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Difference operators via GKLO-type homomorphisms: shuffle approach and application to quantum Q-systems

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Abstract

We present a shuffle realization of the GKLO-type homomorphisms for shifted quantum affine, toroidal, and quiver algebras in the spirit of Feigin and Odesskii (Funktsional. Anal. Prilozhen. 31(3):57–70, 1997), thus generalizing its rational version of Frassek and Tsymbaliuk (Commun. Math. Phys. 392:545–619, 2022) and the type A construction of Finkelberg and Tsymbaliuk (Arnold Math. J. 5(2–3):197–283, 2019). As an application, this allows us to construct large families of commuting and q-commuting difference operators, in particular, providing a convenient approach to the Q-systems where it proves a conjecture of Di Francesco and Kedem (Commun. Math. Phys. 369(3):867–928, 2019).

Keywords Shuffle algebras · GKLO-type homomorphisms · Quantum Q-systems · Generalized Macdonald operators · Quantum loop algebras

Mathematics Subject Classification 17B37 · 81R10

1 Introduction

1.1 Summary

The key result of this note is the shuffle realization of the GKLO-type homomorphisms from various shifted quantum "loop" algebras to the algebras of (localized) difference operators. We use this to reinterpret the recent results of [4, 5] on the quantum Q-systems of type A. In the upcoming work, this will be also used as the main technical ingredient to:

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- prove the regularity of certain trigonometric BCD-type Lax matrices (generalizing the rational counterpart of [13]),
- develop the integral forms of K-theoretic Coulomb branches (generalizing the A-type case of [12]),
- study difference operators arising from large families of q-commuting elements in quantum affine algebras (generalizing [4] with $\mathfrak{sl}_2$ been replaced by any simple $\mathfrak{g}$).

The GKLO-type homomorphisms for the quantum loop algebras $U_q(L\mathfrak{g})$ were first introduced in [14] (hence, their acronym). Their analogues for the "shifted" versions (the shift refers to the fact that Cartan currents $\psi_i^{\pm}(z)$ start not necessarily from z^0 modes, while the defining relations are kept unchanged) arise naturally in the recent study of the quantized Coulomb branches, see [1, 2] and [11], providing algebraic models for the geometric objects.

On the other hand, the shuffle approach provides a convenient combinatorial model for the positive and negative subalgebras of such quantum algebras. An essential benefit of this approach is that it allows to work with various elements of quantum algebras that are provided by complicated formulas in the original loop generators, making it hard to work with them directly. In the present note, we focus on the following cases: quantum affine of any simple $\mathfrak{g}$, quantum toroidal of $\mathfrak{gl}_1$ and $\mathfrak{sl}_n$ ($n \geq 3$) with two parameters, and quantum quiver algebras, for which the shuffle realizations were established in [19], [15, 16], and [18], respectively.

Let $U_L^>$ denote the corresponding positive subalgebra, generated by the loop generators $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ (here, I denotes a labeling set, while the subscript "L" is merely used to remind of the loop realization, in spirit of [3]) subject to the corresponding defining relations. Then, one considers an $\mathbb{N}^I$-graded vector space $\mathbb{S} = \bigoplus_{\underline{k}\in\mathbb{N}^I} \mathbb{S}_{\underline{k}}$, with $\mathbb{S}_{\underline{k}}$ consisting of multisymmetric rational functions in the variables $\{x_{i,r}\}_{i\in I}^{1\leq r\leq k_i}$ subject to rather simple "pole" conditions, equipped with an algebra structure via the shuffle product $\star\colon \mathbb{S}_{\underline{k}} \times \mathbb{S}_{\underline{\ell}} \to \mathbb{S}_{\underline{k}+\underline{\ell}}$ given by

$$F(\ldots, x_{i,1}, \ldots, x_{i,k_i}, \ldots)\star G(\ldots, x_{i,1}, \ldots, x_{i,\ell_i}, \ldots) := \frac{1}{\prod_{i\in I} k_i! \cdot \ell_i!}$$
$$\times \mathrm{Sym}\left(F\left(\{x_{i,r}\}_{i\in I}^{1\leq r\leq k_i}\right) G\left(\{x_{i',r'}\}_{i'\in I}^{k_{i'}<r'\leq k_{i'}+\ell_{i'}}\right) \cdot \prod_{i\in I}^{i'\in I} \prod_{r\leq k_i}^{r'>k_{i'}} \zeta_{ii'}\left(\frac{x_{i,r}}{x_{i',r'}}\right)\right).$$

The rational ζ-factors are specifically chosen to allow for an algebra embedding

$$\Upsilon\colon U_L^> \hookrightarrow \mathbb{S} \quad \text{given by} \quad e_{i,r} \mapsto x_{i,1}^r \quad \text{for all} \quad i \in I, r \in \mathbb{Z}. \tag{1}$$

On the other hand, (the restriction of) the aforementioned GKLO-type homomorphism

$$\widetilde{\Phi}\colon U_L^> \longrightarrow \widetilde{\mathcal{A}}_{\underline{a}} \tag{2}$$

to the algebra $\widetilde{\mathcal{A}}_{\underline{a}}$ of localized difference operators, generated by $\{\mathsf{w}_{i,r}^{\pm 1}, D_{i,r}^{\pm 1}\}_{i\in I}^{1\le r\le a_i}$ as well as $\{(\mathsf{w}_{i,r}-\mathsf{q}_i^m \mathsf{w}_{i,s})^{-1}\}_{r\neq s}^{m\in\mathbb{Z}}$ subject to

$$[\mathsf{w}_{i,r}, \mathsf{w}_{j,s}] = 0 = [D_{i,r}, D_{j,s}] \quad \text{and} \quad D_{i,r}\mathsf{w}_{j,s} = \mathsf{q}_i^{\delta_{ij}\delta_{rs}} \mathsf{w}_{j,s} D_{i,r} \quad \text{for some} \quad \mathsf{q}_i,$$

is explicitly given by specifying $\widetilde{\Phi}(e_{i,r})$, reminiscent of the Gelfand–Tsetlin formulas in type A.

Thus, our main construction is the algebra homomorphism

$$\hat{\Phi}\colon \mathbb{S} \longrightarrow \widetilde{\mathcal{A}}'_{\underline{a}}, \tag{3}$$

where $\widetilde{\mathcal{A}}'_{\underline{a}}$ denotes a localization of $\widetilde{\mathcal{A}}_{\underline{a}}$ at some other elements $\mathsf{w}_{i,r} - \gamma \mathsf{w}_{j,s}$, given by

$$\mathbb{S}_{\underline{k}} \ni E \overset{\hat{\Phi}}{\mapsto} \sum_{\substack{m_1^{(i)}+\ldots+m_{a_i}^{(i)}=k_i \\ m_r^{(i)}\in\mathbb{N}\ \forall\, i\in I}} E\left(\left\{\mathsf{w}_{i,r}\mathsf{q}_i^{-(p-1)}\right\}_{i\in I, r\le a_i}^{1\le p\le m_r^{(i)}}\right) \cdot (\text{rational prefactor}) \cdot \prod_{i\in I}^{r\le a_i} D_{i,r}^{-m_r^{(i)}} \tag{4}$$

and such that its composition with Υ of (1) recovers $\widetilde{\Phi}$ of (2):

$$\widetilde{\Phi} = \hat{\Phi}\circ\Upsilon\colon U_L^{>} \longrightarrow \widetilde{\mathcal{A}}_{\underline{a}}. \tag{5}$$

In particular, the image of $U_L^{>}$ under the composition (5) is in the subalgebra $\widetilde{\mathcal{A}}_{\underline{a}}$ of $\widetilde{\mathcal{A}}'_{\underline{a}}$. This $\hat{\Phi}$ can be perceived as a trigonometric counterpart of a much older construction from [9].

We want to **emphasize** that this construction of $\hat{\Phi}$ is a general phenomenon that applies in a much wider setup. However, if one wishes to remain in the realm of quantum algebras, then one needs to restrict $\hat{\Phi}$ to the image of the embedding Υ of (1). The latter is often described by certain "wheel" conditions, see (16, 42, 63, 89) for the cases treated in the present note, which actually constitutes the core of the aforementioned shuffle algebra isomorphisms.

In the simplest case of quantum affine $\mathfrak{sl}_2$, some of the resulting difference operators can be patched nicely to form a q-commuting family satisfying the quantum Q-system relations of type A. On the other hand, for the case of quantum toroidal $\mathfrak{gl}_1$, we obtain the famous Macdonald difference operators as well as their generalizations from [5]. Finally, for the case of quantum toroidal $\mathfrak{sl}_n$, the images of natural commutative subalgebras of the quantum toroidal $U_L^{>}$ give rise to compelling large families of pairwise commuting difference operators (it is interesting to understand their relation to the recent construction of [20], if any).

1.2 Outline of the paper

The structure of the present paper is the following:

• In Sect. 2, we recall the notion of shifted quantum affine algebras and the GKLO-type homomorphisms $\widetilde{\Phi}^{\lambda,\text{z}}_{\mu}$ of (9), following [11]. The main result of this section is Theorem 2.8, which provides a shuffle realization of $\widetilde{\Phi}^{\lambda,\text{z}}_{\mu}$ restricted to the positive and negative subalgebras (actually, extending it to larger algebras $\mathbb{S}^{(\mathfrak{g})}$ and $\mathbb{S}^{(\mathfrak{g}),\mathrm{op}}$, whose elements are rational functions of (13) that do not necessarily satisfy the wheel conditions (16)). As an application, we construct a natural family of elements in the shifted quantum affine algebras whose $\widetilde{\Phi}^{\lambda,\text{z}}_{\mu}$-images are given by simple and interesting formulas of Lemma 2.12. In Remark 2.10, we explain the resemblance between our Theorem 2.8 and a much older result [9, Proposition 2].

• In Sect. 3, we generalize the results of Sect. 2 to the context of shifted quantum toroidal algebras of $\mathfrak{gl}_1$ (depending on two parameters). The main result of this section is Theorem 3.10, providing shuffle realization of the restrictions of the homomorphisms $\widetilde{\Phi}^{\text{z}}_{a}$ from Proposition 3.4 to the positive and negative subalgebras (again extended to the larger algebras $\mathbb{S}$ and $\mathbb{S}^{\mathrm{op}}$). In Lemma 3.12, we derive interesting difference operators as the images of (52, 53).

• In Sect. 4, we generalize the results of Sect. 2 to the context of shifted quantum toroidal algebras of $\mathfrak{sl}_n$ (depending on two parameters). The main result of this section is Theorem 4.8, providing shuffle realization of the restrictions of the homomorphisms $\widetilde{\Phi}^{\underline{a},\text{z}}_{\underline{b}}$ from Proposition 4.3 to the positive and negative subalgebras (extended to the larger algebras $\mathbb{S}^{[n]}$ and $\mathbb{S}^{[n],\mathrm{op}}$). In Lemma 4.10, we get interesting difference operators as the images of (72, 73). In Example 4.11, we use the shuffle descriptions [10, 21, 22] of the Bethe and horizontal Heisenberg subalgebras to construct large commutative families of difference operators.

• In Sect. 5, we generalize the results of Sect. 2 to the context of (shifted) quantum quiver algebras as recently introduced in [18]. The main result of this section is Theorem 5.7, providing shuffle realization of the restrictions of the new GKLO-type homomorphisms from Proposition 5.3 to the positive and negative subalgebras (extended to the larger algebras $\mathbb{S}^{Q}$ and $\mathbb{S}^{Q,\mathrm{op}}$), in analogy with Theorems 2.8, 3.10, 4.8.

• In Sect. 6, we present a shuffle interpretation of the quantum Q-system of type A, thus simplifying proofs of [4, Theorems 2.10, 2.11], see Propositions 6.3, 6.7, 6.8. We also match the difference operators of [4, §6] with those from Sect. 2 in the simplest case of $\mathfrak{g} = \mathfrak{sl}_2$, see Lemma 6.12 and Proposition 6.13. Finally, in Lemma 6.15, we explain how the images of the Cartan and negative subalgebras can be expressed via the images of finitely many elements in the positive subalgebra, after a localization at two elements.

• In Sect. 7, we provide a shuffle interpretation of the (t, q)-deformed Q-system of type A as recently investigated in [5]. In particular, we identify the generalized Macdonald operators (124) of [5] with the elements of Lemma 3.12, see Proposition 7.13.

This clarifies a shuffle approach in [5] and also establishes [5, Conjecture 1.17], see Theorem 7.14.

2 Shuffle realization of GKLO-type homomorphisms for $U^{\mathrm{sc}}_{\mu^+,\mu^-}$

2.1 Shifted quantum affine algebra

Let $\mathfrak{g}$ be a simple Lie algebra, and $\{\alpha_i^\vee\}_{i\in I}$ (resp. $\{\alpha_i\}_{i\in I}$) be the simple roots (resp. simple coroots) of $\mathfrak{g}$. Let $(\cdot,\cdot)$ denote the corresponding pairing on the root lattice, and set $\mathsf{d}_i := \frac{(\alpha_i^\vee,\alpha_i^\vee)}{2} \in \{1,2,3\}$. Let $(c_{ij})_{i,j\in I}$ be the Cartan matrix of $\mathfrak{g}$, so that $\mathsf{d}_i c_{ij} = (\alpha_i^\vee,\alpha_j^\vee) = \mathsf{d}_j c_{ji}$. Let Λ be the coweight lattice of $\mathfrak{g}$, and $\Lambda^+ \subset \Lambda$ be the submonoid of dominant integral weights.

Given coweights $\mu^\pm \in \Lambda$, set $\underline{b}^\pm = \{b_i^\pm\}_{i\in I} \in \mathbb{Z}^I$ with $b_i^\pm := \alpha_i^\vee(\mu^\pm)$. Following [11, §5(i)], we define the *simply connected version of shifted quantum affine algebra*, denoted by $U^{\mathrm{sc}}_{\mu^+,\mu^-}$ or $U^{\mathrm{sc}}_{\underline{b}^+,\underline{b}^-}$, as the associative $\mathbb{C}(q)$-algebra generated by $\{e_{i,r}, f_{i,r}, \psi^\pm_{i,\pm s_i^\pm}, (\psi^\pm_{i,\mp b_i^\pm})^{-1}\}_{i\in I}^{r\in\mathbb{Z}, s_i^\pm\geq -b_i^\pm}$ with the following defining relations (for all $i,j\in I$ and $\epsilon,\epsilon'\in\{\pm\}$):

$$[\psi_i^\epsilon(z),\psi_j^{\epsilon'}(w)] = 0, \quad \psi^\pm_{i,\mp b_i^\pm}\cdot(\psi^\pm_{i,\mp b_i^\pm})^{-1} = (\psi^\pm_{i,\mp b_i^\pm})^{-1}\cdot\psi^\pm_{i,\mp b_i^\pm} = 1, \tag{U1}$$

$$(z - q_i^{c_{ij}} w)e_i(z)e_j(w) = (q_i^{c_{ij}} z - w)e_j(w)e_i(z), \tag{U2}$$

$$(q_i^{c_{ij}} z - w) f_i(z) f_j(w) = (z - q_i^{c_{ij}} w) f_j(w) f_i(z), \tag{U3}$$

$$(z - q_i^{c_{ij}} w)\psi_i^\epsilon(z)e_j(w) = (q_i^{c_{ij}} z - w)e_j(w)\psi_i^\epsilon(z), \tag{U4}$$

$$(q_i^{c_{ij}} z - w)\psi_i^\epsilon(z) f_j(w) = (z - q_i^{c_{ij}} w) f_j(w)\psi_i^\epsilon(z), \tag{U5}$$

$$[e_i(z), f_j(w)] = \frac{\delta_{ij}}{q_i - q_i^{-1}}\delta\left(\frac{z}{w}\right)\left(\psi_i^+(z) - \psi_i^-(z)\right), \tag{U6}$$

$$\operatorname*{Sym}_{z_1,\ldots,z_{1-c_{ij}}} \sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix}_{q_i} e_i(z_1)\cdots e_i(z_r)e_j(w)e_i(z_{r+1})\cdots e_i(z_{1-c_{ij}}) = 0, \tag{U7}$$

$$\operatorname*{Sym}_{z_1,\ldots,z_{1-c_{ij}}} \sum_{r=0}^{1-c_{ij}} (-1)^r \begin{bmatrix} 1-c_{ij} \\ r \end{bmatrix}_{q_i} f_i(z_1)\cdots f_i(z_r)f_j(w)f_i(z_{r+1})\cdots f_i(z_{1-c_{ij}}) = 0, \tag{U8}$$

where $q_i := q^{d_i}$, $[a,b]_x := ab - x\cdot ba$, $[m]_q := \frac{q^m-q^{-m}}{q-q^{-1}}$, $\begin{bmatrix} m \\ r \end{bmatrix}_q := \frac{[m-r+1]_q\cdots[m]_q}{[1]_q\cdots[r]_q}$, $\operatorname*{Sym}_{z_1,\ldots,z_s}$ stands for the symmetrization in $z_1,\ldots,z_s$, and the generating series are defined as follows:

$$e_i(z) := \sum_{r\in\mathbb{Z}} e_{i,r} z^{-r}, \quad f_i(z) := \sum_{r\in\mathbb{Z}} f_{i,r} z^{-r},$$
$$\psi_i^{\pm}(z) := \sum_{r\geq -b_i^{\pm}} \psi_{i,\pm r}^{\pm} z^{\mp r}, \quad \delta(z) := \sum_{r\in\mathbb{Z}} z^r. \tag{6}$$

Let $U^{\mathrm{sc},<}_{\mu^+,\mu^-}$, $U^{\mathrm{sc},>}_{\mu^+,\mu^-}$, $U^{\mathrm{sc},0}_{\mu^+,\mu^-}$ be the $\mathbb{C}(q)$-subalgebras of $U^{\mathrm{sc}}_{\mu^+,\mu^-}$ generated by $\{f_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$, $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$, $\{\psi^{\pm}_{i,\pm s_i^{\pm}}, (\psi^{\pm}_{i,\mp b_i^{\pm}})^{-1}\}_{i\in I}^{s_i^{\pm}\geq -b_i^{\pm}}$, respectively. The following result is standard:

Proposition 2.2 [11] *(a) (Triangular decomposition of $U^{\mathrm{sc}}_{\mu^+,\mu^-}$) The multiplication map*

$$m\colon U^{\mathrm{sc},<}_{\mu^+,\mu^-} \otimes U^{\mathrm{sc},0}_{\mu^+,\mu^-} \otimes U^{\mathrm{sc},>}_{\mu^+,\mu^-} \longrightarrow U^{\mathrm{sc}}_{\mu^+,\mu^-}$$

is an isomorphism of $\mathbb{C}(q)$-vector spaces.

(b) The algebras $U^{\mathrm{sc},<}_{\mu^+,\mu^-}$, $U^{\mathrm{sc},>}_{\mu^+,\mu^-}$, and $U^{\mathrm{sc},0}_{\mu^+,\mu^-}$ are isomorphic to the $\mathbb{C}(q)$-algebras generated by $\{f_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$, $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$, and $\{\psi^{\pm}_{i,\pm s_i^{\pm}}, (\psi^{\pm}_{i,\mp b_i^{\pm}})^{-1}\}_{i\in I}^{s_i^{\pm}\geq -b_i^{\pm}}$ with the defining relations (U3, U8), (U2, U7), and (U1), respectively. In particular, $U^{\mathrm{sc},<}_{\mu^+,\mu^-}$, $U^{\mathrm{sc},>}_{\mu^+,\mu^-}$ are independent of $\mu^{\pm}\in\Lambda$.

The algebras $U^{\mathrm{sc}}_{\mu^+,\mu^-}$ and $U^{\mathrm{sc}}_{0,\mu^++\mu^-}$ are naturally isomorphic for any $\mu^{\pm}\in\Lambda$, see [11, p. 162]. Therefore, we do not lose generality by considering only $U_q^{(b)} = U_q^{\mu} := U^{\mathrm{sc}}_{0,\mu}$ in the rest of this note. The quantum loop algebra $U_q(L\mathfrak{g})$ is isomorphic to $U^{\mathrm{sc}}_{0,0}/(\psi^+_{i,0}\psi^-_{i,0}-1)_{i\in I}$.

2.3 GKLO-type homomorphisms

Fix an orientation of the graph $\mathrm{Dyn}(\mathfrak{g})$ obtained from the Dynkin diagram of $\mathfrak{g}$ by replacing all multiple edges by simple ones. The notation $j - i$ (resp. $j\to i$ or $j\leftarrow i$) is to indicate an edge (resp. oriented edge pointing towards i or j) between the vertices $i, j\in \mathrm{Dyn}(\mathfrak{g})$. We fix a dominant coweight $\lambda\in\Lambda^+$ and a coweight $\mu\in\Lambda$, such that $\lambda-\mu=\sum_{i\in I} a_i\alpha_i$ with $a_i\in\mathbb{N}$. We also fix a sequence $\underline{\lambda}=(\omega_{i_1},\ldots,\omega_{i_N})$ of fundamental coweights, such that $\sum_{k=1}^N \omega_{i_k}=\lambda$, as well as a sequence $\underline{\mathsf{z}}=(\mathsf{z}_1,\ldots,\mathsf{z}_N)\in(\mathbb{C}^\times)^N$.

Consider the associative $\mathbb{C}(q)$-algebra $\hat{\mathcal{A}}^q_{\mathrm{frac}}$ generated by $\{D^{\pm 1}_{i,r}, \mathsf{w}^{\pm 1/2}_{i,r}\}_{i\in I}^{1\leq r\leq a_i}$ subject to

$$[D_{i,r}, D_{j,s}] = \left[\mathsf{w}^{1/2}_{i,r}, \mathsf{w}^{1/2}_{j,s}\right] = 0, \;\; D^{\pm 1}_{i,r} D^{\mp 1}_{i,r} = \mathsf{w}^{\pm 1/2}_{i,r}\mathsf{w}^{\mp 1/2}_{i,r} = 1,$$
$$D_{i,r}\mathsf{w}^{1/2}_{j,s} = q_i^{\delta_{ij}\delta_{rs}}\mathsf{w}^{1/2}_{j,s} D_{i,r} \tag{7}$$

for all $i, j \in I$, $1 \le r \le a_i$, $1 \le s \le a_j$. Let $\widetilde{\mathcal{A}}^q_{\mathrm{frac}}$ be the localization of $\hat{\mathcal{A}}^q_{\mathrm{frac}}$ by the multiplicative set generated by $\{\mathsf{w}_{i,r} - q_i^m \mathsf{w}_{i,s}\}_{i \in I, m \in \mathbb{Z}}^{1 \le r \ne s \le a_i}$, which obviously satisfies the Ore conditions. We also define:

$$\mathsf{Z}_i(z) := \prod_{1 \le s \le N}^{i_s = i} \left(1 - \frac{q_i \mathsf{z}_s}{z}\right), \ W_i(z) := \prod_{r=1}^{a_i} \left(1 - \frac{\mathsf{w}_{i,r}}{z}\right), \ W_{i,r}(z) := \prod_{1 \le s \le a_i}^{s \ne r} \left(1 - \frac{\mathsf{w}_{i,s}}{z}\right). \tag{8}$$

The following result has been established in [11, Theorem 7.1] (in the *unshifted case* $\mu^+ = \mu^- = 0$, more precisely for $U_q(L\mathfrak{g})$, this result appeared without a proof in [14]):

Proposition 2.4 [11] *There exists a unique $\mathbb{C}(q)$-algebra homomorphism*

$$\widetilde{\Phi}^{\lambda, \mathsf{z}}_{\mu} \colon U_q^{\mu} \longrightarrow \widetilde{\mathcal{A}}^q_{\mathrm{frac}} \tag{9}$$

such that

$$\begin{aligned}
e_i(z) &\mapsto \frac{-q_i}{1 - q_i^2} \prod_{t=1}^{a_i} \mathsf{w}_{i,t} \prod_{j \to i} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{c_{ji}/2} \cdot \sum_{r=1}^{a_i} \delta\left(\frac{\mathsf{w}_{i,r}}{z}\right) \frac{Z_i(\mathsf{w}_{i,r})}{W_{i,r}(\mathsf{w}_{i,r})} \prod_{j \to i} \prod_{p=1}^{-c_{ji}} W_j(q_j^{-c_{ji}-2p} z) D_{i,r}^{-1}, \\
f_i(z) &\mapsto \frac{1}{1 - q_i^2} \prod_{j \leftarrow i} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{c_{ji}/2} \cdot \sum_{r=1}^{a_i} \delta\left(\frac{q_i^2 \mathsf{w}_{i,r}}{z}\right) \frac{1}{W_{i,r}(\mathsf{w}_{i,r})} \prod_{j \leftarrow i} \prod_{p=1}^{-c_{ji}} W_j(q_j^{-c_{ji}-2p} z) D_{i,r}, \\
\psi_i^{\pm}(z) &\mapsto \prod_{t=1}^{a_i} \mathsf{w}_{i,t} \prod_{j - i} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{c_{ji}/2} \cdot \left(\frac{Z_i(z)}{W_i(z) W_i(q_i^{-2} z)} \prod_{j - i} \prod_{p=1}^{-c_{ji}} W_j(q_j^{-c_{ji}-2p} z)\right)^{\pm}.
\end{aligned} \tag{10}$$

We write $\gamma(z)^{\pm}$ for the expansion of a rational function $\gamma(z)$ in $z^{\mp 1}$, respectively.

2.5 Shuffle algebra realization of the positive and negative subalgebras

According to Proposition 2.2(b), we have algebra isomorphisms for any $\mu^+, \mu^- \in \Lambda$:

$$\begin{aligned}
U_q^{\mu,>} &\xrightarrow{\sim} U_q^{>}(L\mathfrak{g}) \quad \text{given by} \quad e_{i,r} \mapsto e_{i,r} \quad \text{for} \quad i \in I, r \in \mathbb{Z}, \\
U_q^{\mu,<} &\xrightarrow{\sim} U_q^{<}(L\mathfrak{g}) \quad \text{given by} \quad f_{i,r} \mapsto f_{i,r} \quad \text{for} \quad i \in I, r \in \mathbb{Z}.
\end{aligned} \tag{11}$$

We also note the algebra isomorphism

$$U_q^{<}(L\mathfrak{g}) \xrightarrow{\sim} U_q^{>}(L\mathfrak{g})^{\mathrm{op}} \quad \text{given by} \quad f_{i,r} \mapsto e_{i,r} \quad \text{for} \quad i \in I, r \in \mathbb{Z}, \tag{12}$$

where for any algebra A we use A^{op} to denote the algebra with the opposite multiplication.

Consider an $\mathbb{N}^I$-graded $\mathbb{C}(q)$-vector space $\mathbb{S}^{(\mathfrak{g})} = \bigoplus_{\underline{k}=(k_i)_{i\in I}\in\mathbb{N}^I} \mathbb{S}^{(\mathfrak{g})}_{\underline{k}}$, with the graded components

$$\mathbb{S}^{(\mathfrak{g})}_{\underline{k}} = \left\{ F = \frac{f(\{x_{i,r}\}_{i\in I}^{1\le r\le k_i})}{\prod_{i-j}^{\text{unordered}} \prod_{r\le k_i}^{s\le k_j} (x_{i,r} - x_{j,s})} \,\Big|\, f \in \mathbb{C}\Big[\{x_{i,r}^{\pm 1}\}_{i\in I}^{1\le r\le k_i}\Big]^{S_{\underline{k}}} \right\}, \qquad (13)$$

where $S_{\underline{k}} := \prod_{i\in I} S(k_i)$ is the product of symmetric groups. We also fix rational functions:

$$\zeta_{ij}\left(\frac{z}{w}\right) = \frac{z - q_i^{-c_{ij}} w}{z - w} \qquad \forall\, i, j \in I. \qquad (14)$$

Let us now introduce the bilinear *shuffle product* $\star$ on $\mathbb{S}^{(\mathfrak{g})}$ as follows:

$$F(\ldots, x_{i,1}, \ldots, x_{i,k_i}, \ldots) \star G(\ldots, x_{i,1}, \ldots, x_{i,\ell_i}, \ldots) := \frac{1}{\underline{k}! \cdot \underline{\ell}!}$$
$$\times \mathrm{Sym}\left(F\left(\{x_{i,r}\}_{i\in I}^{1\le r\le k_i}\right) G\left(\{x_{i',r'}\}_{i'\in I}^{k_{i'}<r'\le k_{i'}+\ell_{i'}}\right) \cdot \prod_{i\in I}^{i'\in I} \prod_{r\le k_i}^{r'>k_{i'}} \zeta_{ii'}\left(\frac{x_{i,r}}{x_{i',r'}}\right)\right). \qquad (15)$$

Here, $\underline{k}! = \prod_{i\in I} k_i!$, while the *symmetrization* of $f \in \mathbb{C}(\{x_{i,1}, \ldots, x_{i,m_i}\}_{i\in I})$ is defined via:

$$\mathrm{Sym}\,(f)\Big(\{x_{i,1}, \ldots, x_{i,m_i}\}_{i\in I}\Big) = \sum_{\sigma_i\in S(m_i)\,\forall\, i\in I} f\Big(\{x_{i,\sigma_i(1)}, \ldots, x_{i,\sigma_i(m_i)}\}_{i\in I}\Big).$$

This endows $\mathbb{S}^{(\mathfrak{g})}$ with a structure of an associative $\mathbb{C}(q)$-algebra with the unit $\mathbf{1} \in \mathbb{S}^{(\mathfrak{g})}_{(0,\ldots,0)}$.

We are interested in an $\mathbb{N}^I$-graded $\mathbb{C}(q)$-subspace of $\mathbb{S}^{(\mathfrak{g})}$ defined by the *wheel conditions*:

$$F\Big(\{x_{i,r}\}\Big)\Big|_{(x_{i,1},x_{i,2},x_{i,3},\ldots,x_{i,1-c_{ij}})\mapsto(w,wq_i^2,wq_i^4,\ldots,wq_i^{-2c_{ij}}),\; x_{j,1}\mapsto wq_i^{-c_{ij}}} = 0 \qquad (16)$$

for any connected vertices $i - j$ in $\mathrm{Dyn}(\mathfrak{g})$. Let $S^{(\mathfrak{g})} \subset \mathbb{S}^{(\mathfrak{g})}$ denote the subspace of all such elements F. It is straightforward to check that $S^{(\mathfrak{g})} \subset \mathbb{S}^{(\mathfrak{g})}$ is $\star$-closed. The resulting algebra $\left(S^{(\mathfrak{g})}, \star\right)$ is called the *(trigonometric Feigin–Odesskii) shuffle algebra of type* $\mathfrak{g}$.

The following result has been recently established in [19, Theorem 1.7]:

Proposition 2.6 [19] *The assignments $e_{i,r} \mapsto x_{i,1}^r$ and $f_{i,r} \mapsto x_{i,1}^r$ for $i \in I, r \in \mathbb{Z}$ give rise to $\mathbb{C}(q)$-algebra isomorphisms:*

$$\Upsilon\colon U_q^>(L\mathfrak{g}) \xrightarrow{\sim} S^{(\mathfrak{g})} \quad \text{and} \quad \Upsilon\colon U_q^<(L\mathfrak{g}) \xrightarrow{\sim} S^{(\mathfrak{g}),\mathrm{op}}. \qquad (17)$$

2.7 Shuffle algebra realization of the GKLO-type homomorphisms

The main new result of this section is the shuffle algebra interpretation of the homomorphisms $\widetilde{\Phi}_{\mu}^{\lambda,\mathsf{z}}$. We note that the type A case of this result is due to [12, Theorem 4.11], while its rational counterpart is due to [13, Theorem B.17], where they played crucial roles.

To this end, for any $i \in I$ and $1 \le r \le a_i$, we define:

$$Y_{i,r}(z) := \frac{1}{q_i - q_i^{-1}} \prod_{t=1}^{a_i} \mathsf{w}_{i,t} \prod_{j \to i} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{c_{ji}/2} \cdot \frac{\mathsf{Z}_i(z) \prod_{j \to i} \prod_{p=1}^{-c_{ji}} W_j(zq_j^{-c_{ji}-2p})}{W_{i,r}(z)},$$

$$Y'_{i,r}(z) := \frac{1}{1 - q_i^2} \prod_{j \leftarrow i} \prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{c_{ji}/2} \cdot \frac{\prod_{j \leftarrow i} \prod_{p=1}^{-c_{ji}} W_j(zq_j^{-c_{ji}-2p})}{W_{i,r}(zq_i^{-2})}. \tag{18}$$

Define the $\mathbb{C}(q)$-algebra $\widetilde{\mathcal{A}}_{\mathrm{frac}}^{q,\prime}$ as the further localization of $\widetilde{\mathcal{A}}_{\mathrm{frac}}^{q}$ by the multiplicative set generated by $\{\mathsf{w}_{i,r} - q^m \mathsf{w}_{j,s}\}_{i-j,m\in\mathbb{Z}}^{r\le a_i, s\le a_j}$. We note that $\widetilde{\mathcal{A}}_{\mathrm{frac}}^{q}$ is naturally embedded into $\widetilde{\mathcal{A}}_{\mathrm{frac}}^{q,\prime}$. Then, we have the following result:

Theorem 2.8 (a) *The assignment*

$$\begin{aligned}
&\mathbb{S}_{\underline{k}}^{(\mathfrak{g})} \ni E \mapsto \prod_{i\in I} q_i^{k_i - k_i^2} \\
&\times \sum_{\substack{m_1^{(i)}+\ldots+m_{a_i}^{(i)}=k_i \\ m_r^{(i)}\in\mathbb{N}\ \forall\, i\in I}} \left\{ \prod_{i\in I} \prod_{r=1}^{a_i} \prod_{p=1}^{m_r^{(i)}} Y_{i,r}\left(\mathsf{w}_{i,r} q_i^{-2(p-1)}\right) \cdot E\left(\left\{\mathsf{w}_{i,r} q_i^{-2(p-1)}\right\}_{i\in I, 1\le r\le a_i}^{1\le p\le m_r^{(i)}}\right)\right. \\
&\times \prod_{i\in I} \prod_{1\le r\le a_i} \prod_{1\le p_1<p_2\le m_r^{(i)}} \zeta_{ii}^{-1}\left(\mathsf{w}_{i,r} q_i^{-2(p_1-1)} \big/ \mathsf{w}_{i,r} q_i^{-2(p_2-1)}\right) \\
&\times \prod_{i\in I} \prod_{1\le r_1\ne r_2\le a_i} \prod_{1\le p_1\le m_{r_1}^{(i)}}^{1\le p_2\le m_{r_2}^{(i)}} \zeta_{ii}^{-1}\left(\mathsf{w}_{i,r_1} q_i^{-2(p_1-1)} \big/ \mathsf{w}_{i,r_2} q_i^{-2(p_2-1)}\right) \\
&\times \left.\prod_{j\to i} \prod_{1\le r_1\le a_i}^{1\le r_2\le a_j} \prod_{1\le p_1\le m_{r_1}^{(i)}}^{1\le p_2\le m_{r_2}^{(j)}} \zeta_{ij}^{-1}\left(\mathsf{w}_{i,r_1} q_i^{-2(p_1-1)} \big/ \mathsf{w}_{j,r_2} q_j^{-2(p_2-1)}\right) \cdot \prod_{i\in I} \prod_{r=1}^{a_i} D_{i,r}^{-m_r^{(i)}} \right\}
\end{aligned} \tag{19}$$

gives rise to the algebra homomorphism

$$\widehat{\Phi}_{\mu}^{\lambda,\mathsf{z}} \colon \mathbb{S}^{(\mathfrak{g})} \longrightarrow \widetilde{\mathcal{A}}_{\mathrm{frac}}^{q,\prime}. \tag{20}$$

Moreover, the composition

$$U_q^{\mu,>} \xrightarrow[\sim]{(11)} U_q^{>}(L\mathfrak{g}) \xrightarrow[\sim]{\Upsilon} S^{(\mathfrak{g})} \xrightarrow{\widehat{\Phi}^{\lambda,\mathsf{z}}_{\mu}} \widetilde{\mathcal{A}}^{q,\prime}_{\mathrm{frac}} \tag{21}$$

coincides with the restriction of the homomorphism $\widetilde{\Phi}^{\lambda,\mathsf{z}}_{\mu}$ *of* (9) *to the subalgebra* $U_q^{\mu,>}$ *of* U_q^{μ}. *In particular, the image of* $U_q^{\mu,>}$ *under the composition* (21) *is in the subalgebra* $\widetilde{\mathcal{A}}^{q}_{\mathrm{frac}}$ *of* $\widetilde{\mathcal{A}}^{q,\prime}_{\mathrm{frac}}$.

(b) *The assignment*

$$\begin{aligned}
&\mathbb{S}^{(\mathfrak{g}),\mathrm{op}}_{\underline{k}} \ni F \mapsto \\
&\sum_{\substack{m_1^{(i)}+\ldots+m_{a_i}^{(i)}=k_i \\ m_r^{(i)}\in\mathbb{N}\ \forall i\in I}} \left\{ \prod_{i\in I}\prod_{r=1}^{a_i}\prod_{p=1}^{m_r^{(i)}} Y'_{i,r}\left(\mathsf{w}_{i,r}q_i^{2p}\right)\cdot F\left(\left\{\mathsf{w}_{i,r}q_i^{2p}\right\}_{i\in I,1\leq r\leq a_i}^{1\leq p\leq m_r^{(i)}}\right)\right. \\
&\times\prod_{i\in I}\prod_{1\leq r\leq a_i}\prod_{1\leq p_1<p_2\leq m_r^{(i)}} \zeta_{ii}^{-1}\left(\mathsf{w}_{i,r}q_i^{2p_2}\big/\mathsf{w}_{i,r}q_i^{2p_1}\right) \\
&\times\prod_{i\in I}\prod_{1\leq r_1\neq r_2\leq a_i}\prod_{1\leq p_1\leq m_{r_1}^{(i)}}^{1\leq p_2\leq m_{r_2}^{(i)}} q_i^{-1}\zeta_{ii}^{-1}\left(\mathsf{w}_{i,r_2}q_i^{2p_2}\big/\mathsf{w}_{i,r_1}q_i^{2p_1}\right) \\
&\left.\times\prod_{j\leftarrow i}\prod_{1\leq r_1\leq a_i}^{1\leq r_2\leq a_j}\prod_{1\leq p_1\leq m_{r_1}^{(i)}}^{1\leq p_2\leq m_{r_2}^{(j)}} \zeta_{ji}^{-1}\left(\mathsf{w}_{j,r_2}q_j^{2p_2}\big/\mathsf{w}_{i,r_1}q_i^{2p_1}\right)\cdot\prod_{i\in I}\prod_{r=1}^{a_i} D_{i,r}^{m_r^{(i)}}\right\}
\end{aligned} \tag{22}$$

gives rise to the algebra homomorphism

$$\widehat{\Phi}^{\lambda,\mathsf{z}}_{\mu}\colon \mathbb{S}^{(\mathfrak{g}),\mathrm{op}} \longrightarrow \widetilde{\mathcal{A}}^{q,\prime}_{\mathrm{frac}}. \tag{23}$$

Moreover, the composition

$$U_q^{\mu,<} \xrightarrow[\sim]{(11)} U_q^{<}(L\mathfrak{g}) \xrightarrow[\sim]{\Upsilon} S^{(\mathfrak{g}),\mathrm{op}} \xrightarrow{\widehat{\Phi}^{\lambda,\mathsf{z}}_{\mu}} \widetilde{\mathcal{A}}^{q,\prime}_{\mathrm{frac}} \tag{24}$$

coincides with the restriction of the homomorphism $\widetilde{\Phi}^{\lambda,\mathsf{z}}_{\mu}$ *of* (9) *to the subalgebra* $U_q^{\mu,<}$ *of* U_q^{μ}. *In particular, the image of* $U_q^{\mu,<}$ *under the composition* (24) *is in the subalgebra* $\widetilde{\mathcal{A}}^{q}_{\mathrm{frac}}$ *of* $\widetilde{\mathcal{A}}^{q,\prime}_{\mathrm{frac}}$.

Proof (a) Let us denote the right-hand side of (19) by $\widehat{\Phi}^{\lambda,\mathsf{z}}_{\mu}(E)$. A tedious straightforward verification proves $\widehat{\Phi}^{\lambda,\mathsf{z}}_{\mu}(E\star E')=\widehat{\Phi}^{\lambda,\mathsf{z}}_{\mu}(E)\widehat{\Phi}^{\lambda,\mathsf{z}}_{\mu}(E')$ for any $E\in\mathbb{S}^{(\mathfrak{g})}_{\underline{k}}$, $E'\in\mathbb{S}^{(\mathfrak{g})}_{\underline{\ell}}$

with arbitrary $\underline{k}, \underline{\ell} \in \mathbb{N}^I$. Thus, $\widehat{\Phi}^{\lambda,\mathsf{z}}_{\mu}\colon \mathbb{S}^{(\mathfrak{g})} \to \widetilde{\mathcal{A}}^{q,\prime}_{\mathrm{frac}}$ is a $\mathbb{C}(q)$-algebra homomorphism, and clearly the images of $\{e_{i,r}\}_{i\in I}^{r\in\mathbb{Z}}$ under (21) and $\widetilde{\Phi}^{\lambda,\mathsf{z}}_{\mu}$ do coincide. This completes our proof of Theorem 2.8(a).

(b) The proof of Theorem 2.8(b) is completely analogous. □

Remark 2.9 We note that Theorem 2.8 can actually be used to simplify our proof of Proposition 2.4. Indeed, it immediately implies the compatibility of the assignment $\widetilde{\Phi}^{\lambda,\mathsf{z}}_{\mu}$ with the defining relations (U2, U3, U7, U8), while the compatibility with (U1, U4, U5) is easily checked. Thus, it remains only to prove the compatibility with (U6), which is verified by expressing $\gamma(z)^+ - \gamma(z)^-$ as a sum of delta-functions in a standard way, see [11, Lemma C.1, §C(vi)].

Remark 2.10 The construction (19) is reminiscent of that from [9, Proposition 2] in the elliptic setting. To this end, we consider the $\mathbb{C}(q)$-algebra $\widetilde{\mathcal{B}}^{(\underline{a}),q}_{\mathrm{frac}}$ generated by $\{\mathsf{w}^{\pm 1}_{i,r}, \mathsf{E}_{i,r}\}_{i\in I}^{1\leq r\leq a_i}$, being further localized by the multiplicative set generated by $\{\mathsf{w}_{i,r} - q^{mc_{ij}}\mathsf{w}_{j,s}\}_{(i,r)\neq(j,s)}^{c_{ij}\neq 0, m\in\mathbb{Z}}$, with:

$$
\begin{gathered}
\mathsf{w}_{i,r}\mathsf{w}_{j,s} = \mathsf{w}_{j,s}\mathsf{w}_{i,r}, \quad \mathsf{E}_{i,r}\mathsf{w}_{j,s} = q_i^{-2\delta_{ij}\delta_{rs}}\mathsf{w}_{j,s}\mathsf{E}_{i,r}, \\
\mathsf{E}_{i,r}\mathsf{E}_{j,s} = \frac{q_i^{c_{ij}}\mathsf{w}_{i,r} - \mathsf{w}_{j,s}}{\mathsf{w}_{i,r} - q_i^{c_{ij}}\mathsf{w}_{j,s}}\mathsf{E}_{j,s}\mathsf{E}_{i,r}.
\end{gathered} \tag{25}
$$

This algebra is equipped with the following homomorphism to the algebra $\widetilde{\mathcal{A}}^{q,\prime}_{\mathrm{frac}}$:

$$
\varsigma\colon \widetilde{\mathcal{B}}^{(\underline{a}),q}_{\mathrm{frac}} \longrightarrow \widetilde{\mathcal{A}}^{q,\prime}_{\mathrm{frac}} \quad \text{given by} \quad \mathsf{w}_{i,r} \mapsto \mathsf{w}_{i,r},\ \mathsf{E}_{i,r} \mapsto Y_{i,r}(\mathsf{w}_{i,r})D^{-1}_{i,r}. \tag{26}
$$

Then:

(a) The restriction of the algebra homomorphism $\widetilde{\Phi}^{\lambda,\mathsf{z}}_{\mu}$ to the positive subalgebra $U^{\mu,>}_{q}$, identified with $U^{>}_{q}(L\mathfrak{g})$ via (11), can be interpreted as a composition of ς from (26) and

$$
\overline{\Phi}_{\underline{a}}\colon U^{>}_{q}(L\mathfrak{g}) \longrightarrow \widetilde{\mathcal{B}}^{(\underline{a}),q}_{\mathrm{frac}} \quad \text{given by} \quad e_i(z) \mapsto \sum_{r=1}^{a_i} \delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\cdot \mathsf{E}_{i,r}. \tag{27}
$$

(b) The homomorphisms $\overline{\Phi}_{\underline{a}}$ of (27) can be obtained from their simplest counterparts with $\underline{a} = (0,\ldots,0,1,\ldots,0)$ via the "twisted tensor product". To this end, for $\underline{a}^{(1)}, \underline{a}^{(2)} \in \mathbb{N}^I$ set $\underline{a}^{(12)} := \underline{a}^{(1)} + \underline{a}^{(2)}$, and consider the corresponding algebras $\widetilde{\mathcal{B}}^{(1),q}_{\mathrm{frac}}, \widetilde{\mathcal{B}}^{(2),q}_{\mathrm{frac}}, \widetilde{\mathcal{B}}^{(12),q}_{\mathrm{frac}}$. Let $U^{\geq}_{q}(L\mathfrak{g})$ be the subalgebra generated by $\{e_{i,r}, \psi^{-}_{i,-k}\}_{i\in I}^{r\in\mathbb{Z},k\in\mathbb{N}}$. It is endowed with the formal coproduct:

$$
\Delta\colon e_i(z) \mapsto e_i(z)\otimes 1 + \psi^{-}_i(z)\otimes e_i(z), \quad \psi^{-}_i(z) \mapsto \psi^{-}_i(z)\otimes\psi^{-}_i(z). \tag{28}
$$

Following (10), let us extend the algebra homomorphism (27) to $\overline{\Phi}_{\underline{a}}\colon U_q^{\geq}(L\mathfrak{g})\to\widetilde{\mathcal{B}}^{(\underline{a}),q}_{\mathrm{frac}}$. We also consider the algebra embedding $\iota\colon \widetilde{\mathcal{B}}^{(12),q}_{\mathrm{frac}}\hookrightarrow \widetilde{\mathcal{B}}^{(1),q}_{\mathrm{frac}}\otimes\widetilde{\mathcal{B}}^{(2),q}_{\mathrm{frac}}$ determined by

$$\mathsf{w}_{i,r}\mapsto\begin{cases}\mathsf{w}^{(1)}_{i,r} & \text{if } r\leq a^{(1)}_i\\ \mathsf{w}^{(2)}_{i,r-a^{(1)}_i} & \text{if } r>a^{(1)}_i\end{cases},\quad \mathsf{E}_{i,r}\mapsto\begin{cases}\mathsf{E}^{(1)}_{i,r} & \text{if } r\leq a^{(1)}_i\\ \overline{\Phi}_{\underline{a}^{(1)}}(\psi_i^-(\mathsf{w}^{(2)}_{i,r-a^{(1)}_i}))\mathsf{E}^{(2)}_{i,r-a^{(1)}_i} & \text{if } r>a^{(1)}_i\end{cases}. \tag{29}$$

Then, $\overline{\Phi}_{\underline{a}^{(1)}+\underline{a}^{(2)}}\colon U_q^{\geq}(L\mathfrak{g})\to\widetilde{\mathcal{B}}^{(12),q}_{\mathrm{frac}}$ factors through the composition $(\overline{\Phi}_{\underline{a}^{(1)}}\otimes\overline{\Phi}_{\underline{a}^{(2)}})\circ\Delta$, that is:

$$\iota\circ\overline{\Phi}_{\underline{a}^{(1)}+\underline{a}^{(2)}}=(\overline{\Phi}_{\underline{a}^{(1)}}\otimes\overline{\Phi}_{\underline{a}^{(2)}})\circ\Delta.$$

2.11 Special difference operators

For any $\underline{k}\in\mathbb{N}^I$ and any multisymmetric Laurent polynomial $g\in\mathbb{C}(q)\left[\{x^{\pm1}_{i,r}\}^{1\leq r\leq k_i}_{i\in I}\right]^{S_{\underline{k}}}$, consider the following shuffle element $\widetilde{E}_{\underline{k}}(g)\in S^{(\mathfrak{g})}_{\underline{k}}$:

$$\widetilde{E}_{\underline{k}}(g):=\prod_{i\in I}\left\{q_i^{k_i^2-k_i}(q_i-q_i^{-1})^{k_i}\right\}\frac{\prod_{i\in I}\prod_{1\leq r\neq s\leq k_i}(x_{i,r}-q_i^{-2}x_{i,s})\cdot g\left(\{x_{i,r}\}^{1\leq r\leq k_i}_{i\in I}\right)}{\prod_{i\to j}\prod^{s\leq k_j}_{r\leq k_i}(x_{j,s}-x_{i,r})}. \tag{30}$$

These elements obviously satisfy the wheel conditions (16), due to the presence of the factor $\prod_{i\in I}\prod_{1\leq r\neq s\leq k_i}(x_{i,r}-q_i^{-2}x_{i,s})$ and thus can be written as $\widetilde{E}_{\underline{k}}(g)=\Upsilon(\widetilde{e}_{\underline{k}}(g))$ for unique $\widetilde{e}_{\underline{k}}(g)\in U_q^{\mu,>}\simeq U_q^{>}(L\mathfrak{g})$ by Proposition 2.6, so that $\widehat{\Phi}^{\lambda,\mathsf{z}}_{\mu}(\widetilde{E}_{\underline{k}}(g))=\widetilde{\Phi}^{\lambda,\mathsf{z}}_{\mu}(\widetilde{e}_{\underline{k}}(g))$ by Theorem 2.8(a). We also consider $\widetilde{F}_{\underline{k}}(g)\in S^{(\mathfrak{g}),\mathrm{op}}_{\underline{k}}$ defined via:

$$\widetilde{F}_{\underline{k}}(g):=\prod_{i\in I}\left\{q_i^{k_i-k_i^2}(1-q_i^2)^{k_i}\right\}\frac{\prod_{i\in I}\prod_{1\leq r\neq s\leq k_i}(x_{i,r}-q_i^{-2}x_{i,s})\cdot g\left(\{x_{i,r}\}^{1\leq r\leq k_i}_{i\in I}\right)}{\prod_{i\to j}\prod^{s\leq k_j}_{r\leq k_i}(x_{i,r}-x_{j,s})}. \tag{31}$$

The following result generalizes its type A case established in [12, Proposition 4.12]:

Lemma 2.12 *(a) For $\widetilde{E}_{\underline{k}}(g)\in S^{(\mathfrak{g})}_{\underline{k}}$ given by (30), we have:*

$$\widehat{\Phi}^{\lambda,\mathsf{z}}_{\mu}(\widetilde{E}_{\underline{k}}(g))=\prod_{i\in I}\Big(\prod_{t=1}^{a_i}\mathsf{w}_{i,t}\Big)^{k_i+\sum_{j\leftarrow i}\frac{c_{ij}}{2}k_j}$$

$$\times \sum_{\substack{J_i \subset \{1,\dots,a_i\} \\ |J_i|=k_i \ \forall i \in I}} \left(\frac{\prod_{j\to i} \prod_{r\in J_i}^{1\le s\le a_j} \prod_{p=1}^{-c_{ji}-\delta_{s\in J_j}} \left(1 - \frac{q_j^{c_{ji}+2p} w_{j,s}}{w_{i,r}}\right)}{\prod_{i\in I} \prod_{r\in J_i}^{s\notin J_i} \left(1 - \frac{w_{i,s}}{w_{i,r}}\right)} \cdot g\left(\{w_{i,r}\}_{i\in I}^{r\in J_i}\right) \right.$$

$$\left. \times \prod_{i\in I} \prod_{r\in J_i} Z_i(w_{i,r}) \cdot \prod_{i\in I} \Big(\prod_{r\in J_i} w_{i,r} \Big)^{k_i - 1 - \sum_{j\to i} k_j} \cdot \prod_{i\in I} \prod_{r\in J_i} D_{i,r}^{-1} \right). \tag{32}$$

(b) For $\widetilde{F}_{\underline{k}}(g) \in S_{\underline{k}}^{(\mathfrak{g}),\mathrm{op}}$ given by (31), we have:

$$\widehat{\Phi}_\mu^{\lambda,z}(\widetilde{F}_{\underline{k}}(g)) = \prod_{i\in I} \Big(\prod_{t=1}^{a_i} w_{i,t} \Big)^{\sum_{j\to i} \frac{c_{ij}}{2} k_j}$$

$$\times \sum_{\substack{J_i \subset \{1,\dots,a_i\} \\ |J_i|=k_i \ \forall i \in I}} \left(\frac{\prod_{j\leftarrow i} \prod_{r\in J_i}^{1\le s\le a_j} \prod_{p=1+\delta_{s\in J_j}}^{-c_{ji}} \left(1 - \frac{q_j^{c_{ji}+2p} q_i^{-2} w_{j,s}}{w_{i,r}}\right)}{\prod_{i\in I} \prod_{r\in J_i}^{s\notin J_i} \left(1 - \frac{w_{i,s}}{w_{i,r}}\right)} \cdot g\left(\{q_i^2 w_{i,r}\}_{i\in I}^{r\in J_i}\right) \right.$$

$$\left. \times \prod_{i\in I} \Big(\prod_{r\in J_i} w_{i,r} \Big)^{k_i - 1 - \sum_{j\leftarrow i} k_j} \cdot \prod_{i\in I} q_i^{\sum_{j\leftarrow i} (c_{ij}-2) k_i k_j} \cdot \prod_{i\in I} \prod_{r\in J_i} D_{i,r} \right). \tag{33}$$

Proof The proof is straightforward and is based on (19, 22). Due to the presence of the factors $\prod_{i\in I} \prod_{1\le r\ne s\le k_i} (x_{i,r} - q_i^{-2} x_{i,s})$, the summands of (19, 22) with at least one $m_r^{(i)} > 1$ do vanish. This explains why the summations over all partitions of k_i into a_i nonnegative terms in (19, 22) are replaced by the summations over all cardinality k_i subsets of $\{1,\dots,a_i\}$ in (32, 33). □

Corollary 2.13 *If $k_i > a_i$ for some $i \in I$, then $\widehat{\Phi}_\mu^{\lambda,z}(\widetilde{E}_{\underline{k}}(g)) = 0 = \widehat{\Phi}_\mu^{\lambda,z}(\widetilde{F}_{\underline{k}}(g))$ for all g.*

3 Generalization to the quantum toroidal $\mathfrak{gl}_1$

The above constructions admit natural generalizations to the case of shifted version of the quantum toroidal algebra $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, related (e.g. via [2]) to the Jordan quiver. We shall state the key results, skipping the proofs when they are similar to those from Sect. 2.

3.1 Shifted quantum toroidal $\mathfrak{gl}_1$

Fix $q_1, q_2, q_3 \in \mathbb{C}^\times$ that are not roots of unity and satisfy $q_1 q_2 q_3 = 1$. For $b^+, b^- \in \mathbb{Z}$, we define the *shifted quantum toroidal algebra of* $\mathfrak{gl}_1$, denoted by $\ddot{U}_{q_1,q_2,q_3}^{(b^+,b^-)}$, to be the associative $\mathbb{C}$-algebra generated by $\{e_r, f_r, \psi^\pm_{\pm s^\pm}, (\psi^\pm_{\mp b^\pm})^{-1}\}_{r\in\mathbb{Z}}^{s^\pm \ge -b^\pm}$ with the following defining relations:

$$[\psi^\epsilon(z),\psi^{\epsilon'}(w)]=0,\quad \psi^\pm_{\mp b^\pm}\cdot(\psi^\pm_{\mp b^\pm})^{-1}=(\psi^\pm_{\mp b^\pm})^{-1}\cdot\psi^\pm_{\mp b^\pm}=1, \tag{t1}$$

$$(z-q_1w)(z-q_2w)(z-q_3w)e(z)e(w)=(q_1z-w)(q_2z-w)(q_3z-w)e(w)e(z), \tag{t2}$$

$$(q_1z-w)(q_2z-w)(q_3z-w)f(z)f(w)=(z-q_1w)(z-q_2w)(z-q_3w)f(w)f(z), \tag{t3}$$

$$(z-q_1w)(z-q_2w)(z-q_3w)\psi^\epsilon(z)e(w)=(q_1z-w)(q_2z-w)(q_3z-w)e(w)\psi^\epsilon(z), \tag{t4}$$

$$(q_1z-w)(q_2z-w)(q_3z-w)\psi^\epsilon(z)f(w)=(z-q_1w)(z-q_2w)(z-q_3w)f(w)\psi^\epsilon(z), \tag{t5}$$

$$[e(z),f(w)]=\frac{1}{\beta_1}\delta\left(\frac{z}{w}\right)\left(\psi^+(z)-\psi^-(z)\right), \tag{t6}$$

$$\underset{z_1,z_2,z_3}{\mathrm{Sym}}\ \frac{z_2}{z_3}[e(z_1),[e(z_2),e(z_3)]]=0, \tag{t7}$$

$$\underset{z_1,z_2,z_3}{\mathrm{Sym}}\ \frac{z_2}{z_3}[f(z_1),[f(z_2),f(z_3)]]=0, \tag{t8}$$

where $\epsilon,\epsilon'\in\{\pm\}$, $\beta_1=(1-q_1)(1-q_2)(1-q_3)$, and the generating series are defined via:

$$e(z):=\sum_{r\in\mathbb{Z}}e_rz^{-r},\quad f(z):=\sum_{r\in\mathbb{Z}}f_rz^{-r},\quad \psi^\pm(z):=\sum_{r\geq -b^\pm}\psi^\pm_{\pm r}z^{\mp r}.$$

Remark 3.2 (a) The original quantum toroidal algebra of $\mathfrak{gl}_1$, denoted by $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, is isomorphic to $\ddot{U}^{(0,0)}_{q_1,q_2,q_3}/(\psi^+_0\psi^-_0-1)$.

(b) We note the $S(3)$-symmetry of $\ddot{U}^{(b^+,b^-)}_{q_1,q_2,q_3}$ with respect to the permutations of q_1,q_2,q_3.

The algebras $\ddot{U}^{(b^+,b^-)}_{q_1,q_2,q_3}$ and $\ddot{U}^{(0,b^++b^-)}_{q_1,q_2,q_3}$ are naturally isomorphic for any $b^\pm$. Hence, we do not lose generality by considering only $\ddot{U}^{(0,b)}_{q_1,q_2,q_3}$, which will be denoted by $\ddot{U}^{(b)}_{q_1,q_2,q_3}$ for simplicity.

3.3 GKLO-type homomorphisms

Fix a pair of integers: $a\geq 1$ and $N\geq 0$ (following [2, §A(iii)], one can interpret them as $a=\dim(V)$ and $N=\dim(W)$ in the Jordan quiver). Let $\hat{\mathcal{A}}^{q_1}$ be the associative $\mathbb{C}$-algebra generated by $\{D_r^{\pm1},\mathsf{w}_r^{\pm1}\}_{1\leq r\leq a}$ with the only nontrivial commutator $D_r\mathsf{w}_s=q_1^{\delta_{rs}}\mathsf{w}_sD_r$, and let $\widetilde{\mathcal{A}}^{q_1}$ be the localization of $\hat{\mathcal{A}}^{q_1}$ by the multiplicative set generated by $\{\mathsf{w}_r-q_1^m\mathsf{w}_s\}^{m\in\mathbb{Z}}_{1\leq r\neq s\leq a}$. We also choose a sequence $\underline{\mathsf{z}}=(\mathsf{z}_1,\ldots,\mathsf{z}_N)\in(\mathbb{C}^\times)^N$ and define $\mathsf{Z}(z):=\prod_{k=1}^N\left(1-\frac{\mathsf{z}_k}{z}\right)$.

Then, we have the following analogue of Proposition 2.4:

Proposition 3.4 *There exists a unique $\mathbb{C}$-algebra homomorphism*

$$\widetilde{\Phi}^{\underline{\mathsf{z}}}_a\colon \ddot{U}^{(N)}_{q_1,q_2,q_3}\longrightarrow\widetilde{\mathcal{A}}^{q_1} \tag{34}$$

such that

$$\begin{aligned} e(z) &\mapsto \tfrac{-1}{1-q_1^{-1}} \sum_{r=1}^{a} \delta\left(\tfrac{\mathsf{w}_r}{z}\right) Z(\mathsf{w}_r) \prod_{1\le s\le a}^{s\neq r} \tfrac{\mathsf{w}_r - q_2^{-1}\mathsf{w}_s}{\mathsf{w}_r-\mathsf{w}_s} D_r^{-1}, \\ f(z) &\mapsto \tfrac{1}{1-q_1} \sum_{r=1}^{a} \delta\left(\tfrac{q_1\mathsf{w}_r}{z}\right) \prod_{1\le s\le a}^{s\neq r} \tfrac{\mathsf{w}_r - q_2\mathsf{w}_s}{\mathsf{w}_r-\mathsf{w}_s} D_r, \\ \psi^{\pm}(z) &\mapsto \left(Z(z)\cdot \prod_{r=1}^{a} \tfrac{(z-q_2^{-1}\mathsf{w}_r)(z-q_3^{-1}\mathsf{w}_r)}{(z-\mathsf{w}_r)(z-q_1\mathsf{w}_r)}\right)^{\pm}. \end{aligned} \tag{35}$$

As before, $\gamma(z)^{\pm}$ denotes the expansion of a rational function $\gamma(z)$ in $z^{\mp 1}$, respectively.

Remark 3.5 Due to the $S(3)$-symmetry of $\ddot{U}^{(N)}_{q_1,q_2,q_3}$ (Remark 3.2(b)), we can replace q_2 by q_3 in (35). Overall, we have six similar homomorphisms: two $\ddot{U}^{(N)}_{q_1,q_2,q_3} \to \widetilde{\mathcal{A}}^{q_i}$ for each $i = 1, 2, 3$.

Remark 3.6 In the unshifted case $N = 0$, (34) factors through $\widetilde{\Phi}_a\colon \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \to \widetilde{\mathcal{A}}^{q_1}$ (see Remark 3.2(a)) that maps:

$$\begin{gathered} \widetilde{\Phi}_a: \quad e_0 \mapsto \tfrac{-1}{1-q_1^{-1}} \sum_{r=1}^{a} \prod_{1\le s\le a}^{s\neq r} \tfrac{\mathsf{w}_r-q_2^{-1}\mathsf{w}_s}{\mathsf{w}_r-\mathsf{w}_s} D_r^{-1}, \quad f_0 \mapsto \tfrac{1}{1-q_1} \sum_{r=1}^{a} \prod_{1\le s\le a}^{s\neq r} \tfrac{\mathsf{w}_r-q_2\mathsf{w}_s}{\mathsf{w}_r-\mathsf{w}_s} D_r, \\ \psi_1^+ \mapsto (1-q_2^{-1})(1-q_3^{-1}) \sum_{r=1}^{a} \mathsf{w}_r, \quad \psi_{-1}^- \mapsto (1-q_2)(1-q_3) \sum_{r=1}^{a} \mathsf{w}_r^{-1}, \quad \psi_0^{\pm} \mapsto 1. \end{gathered} \tag{36}$$

Let us compare this with [7, Proposition 5.1], where a natural $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$-representation of [7, Lemma 3.7] is interpreted as an algebra homomorphism $\bar{\Phi}_a\colon \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \to \widetilde{\mathcal{A}}^{q_1}$ given by (we swap $q_2 \leftrightarrow q_3$ in the formulas of [7]):

$$\begin{aligned} \bar{\Phi}_a\colon\ e_0 &\mapsto \tfrac{1}{1-q_1} \sum_{r=1}^{a} \mathsf{w}_r, \quad f_0 \mapsto \tfrac{-1}{1-q_1^{-1}} \sum_{r=1}^{a} \mathsf{w}_r^{-1}, \quad \psi_0^{\pm} \mapsto 1, \\ \psi_1^+ &\mapsto (1-q_2)(1-q_3) \sum_{r=1}^{a} \prod_{s\neq r} \tfrac{\mathsf{w}_r-q_2\mathsf{w}_s}{\mathsf{w}_r-\mathsf{w}_s} D_r, \\ \psi_{-1}^- &\mapsto (1-q_2^{-1})(1-q_3^{-1}) \sum_{r=1}^{a} \prod_{s\neq r} \tfrac{\mathsf{w}_r-q_2^{-1}\mathsf{w}_s}{\mathsf{w}_r-\mathsf{w}_s} D_r^{-1}. \end{aligned} \tag{37}$$

Both $\widetilde{\Phi}_a$ and $\bar{\Phi}_a$ factor through the central quotient $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)/(\psi_0^{\pm}-1)$ and the resulting homomorphisms $\widetilde{\Phi}_a, \bar{\Phi}_a\colon \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)/(\psi_0^{\pm}-1) \to \widetilde{\mathcal{A}}^{q_1}$ are related via $\bar{\Phi}_a = \widetilde{\Phi}_a \circ \varpi$, where ϖ is an automorphism (a version of the Burban–Schiffmann/Miki's automorphism) of $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)/(\psi_0^{\pm}-1)$ determined by:

$$\varpi\colon \psi_1^+ \mapsto \beta_1 f_0, \quad \psi_{-1}^- \mapsto \beta_1 e_0, \quad e_0 \mapsto q_1^{-1}\beta_1^{-1}\psi_1^+, \quad f_0 \mapsto q_1\beta_1^{-1}\psi_{-1}^-. \tag{38}$$

3.7 Shuffle algebra realization of the positive and negative subalgebras

Similar to (11, 12), we have the following algebra isomorphisms:

$$\begin{gathered} \ddot{U}^{(N),>}_{q_1,q_2,q_3} \xrightarrow{\sim} \ddot{U}^{>}_{q_1,q_2,q_3}(\mathfrak{gl}_1), \ \ddot{U}^{(N),<}_{q_1,q_2,q_3} \xrightarrow{\sim} \ddot{U}^{<}_{q_1,q_2,q_3}(\mathfrak{gl}_1), \\ \ddot{U}^{<}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \xrightarrow{\sim} \ddot{U}^{>}_{q_1,q_2,q_3}(\mathfrak{gl}_1)^{\mathrm{op}}, \end{gathered} \tag{39}$$

with subalgebras $\ddot{U}^{(N),>}_{q_1,q_2,q_3}$, $\ddot{U}^{>}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, $\ddot{U}^{(N),<}_{q_1,q_2,q_3}$, $\ddot{U}^{<}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ defined in a self-explaining way.

Consider an $\mathbb{N}$-graded $\mathbb{C}$-vector space $\mathbb{S} = \bigoplus_{k\in\mathbb{N}} \mathbb{S}_k$, with the graded components

$$\mathbb{S}_k = \left\{ F = \frac{f(x_1,\ldots,x_k)}{\prod_{1\le r\ne s\le k}(x_r - x_s)} \,\middle|\, f \in \mathbb{C}\left[x_1^{\pm 1},\ldots,x_k^{\pm 1}\right]^{S(k)} \right\}. \tag{40}$$

We also fix a rational function

$$\zeta\left(\frac{z}{w}\right) = \frac{(z-q_1^{-1}w)(z-q_2^{-1}w)(z-q_3^{-1}w)}{(z-w)^3}. \tag{41}$$

The bilinear *shuffle product* $\star$ on $\mathbb{S}$ is defined completely analogously to (15), thus endowing $\mathbb{S}$ with a structure of an associative unital $\mathbb{C}$-algebra. As before, we are interested in an $\mathbb{N}$-graded subspace of $\mathbb{S}$ defined by the following *wheel conditions*:

$$F(x_1,\ldots,x_k) = 0 \quad \text{once} \quad \left\{\frac{x_1}{x_2}, \frac{x_2}{x_3}, \frac{x_3}{x_1}\right\} = \{q_1, q_2, q_3\}. \tag{42}$$

Let $S \subset \mathbb{S}$ denote the subspace of all such elements F, which is easily seen to be $\star$-closed. The resulting *shuffle algebra* $(S, \star)$ is related to $\ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$ via the following result of [15]:

Proposition 3.8 [15] *The assignments $e_r \mapsto x_1^r$ and $f_r \mapsto x_1^r$ for $r \in \mathbb{Z}$ give rise to $\mathbb{C}$-algebra isomorphisms*

$$\Upsilon\colon \ddot{U}^{>}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \xrightarrow{\sim} S \quad \text{and} \quad \Upsilon\colon \ddot{U}^{<}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \xrightarrow{\sim} S^{\mathrm{op}}. \tag{43}$$

3.9 Shuffle algebra realization of the GKLO-type homomorphisms

For $1 \le r \le a$, we define:

$$Y_r(z) := \frac{-1}{1-q_1^{-1}} \mathsf{Z}(z) \prod_{1\le s\le a}^{s\ne r} \frac{z - \mathsf{w}_s q_2^{-1}}{z-\mathsf{w}_s}, \quad Y'_r(z) := \frac{1}{1-q_1} \prod_{1\le s\le a}^{s\ne r} \frac{zq_1^{-1} - \mathsf{w}_s q_2}{zq_1^{-1} - \mathsf{w}_s}. \tag{44}$$

We also define

$$\varphi\left(\frac{z}{w}\right) := \frac{\left(q_1^{1/2}z - q_1^{-1/2}w\right)\left(q_2^{1/2}z - q_2^{-1/2}w\right)}{(z-w)^2}. \tag{45}$$

Let $\widetilde{\mathcal{A}}^{q_1,\prime}$ be the localization of $\widetilde{\mathcal{A}}^{q_1}$ by the multiplicative set generated by $\{\mathsf{w}_r - q_1^m q_2 \mathsf{w}_s\}_{r\ne s}^{m\in\mathbb{Z}}$. The following is our key result and is proved completely analogously to Theorem 2.8:

Theorem 3.10 (a) *The assignment*

$$\begin{aligned}
\mathbb{S}_k \ni E \mapsto \sum_{m_1+\ldots+m_a=k} & \left\{ \prod_{r=1}^{a} \prod_{p=1}^{m_r} Y_r\left(w_r q_1^{-(p-1)}\right) \cdot E\left(\left\{w_r q_1^{-(p-1)}\right\}_{1\le r\le a}^{1\le p\le m_r}\right) \right. \\
& \times \prod_{1\le r\le a} \prod_{1\le p_1<p_2\le m_r} \zeta^{-1}\left(w_r q_1^{-(p_1-1)} \Big/ w_r q_1^{-(p_2-1)}\right) \\
& \left. \times \prod_{1\le r_1\ne r_2\le a} \prod_{1\le p_1\le m_{r_1}}^{1\le p_2\le m_{r_2}} \varphi^{-1}\left(w_{r_1} q_1^{-(p_1-1)} \Big/ w_{r_2} q_1^{-(p_2-1)}\right) \cdot \prod_{r=1}^{a} D_r^{-m_r} \right\}
\end{aligned} \tag{46}$$

gives rise to the algebra homomorphism

$$\widehat{\Phi}^{\mathbf{z}}_a \colon \mathbb{S} \longrightarrow \widetilde{\mathcal{A}}^{q_1,\prime}. \tag{47}$$

Moreover, the composition

$$\ddot{U}^{(N),>}_{q_1,q_2,q_3} \xrightarrow[\sim]{(39)} \ddot{U}^{>}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \xrightarrow[\sim]{\Upsilon} S \xrightarrow{\hat{\Phi}^{\mathbf{z}}_a} \widetilde{\mathcal{A}}^{q_1,\prime} \tag{48}$$

coincides with the restriction of the homomorphism $\widetilde{\Phi}^{\mathbf{z}}_a$ *of* (34) *to the subalgebra* $\ddot{U}^{(N),>}_{q_1,q_2,q_3}$.

(b) *The assignment*

$$\begin{aligned}
\mathbb{S}^{\mathrm{op}}_k \ni F \mapsto \sum_{m_1+\ldots+m_a=k} & \left\{ \prod_{r=1}^{a} \prod_{p=1}^{m_r} Y'_r\left(w_r q_1^{p}\right) \cdot F\left(\left\{w_r q_1^{p}\right\}_{1\le r\le a}^{1\le p\le m_r}\right) \right. \\
& \times \prod_{1\le r\le a} \prod_{1\le p_1<p_2\le m_r} \zeta^{-1}\left(w_r q_1^{p_2} \Big/ w_r q_1^{p_1}\right) \\
& \left. \times \prod_{1\le r_1\ne r_2\le a} \prod_{1\le p_1\le m_{r_1}}^{1\le p_2\le m_{r_2}} \varphi^{-1}\left(w_{r_2} q_1^{p_2} \Big/ w_{r_1} q_1^{p_1}\right) \cdot \prod_{r=1}^{a} D_r^{m_r} \right\}
\end{aligned} \tag{49}$$

gives rise to the algebra homomorphism

$$\widehat{\Phi}^{\mathbf{z}}_a \colon \mathbb{S}^{\mathrm{op}} \longrightarrow \widetilde{\mathcal{A}}^{q_1,\prime}. \tag{50}$$

Moreover, the composition

$$\ddot{U}^{(N),<}_{q_1,q_2,q_3} \xrightarrow[\sim]{(39)} \ddot{U}^{<}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \xrightarrow[\sim]{\Upsilon} S^{\mathrm{op}} \xrightarrow{\hat{\Phi}^{\mathbf{z}}_a} \widetilde{\mathcal{A}}^{q_1,\prime} \tag{51}$$

coincides with the restriction of the homomorphism $\widetilde{\Phi}^{\mathbf{z}}_a$ *of* (34) *to the subalgebra* $\ddot{U}^{(N),<}_{q_1,q_2,q_3}$.

3.11 Special difference operators

For any $g \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_k^{\pm 1}]^{S(k)}$, consider the following shuffle elements $\widetilde{E}_k(g) \in S_k$:

$$\widetilde{E}_k(g) := q_3^{\frac{k-k^2}{2}} (q_1^{-1} - 1)^k \cdot \frac{\prod_{1 \le r \ne s \le k} (x_r - q_1^{-1} x_s) \cdot g(x_1, \ldots, x_k)}{\prod_{1 \le r \ne s \le k} (x_r - x_s)}, \tag{52}$$

which obviously satisfy the wheel conditions (42). Due to Proposition 3.8, $\widetilde{E}_k(g) = \Upsilon(\widetilde{e}_k(g))$ for unique elements $\widetilde{e}_k(g) \in \ddot{U}^{(N),>}_{q_1,q_2,q_3} \simeq \ddot{U}^{>}_{q_1,q_2,q_3}(\mathfrak{gl}_1)$, so that $\widehat{\Phi}^{\mathsf{z}}_a(\widetilde{E}_k(g)) = \widetilde{\Phi}^{\mathsf{z}}_a(\widetilde{e}_k(g))$ by Theorem 3.10(a). We also consider $\widetilde{F}_k(g) \in S_k^{\mathrm{op}}$ defined via:

$$\widetilde{F}_k(g) := (q_2/q_1)^{\frac{k-k^2}{2}} (1 - q_1)^k \cdot \frac{\prod_{1 \le r \ne s \le k} (x_r - q_1^{-1} x_s) \cdot g(x_1, \ldots, x_k)}{\prod_{1 \le r \ne s \le k} (x_r - x_s)}. \tag{53}$$

The following result is established completely analogously to Lemma 2.12:

Lemma 3.12 (a) *For $\widetilde{E}_k(g) \in S_k$ given by (52), we have:*

$$\widehat{\Phi}^{\mathsf{z}}_a(\widetilde{E}_k(g)) = \sum_{J \subset \{1,\ldots,a\}}^{|J|=k} \left\{ \prod_{r \in J}^{s \notin J} \frac{\mathsf{w}_r - q_2^{-1}\mathsf{w}_s}{\mathsf{w}_r - \mathsf{w}_s} \cdot \prod_{r \in J} Z(\mathsf{w}_r) \cdot g\Big(\{\mathsf{w}_r\}_{r \in J}\Big) \cdot \prod_{r \in J} D_r^{-1} \right\}. \tag{54}$$

(b) *For $\widetilde{F}_k(g) \in S_k^{\mathrm{op}}$ given by (53), we have:*

$$\widehat{\Phi}^{\mathsf{z}}_a(\widetilde{F}_k(g)) = \sum_{J \subset \{1,\ldots,a\}}^{|J|=k} \left\{ \prod_{r \in J}^{s \notin J} \frac{\mathsf{w}_r - q_2\mathsf{w}_s}{\mathsf{w}_r - \mathsf{w}_s} \cdot g\Big(\{q_1\mathsf{w}_r\}_{r \in J}\Big) \cdot \prod_{r \in J} D_r \right\}. \tag{55}$$

Example 3.13 For $N = 0$ and $g = 1$, we recover the famous Macdonald difference operators:

$$\begin{aligned} \widehat{\Phi}^{\mathsf{z}}_a(\widetilde{E}_k(1)) &= \sum_{J \subset \{1,\ldots,a\}}^{|J|=k} \prod_{r \in J}^{s \notin J} \frac{\mathsf{w}_r - q_2^{-1}\mathsf{w}_s}{\mathsf{w}_r - \mathsf{w}_s} \cdot \prod_{r \in J} D_r^{-1} =: \mathcal{D}^k_a(q_1, q_2), \\ \widehat{\Phi}^{\mathsf{z}}_a(\widetilde{F}_k(1)) &= \sum_{J \subset \{1,\ldots,a\}}^{|J|=k} \prod_{r \in J}^{s \notin J} \frac{\mathsf{w}_r - q_2\mathsf{w}_s}{\mathsf{w}_r - \mathsf{w}_s} \cdot \prod_{r \in J} D_r =: \mathcal{D}^k_a(q_1^{-1}, q_2^{-1}). \end{aligned} \tag{56}$$

Remark 3.14 We note that the crucial and rather nontrivial commutativity

$$\Big[\mathcal{D}^k_a(q_1, q_2), \, \mathcal{D}^{k'}_a(q_1, q_2)\Big] = 0 \quad \text{for all} \quad 1 \le k, k' \le a$$

thus arises as an immediate consequence of a simple equality $[\widetilde{E}_k(1), \widetilde{E}_{k'}(1)] = 0$ in the shuffle algebra S, see [8, Proposition 2.21].

4 Generalization to the quantum toroidal $\mathfrak{sl}_n$ ($n \geq 3$)

The above constructions admit natural generalizations to the case of shifted version of the quantum toroidal algebra $\ddot{U}_{q,d}(\mathfrak{sl}_n)$, related (e.g. via [2]) to the cyclic n-vertex quiver. We shall state the key results, skipping the proofs when they are similar to those from Sect. 2.

4.1 Shifted quantum toroidal $\mathfrak{sl}_n$

For $n \geq 3$, consider an index set $[n] := \{0, 1, \ldots, n-1\}$ (also viewed as a set of residues modulo n). We define two matrices $(c_{ij})_{i,j\in[n]}$ (the Cartan matrix of $\widehat{\mathfrak{sl}}_n$) and $(m_{ij})_{i,j\in[n]}$ via:

$$c_{ii} = 2, \quad c_{i,i\pm 1} = -1, \quad m_{i,i\pm 1} = \mp 1, \quad \text{and} \quad c_{ij} = 0 = m_{ij} \quad \text{otherwise.} \tag{57}$$

Fix $q, d \in \mathbb{C}^\times$ such that $q, qd^{\pm 1}$ are not roots of unity. Given $\underline{b}^\pm = \{b_i^\pm\}_{i\in[n]} \in \mathbb{Z}^{[n]}$, we define the *shifted quantum toroidal algebra of* $\mathfrak{sl}_n$, denoted by $\ddot{U}_{q,d}^{(\underline{b}^+,\underline{b}^-)}$, to be the associative $\mathbb{C}$-algebra generated by $\{e_{i,r}, f_{i,r}, \psi^\pm_{i,\pm s_i^\pm}, (\psi^\pm_{i,\mp b_i^\pm})^{-1}\}_{i\in[n]}^{r\in\mathbb{Z}, s_i^\pm \geq -b_i^\pm}$ with the following defining relations (for all $i, j \in [n]$ and $\epsilon, \epsilon' \in \{\pm\}$):

$$[\psi_i^\epsilon(z), \psi_j^{\epsilon'}(w)] = 0, \quad \psi^\pm_{i,\mp b_i^\pm} \cdot (\psi^\pm_{i,\mp b_i^\pm})^{-1} = (\psi^\pm_{i,\mp b_i^\pm})^{-1} \cdot \psi^\pm_{i,\mp b_i^\pm} = 1, \tag{T1}$$

$$(d^{m_{ij}} z - q^{c_{ij}} w) e_i(z) e_j(w) = (q^{c_{ij}} d^{m_{ij}} z - w) e_j(w) e_i(z), \tag{T2}$$

$$(q^{c_{ij}} d^{m_{ij}} z - w) f_i(z) f_j(w) = (d^{m_{ij}} z - q^{c_{ij}} w) f_j(w) f_i(z), \tag{T3}$$

$$(d^{m_{ij}} z - q^{c_{ij}} w) \psi_i^\epsilon(z) e_j(w) = (q^{c_{ij}} d^{m_{ij}} z - w) e_j(w) \psi_i^\epsilon(z), \tag{T4}$$

$$(q^{c_{ij}} d^{m_{ij}} z - w) \psi_i^\epsilon(z) f_j(w) = (d^{m_{ij}} z - q^{c_{ij}} w) f_j(w) \psi_i^\epsilon(z), \tag{T5}$$

$$[e_i(z), f_j(w)] = \frac{\delta_{ij}}{q - q^{-1}} \delta\left(\frac{z}{w}\right) \left(\psi_i^+(z) - \psi_i^-(z)\right), \tag{T6}$$

$$\underset{z_1,z_2}{\mathrm{Sym}} \left(e_i(z_1)e_i(z_2)e_{i\pm 1}(w) - (q+q^{-1}) e_i(z_1) e_{i\pm 1}(w) e_i(z_2) + e_{i\pm 1}(w) e_i(z_1) e_i(z_2)\right) = 0, \tag{T7}$$

$$\underset{z_1,z_2}{\mathrm{Sym}} \left(f_i(z_1)f_i(z_2)f_{i\pm 1}(w) - (q+q^{-1}) f_i(z_1) f_{i\pm 1}(w) f_i(z_2) + f_{i\pm 1}(w) f_i(z_1) f_i(z_2)\right) = 0, \tag{T8}$$

where the generating series $\{e_i(z), f_i(z), \psi_i^\pm(z)\}_{i\in[n]}$ are defined as in (6).

The algebras $\ddot{U}_{q,d}^{(\underline{b}^+,\underline{b}^-)}$ and $\ddot{U}_{q,d}^{(0,\underline{b}^+ + \underline{b}^-)}$ are naturally isomorphic for any $\underline{b}^\pm \in \mathbb{Z}^{[n]}$. Thus, we do not lose generality by considering only $\ddot{U}_{q,d}^{(0,\underline{b})}$, which will be denoted by $\ddot{U}_{q,d}^{(\underline{b})}$ for simplicity. The original quantum toroidal algebra $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ is isomorphic to $\ddot{U}_{q,d}^{(0,0)}/(\psi^+_{i,0}\psi^-_{i,0} - 1)_{i\in[n]}$.

4.2 GKLO-type homomorphisms

Fix $\underline{b} \in \mathbb{Z}^{[n]}$ and let $\underline{a} \in \mathbb{N}^{[n]}$ be such that $N_i := b_i + 2a_i - a_{i-1} - a_{i+1} \geq 0$ for all $i \in [n]$ (in particular, existence of such $\underline{a}$ forces $\sum_{i\in[n]} b_i \geq 0$). We pick $\underline{\mathsf{z}} = (\{\mathsf{z}_{i,r}\}_{i\in[n]}^{1\le r\le N_i})$ with $\mathsf{z}_{i,r} \in \mathbb{C}^{\times}$, as well as an orientation of the cyclic quiver $\mathrm{Dyn}(\widehat{\mathfrak{sl}}_n)$ with the vertex set $[n]$ and the vertex i connected to the vertices $i+1, i-1$. We define the $\mathbb{C}$-algebra $\widetilde{\mathcal{A}}^q$ as in Sect. 2.3 (note that we omit the subscript "frac" as it is now a $\mathbb{C}$-algebra) and follow the notations (8).

Then, we have the following analogue of Proposition 2.4:

Proposition 4.3 *There exists a unique $\mathbb{C}$-algebra homomorphism*

$$\widetilde{\Phi}^{\underline{a},\underline{\mathsf{z}}}_{\underline{b}} \colon \ddot{U}^{(\underline{b})}_{q,d} \longrightarrow \widetilde{\mathcal{A}}^q \tag{58}$$

such that

$$\begin{aligned}
e_i(z) &\mapsto \frac{-q}{1-q^2}\prod_{t=1}^{a_i}\mathsf{w}_{i,t}\prod_{j\to i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{-1/2}\cdot\sum_{r=1}^{a_i}\delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\frac{Z_i(\mathsf{w}_{i,r})}{W_{i,r}(\mathsf{w}_{i,r})}\prod_{j\to i}W_j(q^{-1}d^{m_{ij}}z)D_{i,r}^{-1},\\
f_i(z) &\mapsto \frac{1}{1-q^2}\prod_{j\leftarrow i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{-1/2}\cdot\sum_{r=1}^{a_i}\delta\left(\frac{q^2\mathsf{w}_{i,r}}{z}\right)\frac{1}{W_{i,r}(\mathsf{w}_{i,r})}\prod_{j\leftarrow i}W_j(q^{-1}d^{m_{ij}}z)D_{i,r},\\
\psi_i^{\pm}(z) &\mapsto \prod_{t=1}^{a_i}\mathsf{w}_{i,t}\prod_{j-i}\prod_{t=1}^{a_j}\mathsf{w}_{j,t}^{-1/2}\cdot\left(\frac{Z_i(z)}{W_i(z)W_i(q^{-2}z)}\prod_{j-i}W_j(q^{-1}d^{m_{ij}}z)\right)^{\pm}.
\end{aligned}\tag{59}$$

As before, $\gamma(z)^{\pm}$ denotes the expansion of a rational function $\gamma(z)$ in $z^{\mp 1}$, respectively.

Remark 4.4 We note that the unshifted case $\underline{b}=\underline{0}$ corresponds to $a_0 = a_1 = \ldots = a_{n-1}$.

4.5 Shuffle algebra realization of the positive and negative subalgebras

Similar to (11, 12, 39), we have the following algebra isomorphisms:

$$\ddot{U}^{(\underline{b}),>}_{q,d} \xrightarrow{\sim} \ddot{U}^{>}_{q,d}(\mathfrak{sl}_n),\quad \ddot{U}^{(\underline{b}),<}_{q,d} \xrightarrow{\sim} \ddot{U}^{<}_{q,d}(\mathfrak{sl}_n),\quad \ddot{U}^{<}_{q,d}(\mathfrak{sl}_n) \xrightarrow{\sim} \ddot{U}^{>}_{q,d}(\mathfrak{sl}_n)^{\mathrm{op}}, \tag{60}$$

with the subalgebras $\ddot{U}^{(\underline{b}),>}_{q,d}$, $\ddot{U}^{>}_{q,d}(\mathfrak{sl}_n)$, $\ddot{U}^{(\underline{b}),<}_{q,d}(\mathfrak{sl}_n)$, $\ddot{U}^{<}_{q,d}(\mathfrak{sl}_n)$ defined in a self-explaining way.

Consider an $\mathbb{N}^{[n]}$-graded $\mathbb{C}$-vector space $\mathbb{S}^{[n]} = \bigoplus_{\underline{k}=(k_i)_{i\in[n]}\in\mathbb{N}^{[n]}}\mathbb{S}^{[n]}_{\underline{k}}$, with the graded components

$$\mathbb{S}^{[n]}_{\underline{k}} = \left\{ F = \frac{f(\{x_{i,r}\}_{i\in[n]}^{1\le r\le k_i})}{\prod_{i\in[n]}\prod_{r\le k_i}^{s\le k_{i+1}}(x_{i,r}-x_{i+1,s})} \,\middle|\, f \in \mathbb{C}\Big[\{x_{i,r}^{\pm 1}\}_{i\in[n]}^{1\le r\le k_i}\Big]^{S_{\underline{k}}} \right\}, \tag{61}$$

where $S_{\underline{k}} := \prod_{i\in[n]} S(k_i)$. We also fix rational functions $\{\zeta_{ij}(z)\}_{i,j\in[n]}$ via:

$$\zeta_{i,i+1}\left(\frac{z}{w}\right) = \frac{d^{-1}z - qw}{z-w}, \quad \zeta_{i,i-1}\left(\frac{z}{w}\right) = \frac{z - qd^{-1}w}{z-w},$$
$$\zeta_{ii}\left(\frac{z}{w}\right) = \frac{z - q^{-2}w}{z-w}, \quad \zeta_{ij}\left(\frac{z}{w}\right) = 1 \quad \text{if} \quad j \neq i, i \pm 1. \tag{62}$$

The bilinear *shuffle product* $\star$ on $\mathbb{S}^{[n]}$ is defined completely analogously to (15), thus endowing $\mathbb{S}^{[n]}$ with a structure of an associative unital $\mathbb{C}$-algebra. As before, we are interested in an $\mathbb{N}^{[n]}$-graded subspace of $\mathbb{S}^{[n]}$ defined by the following *wheel conditions*:

$$F\left(\{x_{i,r}\}\right) = 0 \;\text{ once }\; x_{i,2} = q^2 x_{i,1} \text{ and } x_{i+\epsilon,1} = qd^{-\epsilon}x_{i,1} \;\text{ for }\; i \in [n],\; \epsilon = \pm 1. \tag{63}$$

Let $S^{[n]} \subset \mathbb{S}^{[n]}$ denote the subspace of all such elements F, which is easily seen to be $\star$-closed. The resulting *shuffle algebra* $\left(S^{[n]}, \star\right)$ is related to $\ddot{U}_{q,d}(\mathfrak{sl}_n)$ via the following result of [16]:

Proposition 4.6 [16] *The assignments* $e_{i,r} \mapsto x^r_{i,1}$ *and* $f_{i,r} \mapsto x^r_{i,1}$ *for* $i \in [n], r \in \mathbb{Z}$ *give rise to* $\mathbb{C}$*-algebra isomorphisms*

$$\Upsilon\colon \ddot{U}^{>}_{q,d}(\mathfrak{sl}_n) \xrightarrow{\sim} S^{[n]} \quad \text{and} \quad \Upsilon\colon \ddot{U}^{<}_{q,d}(\mathfrak{sl}_n) \xrightarrow{\sim} S^{[n],\mathrm{op}}. \tag{64}$$

4.7 Shuffle algebra realization of the GKLO-type homomorphisms

For any $i \in [n]$ and $1 \le r \le a_i$, we define:

$$Y_{i,r}(z) := \frac{1}{q-q^{-1}} \prod_{t=1}^{a_i} \mathsf{w}_{i,t} \prod_{j\to i}\prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{-1/2} \cdot \frac{\mathsf{Z}_i(z)\prod_{j\to i} W_j(zq^{-1}d^{m_{ij}})}{W_{i,r}(z)},$$
$$Y'_{i,r}(z) := \frac{1}{1-q^2} \prod_{j\leftarrow i}\prod_{t=1}^{a_j} \mathsf{w}_{j,t}^{-1/2} \cdot \frac{\prod_{j\leftarrow i} W_j(zq^{-1}d^{m_{ij}})}{W_{i,r}(zq^{-2})}. \tag{65}$$

Define the $\mathbb{C}$-algebra $\widetilde{\mathcal{A}}^{q,'}$ as the further localization of $\widetilde{\mathcal{A}}^q$ by the multiplicative set generated by $\{d^{m_{ij}}q^{c_{ij}}\mathsf{w}_{i,r} - q^{2m}\mathsf{w}_{j,s}\}^{r\le a_i, s\le a_j}_{j=i\pm 1, m\in\mathbb{Z}}$. We note that $\widetilde{\mathcal{A}}^q$ is naturally embedded into $\widetilde{\mathcal{A}}^{q,'}$.

The following is our key result and is proved completely analogously to Theorem 2.8:

Theorem 4.8 (a) *The assignment*

$$\mathbb{S}^{[n]}_{\underline{k}} \ni E \mapsto q^{\sum_{i\in[n]}(k_i - k_i^2)}$$

$$\times \sum_{\substack{m_1^{(i)}+\ldots+m_{a_i}^{(i)}=k_i \\ m_r^{(i)}\in\mathbb{N}\ \forall\, i\in[n]}} \left\{ \prod_{i\in[n]} \prod_{r=1}^{a_i} \prod_{p=1}^{m_r^{(i)}} Y_{i,r}\left(\mathsf{w}_{i,r}q^{-2(p-1)}\right) \cdot E\left(\left\{\mathsf{w}_{i,r}q^{-2(p-1)}\right\}_{i\in[n],1\le r\le a_i}^{1\le p\le m_r^{(i)}}\right)\right.$$

$$\times \prod_{i\in[n]} \prod_{1\le r\le a_i} \prod_{1\le p_1<p_2\le m_r^{(i)}} \zeta_{ii}^{-1}\left(\mathsf{w}_{i,r}q^{-2(p_1-1)}\Big/\mathsf{w}_{i,r}q^{-2(p_2-1)}\right)$$

$$\times \prod_{i\in[n]} \prod_{1\le r_1\ne r_2\le a_i} \prod_{1\le p_1\le m_{r_1}^{(i)}}^{1\le p_2\le m_{r_2}^{(i)}} \zeta_{ii}^{-1}\left(\mathsf{w}_{i,r_1}q^{-2(p_1-1)}\Big/\mathsf{w}_{i,r_2}q^{-2(p_2-1)}\right)$$

$$\left.\times \prod_{j\to i} \prod_{1\le r_1\le a_i}^{1\le r_2\le a_j} \prod_{1\le p_1\le m_{r_1}^{(i)}}^{1\le p_2\le m_{r_2}^{(j)}} \zeta_{ij}^{-1}\left(\mathsf{w}_{i,r_1}q^{-2(p_1-1)}\Big/\mathsf{w}_{j,r_2}q^{-2(p_2-1)}\right) \cdot \prod_{i\in[n]} \prod_{r=1}^{a_i} D_{i,r}^{-m_r^{(i)}} \right\} \tag{66}$$

gives rise to the algebra homomorphism

$$\widehat{\Phi}_{\underline{b}}^{\underline{a},\underline{z}} \colon \mathbb{S}^{[n]} \longrightarrow \widetilde{\mathcal{A}}^{q,\prime}. \tag{67}$$

Moreover, the composition

$$\ddot{U}_{q,d}^{(\underline{b}),>} \xrightarrow[\sim]{(60)} \ddot{U}_{q,d}^{>}(\mathfrak{sl}_n) \xrightarrow[\sim]{\Upsilon} S^{[n]} \xrightarrow{\widehat{\Phi}_{\underline{b}}^{\underline{a},\underline{z}}} \widetilde{\mathcal{A}}^{q,\prime} \tag{68}$$

coincides with the restriction of the homomorphism $\widetilde{\Phi}_{\underline{b}}^{\underline{a},\underline{z}}$ *of* (58) *to the subalgebra* $\ddot{U}_{q,d}^{(\underline{b}),>}$. *In particular, the image of* $\ddot{U}_{q,d}^{(\underline{b}),>}$ *under the composition* (68) *is in the subalgebra* $\widetilde{\mathcal{A}}^{q}$ *of* $\widetilde{\mathcal{A}}^{q,\prime}$.

(b) *The assignment*

$$\mathbb{S}_{\underline{k}}^{[n],\mathrm{op}} \ni F \mapsto$$

$$\sum_{\substack{m_1^{(i)}+\ldots+m_{a_i}^{(i)}=k_i \\ m_r^{(i)}\in\mathbb{N}\ \forall\, i\in[n]}} \left\{ \prod_{i\in[n]} \prod_{r=1}^{a_i} \prod_{p=1}^{m_r^{(i)}} Y'_{i,r}\left(\mathsf{w}_{i,r}q^{2p}\right) \cdot F\left(\left\{\mathsf{w}_{i,r}q^{2p}\right\}_{i\in[n],1\le r\le a_i}^{1\le p\le m_r^{(i)}}\right)\right.$$

$$\times \prod_{i\in[n]} \prod_{1\le r\le a_i} \prod_{1\le p_1<p_2\le m_r^{(i)}} \zeta_{ii}^{-1}\left(\mathsf{w}_{i,r}q^{2p_2}\Big/\mathsf{w}_{i,r}q^{2p_1}\right)$$

$$\times \prod_{i\in[n]} \prod_{1\le r_1\ne r_2\le a_i} \prod_{1\le p_1\le m_{r_1}^{(i)}}^{1\le p_2\le m_{r_2}^{(i)}} q^{-1}\zeta_{ii}^{-1}\left(\mathsf{w}_{i,r_2}q^{2p_2}\Big/\mathsf{w}_{i,r_1}q^{2p_1}\right)$$

$$\left.\times \prod_{j\leftarrow i} \prod_{1\le r_1\le a_i}^{1\le r_2\le a_j} \prod_{1\le p_1\le m_{r_1}^{(i)}}^{1\le p_2\le m_{r_2}^{(j)}} \zeta_{ji}^{-1}\left(\mathsf{w}_{j,r_2}q^{2p_2}\Big/\mathsf{w}_{i,r_1}q^{2p_1}\right) \cdot \prod_{i\in[n]} \prod_{r=1}^{a_i} D_{i,r}^{m_r^{(i)}} \right\} \tag{69}$$

gives rise to the algebra homomorphism

$$\widehat{\Phi}^{\underline{a},\underline{z}}_{\underline{b}} \colon \mathbb{S}^{[n],\mathrm{op}} \longrightarrow \widetilde{\mathcal{A}}^{q,\prime}. \tag{70}$$

Moreover, the composition

$$\ddot{U}^{(\underline{b}),<}_{q,d} \overset{(60)}{\underset{\sim}{\longrightarrow}} \ddot{U}^{<}_{q,d}(\mathfrak{sl}_n) \overset{\Upsilon}{\underset{\sim}{\longrightarrow}} S^{[n],\mathrm{op}} \xrightarrow{\widehat{\Phi}^{\underline{a},\underline{z}}_{\underline{b}}} \widetilde{\mathcal{A}}^{q,\prime} \tag{71}$$

coincides with the restriction of the homomorphism $\widetilde{\Phi}^{\underline{a},\underline{z}}_{\underline{b}}$ *of* (58) *to the subalgebra* $\ddot{U}^{(\underline{b}),<}_{q,d}$. *In particular, the image of* $\ddot{U}^{(\underline{b}),<}_{q,d}$ *under the composition* (71) *is in the subalgebra* $\widetilde{\mathcal{A}}^{q}$ *of* $\widetilde{\mathcal{A}}^{q,\prime}$.

4.9 Special difference operators

For any $\underline{k} \in \mathbb{N}^{[n]}$ and any multisymmetric Laurent polynomial $g \in \mathbb{C}(q)\left[\{x^{\pm 1}_{i,r}\}^{r\le k_i}_{i\in[n]}\right]^{S_{\underline{k}}}$, consider the following shuffle elements $\widetilde{E}_{\underline{k}}(g) \in S^{[n]}_{\underline{k}}$:

$$\widetilde{E}_{\underline{k}}(g) := \prod_{i\in[n]} \left\{q^{k_i^2-k_i}(q-q^{-1})^{k_i}\right\} \cdot \frac{\prod_{i\in[n]}\prod_{1\le r\ne s\le k_i}(x_{i,r}-q^{-2}x_{i,s}) \cdot g\left(\{x_{i,r}\}^{1\le r\le k_i}_{i\in[n]}\right)}{\prod_{i\to j}\prod^{s\le k_j}_{r\le k_i}(x_{j,s}-x_{i,r})}, \tag{72}$$

which obviously satisfy the wheel conditions (63). Due to Proposition 4.6, $\widetilde{E}_{\underline{k}}(g) = \Upsilon(\widetilde{e}_{\underline{k}}(g))$ for unique elements $\widetilde{e}_{\underline{k}}(g) \in \ddot{U}^{(\underline{b}),>}_{q,d} \simeq \ddot{U}^{>}_{q,d}(\mathfrak{sl}_n)$, so that $\widehat{\Phi}^{\underline{a},\underline{z}}_{\underline{b}}(\widetilde{E}_{\underline{k}}(g)) = \widetilde{\Phi}^{\underline{a},\underline{z}}_{\underline{b}}(\widetilde{e}_{\underline{k}}(g))$ by Theorem 4.8(a). We also consider $\widetilde{F}_{\underline{k}}(g) \in S^{[n],\mathrm{op}}_{\underline{k}}$ defined via:

$$\widetilde{F}_{\underline{k}}(g) := \prod_{i\in[n]} \left\{q^{k_i-k_i^2}(1-q^2)^{k_i}\right\} \cdot \frac{\prod_{i\in[n]}\prod_{1\le r\ne s\le k_i}(x_{i,r}-q^{-2}x_{i,s}) \cdot g\left(\{x_{i,r}\}^{1\le r\le k_i}_{i\in[n]}\right)}{\prod_{i\to j}\prod^{s\le k_j}_{r\le k_i}(x_{i,r}-x_{j,s})}. \tag{73}$$

The following result is established completely analogously to Lemma 2.12:

Lemma 4.10 (a) *For* $\widetilde{E}_{\underline{k}}(g) \in S^{[n]}_{\underline{k}}$ *given by* (72), *we have:*

$$\widehat{\Phi}^{\underline{a},\underline{z}}_{\underline{b}}(\widetilde{E}_{\underline{k}}(g)) = d^{\sum_{i\in[n]} k_i k_{i+1}\delta_{i+1\to i}} \prod_{i\in[n]} \left(\prod_{t=1}^{a_i} \mathsf{w}_{i,t}\right)^{k_i-\frac{1}{2}\sum_{j\leftarrow i} k_j}$$
$$\times \sum_{\substack{J_i\subset\{1,\dots,a_i\}\\ |J_i|=k_i\ \forall i\in[n]}} \left(\frac{\prod_{j\to i}\prod^{s\notin J_j}_{r\in J_i}\left(1-\frac{qd^{m_{ji}}\mathsf{w}_{j,s}}{\mathsf{w}_{i,r}}\right)}{\prod_{i\in[n]}\prod^{s\notin J_i}_{r\in J_i}\left(1-\frac{\mathsf{w}_{i,s}}{\mathsf{w}_{i,r}}\right)} \cdot g\left(\{\mathsf{w}_{i,r}\}^{r\in J_i}_{i\in[n]}\right)\right.$$

$$\times \prod_{i\in[n]}\prod_{r\in J_i} Z_i(\mathsf{w}_{i,r})\cdot \prod_{i\in[n]}\left(\prod_{r\in J_i}\mathsf{w}_{i,r}\right)^{k_i-1-\sum_{j\to i}k_j}\cdot \prod_{i\in[n]}\prod_{r\in J_i} D_{i,r}^{-1}\Bigg). \tag{74}$$

(b) *For* $\widetilde{F}_{\underline{k}}(g)\in S_{\underline{k}}^{[n],\mathrm{op}}$ *given by* (73), *we have:*

$$\widehat{\Phi}_{\underline{b}}^{\underline{a},\mathsf{z}}(\widetilde{F}_{\underline{k}}(g)) = d^{\sum_{i\in[n]}k_ik_{i+1}\delta_{i+1\leftarrow i}}q^{-3\sum_{i\in[n]}k_ik_{i+1}}\prod_{i\in[n]}\left(\prod_{t=1}^{a_i}\mathsf{w}_{i,t}\right)^{-\frac{1}{2}\sum_{j\to i}k_j}$$
$$\times \sum_{\substack{J_i\subset\{1,\ldots,a_i\}\\ |J_i|=k_i\ \forall i\in[n]}}\left(\frac{\prod_{j\leftarrow i}\prod_{r\in J_i}^{s\notin J_j}\left(1-\frac{q^{-1}d^{m_{ji}}\mathsf{w}_{j,s}}{\mathsf{w}_{i,r}}\right)}{\prod_{i\in[n]}\prod_{r\in J_i}^{s\notin J_i}\left(1-\frac{\mathsf{w}_{i,s}}{\mathsf{w}_{i,r}}\right)}\cdot g\left(\{q^2\mathsf{w}_{i,r}\}_{i\in[n]}^{r\in J_i}\right)\right.$$
$$\times \prod_{i\in[n]}\left(\prod_{r\in J_i}\mathsf{w}_{i,r}\right)^{k_i-1-\sum_{j\leftarrow i}k_j}\cdot \prod_{i\in[n]}\prod_{r\in J_i} D_{i,r}\Bigg). \tag{75}$$

Example 4.11 Consider the orientation of the cyclic quiver with arrows $i\to i+1\,(i\in[n])$.

(a) For $p\in[n]$ and $k\geq 1$, consider the degree $\underline{k}=(k,k,\ldots,k)\in\mathbb{N}^{[n]}$ elements

$$\Gamma^0_{p;k} := \widetilde{E}_{\underline{k}}\left(\prod_{i\in[n]}(x_{i,1}\cdots x_{i,k})^{1+\delta_{i0}-\delta_{ip}}\right)$$
$$= q^{n(k^2-k)}(q-q^{-1})^{nk}\cdot\frac{\prod_{i\in[n]}\prod_{1\leq r\neq s\leq k}(x_{i,r}-q^{-2}x_{i,s})\cdot\prod_{i\in[n]}\prod_{r=1}^{k}x_{i,r}}{\prod_{i\in[n]}\prod_{1\leq r,s\leq k}(x_{i,r}-x_{i-1,s})}\cdot\prod_{r=1}^{k}\frac{x_{0,r}}{x_{p,r}}. \tag{76}$$

Their images under $\widehat{\Phi}_{\underline{b}}^{\underline{a},\mathsf{z}}$ of (20) vanish if $k>\min\{a_i\}$ and otherwise are given by:

$$\widehat{\Phi}_{\underline{b}}^{\underline{a},\mathsf{z}}(\Gamma^0_{p;k}) = \prod_{i\in[n]}\left(\prod_{t=1}^{a_i}\mathsf{w}_{i,t}\right)^{k_i-\frac{1}{2}k_{i+1}}\times\sum_{\substack{J_i\subset\{1,\ldots,a_i\}\\ |J_i|=k\ \forall i\in[n]}}\left(\frac{\prod_{r\in J_i}^{s\notin J_{i-1}}\left(1-\frac{qd^{-1}\mathsf{w}_{i-1,s}}{\mathsf{w}_{i,r}}\right)}{\prod_{i\in[n]}\prod_{r\in J_i}^{s\notin J_i}\left(1-\frac{\mathsf{w}_{i,s}}{\mathsf{w}_{i,r}}\right)}\right.$$
$$\times\prod_{i\in[n]}\prod_{r\in J_i}\mathsf{Z}_i(\mathsf{w}_{i,r})\cdot\prod_{i\in[n]}\left(\prod_{r\in J_i}\mathsf{w}_{i,r}\right)^{k_i-k_{i-1}+\delta_{i0}-\delta_{ip}}\cdot\prod_{i\in[n]}\prod_{r\in J_i}D_{i,r}^{-1}\Bigg). \tag{77}$$

Similar to Remark 3.14, the difference operators (77) pairwise commute, due to the equality $[\Gamma^0_{p;k},\Gamma^0_{p';k'}]=0$ in the shuffle algebra $S^{[n]}$ established in [10, Remark 4.11(a)] (the limit case of [10, Theorem 3.3], see part (b) below). According to [21, 22], the elements $\{\Upsilon^{-1}(\Gamma^0_{p;k})\}_{p\in[n]}^{k\geq1}$ generate the "positive half of the horizontal" Heisenberg subalgebra of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.

(b) For $\mu \in \mathbb{C}$, $k \geq 1$, and $\underline{s} = (s_0, s_1, \ldots, s_{n-1}) \in (\mathbb{C}^\times)^n$ satisfying $s_0 s_1 \cdots s_{n-1} = 1$, consider:

$$F_k^\mu(\underline{s}) := \widetilde{E}_{\underline{k}}\left(\prod_{i\in[n]}\left(s_0\cdots s_i \prod_{r=1}^{k} x_{i,r} - \mu \prod_{r=1}^{k} x_{i+1,r}\right)\right) = q^{n(k^2-k)}(q-q^{-1})^{nk} \\ \times \frac{\prod_{i\in[n]}\prod_{1\leq r\neq s\leq k}(x_{i,r}-q^{-2}x_{i,s})\cdot\prod_{i\in[n]}(s_0\cdots s_i\prod_{r=1}^{k}x_{i,r}-\mu\prod_{r=1}^{k}x_{i+1,r})}{\prod_{i\in[n]}\prod_{1\leq r,s\leq k}(x_{i,r}-x_{i-1,s})}. \tag{78}$$

Their images under $\widehat{\Phi}^{\underline{a},\underline{\mathsf{z}}}_{\underline{b}}$ of (67) vanish if $k > \min\{a_i\}$ and otherwise are given by:

$$\widehat{\Phi}^{\underline{a},\underline{\mathsf{z}}}_{\underline{b}}(F_k^\mu(\underline{s})) = \prod_{i\in[n]}\left(\prod_{t=1}^{a_i}\mathsf{w}_{i,t}\right)^{k_i-\frac{1}{2}k_{i+1}} \\ \times \sum_{\substack{J_i\subset\{1,\ldots,a_i\}\\ |J_i|=k\ \forall i\in[n]}}\left(\frac{\prod_{r\in J_i}^{s\notin J_{i-1}}\left(1-\frac{qd^{-1}\mathsf{w}_{i-1,s}}{\mathsf{w}_{i,r}}\right)}{\prod_{i\in[n]}\prod_{r\in J_i}^{s\notin J_i}\left(1-\frac{\mathsf{w}_{i,s}}{\mathsf{w}_{i,r}}\right)}\cdot\prod_{i\in[n]}\left(s_0\cdots s_i-\mu\frac{\prod_{r\in J_{i+1}}\mathsf{w}_{i+1,r}}{\prod_{r\in J_i}\mathsf{w}_{i,r}}\right)\right. \\ \left.\times\prod_{i\in[n]}\prod_{r\in J_i}\mathsf{Z}_i(\mathsf{w}_{i,r})\cdot\prod_{i\in[n]}\left(\prod_{r\in J_i}\mathsf{w}_{i,r}\right)^{k_i-k_{i-1}}\cdot\prod_{i\in[n]}\prod_{r\in J_i}D_{i,r}^{-1}\right). \tag{79}$$

Similar to part (a), the difference operators (79) pairwise commute, due to the equality $[F_k^\mu(\underline{s}), F_{k'}^{\mu'}(\underline{s})] = 0$ in the shuffle algebra $S^{[n]}$ established in [10, Theorem 3.3]. According to [10, Theorem 4.10], we note that the elements $\{\Upsilon^{-1}(F_k^\mu(\underline{s}))\}$ in fact generate the Bethe commutative subalgebra of the "horizontal" quantum affine subalgebra $U_q(\widehat{\mathfrak{gl}}_n)$ of $\ddot{U}_{q,d}(\mathfrak{sl}_n)$.

5 Generalization to the quantum quiver algebras

The above constructions admit natural generalizations to the case of quantum algebras associated with quivers as recently introduced in [18] following [17]. We shall state the key results, skipping the proofs when they are similar to those from Sect. 2.

5.1 Shifted quantum algebras associated with quivers

Let E be a finite quiver, with a vertex set I and an edge set E (here, multiple edges and edge loops are allowed). Any edge e of E from a vertex $i \in I$ to a vertex $j \in I$ shall be written as $e = \vec{ij} \in E$. We fix $q \in \mathbb{C}^{\times}$ and equip every edge $e \in E$ with a weight $t_e \in \mathbb{C}^{\times}$. Furthermore, following [17, 18], we shall make the following assumption (cf. [18, Definition 5.2]):

$$|q| < |t_e| < 1 \quad \text{for all} \quad e \in E. \tag{$\dagger$}$$

We define rational functions $\{\zeta_{ij}(z)\}_{i,j\in I}$ via:

$$\zeta_{ij}\left(\frac{z}{w}\right) = \left(\frac{zq^{-1}-w}{z-w}\right)^{\delta_{ij}} \prod_{e=\vec{ij}\in E}\left(\frac{1}{t_e}-\frac{z}{w}\right) \prod_{e=\vec{ji}\in E}\left(1-\frac{wt_e}{zq}\right). \tag{80}$$

Let $\overline{E}$ be the "double" of the edge set E, i.e. there are two edges $e = \vec{ij}$, $e^* = \vec{ji} \in \overline{E}$ for every $e = \vec{ij} \in E$. Note the canonical involution $e \leftrightarrow e^*$ on $\overline{E}$ and extend the notation t_e to $\overline{E}$ via:

$$t_{e^*} := q/t_e. \tag{81}$$

For any $\underline{b}^{\pm} = \{b_i^{\pm}\}_{i\in I} \in \mathbb{Z}^I$, we define the *shifted quantum quiver algebra*, denoted by $U_Q^{(\underline{b}^+,\underline{b}^-)}$, to be the associative $\mathbb{C}$-algebra generated by $\{e_{i,r}, f_{i,r}, \psi^{\pm}_{i,\pm s_i^{\pm}}, (\psi^{\pm}_{i,\mp b_i^{\pm}})^{-1}\}_{i\in I}^{r\in\mathbb{Z}, s_i^{\pm}\geq -b_i^{\pm}}$ with the following defining relations (for all $i, j \in I$ and $\epsilon, \epsilon' \in \{\pm\}$):

$$[\psi_i^{\epsilon}(z), \psi_j^{\epsilon'}(w)] = 0, \quad \psi^{\pm}_{i,\mp b_i^{\pm}} \cdot (\psi^{\pm}_{i,\mp b_i^{\pm}})^{-1} = (\psi^{\pm}_{i,\mp b_i^{\pm}})^{-1} \cdot \psi^{\pm}_{i,\mp b_i^{\pm}} = 1, \tag{Q1}$$

$$\zeta_{ji}\left(\frac{w}{z}\right) e_i(z)e_j(w) = \zeta_{ij}\left(\frac{z}{w}\right) e_j(w)e_i(z), \tag{Q2}$$

$$\zeta_{ij}\left(\frac{z}{w}\right) f_i(z)f_j(w) = \zeta_{ji}\left(\frac{w}{z}\right) f_j(w)f_i(z), \tag{Q3}$$

$$\zeta_{ji}\left(\frac{w}{z}\right) \psi_i^{\epsilon}(z)e_j(w) = \zeta_{ij}\left(\frac{z}{w}\right) e_j(w)\psi_i^{\epsilon}(z), \tag{Q4}$$

$$\zeta_{ij}\left(\frac{z}{w}\right) \psi_i^{\epsilon}(z)f_j(w) = \zeta_{ji}\left(\frac{w}{z}\right) f_j(w)\psi_i^{\epsilon}(z), \tag{Q5}$$

$$[e_i(z), f_j(w)] = \delta_{ij}\delta\left(\frac{z}{w}\right)\left(\psi_i^+(z) - \psi_i^-(z)\right), \tag{Q6}$$

and more complicated cubic Serre relations of [18, §5.4] that shall be omitted for brevity. Here, the generating series $\{e_i(z), f_i(z), \psi_i^{\pm}(z)\}_{i\in I}$ are defined as in (6). The original quantum quiver algebra U_Q of [18] is isomorphic to $U_Q^{(0,0)}/(\psi^+_{i,0}\psi^-_{i,0}-1)_{i\in I}$.

5.2 GKLO-type homomorphisms

Fix $\underline{a} = (a_i)_{i\in I} \in \mathbb{N}^I$, $\underline{N} = (N_i)_{i\in I} \in \mathbb{N}^I$, and a collection $\underline{\mathsf{z}} = \{\mathsf{z}_{i,r}\}_{i\in I}^{1\le r\le N_i}$ with $\mathsf{z}_{i,r} \in \mathbb{C}^\times$. We define $\mathsf{Z}_i(z) := \prod_{r=1}^{N_i}\left(1 - \frac{\mathsf{z}_{i,r}}{z}\right)$. Finally, we consider the following particular $\underline{b}^\pm \in \mathbb{Z}^I$:

$$b_i^+ = \sum_{j\in I} a_j \cdot \#\left\{e = \vec{ij} \in E\right\} - a_i, \quad b_i^- = -N_i - \sum_{j\in I} a_j \cdot \#\left\{e = \vec{ji} \in E\right\} + a_i. \tag{82}$$

For any $i, j \in I$, we also define constants $\gamma_{ij}^+, \gamma_{ij}^-, \gamma_{ij}^0$ via:

$$\gamma_{ij}^+ = \sum_{e=\vec{ij}\in E} \log_q(t_e), \quad \gamma_{ij}^- = -\sum_{e=\vec{ji}\in E} \log_q(t_e), \quad \gamma_{ij}^0 = \gamma_{ij}^+ + \gamma_{ij}^-. \tag{83}$$

Let $\hat{\mathcal{A}}^q$ be the associative $\mathbb{C}$-algebra generated by $\{\mathsf{w}_{i,r}^{\pm 1}, D_{i,r}^{\pm 1}\}_{i\in I}^{1\le r\le a_i}$ satisfying the relations $[\mathsf{w}_{i,r}, \mathsf{w}_{j,s}] = 0 = [D_{i,r}, D_{j,s}]$ and $D_{i,r}\mathsf{w}_{i,r} = q^{-\delta_{ij}\delta_{rs}}\mathsf{w}_{i,r}D_{i,r}$. Let $\mathcal{A}^q$ be obtained from $\hat{\mathcal{A}}^q$ by formally adjoining $\{\left(\prod_{r=1}^{a_i}\mathsf{w}_{i,r}\right)^{\gamma_{ji}^\pm}\}_{i,j\in I}$ satisfying the relations $\mathsf{w}_{\iota,s}\left(\prod_{r=1}^{a_i}\mathsf{w}_{i,r}\right)^{\gamma_{ji}^\pm} = \left(\prod_{r=1}^{a_i}\mathsf{w}_{i,r}\right)^{\gamma_{ji}^\pm}\mathsf{w}_{\iota,s}$ and $D_{\iota,s}\left(\prod_{r=1}^{a_i}\mathsf{w}_{i,r}\right)^{\gamma_{ji}^\pm} = q^{-\delta_{\iota i}\gamma_{ji}^\pm}\left(\prod_{r=1}^{a_i}\mathsf{w}_{i,r}\right)^{\gamma_{ji}^\pm}D_{\iota,s}$, for all i, j, ι, s. We define $\widetilde{\mathcal{A}}^q$ as the localization of $\mathcal{A}^q$ by the multiplicative set generated by $\{\mathsf{w}_{i,r} - q^m\mathsf{w}_{i,s}\}_{i\in I, r\ne s}^{m\in\mathbb{Z}}$.

Then, we have the following analogue of Proposition 2.4:

Proposition 5.3 *There exists a unique $\mathbb{C}$-algebra homomorphism*

$$\widetilde{\Phi}_{\underline{a}}^{\underline{\mathsf{z}}}\colon U_Q^{(\underline{b}^+,\underline{b}^-)} \longrightarrow \widetilde{\mathcal{A}}^q \tag{84}$$

for any $\underline{a}$ and $\underline{\mathsf{z}}$ as above, with $\underline{b}^\pm \in \mathbb{Z}^I$ defined via (82), such that

$$e_i(z) \mapsto \prod_{j\ne i}\left(\prod_{s=1}^{a_j}\mathsf{w}_{j,s}\right)^{\gamma_{ij}^+} \cdot \sum_{r=1}^{a_i}\delta\left(\frac{\mathsf{w}_{i,r}}{z}\right)\frac{\mathsf{Z}_i(\mathsf{w}_{i,r})\prod_{j\ne i}^{s\le a_j}\prod_{e=\vec{ij}}\left(\frac{1}{t_e} - \frac{z}{\mathsf{w}_{j,s}}\right)\prod_{e=\vec{ii}}^{s\ne r}\left(\frac{1}{t_e} - \frac{z}{\mathsf{w}_{i,s}}\right)}{\prod_{s\ne r}\left(1 - \frac{z}{\mathsf{w}_{i,s}}\right)}D_{i,r}^{-1},$$

$$f_i(z) \mapsto \prod_{j\ne i}\left(\prod_{s=1}^{a_j}\mathsf{w}_{j,s}\right)^{\gamma_{ij}^-} \cdot \sum_{r=1}^{a_i}\delta\left(\frac{\mathsf{w}_{i,r}}{qz}\right)\frac{\prod_{j\ne i}^{s\le a_j}\prod_{e=\vec{ji}}\left(1 - \frac{\mathsf{w}_{j,s}t_e}{zq}\right)\prod_{e=\vec{ii}}^{s\ne r}\left(1 - \frac{\mathsf{w}_{i,s}t_e}{zq}\right)}{\prod_{s\ne r}\left(1 - \frac{\mathsf{w}_{i,s}}{zq}\right)}D_{i,r},$$

$$\psi_i^\pm(z) \mapsto \frac{q^{-1} - 1}{\prod_{e=\vec{ii}}\left\{\left(\frac{1}{t_e} - 1\right)\left(1 - \frac{t_e}{q}\right)\right\}} \cdot \prod_{j\ne i}\left(\prod_{s=1}^{a_j}\mathsf{w}_{j,s}\right)^{\gamma_{ij}^0}$$
$$\times\left(\mathsf{Z}_i(z)\cdot\frac{\prod_{j\in I}\prod_{s=1}^{a_j}\left\{\prod_{e=\vec{ij}}\left(\frac{1}{t_e} - \frac{z}{\mathsf{w}_{j,s}}\right)\cdot\prod_{e=\vec{ji}}\left(1 - \frac{\mathsf{w}_{j,s}t_e}{zq}\right)\right\}}{\prod_{r=1}^{a_i}\left\{\left(1 - \frac{z}{\mathsf{w}_{i,r}}\right)\left(1 - \frac{\mathsf{w}_{i,r}}{zq}\right)\right\}}\right)^{\pm}.$$

Here, $e \in E$ and $\gamma(z)^{\pm}$ denotes the expansion of a rational function $\gamma(z)$ in $z^{\mp 1}$, respectively.

5.4 Shuffle algebra realization of the positive and negative subalgebras

Similar to (11, 12, 39, 60), we have the following algebra isomorphisms:

$$U_Q^{(\underline{b}^+,\underline{b}^-),>} \xrightarrow{\sim} U_Q^>, \quad U_Q^{(\underline{b}^+,\underline{b}^-),<} \xrightarrow{\sim} U_Q^<, \quad U_Q^< \xrightarrow{\sim} U_Q^{>,\mathrm{op}}, \tag{85}$$

with the subalgebras $U_Q^{(\underline{b}^+,\underline{b}^-),>}, U_Q^>, U_Q^{(\underline{b}^+,\underline{b}^-),<}, U_Q^<$ defined in a self-explaining way.

Consider an $\mathbb{N}^I$-graded $\mathbb{C}$-vector space $\mathbb{S}^Q = \bigoplus_{\underline{k}=(k_i)_{i\in I}\in\mathbb{N}^I} \mathbb{S}^Q_{\underline{k}}$, with the graded components

$$\mathbb{S}^Q_{\underline{k}} = \left\{ F \in \mathbb{C}\left[\{x_{i,r}^{\pm 1}\}_{i\in I}^{1\le r\le k_i} \right]^{S_{\underline{k}}} \right\}. \tag{86}$$

Evoking the rational functions of (80), we equip $\mathbb{S}^Q$ with the bilinear *shuffle product* $\star$ completely analogously to (15), thus making $\mathbb{S}^Q$ into an associative unital $\mathbb{C}$-algebra. As before, we are interested in an $\mathbb{N}^I$-graded subspace of $\mathbb{S}^Q$ defined by the following *wheel conditions*:

$$F|_{x_{i,2}=qx_{i,1}} \quad \text{is divisible by} \quad (x_{j,1}-\gamma x_{i,1})^{\flat_{ij}(\gamma)} \tag{87}$$

for any $\gamma \in \mathbb{C}^\times$ and $j \in I$, where

$$\flat_{ij}(\gamma) = \#\left\{ e = \vec{ij} \in \overline{E} \,\middle|\, t_e = \gamma \right\}. \tag{88}$$

In particular, as pointed out in [17, 18], if for any $i, j \in I$ all the weights $\{t_e | e = \vec{ij} \in \overline{E}\}$ are pairwise distinct, then (87) may be written in a more familiar form, cf. (16, 42, 63), as:

$$\begin{gathered} F\left(\{x_{i,r}\}\right) = 0 \quad \text{once} \quad x_{i,a} = qt_e^{-1}x_{j,b} = qx_{i,c} \\ \text{for any edge } \overline{E} \ni e = \vec{ij} \text{ and } a \neq c, \text{ where } a \neq b \neq c \text{ if } i = j. \end{gathered} \tag{89}$$

Let $S^Q \subset \mathbb{S}^Q$ denote the subspace of all such elements F, which is easily seen to be $\star$-closed. The resulting *shuffle algebra* $(S^Q, \star)$ is related to U_Q via [18, Theorem 5.8]:

Proposition 5.5 [18] *The assignments $e_{i,r} \mapsto x_{i,1}^r$ and $f_{i,r} \mapsto x_{i,1}^r$ for $i \in I, r \in \mathbb{Z}$ give rise to $\mathbb{C}$-algebra isomorphisms*

$$\Upsilon\colon U_Q^> \xrightarrow{\sim} S^Q \quad \text{and} \quad \Upsilon\colon U_Q^< \xrightarrow{\sim} S^{Q,\mathrm{op}}. \tag{90}$$

5.6 Shuffle algebra realization of the GKLO-type homomorphisms

For any $i \in I$ an $1 \leq r \leq a_i$, we define:

$$Y_{i,r}(z) := \prod_{j\neq i}\left(\prod_{j=1}^{a_j} \mathsf{w}_{j,s}\right)^{\gamma_{ij}^{+}} \cdot \frac{\mathsf{Z}_i(z)\prod_{j\neq i}^{s\leq a_j}\prod_{e=\vec{ij}\in E}\left(\frac{1}{t_e}-\frac{z}{\mathsf{w}_{j,s}}\right)\prod_{e=\vec{ii}\in E}^{s\neq r}\left(\frac{1}{t_e}-\frac{z}{\mathsf{w}_{i,s}}\right)}{\prod_{s\neq r}\left(1-\frac{z}{\mathsf{w}_{i,s}}\right)},$$
$$Y'_{i,r}(z) := \prod_{j\neq i}\left(\prod_{j=1}^{a_j} \mathsf{w}_{j,s}\right)^{\gamma_{ij}^{-}} \cdot \frac{\prod_{j\neq i}^{s\leq a_j}\prod_{e=\vec{ji}\in E}\left(1-\frac{\mathsf{w}_{j,s}t_e}{zq}\right)\prod_{e=\vec{ii}\in E}^{s\neq r}\left(1-\frac{\mathsf{w}_{i,s}t_e}{zq}\right)}{\prod_{s\neq r}\left(1-\frac{\mathsf{w}_{i,s}}{zq}\right)}. \tag{91}$$

We also define

$$\varphi_{ij}\left(\frac{z}{w}\right) = \left(\frac{z-w}{zq^{-1}-w}\right)^{\delta_{ij}} \prod_{e=\vec{ij}\in E}\left(\frac{1}{t_e}-\frac{z}{w}\right)^{-1}. \tag{92}$$

Define the $\mathbb{C}$-algebra $\widetilde{\mathcal{A}}^{q,'}$ as the further localization of $\widetilde{\mathcal{A}}^{q}$ by the multiplicative set generated by $\{\mathsf{w}_{i,r} - t_e^{-1}q^m\mathsf{w}_{j,s}\}_{e=\vec{ij}\in E, m\in\mathbb{Z}}^{r\leq a_i, s\leq a_j}$. We note that $\widetilde{\mathcal{A}}^{q}$ is naturally embedded into $\widetilde{\mathcal{A}}^{q,'}$.

The following result is proved completely analogously to Theorem 2.8:

Theorem 5.7 (a) *The assignment*

$$\begin{aligned}
\mathbb{S}^{Q}_{\underline{k}} \ni E \mapsto & \prod_{i\in I}\prod_{e=\vec{ii}\in E} t_e^{\frac{k_i-k_i^2}{2}} \\
\times & \sum_{\substack{m_1^{(i)}+\ldots+m_{a_i}^{(i)}=k_i \\ m_r^{(i)}\in\mathbb{N}\ \forall i\in I}} \left\{ \prod_{i\in I}\prod_{r=1}^{a_i}\prod_{p=1}^{m_r^{(i)}} Y_{i,r}\left(\mathsf{w}_{i,r}q^{p-1}\right) \cdot E\left(\left\{\mathsf{w}_{i,r}q^{p-1}\right\}_{i\in I, 1\leq r\leq a_i}^{1\leq p\leq m_r^{(i)}}\right)\right. \\
\times & \prod_{i\in I}\prod_{1\leq r\leq a_i}\prod_{1\leq p_1<p_2\leq m_r^{(i)}}\left(\zeta_{ii}^{-1}\left(\mathsf{w}_{i,r}q^{p_1-1}\big/\mathsf{w}_{i,r}q^{p_2-1}\right)\cdot\prod_{e=\vec{ii}\in E} t_e\right) \\
\times & \left.\prod_{i,j\in I}\prod_{\substack{(i,r_1)\neq(j,r_2) \\ 1\leq r_1\leq a_i \\ 1\leq r_2\leq a_j}}\prod_{\substack{1\leq p_2\leq m_{r_2}^{(j)} \\ 1\leq p_1\leq m_{r_1}^{(i)}}} \varphi_{ij}\left(\mathsf{w}_{i,r_1}q^{p_1-1}\big/\mathsf{w}_{j,r_2}q^{p_2-1}\right)\cdot\prod_{i\in I}\prod_{r=1}^{a_i} D_{i,r}^{-m_r^{(i)}}\right\}
\end{aligned} \tag{93}$$

gives rise to the algebra homomorphism

$$\widehat{\Phi}^{\underline{z}}_{\underline{a}} \colon \mathbb{S}^{Q} \longrightarrow \widetilde{\mathcal{A}}^{q,'}. \tag{94}$$

Moreover, for $\underline{b}^{\pm} \in \mathbb{Z}^I$ *defined via* (82)*, the composition*

$$U_Q^{(\underline{b}^+,\underline{b}^-),>} \overset{(85)}{\underset{\sim}{\longrightarrow}} U_Q^{>} \overset{\Upsilon}{\underset{\sim}{\longrightarrow}} S^Q \overset{\widehat{\Phi}^{\underline{z}}_{\underline{a}}}{\longrightarrow} \widetilde{\mathcal{A}}^{q,'} \tag{95}$$

coincides with the restriction of the homomorphism $\widetilde{\Phi}^{\underline{z}}_{\underline{a}}$ *of* (84) *to the subalgebra* $U_Q^{(\underline{b}^+,\underline{b}^-),>}$*. In particular, the image of* $U_Q^{(\underline{b}^+,\underline{b}^-),>}$ *under* (95) *is in the subalgebra* $\widetilde{\mathcal{A}}^q$ *of* $\widetilde{\mathcal{A}}^{q,'}$.

(b) *The assignment*

$$\begin{aligned}
&\mathbb{S}^{Q,\mathrm{op}}_{\underline{k}} \ni F \mapsto \prod_{i\in I}\prod_{e=\vec{ii}\in E} t_e^{\frac{k_i-k_i^2}{2}} \\
&\times \sum_{\substack{m_1^{(i)}+\ldots+m_{a_i}^{(i)}=k_i\\ m_r^{(i)}\in\mathbb{N}\ \forall i\in I}} \left\{ \prod_{i\in I}\prod_{r=1}^{a_i}\prod_{p=1}^{m_r^{(i)}} Y'_{i,r}\left(\mathsf{w}_{i,r}q^{-p}\right)\cdot F\left(\left\{\mathsf{w}_{i,r}q^{-p}\right\}_{i\in I, 1\le r\le a_i}^{1\le p\le m_r^{(i)}}\right)\right. \\
&\times \prod_{i\in I}\prod_{1\le r\le a_i}\prod_{1\le p_1<p_2\le m_r^{(i)}} \left(\zeta_{ii}^{-1}\left(\mathsf{w}_{i,r}q^{-p_2}\big/\mathsf{w}_{i,r}q^{-p_1}\right)\cdot\prod_{e=\vec{ii}\in E} t_e\right) \\
&\left.\times \prod_{i,j\in I}\prod_{\substack{1\le r_1\le a_i\\ 1\le r_2\le a_j}}^{(i,r_1)\ne(j,r_2)}\prod_{1\le p_1\le m_{r_1}^{(i)}}^{1\le p_2\le m_{r_2}^{(j)}} \varphi_{ji}\left(\mathsf{w}_{j,r_2}q^{-p_2}\big/\mathsf{w}_{i,r_1}q^{-p_1}\right)\cdot\prod_{i\in I}\prod_{r=1}^{a_i} D_{i,r}^{m_r^{(i)}}\right\}
\end{aligned} \tag{96}$$

gives rise to the algebra homomorphism

$$\widehat{\Phi}^{\underline{z}}_{\underline{a}} \colon \mathbb{S}^{Q,\mathrm{op}} \longrightarrow \widetilde{\mathcal{A}}^{q,'}. \tag{97}$$

Moreover, for $\underline{b}^{\pm} \in \mathbb{Z}^I$ *defined via* (82)*, the composition*

$$U_Q^{(\underline{b}^+,\underline{b}^-),<} \overset{(85)}{\underset{\sim}{\longrightarrow}} U_Q^{<} \overset{\Upsilon}{\underset{\sim}{\longrightarrow}} S^{Q,\mathrm{op}} \overset{\widehat{\Phi}^{\underline{z}}_{\underline{a}}}{\longrightarrow} \widetilde{\mathcal{A}}^{q,'} \tag{98}$$

coincides with the restriction of the homomorphism $\widetilde{\Phi}^{\underline{z}}_{\underline{a}}$ *of* (84) *to the subalgebra* $U_Q^{(\underline{b}^+,\underline{b}^-),<}$*. In particular, the image of* $U_Q^{(\underline{b}^+,\underline{b}^-),<}$ *under* (98) *is in the subalgebra* $\widetilde{\mathcal{A}}^q$ *of* $\widetilde{\mathcal{A}}^{q,'}$.

Remark 5.8 This theorem immediately implies that the assignment of Proposition 5.3 is indeed compatible with the cubic Serre relations of [18, §5.4] which we omitted, cf. Remark 2.9.

6 Relation to quantum *Q*-systems of type *A*

In this section, we explain how the shuffle approach from Sect. 2 in the simplest case of $\mathfrak{g} = \mathfrak{sl}_2$ simplifies some of the tedious arguments of [4] in their study of A-type Q-systems. We also match their difference operators representing the M-system with those of Sect. 2.

6.1 Elements $E_{k,n}$ and $M_{k,n}$ for $\mathfrak{g} = \mathfrak{sl}_2$

For any $k \geq 1$ and $n \in \mathbb{Z}$, consider the elements $E_{k,n} \in S_k = S_k^{(\mathfrak{sl}_2)}$ defined via:

$$E_{k,n}(x_1, \ldots, x_k) := \prod_{1 \leq r \leq k} x_r^n \prod_{1 \leq r \neq s \leq k} (x_r - q^{-2} x_s). \tag{99}$$

The following result identifies these elements with those featuring in [11, (9.2)]:

Lemma 6.2 *The elements $E_{k,n}$ correspond to explicit q-commutators in $U_q^{>}(L\mathfrak{sl}_2)$:*

$$E_{k,n} = \frac{(-1)^{\frac{k(k-1)}{2}}}{(1-q^{-2})^{k-1}} \cdot \Upsilon\Big([e_n, [e_{n+2}, \cdots, [e_{n+2(k-2)}, e_{n+2(k-1)}]_{q^{-4}} \cdots]_{q^{-2(k-1)}}]_{q^{-2k}}\Big), \tag{100}$$

where $[x, y]_{q^r} = xy - q^r \cdot yx$ as before.

Proof It suffices to prove (100) for $n = 0$. The proof is by induction on $k \geq 1$, the base case $k = 1$ being obvious. For a step of induction, deducing the $k = \ell + 1$ case of (100) from its validity for $k \leq \ell$, we first note (by direct computations) that $[x^0, E_{\ell,2}]_{q^{-2(\ell+1)}} = x^0 \star E_{\ell,2} - q^{-2(\ell+1)} E_{\ell,2} \star x^0 \in S_{\ell+1}$ vanishes under the specialization $x_{\ell+1} = q^2 x_\ell$; hence, it is divisible by the product $\prod_{1 \leq r \neq s \leq \ell+1} (x_r - q^{-2} x_s)$. As $[x^0, E_{\ell,2}]_{q^{-2(\ell+1)}}$ is a polynomial in $x_1, \ldots, x_{\ell+1}$ of the total degree $\ell(\ell+1)$, we get:

$$\Upsilon\Big([e_0, [e_2, \cdots, [e_{2(\ell-1)}, e_{2\ell}]_{q^{-4}} \cdots]_{q^{-2\ell}}]_{q^{-2(\ell+1)}}\Big) = c_{\ell+1} \cdot E_{\ell+1,0} \tag{101}$$

for some constant $c_{\ell+1}$. To determine this constant, we plug $x_{\ell+1} = t$ into (101), divide both sides by $t^{2\ell}$, and consider the $t \to \infty$ limit to obtain:

$$\begin{aligned}
(-q^{-2})^\ell c_{\ell+1} E_{\ell,0} &= (-1)^{\ell-1} q^{-2\ell} c_\ell [x^0, E_{\ell-1,2}]_{q^{-2\ell}} \\
&= (-1)^{\ell-1} q^{-2\ell} \frac{c_\ell}{c_{\ell-1}} \Upsilon\big([e_0, [e_2, \cdots, [e_{2(\ell-2)}, e_{2\ell-2}]_{q^{-4}} \cdots]_{q^{-2\ell+2}}]_{q^{-2\ell}}\big) \\
&= (-1)^{\ell-1} q^{-2\ell} \frac{c_\ell^2}{c_{\ell-1}} E_{\ell,0},
\end{aligned}$$

where we used the induction assumption for $k = \ell - 1$ and $k = \ell$. Combining the resulting equality $c_{\ell+1} = -\frac{c_\ell^2}{c_{\ell-1}}$ with $c_1 = 1$ and $c_2 = q^{-2} - 1$, we get $c_{\ell+1} = (-1)^{\frac{\ell(\ell+1)}{2}}(1-q^{-2})^\ell$. □

Let us now compare this with [4]. To this end, we define $M_{k,n}$ via [4, (2.23)]:[1]

$$M_{k,n} := \frac{(-1)^{\frac{k(k-1)}{2}}}{(1-\mathfrak{q})^{k-1}} \cdot [\cdots[[M_{1,n-k+1}, M_{1,n-k+3}]_{\mathfrak{q}^2}, M_{1,n-k+5}]_{\mathfrak{q}^3}, \cdots, M_{1,n+k-1}]_{\mathfrak{q}^k}, \tag{102}$$

where we identify $M_{1,n}$ with our e_{-n} and their parameter $\mathfrak{q}$ with our q^2, in accordance with [4, (2.20)]. Due to (100), we get:

$$\begin{aligned} \Upsilon(M_{k,n}) &= (-1)^{\frac{k(k-1)}{2}}(1-q^{-2})^{1-k}q^{k(k-1)} \cdot \Upsilon\left([e_{-n-k+1}, \cdots, [e_{n+k-3}, e_{n+k-1}]_{q^{-4}} \cdots]_{q^{-2k}}\right) \\ &= q^{k(k-1)} \cdot E_{k,1-k-n}(x_1, \ldots, x_k) = q^{k(k-1)} \cdot \prod_{1\le r\le k} x_r^{1-k-n} \prod_{1\le r\ne s\le k}(x_r - q^{-2}x_s). \end{aligned} \tag{103}$$

Thus, the generating series $\mathfrak{m}_k(z) := \sum_{n\in\mathbb{Z}} M_{k,n}z^n$ of [4, (2.13)] is identified with:

$$\Upsilon(\mathfrak{m}_k(z)) = q^{k(k-1)} \cdot \prod_{1\le r\le k} x_r^{1-k} \prod_{1\le r\ne s\le k}(x_r - q^{-2}x_s) \cdot \delta\left(\frac{x_1\cdots x_k}{z}\right), \tag{104}$$

where $\delta(z)$ is the delta-function of (6). This immediately implies [4, Theorem 2.10] (expressing $M_{k,n}$ as a noncommutative polynomial in $M_{1,m}$'s with coefficients in $\mathbb{Z}[\mathfrak{q}, \mathfrak{q}^{-1}]$):

Proposition 6.3 *Let* $\Delta_{\mathfrak{q}}(u_1, \ldots, u_k) = \prod_{1\le r<s\le k}(1 - \mathfrak{q}\frac{u_s}{u_r})$. *Then, we have:*

$$\mathfrak{m}_k(z) = \mathrm{CT}_{u_1,\ldots,u_k}\left(\Delta_{\mathfrak{q}}(u_1, \ldots, u_k)\mathfrak{m}_1(u_1)\cdots\mathfrak{m}_1(u_k)\delta\Big(\frac{u_1\cdots u_k}{z}\Big)\right), \tag{105}$$

where $\mathrm{CT}_{u_1,\ldots,u_k}$ *denotes the "constant term" (i.e.* $u_1^0\cdots u_k^0$*-coefficient) of any series in* u_r*'s.*

Proof Combining the key property $f(u)\delta(u/z) = f(z)\delta(u/z)$ of the delta-functions (6) with $\Upsilon(\mathfrak{m}_1(z)) = \delta(x_1/z)$ and evoking the definition of the shuffle product (15), we obtain:

$$\Upsilon\left(\Delta_{\mathfrak{q}}(u_1, \ldots, u_k)\mathfrak{m}_1(u_1)\cdots\mathfrak{m}_1(u_k)\delta\Big(\frac{u_1\cdots u_k}{z}\Big)\right) = (-q^2)^{\frac{k(k-1)}{2}}$$

[1] There seems to be a sign typo in [4, (2.23)] making it actually incompatible with [4, (2.25)].

$$\times \prod_{1\le r\ne s\le k} (x_r - q^{-2}x_s)\delta\Big(\frac{x_1\cdots x_k}{z}\Big) \mathop{\mathrm{Sym}}_{x_1,\ldots,x_k} \left(\delta\left(\frac{x_1}{u_1}\right)\cdots\delta\left(\frac{x_k}{u_k}\right) \prod_{1\le r<s\le k} \frac{1}{x_r(x_r-x_s)}\right).$$

Comparing the constant terms of both sides in the above equality, we get:

$$\mathrm{CT}_{u_1,\ldots,u_k}\left\{\Upsilon\left(\Delta_{\mathfrak{q}}(u_1,\ldots,u_k)\mathfrak{m}_1(u_1)\cdots\mathfrak{m}_1(u_k)\delta\Big(\frac{u_1\cdots u_k}{z}\Big)\right)\right\}$$
$$= (-1)^{\frac{k(k-1)}{2}}q^{k(k-1)} \cdot \prod_{1\le r\ne s\le k}(x_r - q^{-2}x_s)\cdot\delta\Big(\frac{x_1\cdots x_k}{z}\Big)\cdot \mathop{\mathrm{Sym}}_{x_1,\ldots,x_k}\left(\prod_{1\le r<s\le k}\frac{1}{x_r(x_r-x_s)}\right).$$

Combining this equality with the simple identity

$$\mathop{\mathrm{Sym}}_{x_1,\ldots,x_k}\left\{\prod_{1\le r<s\le k}\frac{1}{x_r(x_r-x_s)}\right\} = (-1)^{\frac{k(k-1)}{2}}\prod_{1\le r\le k} x_r^{1-k}, \tag{106}$$

we obtain (105) as a direct consequence of the shuffle realization (104) of $\mathfrak{m}_k(z)$. □

Remark 6.4 The equality (106) is equivalent to $\mathop{\mathrm{Sym}}_{x_1,\ldots,x_k}\left\{\prod_{1\le r<s\le k}\frac{x_s}{x_s-x_r}\right\} = 1$, which is nothing but the standard Vandermonde determinant formula.

6.5 Verifying the *M*-system relations through the shuffle algebra

Let us now explain how the shuffle approach also allows to establish the key relations of [4, (2.1, 2.2)] satisfied by $M_{k,n}$ of (102), thus providing a simple proof of [4, Theorem 2.11].

We start with the following q-commutativity property:

Lemma 6.6 (a) *For any $k \ge 1$ and $m, n \in \mathbb{Z}$ such that $-1 \le m-n \le 2k-1$, we have:*

$$[x^m, E_{k,n}]_{q^{2(m-n-k+1)}} = 0. \tag{107}$$

(b) *For any $k \ge \ell \ge 1$ and $a, b \in \mathbb{Z}$ such that $-1 \le a-b \le 2k-2\ell+1$, we have:*

$$[E_{\ell,a}, E_{k,b}]_{q^{2\ell(a-b+\ell-k)}} = 0. \tag{108}$$

(c) *For any $k \ge 1$, $n \in \mathbb{Z}$, and a collection $\epsilon_1,\ldots,\epsilon_{k-1} \in \{0,1,2\}$, the following $2k$ elements:*

$$E_{k,n},\ E_{k,n+1},\ E_{k-1,n+\epsilon_1},\ E_{k-1,n+\epsilon_1+1},\ \ldots,\ E_{1,n+\epsilon_1+\ldots+\epsilon_{k-1}},\ E_{1,n+\epsilon_1+\ldots+\epsilon_{k-1}+1} \tag{109}$$

pairwise q-commute and are in the Υ-image of the subalgebra generated by $\{e_r\}_{r=n}^{n+2k-1}$.

Proof (a) It suffices to prove (107) for $n=0$. We note that $[x^m, E_{k,0}]_{q^{2(m-k+1)}} \in S_{k+1}$ vanishes under the specialization $x_{k+1} = q^2 x_k$, and thus, it is divisible by $\prod_{1\le r\ne s\le k+1}(x_r - q^{-2}x_s)$. If $0 \le m \le 2k-1$, then $[x^m, E_{k,0}]_{q^{2(m-k+1)}}$ is a polynomial in $x_1, \ldots, x_{k+1}$ of the total degree $m + k(k-1)$. This implies (107) as $\deg(\prod_{1\le r\ne s\le k+1}(x_r - q^{-2}x_s)) = k(k+1) > m + k(k-1)$. If $m=-1$, then similarly $x_1 \cdots x_{k+1} \cdot [x^{-1}, E_{k,0}]_{q^{-2k}} \in S_{k+1}$ is a polynomial in $x_1, \ldots, x_{k+1}$ of the total degree k^2 which is divisible by the product $\prod_{1\le r\ne s\le k+1}(x_r - q^{-2}x_s)$ of the total degree $k(k+1) > k^2$. Therefore, $[x^{-1}, E_{k,0}]_{q^{-2k}} = 0$ as well.

(b) As $-1 \le a-b, a+2-b, \ldots, a+2(\ell-1)-b \le 2k-1$, (108) is in fact an immediate corollary of (107), due to (100) that can be written as:

$$E_{\ell,a} = (-1)^{\frac{\ell(\ell-1)}{2}}(1-q^{-2})^{1-\ell} \times [x^a, [x^{a+2}, \cdots, [x^{a+2(\ell-2)}, x^{a+2(\ell-1)}]_{q^{-4}} \cdots]_{q^{-2(\ell-1)}}]_{q^{-2\ell}}. \qquad (110)$$

(c) The q-commutativity part follows from (b), while the second part is a consequence of (110). □

As particular cases of (108), we obtain the following equalities:

$$[E_{k,1}, E_{k,0}]_{q^{2k}} = 0 \quad \text{and}$$
$$[E_{\ell,k-\ell+\epsilon}, E_{k,0}]_{q^{2\ell\epsilon}} = 0 \quad \text{for} \quad 1 \le \ell \le k,\ \epsilon \in \{-1, 0, 1\}.$$

Since $\Upsilon(M_{\alpha,n}) \in S_\alpha$ is a multiple of $E_{\alpha,1-\alpha-n}$, due to (103), we thus recover [4, (2.2)]:

Proposition 6.7 *For any $\alpha, \beta \in \mathbb{N}$, $n \in \mathbb{Z}$, $\epsilon \in \{0, 1\}$, the elements $M_{k,n}$ of* (102) *satisfy:*

$$M_{\alpha,n} M_{\beta,n+\epsilon} = \mathfrak{q}^{\min(\alpha,\beta)\epsilon} M_{\beta,n+\epsilon} M_{\alpha,n}. \qquad (111)$$

We also have the following result (which together with Proposition 6.7 constitute the content of [4, Theorem 4.18], thus providing a simple proof of [4, Theorem 2.11]):

Proposition 6.8 *The elements* (102) *satisfy the following M-system relation [4, (2.1)]:*

$$M_{\alpha,n}^2 - \mathfrak{q}^\alpha M_{\alpha,n+1} M_{\alpha,n-1} = M_{\alpha+1,n} M_{\alpha-1,n} \quad \text{for any} \quad \alpha \ge 1,\ n \in \mathbb{Z}. \qquad (112)$$

Due to (103), this is a direct consequence of the corresponding relation for $E_{k,n}$ of (99):

Lemma 6.9 *For any $k \ge 1$ and $n \in \mathbb{Z}$, the following quadratic relation holds in $S = S^{(\mathfrak{sl}_2)}$:*

$$E_{k,n}^2 - q^{2k} E_{k,n-1} \star E_{k,n+1} = q^2 E_{k+1,n-1} \star E_{k-1,n+1}. \qquad (113)$$

Proof It suffices to prove (113) for $n = 0$, that is, to show that the shuffle element

$$E'_k := E_{k,0}\star E_{k,0} - q^{2k} E_{k,-1}\star E_{k,1} - q^2 E_{k+1,-1}\star E_{k-1,1} \in S_{2k} \tag{114}$$

vanishes. We prove (114) by induction on $k \geq 1$, the base case $k = 1$ following immediately from Proposition 6.3 (applied to $k = 2$).

For the step of induction (assuming that (114) holds for all $k < \ell$), it suffices to prove

$$E'_\ell(x_1, \ldots, x_{2\ell-2}, y, q^2 y) = 0. \tag{115}$$

Indeed, (115) implies that $x_1 \cdots x_{2\ell} \cdot E'_\ell(x_1, \ldots, x_{2\ell})$ is a polynomial in $x_1, \ldots, x_{2\ell}$ of the total degree $2\ell^2$ which is divisible by the product $\prod_{1\leq r\neq s\leq 2\ell}(x_r - q^{-2}x_s)$ of degree $2\ell(2\ell-1)$. As $4\ell^2 - 2\ell > 2\ell^2$ for $\ell > 1$, we thus obtain $E'_\ell(x_1, \ldots, x_{2\ell}) = 0$ which establishes the step of induction. Finally, the equality (115) follows from the following straightforward computation:

$$\begin{aligned} &E'_\ell(x_1, \ldots, x_{2\ell-2}, y, q^2 y) = (1 + q^{-2})q^{-6(\ell-1)} \\ &\times \prod_{r=1}^{2\ell-2} (x_r - q^{-2}y)(x_r - q^4 y) \cdot E'_{\ell-1}(x_1, \ldots, x_{2\ell-2}) = 0 \end{aligned}$$

with the latter equality due to the induction hypothesis. □

Remark 6.10 We note that similar shuffle interpretations of the relations (111, 112) were suggested (without a proof) in [5, Lemma 8.5].

6.11 Comparison of the difference operators I

Let us now compare the realization of the M-system by difference operators as presented in [4, §6] with the construction of Sect. 2. To this end, we fix $r \in \mathbb{N}$ and let $\mathcal{B}^{\mathfrak{q}}_{\mathrm{frac}}$ denote the $\mathbb{C}(\mathfrak{q}^{\pm 1/2})$-algebra generated by $\{x_i^{\pm 1}, \Gamma_i^{\pm 1}\}_{i=1}^{r+1}$, being further localized by the multiplicative set generated by $\{x_i - \mathfrak{q}^m x_j\}_{i\neq j}^{m\in\mathbb{Z}}$, with all elements pairwise commuting except for $\Gamma_i x_i = \mathfrak{q} x_i \Gamma_i$. Following [4, §6], consider the following series in z with coefficients in $\mathcal{B}^{\mathfrak{q}}_{\mathrm{frac}}$:

$$\begin{aligned} \mathfrak{e}(z)^{\mathrm{DFK}} &= \sum_{i=1}^{r+1} \delta\left(\mathfrak{q}^{1/2} x_i z\right) \prod_{1\leq j\leq r+1}^{j\neq i} \frac{x_i}{x_i - x_j} \Gamma_i, \\ \mathfrak{f}(z)^{\mathrm{DFK}} &= \sum_{i=1}^{r+1} \delta\left(\mathfrak{q}^{-1/2} x_i z\right) \prod_{1\leq j\leq r+1}^{j\neq i} \frac{x_j}{x_j - x_i} \Gamma_i^{-1}, \\ \psi^+(z)^{\mathrm{DFK}} &= (-\mathfrak{q}^{-1/2} z)^{r+1} \cdot \prod_{i=1}^{r+1} x_i \cdot \prod_{i=1}^{r+1} \left(1 - \mathfrak{q}^{1/2} x_i z\right)^{-1} \left(1 - \mathfrak{q}^{-1/2} x_i z\right)^{-1}, \end{aligned}$$

$$\psi^-(z)^{\mathrm{DFK}} = (-\mathfrak{q}^{1/2}z)^{-r-1} \cdot \prod_{i=1}^{r+1} x_i^{-1} \cdot \prod_{i=1}^{r+1} \left(1 - \mathfrak{q}^{1/2}x_i^{-1}z^{-1}\right)^{-1} \left(1 - \mathfrak{q}^{-1/2}x_i^{-1}z^{-1}\right)^{-1}. \tag{116}$$

We shall now identify these currents and those in the construction from Sect. 2 in the special case of $\mathfrak{g} = \mathfrak{sl}_2$, $\mu = -(2r+2)\omega$ with ω being the fundamental coweight of $\mathfrak{sl}_2$, $\lambda = 0$, so that $a = r+1$. To this end, we identify $\iota\colon \widetilde{\mathcal{A}}^q_{\mathrm{frac}} \xrightarrow{\sim} \mathcal{B}^{\mathfrak{q}}_{\mathrm{frac}}$ via

$$\iota\colon q \mapsto \mathfrak{q}^{1/2}, \quad \mathsf{w}_i^{\pm 1} \mapsto x_i^{\mp 1}\mathfrak{q}^{\mp 1/2}, \quad D_i^{\pm 1} \mapsto \Gamma_i^{\mp 1}, \qquad 1 \leq i \leq r+1, \tag{117}$$

and the corresponding shifted quantum affine algebras $\jmath\colon U^{\mathrm{sc}}_{-r-1,-r-1} \xrightarrow{\sim} U^{\mathrm{sc}}_{0,-2r-2}$ via

$$\jmath\colon e(z) \mapsto z^{-r-1}e(z), \quad f(z) \mapsto f(z), \quad \psi^{\pm}(z) \mapsto z^{-r-1}\psi^{\pm}(z).$$

Define the composition:

$$\bar{\Phi}_{r+1}\colon U^{\mathrm{sc}}_{-r-1,-r-1} \overset{\jmath}{\underset{\sim}{\longrightarrow}} U^{\mathrm{sc}}_{0,-2r-2} \xrightarrow{\widetilde{\Phi}^0_{-2r-2}} \widetilde{\mathcal{A}}^q_{\mathrm{frac}} \overset{\iota}{\underset{\sim}{\longrightarrow}} \mathcal{B}^{\mathfrak{q}}_{\mathrm{frac}}. \tag{118}$$

The following is straightforward:

Lemma 6.12 *The currents* (116) *can be expressed as:*

$$\begin{aligned}
\mathfrak{e}(z)^{\mathrm{DFK}} &= (-1)^r(\mathfrak{q}^{1/2} - \mathfrak{q}^{-1/2})\bar{\Phi}_{r+1}(e(z)),\\
\mathfrak{f}(z)^{\mathrm{DFK}} &= (1-\mathfrak{q})\bar{\Phi}_{r+1}(f(z)),\\
\psi^+(z)^{\mathrm{DFK}} &= (-1)^{r+1}\bar{\Phi}_{r+1}(\psi^-(z)),\\
\psi^-(z)^{\mathrm{DFK}} &= (-1)^{r+1}\bar{\Phi}_{r+1}(\psi^+(z)).
\end{aligned}$$

In particular, this immediately shows that the currents (116) indeed satisfy the relations of [4, (5.7)–(5.11)]. Furthermore, we also immediately obtain [4, (6.1)]:

Proposition 6.13 *Under the assignment* $\sum_{n\in\mathbb{Z}} M_{1,n}z^n = \mathfrak{m}_1(z) \mapsto \mathfrak{e}(\mathfrak{q}^{-1/2}z)^{\mathrm{DFK}}$, *the elements* $\{M_{k,n}\}_{k\geq 1}^{n\in\mathbb{Z}}$ *of* (102) *are mapped to:*

$$M_{k,n} \mapsto \sum_{J\subset\{1,\ldots,r+1\}}^{|J|=k} \prod_{i\in J} x_i^n \cdot \prod_{i\in J}^{j\notin J} \frac{x_i}{x_i - x_j} \cdot \prod_{i\in J} \Gamma_i. \tag{119}$$

Proof Formula (119) immediately follows by combining Lemma 6.12 with the shuffle realization (103) of the elements $M_{k,n}$ and the shuffle realization of $\widetilde{\Phi}^0_{-2r-2}$ from Theorem 2.8(a). □

6.14 Finite set of generators

We shall follow the setup of the previous subsection, that is, $\mathfrak{g} = \mathfrak{sl}_2$, $\lambda = 0$, $\mu = -(2r+2)\omega$. The last result of this section explains why it essentially suffices to consider only $\widetilde{\Phi}^0_{-2r-2}(E_{k,n})$:

Lemma 6.15 *For any $n \in \mathbb{Z}$, the $\mathbb{C}(q)$-subalgebra of $\widetilde{\mathcal{A}}^q_{\mathrm{frac}}$ generated by $\{\widetilde{\Phi}^0_{-2r-2}(e_p)\}_{p=n}^{n+2r+1}$ and further localized at $\{\widetilde{\Phi}^0_{-2r-2}(\Upsilon^{-1}(E_{r+1,p}))\}_{p=n}^{n+1}$ coincides with all image $\widetilde{\Phi}^0_{-2r-2}(U_q^{(-2r-2)})$.*

Proof Let $\mathcal{C}_n$ denote the $\mathbb{C}(q)$-subalgebra of $\widetilde{\mathcal{A}}^q_{\mathrm{frac}}$ generated by the above $2r+4$ elements. Since the $\widetilde{\Phi}^0_{-2r-2}$-images of $\psi_s^\pm$ are symmetric Laurent polynomials in $\{\mathsf{w}_k\}_{k=1}^{r+1}$, to prove the inclusions $\widetilde{\Phi}^0_{-2r-2}(\psi_s^\pm) \in \mathcal{C}_n$, it suffices to show that the elementary symmetric polynomials $\{e_k(\mathsf{w}_1, \ldots, \mathsf{w}_{r+1})\}_{k=1}^{r+1}$ as well as $\{e_k(\mathsf{w}_1^{-1}, \ldots, \mathsf{w}_{r+1}^{-1})\}_{k=1}^{r+1}$ belong to $\mathcal{C}_n$. To this end, we define

$$X_{r+1,n}^{(k),\pm} := \sum_{s_0,\ldots,s_r \in \{0,1\}}^{s_0+\ldots+s_r=k} [e_{n\pm s_0}, [e_{n+2\pm s_1}, \cdots, [e_{n+2r-2\pm s_{r-1}}, e_{n+2r\pm s_r}]_{q^{-4}} \cdots]_{q^{-2r}}]_{q^{-2r-2}}. \tag{120}$$

We note that $X_{r+1,n}^{(k),+}, X_{r+1,n+1}^{(k),-}$ are generated by $\{e_p\}_{p=n}^{n+2r+1}$. It is also clear that

$$\begin{aligned}\Upsilon(X_{r+1,n}^{(k),\pm}) &= \Upsilon([e_n, [e_{n+2}, \cdots, [e_{n+2r-2}, e_{n+2r}]_{q^{-4}} \cdots]_{q^{-2r}}]_{q^{-2r-2}}) \cdot e_k(x_1^{\pm 1}, \ldots, x_{r+1}^{\pm 1}) \\ &= (-1)^{\frac{r(r+1)}{2}} (1-q^{-2})^r \cdot E_{r+1,n}(x_1, \ldots, x_{r+1}) \cdot e_k(x_1^{\pm 1}, \ldots, x_{r+1}^{\pm 1}),\end{aligned}$$

with the latter equality due to Lemma 6.2. Applying Lemma 2.12(a), we find:

$$\begin{aligned}e_k(\mathsf{w}_1, \ldots, \mathsf{w}_{r+1}) &= (-1)^{\frac{r(r+1)}{2}} (1-q^{-2})^r q^{-2k} \\ &\quad \times \widetilde{\Phi}^0_{-2r-2}(\Upsilon^{-1}(E_{r+1,n}))^{-1} \cdot \widetilde{\Phi}^0_{-2r-2}(X_{r+1,n}^{(k),+})\end{aligned}$$

and similarly:

$$\begin{aligned}e_k(\mathsf{w}_1^{-1}, \ldots, \mathsf{w}_{r+1}^{-1}) &= (-1)^{\frac{r(r+1)}{2}} (1-q^{-2})^r q^{2k} \\ &\quad \times \widetilde{\Phi}^0_{-2r-2}(\Upsilon^{-1}(E_{r+1,n+1}))^{-1} \cdot \widetilde{\Phi}^0_{-2r-2}(X_{r+1,n+1}^{(k),-}).\end{aligned}$$

This proves $e_k(\mathsf{w}_1^{\pm 1}, \ldots, \mathsf{w}_{r+1}^{\pm 1}) \in \mathcal{C}_n$ for $k \leq r+1$, hence, $\widetilde{\Phi}^0_{-2r-2}(\psi_s^\pm) \in \mathcal{C}_n$ for all possible s.

The inclusions $\widetilde{\Phi}^0_{-2r-2}(e_p) \in \mathcal{C}_n$, for all $p \in \mathbb{Z}$, follow now by induction from the equalities:

$$\widetilde{\Phi}^0_{-2r-2}(e_{p\pm 1}) = (1-q^{\mp 2})^{-1} \cdot \left[e_1(\mathsf{w}_1^{\pm 1}, \ldots, \mathsf{w}_{r+1}^{\pm 1}), \widetilde{\Phi}^0_{-2r-2}(e_p) \right].$$

Finally, the inclusions $\widetilde{\Phi}^0_{-2r-2}(f_p) \in \mathcal{C}_n$, for all $p \in \mathbb{Z}$, follow from the equality:

$$\widetilde{\Phi}^0_{-2r-2}(f_p) = (-1)^{r+1} q^{-2r-1}(q - q^{-1})^{-2} \\ \times \hat{\Phi}^0_{-2r-2}(E_{r+1,-2r-1-p})^{-1} \cdot \hat{\Phi}^0_{-2r-2}(E_{r,-2r-p}),$$

whose right-hand side belongs to $\mathcal{C}_n$, due to Theorem 2.8(a) and Lemma 6.2. □

7 Relation to (t, q)-deformed Q-systems of type A

In this section, we discuss the (t, q)-deformation of the construction and results of Sect. 6.11. In particular, we use the results of Sect. 3 to establish [5, Conjecture 1.17].

7.1 Comparison of the difference operators II

We start by recalling the setup of [5, §3]. To this end, choose two generic complex parameters $\mathfrak{q}$ and $\mathfrak{t} = \theta^2$, as well as $\mathsf{N} \geq 1$. Define the $\mathbb{C}$-algebra $\mathcal{B}^{\mathfrak{q}}$ as in Sect. 6.11 with $r + 1 = \mathsf{N}$ (the subscript "frac" is omitted as it is now a $\mathbb{C}$-algebra). Following [5, (3.6, 3.10)], consider the following series in z with coefficients in $\mathcal{B}^{\mathfrak{q}}$:

$$\mathfrak{e}_1(z)^{\mathrm{DFK}} = \frac{\mathfrak{q}^{1/2}}{1-\mathfrak{q}} \sum_{i=1}^{\mathsf{N}} \delta\left(\mathfrak{q}^{1/2} x_i z\right) \prod_{1 \leq j \leq \mathsf{N}}^{j \neq i} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} \Gamma_i,$$
$$\mathfrak{f}_1(z)^{\mathrm{DFK}} = \frac{\mathfrak{q}^{-1/2}}{1-\mathfrak{q}^{-1}} \sum_{i=1}^{\mathsf{N}} \delta\left(\mathfrak{q}^{-1/2} x_i z\right) \prod_{1 \leq j \leq \mathsf{N}}^{j \neq i} \frac{\theta^{-1} x_i - \theta x_j}{x_i - x_j} \Gamma_i^{-1},$$
$$\psi^{\pm}(z)^{\mathrm{DFK}} = \left(\prod_{i=1}^{\mathsf{N}} \frac{(1-\mathfrak{q}^{-1/2}\mathfrak{t} x_i z)(1-\mathfrak{q}^{1/2}\mathfrak{t}^{-1} x_i z)}{(1-\mathfrak{q}^{-1/2} x_i z)(1-\mathfrak{q}^{1/2} x_i z)} \right)^{\mp}. \tag{121}$$

Let us now match these currents to those arising for the quantum toroidal algebra of $\mathfrak{gl}_1$ in Sect. 3. To this end, let us first relate our former parameters to the above ones via:

$$q_1 = \mathfrak{q},\ q_2 = 1/\mathfrak{t},\ q_3 = 1/q_1 q_2 = \mathfrak{t}/\mathfrak{q} \qquad \text{as well as} \qquad N = 0,\ a = \mathsf{N}. \tag{122}$$

We identify $\iota \colon \widetilde{\mathcal{A}}^{q_1} \xrightarrow{\sim} \mathcal{B}^{\mathfrak{q}}$ via $\mathsf{w}_i^{\pm 1} \mapsto x_i^{\mp 1} \mathfrak{q}^{\mp 1/2}$, $D_i^{\pm 1} \mapsto \Gamma_i^{\mp 1}$, cf. (117). Define the composition:

$$\bar{\Phi}_{\mathsf{N}} \colon \ddot{U}_{q_1,q_2,q_3}(\mathfrak{gl}_1) \xrightarrow{\widetilde{\Phi}_{\mathsf{N}}} \widetilde{\mathcal{A}}^{q_1} \xrightarrow[\sim]{\iota} \mathcal{B}^{\mathfrak{q}}. \tag{123}$$

The following is straightforward:

Lemma 7.2 *The currents* (121) *can be expressed as (recall that* $\theta = \mathfrak{t}^{1/2}$*):*

$$\begin{aligned}
\mathfrak{e}_1(z)^{\mathrm{DFK}} &= \mathfrak{q}^{-\frac{1}{2}}\mathfrak{t}^{\frac{1-N}{2}}\,\bar{\Phi}_N(e(z)),\\
\mathfrak{f}_1(z)^{\mathrm{DFK}} &= -\mathfrak{q}^{\frac{1}{2}}\mathfrak{t}^{\frac{N-1}{2}}\,\bar{\Phi}_N(f(z)),\\
\psi^+(z)^{\mathrm{DFK}} &= \bar{\Phi}_N(\psi^-(z)),\\
\psi^-(z)^{\mathrm{DFK}} &= \bar{\Phi}_N(\psi^+(z)).
\end{aligned}$$

In particular, this immediately shows that the currents (121) indeed satisfy the defining relations (t1–t8) with the parameters q_1, q_2, q_3 as in (122), thus implying [5, Theorem 3.5].

7.3 Generalized Macdonald operators

Following [5, Definition 1.13], for any $1 \le \alpha \le \mathsf{N}$ and any symmetric Laurent polynomial $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_\alpha^{\pm 1}]^{S(\alpha)}$, define the *generalized Macdonald operator* $\mathcal{A}_\alpha(P) \in \mathcal{B}^{\mathfrak{q}}$ via:

$$\mathcal{A}_\alpha(P) := \frac{1}{\alpha! \cdot (\mathsf{N}-\alpha)!} \cdot \underset{x_1,\ldots,x_{\mathsf{N}}}{\mathrm{Sym}} \left(P(x_1,\ldots,x_\alpha) \prod_{1\le i\le \alpha < j \le \mathsf{N}} \frac{\theta x_i - \theta^{-1} x_j}{x_i - x_j} \cdot \Gamma_1 \cdots \Gamma_\alpha \right). \tag{124}$$

In particular, $\iota^{-1}(\mathcal{A}_\alpha(1)) \in \widetilde{\mathcal{A}}^{q_1}$ is a multiple of the Macdonald operator $\mathcal{D}^{\alpha}_{\mathsf{N}}(q_1, q_2)$ from (56).

Remark 7.4 We note that the definition (124) is made in [5] for any symmetric rational function $P \in \mathbb{C}(x_1, \ldots, x_\alpha)^{S(\alpha)}$. However, some of the key results below seem to fail in this generality, see Remarks 7.6, 7.15.

Following [5, Definition 1.15], we also define the difference operator $\mathcal{B}_\alpha(P) \in \mathcal{B}^{\mathfrak{q}}$ via:

$$\mathcal{B}_\alpha(P) := \frac{1}{\alpha!}\mathrm{CT}_{u_1,\ldots,u_\alpha} \left(P(u_1^{-1},\ldots,u_\alpha^{-1}) \prod_{1\le i<j\le\alpha} \frac{(u_i-u_j)(u_i-\mathfrak{q}u_j)}{(u_i-\mathfrak{t}u_j)(u_i-\mathfrak{q}\mathfrak{t}^{-1}u_j)} \mathfrak{d}(u_1)\cdots\mathfrak{d}(u_\alpha) \right), \tag{125}$$

where the constant term $\mathrm{CT}_{u_1,\ldots,u_\alpha}$ is defined as in Proposition 6.3, and $\mathfrak{d}(z)$ is defined via:

$$\mathfrak{d}(z) = \sum_{n\in\mathbb{Z}} \mathcal{D}_{1;n} z^n := (\mathfrak{q}^{-1/2} - \mathfrak{q}^{1/2})\mathfrak{e}_1(\mathfrak{q}^{-1/2}z)^{\mathrm{DFK}}. \tag{126}$$

The above two constructions (124) and (125) are related via [5, Theorem 1.16]:

Proposition 7.5 [5] *For any* $1 \leq \alpha \leq N$ *and* $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_\alpha^{\pm 1}]^{S(\alpha)}$*, we have:*

$$\mathcal{A}_\alpha(P) = \mathcal{B}_\alpha(P). \tag{127}$$

Remark 7.6 We note that this result is stated in [5] for any $P \in \mathbb{C}(x_1, \ldots, x_\alpha)^{S(\alpha)}$. However, this does not look true in that generality as $\mathcal{B}_\alpha(P)$ will involve terms with some powers $\Gamma_i^{>1}$, unlike $\mathcal{A}_\alpha(P)$. For one thing, the constant term $\mathrm{CT}_{u_1,\ldots,u_\alpha}(\cdots)$ should be treated carefully for rational functions by specifying the region in which they are expanded as series.

7.7 Comparing the shuffle algebras

In order to relate the above construction to our Sect. 3, we shall first clarify the shuffle algebra considered in [5, §7] and its relation to the one from Sect. 3.7. To this end, consider an $\mathbb{N}$-graded $\mathbb{C}$-vector space $\mathbb{S}^{\mathrm{DFK}} = \bigoplus_{k\in\mathbb{N}} \mathbb{S}_k^{\mathrm{DFK}}$, with the graded components

$$\mathbb{S}_k^{\mathrm{DFK}} = \left\{ F = \frac{f(x_1, \ldots, x_k)}{\prod_{1\leq r\neq s\leq k}(x_r - \mathfrak{q}^{-1}x_s)} \,\middle|\, f \in \mathbb{C}\left[x_1^{\pm 1}, \ldots, x_k^{\pm 1}\right]^{S(k)} \right\}. \tag{128}$$

We also choose a rational function of [5, §7.1]:

$$\zeta^{\mathrm{DFK}}(x) = \frac{(1-\mathfrak{t}x)(1-\mathfrak{q}\mathfrak{t}^{-1}x)}{(1-x)(1-\mathfrak{q}x)}. \tag{129}$$

The bilinear *shuffle product* $\star$ on $\mathbb{S}^{\mathrm{DFK}}$ is defined completely analogously to (15), thus making $\mathbb{S}^{\mathrm{DFK}}$ into an associative unital $\mathbb{C}$-algebra. As before, consider an $\mathbb{N}$-graded subspace of $\mathbb{S}^{\mathrm{DFK}}$ defined by the same *wheel conditions* (but now on the numerators appearing in (128)):

$$f(x_1, \ldots, x_k) = 0 \quad \text{once} \quad \left\{\frac{x_1}{x_2}, \frac{x_2}{x_3}, \frac{x_3}{x_1}\right\} = \left\{\mathfrak{q}, \frac{1}{\mathfrak{t}}, \frac{\mathfrak{t}}{\mathfrak{q}}\right\}. \tag{130}$$

Let $S^{\mathrm{DFK}} \subset \mathbb{S}^{\mathrm{DFK}}$ denote the subspace of all such elements F, which is easily seen to be $\star$-closed. This construction is related to that of Sect. 3.7 via:

Lemma 7.8 *For* $q_1 = \mathfrak{q}, q_2 = 1/\mathfrak{t}, q_3 = \mathfrak{t}/\mathfrak{q}$ *as in* (122)*, the assignment*

$$P(x_1, \ldots, x_k) \mapsto \mathfrak{q}^{-\frac{k(k-1)}{2}} \cdot \prod_{1\leq r\neq s\leq k} \frac{x_r - x_s}{x_r - \mathfrak{q}^{-1}x_s} \cdot P(x_1^{-1}, \ldots, x_k^{-1}) \tag{131}$$

gives rise to the algebra isomorphism

$$\eta\colon \mathbb{S} \xrightarrow{\sim} \mathbb{S}^{\mathrm{DFK}}, \tag{132}$$

which further restricts to the shuffle algebra isomorphism

$$\eta\colon S \xrightarrow{\sim} S^{\mathrm{DFK}}. \tag{133}$$

Proof Straightforward. □

Combining this with Proposition 3.8, we obtain:

Corollary 7.9 *The assignments $e_r \mapsto x_1^{-r}$ and $f_r \mapsto x_1^{-r}$ give rise to $\mathbb{C}$-algebra isomorphisms*

$$\bar{\Upsilon}\colon \ddot{U}^{>}_{\mathfrak{q},1/\mathfrak{t},\mathfrak{t}/\mathfrak{q}}(\mathfrak{gl}_1) \xrightarrow{\sim} S^{\mathrm{DFK}} \quad \text{and} \quad \bar{\Upsilon}\colon \ddot{U}^{<}_{\mathfrak{q},1/\mathfrak{t},\mathfrak{t}/\mathfrak{q}}(\mathfrak{gl}_1) \xrightarrow{\sim} S^{\mathrm{DFK,op}}. \tag{134}$$

Remark 7.10 In [5], neither pole (128) nor wheel (130) conditions were imposed.

7.11 Generalized Macdonald operators via GKLO-type homomorphisms

Now we are finally ready to relate the aforementioned constructions to those of Sect. 3. To this end, for any $1 \le \alpha \le \mathsf{N}$ and $g \in \mathbb{C}[x_1^{\pm 1}, \dots, x_\alpha^{\pm 1}]^{S(\alpha)}$, recall $\widetilde{E}_\alpha(g) \in S_\alpha$ defined in (52) with the parameters $q_1 = \mathfrak{q}, q_2 = 1/\mathfrak{t}, q_3 = \mathfrak{t}/\mathfrak{q}$ as in (122). The following is straightforward:

Lemma 7.12 $\eta(\widetilde{E}_\alpha(g)) = \mathfrak{t}^{\frac{\alpha-\alpha^2}{2}}(\mathfrak{q}^{-1}-1)^\alpha \cdot g(x_1^{-1}, \dots, x_\alpha^{-1}) \in S^{\mathrm{DFK}}_\alpha.$

Therefore, the span of $\widetilde{E}_\alpha(g) \in \mathbb{S}$ is matched under (132) with the subspace of all symmetric Laurent polynomials in $\mathbb{S}^{\mathrm{DFK}}$, for which the constructions and results of Sect. 7.3 apply. In particular, comparing our Lemma 3.12 with the definition (124), we immediately obtain:

Proposition 7.13 *For any $1 \le \alpha \le N$ and $g \in \mathbb{C}[x_1^{\pm 1}, \dots, x_\alpha^{\pm 1}]^{S(\alpha)}$, we have:*

$$\begin{aligned} &\iota(\hat{\Phi}_N(\widetilde{E}_\alpha(g))) = \theta^{\alpha(N-\alpha)} \cdot \mathcal{A}_\alpha(P) \quad \text{with} \\ &P(x_1, \dots, x_\alpha) = g(\mathfrak{q}^{-1/2}x_1^{-1}, \dots, \mathfrak{q}^{-1/2}x_\alpha^{-1}) \end{aligned} \tag{135}$$

and the identification $\iota\colon \widetilde{\mathcal{A}}^{q_1} \xrightarrow{\sim} \mathcal{B}^{\mathfrak{q}}$ being defined right after (122).

As an immediate corollary, we obtain the following result:

Theorem 7.14 *All* generalized Macdonald operators $\mathcal{A}_\alpha(P) \in \mathcal{B}^{\mathfrak{q}}$ *of* (124) *can be expressed as polynomials in $\mathcal{D}_{1;n}$'s of* (126).

This establishes [5, Conjecture 1.17] by choosing P to be a *generalized Schur function*:

$$P(x_1, \dots, x_\alpha) = s_{a_1,\dots,a_\alpha}(x_1, \dots, x_\alpha) = \frac{\det(x_i^{a_j+\alpha-j})_{1\le i,j\le\alpha}}{\det(x_i^{\alpha-j})_{1\le i,j\le\alpha}}, \quad a_1, \dots, a_\alpha \in \mathbb{Z}. \tag{136}$$

Proof (Proof of Theorem 7.14) Due to (135) and the equality $D_{1;n} = \mathcal{A}_1(x^n)$, it suffices to show that $\widetilde{E}_\alpha(g) \in S_\alpha$ can be expressed as a polynomial in $x^n \in S_1$. This immediately follows from Proposition 3.8 identifying S with $\ddot{U}^{>}_{\mathfrak{q},1/\mathfrak{t},\mathfrak{t}/\mathfrak{q}}(\mathfrak{gl}_1)$, the latter generated by $e_r = \Upsilon^{-1}(x^r)$. □

Remark 7.15 Interpreting the restriction of GKLO-homomorphism $\widetilde{\Phi}_{\mathsf{N}} \colon \ddot{U}_{\mathfrak{q},1/\mathfrak{t},\mathfrak{t}/\mathfrak{q}}(\mathfrak{gl}_1) \to \widetilde{\mathcal{A}}^{\mathfrak{q}}$ as $\hat{\Phi}_{\mathsf{N}} \colon S^{\mathrm{DFK}} \to \mathcal{B}^{\mathfrak{q}}$, we thus see that the images of symmetric Laurent polynomials recover the *generalized Macdonald operators* of (124), while the image of any nonpolynomial $F \in S^{\mathrm{DFK}}$ will necessarily contain terms with at least one $\Gamma_i^{>1}$, due to our explicit formula (46).

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Data availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The author states that there is no conflict of interest.

References

1. Braverman, A., Finkelberg, M., Nakajima, H.: Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II. Adv. Theor. Math. Phys. **22**(5), 1071–1147 (2018)
2. Braverman, A., Finkelberg, M., Nakajima, H.; Coulomb branches of $3d$ $\mathcal{N} = 4$ quiver gauge theories and slices in the affine Grassmannian (with appendices by A. Braverman, M. Finkelberg, J. Kamnitzer, R. Kodera, H. Nakajima, B. Webster, A. Weekes), Adv. Theor. Math. Phys. **23**(1), 75–166 (2019)
3. Drinfeld, V.: A new realization of Yangians and quantized affine algebras. Sov. Math. Dokl. **36**(2), 212–216 (1988)
4. Di Francesco, P., Kedem, R.: Quantum Q-systems: from cluster algebras to quantum current algebras. Lett. Math. Phys. **107**(2), 301–341 (2017)
5. Di Francesco, P., Kedem, R.: (t, q)-deformed Q-systems, DAHA and quantum toroidal algebras via generalized Macdonald operators. Commun. Math. Phys. **369**(3), 867–928 (2019)
6. Di Francesco, P., Kedem, R.: Macdonald operators and quantum Q-systems for classical types. Representation theory, mathematical physics, and integrable systems. Prog. Math. **340**, 163–199 (2021)
7. Feigin, B., Feigin, E., Jimbo, M., Miwa, T., Mukhin, E.: Quantum continuous $\mathfrak{gl}_\infty$: semiinfinite construction of representations. Kyoto J. Math. **51**(2), 337–364 (2011)
8. Feigin, B., Hashizume, K., Hoshino, A., Shiraishi, J., Yanagida, S.: A commutative algebra on degenerate $\mathbb{CP}^1$ and Macdonald polynomials. J. Math. Phys. **50**(9), 095215 (2009)
9. Feigin, B., Odesskii, A.: Elliptic deformations of current algebras and their representations by difference operators (Russian). Funktsional. Anal. i Prilozhen. **31**(3), 57–70 (1997); translation in Funct. Anal. Appl. **31**(3), 193–203 (1998)
10. Feigin, B., Tsymbaliuk, A.: Bethe subalgebras of $U_q(\widehat{\mathfrak{gl}}_n)$ via shuffle algebras. Sel. Math. (N. S.) **22**(2), 979–1011 (2016)
11. Finkelberg, M., Tsymbaliuk, A.: Multiplicative slices, relativistic Toda and shifted quantum affine algebras. In: Gorelik, M., Hinich, V., Melnikov, A. (eds.) Representations and Nilpotent Orbits of Lie Algebraic Systems (Special volume in honour of the 75th birthday of Anthony Joseph). Progress in Mathematics, vol. 330, pp. 133–304. Birkhäuser, Cham (2019)

12. Finkelberg, M., Tsymbaliuk, A.: Shifted quantum affine algebras: integral forms in type *A* (with appendices by A. Tsymbaliuk, A. Weekes). Arnold Math. J. **5**(2–3), 197–283 (2019)
13. Frassek, R., Tsymbaliuk, A.: Rational Lax matrices from antidominantly shifted extended Yangians: BCD types. Commun. Math. Phys. **392**, 545–619 (2022)
14. Gerasimov, A., Kharchev, S., Lebedev, D., Oblezin, S.: On a Class of Representations of Quantum Groups, Noncommutative Geometry and Representation Theory in Mathematical Physics. Contemporary Mathematics, vol. 391, pp. 101–110. American Mathematical Society, Providence (2005)
15. Neguţ, A.: The shuffle algebra revisited. Int. Math. Res. Not. IMRN **22**, 6242–6275 (2014)
16. Neguţ, A.: Quantum toroidal and shuffle algebras. Adv. Math. **372**, 107288 (2020)
17. Neguţ, A.: Shuffle algebras for quivers and wheel conditions. arXiv:2108.08779
18. Neguţ, A., Sala, F., Schiffmann, O.: Shuffle algebras for quivers as quantum groups. arXiv:2111.00249
19. Neguţ, A., Tsymbaliuk, A.: Quantum loop groups and shuffle algebras via Lyndon words. arXiv:2102.11269
20. Orr, D., Shimozono, M.: Difference operators for wreath Macdonald polynomials. arXiv:2110.08808
21. Tsymbaliuk, A.: Several realizations of Fock modules for toroidal $\ddot{U}_{q,d}(\mathfrak{sl}_n)$. Algebr. Represent. Theory **22**(1), 177–209 (2019)
22. Tsymbaliuk, A.: Shuffle approach towards quantum affine and toroidal algebras. arXiv:2209.04294