{ji}^{(k)}z^{-k}.$$

We conclude this Appendix by presenting explicit formulas for $\Upsilon(\mathsf{E}_{ij}(z))$ and $\Upsilon(\mathsf{F}_{ji}(z))$ (combining which with Theorem B.17 recovers the Lax matrices $T_D(z)$ of (2.175)). In what follows, ς_i will denote the i-th coordinate vector: $\varsigma_i = (0,\ldots,0,1,0,\ldots,0)\in\mathbb{N}^r$ with 1 at the spot i.

Lemma B.25 (Type C_r). *Define the polynomial* $Q(z_1,z_2;w_1,w_2)$ *via:*[6]

$$Q(z_1,z_2;w_1,w_2) = 2z_1z_2 + 2w_1w_2 - (z_1+z_2)(w_1+w_2) + \tfrac{1}{2}. \tag{B.26}$$

(a) *We have the following equalities:*

$$\Upsilon\Big(\mathsf{E}_{ij}(z)\Big) = \frac{1}{(z-\frac{i-1}{2}-x_{i,1})\prod_{k=i}^{j-2}(x_{k,1}-x_{k+1,1})} \in S^{(\mathfrak{sp}_{2r})}_{\varsigma_i+\ldots+\varsigma_{j-1}} \quad \text{for } 1\le i<j\le r,$$

$$\Upsilon\Big(\mathsf{E}_{ir'}(z)\Big) = \frac{2}{(z-\frac{i-1}{2}-x_{i,1})\prod_{k=i}^{r-1}(x_{k,1}-x_{k+1,1})} \in S^{(\mathfrak{sp}_{2r})}_{\varsigma_i+\ldots+\varsigma_r} \quad \text{for } 1\le i<r,$$

$$\Upsilon\Big(\mathsf{E}_{ij'}(z)\Big) = \frac{2(2x_{j-1,1}-x_{j,1}-x_{j,2})\prod_{k=j}^{r-2}Q(x_{k,1},x_{k,2};x_{k+1,1},x_{k+1,2})}{(z-\frac{i-1}{2}-x_{i,1})\prod_{k=i}^{r-1}\prod_{p\le 1+\delta_{j\le k<r}}^{p'\le 1+\delta_{j\le k+1<r}}(x_{k,p}-x_{k+1,p'})}$$
$$\in S^{(\mathfrak{sp}_{2r})}_{\varsigma_i+\ldots+\varsigma_{j-1}+2\varsigma_j+\ldots+2\varsigma_{r-1}+\varsigma_r} \quad \text{for } 1\le i<j<r,$$

$$\Upsilon\Big(\mathsf{E}_{ii'}(z)\Big) = \frac{2(2z-i+2-x_{i,1}-x_{i,2})\prod_{k=i}^{r-2}Q(x_{k,1},x_{k,2};x_{k+1,1},x_{k+1,2})}{(z-\frac{i-1}{2}-x_{i,1})(z-\frac{i-1}{2}-x_{i,2})\prod_{k=i}^{r-1}\prod_{p\le 1+\delta_{k<r}}^{p'\le 1+\delta_{k+1<r}}(x_{k,p}-x_{k+1,p'})}$$
$$\in S^{(\mathfrak{sp}_{2r})}_{2\varsigma_i+\ldots+2\varsigma_{r-1}+\varsigma_r} \quad \text{for } 1\le i\le r,$$

$$\Upsilon\Big(\mathsf{E}_{ij'}(z)\Big) =$$
$$\frac{2(Q(z-\frac{i-1}{2},x_{i-1,1};x_{i,1},x_{i,2})+\frac{1}{2}(2x_{i-1,1}-x_{i,1}-x_{i,2}))\prod_{k=i}^{r-2}Q(x_{k,1},x_{k,2};x_{k+1,1},x_{k+1,2})}{(z-\frac{i-1}{2}-x_{i,1})(z-\frac{i-1}{2}-x_{i,2})\prod_{k=j}^{r-1}\prod_{p\le 1+\delta_{i\le k<r}}^{p'\le 1+\delta_{i\le k+1<r}}(x_{k,p}-x_{k+1,p'})}$$
$$\in S^{(\mathfrak{sp}_{2r})}_{\varsigma_j+\ldots+\varsigma_{i-1}+2\varsigma_i+\ldots+2\varsigma_{r-1}+\varsigma_r} \quad \text{for } 1\le j<i<r,$$

$$\Upsilon\Big(\mathsf{E}_{rj'}(z)\Big) = \frac{2}{(z-\frac{r}{2}-x_{r,1})\prod_{k=j}^{r-1}(x_{k,1}-x_{k+1,1})} \in S^{(\mathfrak{sp}_{2r})}_{\varsigma_j+\ldots+\varsigma_r} \quad \text{for } 1\le j<r,$$

$$\Upsilon\Big(\mathsf{E}_{i'j'}(z)\Big) = -\frac{1}{(z+\frac{i-2}{2}-r-x_{i-1,1})\prod_{k=j}^{i-2}(x_{k,1}-x_{k+1,1})} \in S^{(\mathfrak{sp}_{2r})}_{\varsigma_j+\ldots+\varsigma_{i-1}} \quad \text{for } 1\le j<i\le r.$$

(b) *For any* $1\le i<j\le 2r$, $\Upsilon(\mathsf{F}_{ji}(z))\in S^{(\mathfrak{sp}_{2r}),\mathrm{op}}[[z^{-1}]]$ *is given by the same formula (the expansion in* z^{-1} *of the corresponding rational function in (a)) as* $\Upsilon(\mathsf{E}_{ij}(z))\in S^{(\mathfrak{sp}_{2r})}[[z^{-1}]]$.

[6] Note that $Q(w,w-1;w-1/2,z)=0$ in accordance with the wheel conditions (B.11).

Lemma B.27 (Type B_r).

(a) *We have the following equalities:*

$$\Upsilon\Big(\mathsf{E}_{ij}(z)\Big) = -\frac{1}{(z-\frac{i-1}{2}-x_{i,1})\prod_{k=i}^{j-2}(x_{k,1}-x_{k+1,1})} \in S^{(\mathfrak{so}_{2r+1})}_{\varsigma_i+\ldots+\varsigma_{j-1}} \quad \text{for } 1\le i<j\le r+1,$$

$$\Upsilon\Big(\mathsf{E}_{ij'}(z)\Big) = -\frac{\prod_{k=j}^{r-1}(x_{k,1}-x_{k,2}-1)(x_{k,2}-x_{k,1}-1)}{(z-\frac{i-1}{2}-x_{i,1})\prod_{k=i}^{r-1}\prod_{p\le 1+\delta_{k\ge j}}^{p'\le 1+\delta_{k+1\ge j}}(x_{k,p}-x_{k+1,p'})}$$
$$\in S^{(\mathfrak{so}_{2r+1})}_{\varsigma_i+\ldots+\varsigma_{j-1}+2(\varsigma_j+\ldots+\varsigma_r)} \quad \text{for } 1\le i<j\le r,$$

$$\Upsilon\Big(\mathsf{E}_{ii'}(z)\Big) = -\frac{\prod_{k=i}^{r-1}(x_{k,1}-x_{k,2}-1)(x_{k,2}-x_{k,1}-1)}{(z-\frac{i-1}{2}-x_{i,1})(z-\frac{i-1}{2}-x_{i,2})\prod_{k=i}^{r-1}\prod_{p\le 2}^{p'\le 2}(x_{k,p}-x_{k+1,p'})}$$
$$\in S^{(\mathfrak{so}_{2r+1})}_{2(\varsigma_i+\ldots+\varsigma_r)} \quad \text{for } 1\le i\le r,$$

$$\Upsilon\Big(\mathsf{E}_{ij'}(z)\Big) = \frac{(z-\frac{i}{2}-x_{i-1,1})\prod_{k=i}^{r-1}(x_{k,1}-x_{k,2}-1)(x_{k,2}-x_{k,1}-1)}{(z-\frac{i-1}{2}-x_{i,1})(z-\frac{i-1}{2}-x_{i,2})\prod_{k=j}^{r-1}\prod_{p\le 1+\delta_{k\ge i}}^{p'\le 1+\delta_{k+1\ge i}}(x_{k,p}-x_{k+1,p'})}$$
$$\in S^{(\mathfrak{so}_{2r+1})}_{\varsigma_j+\ldots+\varsigma_{i-1}+2(\varsigma_i+\ldots+\varsigma_r)} \quad \text{for } 1\le j<i\le r,$$

$$\Upsilon\Big(\mathsf{E}_{i'j'}(z)\Big) = -\frac{1}{(z+\frac{i+1}{2}-r-x_{i-1,1})\prod_{k=j}^{i-2}(x_{k,1}-x_{k+1,1})}$$
$$\in S^{(\mathfrak{so}_{2r+1})}_{\varsigma_j+\ldots+\varsigma_{i-1}} \quad \text{for } 1\le j<i\le r+1.$$

(b) *For any* $1\le i<j\le 2r+1$, $\Upsilon(\mathsf{F}_{ji}(z))\in S^{(\mathfrak{so}_{2r+1}),\mathrm{op}}[[z^{-1}]]$ *is given by the same formula (the expansion in* z^{-1} *of the corresponding rational function in (a)) as* $\Upsilon(\mathsf{E}_{ij}(z))\in S^{(\mathfrak{so}_{2r+1})}[[z^{-1}]]$.

Inspecting the explicit formulas above, we obtain (cf. Remarks A.3, A.5 for the type D_r):

Corollary B.28. *(a) In the type* C_r*, we have (with* $\varepsilon_i\in\{\pm 1\}$ *defined as in (3.5)):*

$$\mathsf{E}^{(1)}_{ij} = -\varepsilon_i\varepsilon_j\mathsf{E}^{(1)}_{j'i'}, \qquad \mathsf{F}^{(1)}_{ji} = -\varepsilon_i\varepsilon_j\mathsf{F}^{(1)}_{i'j'}, \qquad \forall\, 1\le i<j\le 2r, \tag{B.29}$$

which imply the corresponding equalities for the matrix coefficients of $E^D(z)$, $F^D(z)$*:*

$$e^{(D)1}_{i,j} = -\varepsilon_i\varepsilon_j e^{(D)1}_{j',i'}, \qquad f^{(D)1}_{j,i} = -\varepsilon_i\varepsilon_j f^{(D)1}_{i',j'}, \qquad \forall\, 1\le i<j\le 2r.$$

(b) In the type B_r*, we have:*

$$\mathsf{E}^{(1)}_{ij} = -\mathsf{E}^{(1)}_{j'i'}, \qquad \mathsf{F}^{(1)}_{ji} = -\mathsf{F}^{(1)}_{i'j'}, \qquad \forall\, 1\le i<j\le 2r+1, \tag{B.30}$$

which imply the corresponding equalities for the matrix coefficients of $E^D(z)$, $F^D(z)$*:*

$$e^{(D)1}_{i,j} = -e^{(D)1}_{j',i'}, \qquad f^{(D)1}_{j,i} = -f^{(D)1}_{i',j'}, \qquad \forall\, 1\le i<j\le 2r+1.$$

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