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Shuffle Algebras and Their Integral Forms: Specialization Map Approach in Types B_n and G_2

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We construct a family of PBWD (Poincaré-Birkhoff-Witt-Drinfeld) bases for the positive subalgebras of quantum loop algebras of type B_n and G_2, as well as their Lusztig and RTT (for type B_n only) integral forms, in the new Drinfeld realization. We also establish a shuffle algebra realization of these $\mathbb{Q}(v)$-algebras (proved earlier in [26] by completely different tools) and generalize the latter to the above $\mathbb{Z}[v, v^{-1}]$-forms. The rational counterparts provide shuffle algebra realizations of positive subalgebras of type B_n and G_2 Yangians and their Drinfeld-Gavarini duals. All of this generalizes the type A_n results of [30].

1 Introduction
1.1 Summary

The quantum loop algebras (aka quantum affine algebras with a trivial central charge) associated to a simple finite dimensional Lie algebra $\mathfrak{g}$ admit two well-known presentations: the original Drinfeld-Jimbo realization $U_v^{DJ}(L\mathfrak{g})$ and the new Drinfeld realization $U_v(L\mathfrak{g})$, the latter introduced in [3]. The explicit isomorphism $U_v^{DJ}(L\mathfrak{g}) \simeq U_v(L\mathfrak{g})$ was actually upgraded in [3, Theorem 3] to the isomorphism of the corresponding quantum affine algebras

$$U_v^{DJ}(\widehat{\mathfrak{g}}) \simeq U_v(\widehat{\mathfrak{g}}). \tag{1.1}$$

Many intrinsic properties of quantum affine algebras have been developed in the Drinfeld-Jimbo realization. For example, the classical Poincaré-Birkhoff-Witt theorem for Lie algebras was generalized by Beck in [1] to the case of $U_v^{DJ}(\widehat{\mathfrak{g}})$. More precisely, he constructed the bases of each of the subalgebras featuring in the triangular decomposition

$$U_v^{DJ}(\widehat{\mathfrak{g}}) \simeq U_v^{DJ,>}(\widehat{\mathfrak{g}}) \otimes U_v^{DJ,0}(\widehat{\mathfrak{g}}) \otimes U_v^{DJ,<}(\widehat{\mathfrak{g}}). \tag{1.2}$$

This result is a natural upgrade of Lusztig's PBW theorem for finite quantum groups $U_q(\mathfrak{g})$.

On the other hand, the new Drinfeld realization $U_v(\widehat{\mathfrak{g}})$ is essential to develop the representation theory of quantum affine algebras. In this realization, there are infinitely many generators which can be conveniently encoded by currents $e_i(z), f_i(z), \varphi_i^{\pm}(z)$. Already in the classical case, this approach played the prominent role manifestly featuring affine Lie algebras $\widehat{\mathfrak{g}}$ in the conformal field theory. It is thus

natural to develop algebraic aspects of $U_v(\widehat{\mathfrak{g}})$ intrinsic to the loop realization (with the hope to generalize this to generalized Kac-Moody Lie algebras $\mathfrak{g}$). Let us note right away that $U_v(\widehat{\mathfrak{g}})$ also has a triangular decomposition (a vector space isomorphism)

$$U_v(\widehat{\mathfrak{g}}) \simeq U_v^>(\widehat{\mathfrak{g}}) \otimes U_v^0(\widehat{\mathfrak{g}}) \otimes U_v^<(\widehat{\mathfrak{g}}). \tag{1.3}$$

However, the isomorphism (1.1) <u>does not</u> intertwine the triangular decompositions (1.2)–(1.3). To this end, we note that specific PBW-type bases of $U_v^>(\widehat{\mathfrak{g}}), U_v^<(\widehat{\mathfrak{g}})$ were constructed in [26].

While most often quantum groups are defined by generators and relations, there is an alternative combinatorial approach sweeping the defining relations under the rug. For finite quantum groups, this manifests in the algebra embedding (observed independently in [15, 27, 28]):

$$U_v^>(\mathfrak{g}) \hookrightarrow \mathcal{F} = \bigoplus_{\substack{k \in \mathbb{N} \\ i_1,\ldots,i_k \in I}} \mathbb{Q}(v) \cdot [i_1 \ldots i_k]. \tag{1.4}$$

Here, I denotes the set of simple roots of $\mathfrak{g}$, $\mathcal{F}$ has a basis labeled by finite length words in I and is endowed with the *quantum shuffle* product. As shown by Lalonde-Ram in [20], there is a bijection between the set Δ^+ of positive roots of $\mathfrak{g}$ and so-called *standard Lyndon* words in I:

$$\ell \colon \Delta^+ \overset{\sim}{\longrightarrow} \left\{\text{standard Lyndon words}\right\}. \tag{1.5}$$

In this case, every order on the alphabet I gives rise to a convex order on Δ^+, and the corresponding Lusztig's PBW basis of $U_v^>(\mathfrak{g})$ can be constructed combinatorially via iterated v-commutators, due to Levendorskii-Soibelman convexity property of [21]; see [18, 26].

Using similar ideas, Feigin-Odesskii introduced the elliptic shuffle algebras in [8]–[11]. Their trigonometric counterpart (but in the formal setup with $\mathbb{Q}(v)$ been replaced by $\mathbb{Q}[[\hbar]]$) was further studied by Enriquez in [4, 5]. Explicitly, this manifests in the algebra embedding:

$$\Psi \colon U_v^>(L\mathfrak{g}) \hookrightarrow S. \tag{1.6}$$

Here, S consists of symmetric rational functions in $\{x_{i,r}\}_{i \in I}^{r \in \mathbb{Z}}$ subject to so-called *pole* and *wheel* conditions, and endowed with the shuffle product. Thus, (1.6) is a *functional version* of (1.4).

The major benefit of (1.6) is that it allows to conveniently work with the elements of $U_v(L\mathfrak{g})$ given by rather complicated non-commutative polynomials in the original generators. Within the last decade, this realization has already found major applications in the geometric representation theory and quantum integrable systems. To make this approach self-contained, it is important to have an explicit description of the image $\mathrm{Im}(\Psi)$. In fact, Enriquez conjectured:

$$\Psi \colon U_v^>(L\mathfrak{g}) \overset{\sim}{\longrightarrow} S. \tag{1.7}$$

To prove (1.7), it is crucial to "compare the size" of $U_v^>(L\mathfrak{g})$ and S. For types A_1 and $\hat{A}_1$, this was accomplished in [24] by crucially utilizing *specialization maps* analogous to those from [7, 13]. The same approach was later used to prove (1.7) for types A_n and $\hat{A}_n$ in [25], for two-parameter and super counterparts of type A_n in [30], and for type $D(2, 1; \theta)$ in [6].

In the present note, we generalize most of the results from [30] to types G_2 and B_n, thus establishing the isomorphism (1.7) and constructing families of PBWD (Poincaré-Birkhoff-Witt-Drinfeld)-bases of $U_v^>(L\mathfrak{g})$ in these types. To do so, we introduce the corresponding specialization maps on the shuffle algebra of the associated type, and establish their key properties similarly to type A_n from [30]. Let us note that these specialization maps arise naturally from the specific convex orders on Δ^+.

It should be emphasized right away that Enriquez's conjecture (1.7) has been recently proved for all finite $\mathfrak{g}$ in [26], the joint work of Neguţ and the second author. However, the approach of *loc.cit.* is completely different, as it crucially uses a loop version of (1.4) instead of the specialization maps. The present approach has its own benefits as it can also be used to upgrade both results (the shuffle algebra realization and the PBWD-type bases) to the integral $\mathbb{Z}[v, v^{-1}]$-forms of $U_v^>(L\mathfrak{g})$ and the Yangian version $Y_h^>(\mathfrak{g})$, generalizing type A_n from [29, 30]. We conclude this introduction by noting that similar specialization maps actually exist for all finite types (which was already known to [26]), though their definition is more involved.

1.2 Outline of the paper

The structure of the present paper is the following:

- In Section 2, we recall the notion of quantum loop algebras $U_v^>(L\mathfrak{g})$ in the new Drinfeld realization as well as shuffle algebras S, introduce certain families of quantum root vectors (associated to specific convex orders on the set Δ^+ of positive roots), and state the key results (PBWD bases and shuffle algebra isomorphism) for $U_v^>(L\mathfrak{g})$ of types B_n and G_2. We briefly recall how such results were proved in [30] for type A_n using specialization maps on S, and summarize their key properties in Lemmas 2.6–2.8.

- In Section 3, we define the specialization maps for the shuffle algebra S of exceptional type G_2, establish the counterparts of Lemmas 2.6–2.8 in that setup, and use the latter to prove Theorems 2.4 and 2.5 for type G_2; see Theorem 3.7. We upgrade both results to the Lusztig/Chari-Pressley/Grojnowski integral form $\mathbf{U}_v^>(L\mathfrak{g}_2)$ in Theorem 3.12.

- In Section 4, we define the specialization maps for the shuffle algebra S of type B_n, establish the counterparts of Lemmas 2.6–2.8 in that setup, and use the latter to prove Theorems 2.4 and 2.5 for type B_n; see Theorem 4.8. We upgrade both results to the Lusztig/Chari-Pressley/Grojnowski integral form $\mathbf{U}_v^>(L\mathfrak{o}_{2n+1})$ in Theorem 4.19. Likewise, we upgrade both results to the RTT integral form $\mathcal{U}_v^>(L\mathfrak{o}_{2n+1})$ in Theorem 4.14.

- In Section 5, we generalize the results of Sections 3–4 to the rational setup by providing the shuffle realization and constructing PBWD bases for the positive subalgebras of the Yangians and their Drinfeld-Gavarini duals in types G_2 and B_n; see Theorems 5.14, 5.20.

- In Appendix A, we use the RTT realization of $U_v(L\mathfrak{o}_{2n+1})$ from [17] to explain the natural origin and the name of the RTT integral form $\mathcal{U}_v^>(L\mathfrak{o}_{2n+1})$ from Subsection 4.2.

2 Preliminaries

2.1 Quantum loop algebras and shuffle algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra with simple positive roots $\{\alpha_i\}_{i \in I}$. We denote the set of positive roots by Δ^+. Each $\beta \in \Delta^+$ can be uniquely expressed as a sum of simple roots: $\beta = \sum_{i \in I} v_{\beta,i} \alpha_i$ with $v_{\beta,i} \in \mathbb{N}$ (the set $\mathbb{N}$ will be assumed to include 0). We shall refer to $v_{\beta,i}$ as the *coefficient of α_i in β*, and we shall use the following notation:

$$i \in \beta \iff v_{\beta,i} \neq 0. \tag{2.1}$$

We fix a nondegenerate invariant bilinear form on the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. This gives rise to a nondegenerate form on the dual $\mathfrak{h}^*$, and we set $d_i := \frac{(\alpha_i, \alpha_i)}{2}$. The choice of the form is such that $d_i = 1$ for short roots α_i. Let $A = (a_{ij})_{i,j \in I}$ be the Cartan matrix of $\mathfrak{g}$, so that $d_i a_{ij} = (\alpha_i, \alpha_j) = d_j a_{ji}$. In this paper, we consider simple Lie algebras of types A_n, B_n, G_2. The corresponding Dynkin diagrams look as follows:

$$A_n\text{-type } (n \geq 1): \quad \underset{\alpha_1}{\circ} \!-\! \underset{\alpha_2}{\circ} \!-\! \cdots \!-\! \underset{\alpha_{n-1}}{\circ} \!-\! \underset{\alpha_n}{\circ} \tag{2.2}$$

$$B_n\text{-type } (n \geq 2): \quad \underset{\alpha_1}{\circ} \!-\! \underset{\alpha_2}{\circ} \!-\! \cdots \!-\! \underset{\alpha_{n-1}}{\circ} \!\Rightarrow\! \underset{\alpha_n}{\circ} \tag{2.3}$$

$$G_2\text{-type}: \quad \underset{\alpha_1}{\circ} \!\Rightarrow\! \underset{\alpha_2}{\circ} \tag{2.4}$$

For these types, we have

$$A_n\text{-type } (n \geq 1): \quad d_i = 1 \ (1 \leq i \leq n), \tag{2.5}$$

$$B_n\text{-type } (n \geq 2): \quad d_i = 2 \ (1 \leq i \leq n-1), \ d_n = 1, \tag{2.6}$$

$$G_2\text{-type}: \quad d_1 = 3, \ d_2 = 1. \tag{2.7}$$

Let v be a formal variable. We define $v_\alpha = v^{(\alpha,\alpha)/2}$ for any $\alpha \in \Delta^+$, and denote $v_{\alpha_i} = v^{d_i}$ simply by v_i for any $i \in I$. Let $\mathfrak{S}_m$ denote the symmetric group of degree m. Let $U_v^>(L\mathfrak{g})$ be the **"positive subalgebra"** **of the quantum loop algebra** $U_v(L\mathfrak{g})$ associated to $\mathfrak{g}$ in the new Drinfeld realization. Explicitly, $U_v^>(L\mathfrak{g})$ is the $\mathbb{Q}(v)$-algebra generated by $\{e_{i,r}\}_{i \in I}^{r \in \mathbb{Z}}$ subject to the following defining relations:

$$(z - v_i^{a_{ij}} w)e_i(z)e_j(w) = (v_i^{a_{ij}} z - w)e_j(w)e_i(z) \qquad \forall \, i,j \in I, \tag{2.8}$$

$$\operatorname*{Sym}_{z_1,\ldots,z_{1-a_{ij}}} \sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{v_i} e_i(z_1)\cdots e_i(z_k)e_j(w)e_i(z_{k+1})\cdots e_i(z_{1-a_{ij}}) = 0 \qquad \forall \, i \neq j. \tag{2.9}$$

Here, we use the following notations:

$$[\ell]_u := \frac{u^\ell - u^{-\ell}}{u - u^{-1}}, \quad [\ell]_u! := \prod_{k=1}^{\ell} [k]_u, \quad \begin{bmatrix} \ell \\ m \end{bmatrix}_u := \frac{[\ell]_u!}{[\ell-m]_u! \, [m]_u!},$$
$$e_i(z) := \sum_{r \in \mathbb{Z}} e_{i,r} z^{-r}, \quad \operatorname*{Sym}_{z_1,\ldots,z_m} V(z_1,\ldots,z_m) := \sum_{\sigma \in \mathfrak{S}_m} V(z_{\sigma(1)},\ldots,z_{\sigma(m)}). \tag{2.10}$$

We shall also need the following notation later:

$$\langle m \rangle_u := u^m - u^{-m} \qquad \forall \, m \in \mathbb{N}. \tag{2.11}$$

We define $\mathfrak{S}_{\underline{k}} := \prod_{i \in I} \mathfrak{S}_{k_i}$ for any $\underline{k} = (k_1,\ldots,k_{|I|}) \in \mathbb{N}^I$. Associated to the Cartan matrix $A = (a_{ij})_{i,j \in I}$, we also have the trigonometric version of the Feigin-Odesskii shuffle algebra. To this end, consider the following $\mathbb{N}^I$-graded $\mathbb{Q}(v)$-vector space

$$\mathbb{S} = \bigoplus_{\underline{k} \in \mathbb{N}^I} \mathbb{S}_{\underline{k}},$$

where $\mathbb{S}_{\underline{k}}$ consists of rational functions F in the variables $\{x_{i,r}\}_{i \in I}^{1 \leq r \leq k_i}$ such that:

- F is $\mathfrak{S}_{\underline{k}}$-symmetric, that is, symmetric in $\{x_{i,r}\}_{r=1}^{k_i}$ for each $i \in I$,
- (*pole conditions*) F has the form

$$F = \frac{f(\{x_{i,r}\}_{i \in I}^{1 \leq r \leq k_i})}{\prod_{i<j}^{a_{ij} \neq 0} \prod_{1 \leq r \leq k_i}^{1 \leq s \leq k_j} (x_{i,r} - x_{j,s})}, \tag{2.12}$$

where $f \in \mathbb{Q}(v)[\{x_{i,r}^{\pm 1}\}_{i \in I}^{1 \leq r \leq k_i}]^{\mathfrak{S}_{\underline{k}}}$ and an arbitrary order $<$ is chosen on I to make sense of $i < j$ (though the space $\mathbb{S}_{\underline{k}}$ is clearly independent of this order).

Let $(\zeta_{i,j}(z))_{i,j \in I}$ be the matrix of rational functions in z given by

$$\zeta_{i,j}(z) = \frac{z - v^{-(\alpha_i,\alpha_j)}}{z - 1}. \tag{2.13}$$

For $\underline{k}, \underline{\ell} \in \mathbb{N}^I$, let

$$\underline{k} + \underline{\ell} = (k_i + \ell_i)_{i \in I} \in \mathbb{N}^I.$$

Let us introduce the bilinear *shuffle product* $\star$ on $\mathbb{S}$ as follows: for $F \in \mathbb{S}_{\underline{k}}$ and $G \in \mathbb{S}_{\underline{\ell}}$, we set

$$F \star G\big(\{x_{i,r}\}_{i \in I}^{1 \le r \le k_i + \ell_i}\big) =$$

$$\frac{1}{\underline{k}! \cdot \underline{\ell}!} \cdot Sym_{\mathfrak{S}_{\underline{k}+\underline{\ell}}}\left(F\big(\{x_{i,r}\}_{i \in I}^{1 \le r \le k_i}\big) \cdot G\big(\{x_{j,s}\}_{j \in I}^{k_j < s \le k_j + \ell_j}\big) \prod_{i,j \in I} \prod_{r \le k_i}^{s > k_j} \zeta_{i,j}\Big(\frac{x_{i,r}}{x_{j,s}}\Big)\right). \tag{2.14}$$

Here, for $\underline{k} \in \mathbb{N}^I$, we set $\underline{k}! = \prod_{i \in I} k_i!$, and define the *symmetrization*

$$Sym_{\mathfrak{S}_{\underline{k}}}\big(F(\{x_{i,r}\}_{i \in I}^{1 \le r \le k_i})\big) := \sum_{(\sigma_1, \dots, \sigma_{|I|}) \in \mathfrak{S}_{\underline{k}}} F(\{x_{i,\sigma_i(r)}\}_{i \in I}^{1 \le r \le k_i}). \tag{2.15}$$

This endows $\mathbb{S}$ with a structure of an associative unital algebra. The resulting algebra $(\mathbb{S}, \star)$ is related to $U_v^>(L\mathfrak{g})$ via the following result (cf. [4, Theorem 3], [5, Proposition 1.2]):

Proposition 2.1. The assignment $e_{i,r} \mapsto x_{i,1}^r \in \mathbb{S}_{\mathbf{1}_i}$ ($i \in I, r \in \mathbb{Z}$), where $\mathbf{1}_i = (0, \dots, 1, \dots, 0)$ with 1 at the i-th coordinate, gives rise to a $\mathbb{Q}(v)$-algebra homomorphism

$$\Psi : U_v^>(L\mathfrak{g}) \longrightarrow \mathbb{S}. \tag{2.16}$$

Moreover, for any $F \in \mathrm{Im}(\Psi)$, its numerator f from (2.12) satisfies:

$$f(\{x_{i,r}\}_{i \in I}^{1 \le r \le k_i}) = 0 \quad \text{once} \quad x_{i,s_1} = v_i^2 x_{i,s_2} = \cdots = v_i^{-2a_{ij}} x_{i,s_{1-a_{ij}}} = v_i^{-a_{ij}} x_{j,r} \tag{2.17}$$

for any $i \ne j$ such that $a_{ij} \ne 0$, pairwise distinct $1 \le s_1, \dots, s_{1-a_{ij}} \le k_i$, and $1 \le r \le k_j$.

The vanishing conditions (2.17) are usually called *wheel conditions*. Let $S_{\underline{k}}$ denote the subspace of all elements of $\mathbb{S}_{\underline{k}}$ satisfying the wheel conditions, and set $S = \bigoplus_{\underline{k} \in \mathbb{N}^I} S_{\underline{k}}$. The following is straightforward:

Lemma 2.2. S is a subalgebra of $\mathbb{S}$ under the shuffle product $\star$ determined by (2.14).

From now on, we will refer to $(S, \star)$ as the **(trigonometric Feigin-Odesskii) shuffle algebra** (of type $\mathfrak{g}$). According to [26, Proposition 5.7] (cf. [5, Corollary 1.4]), we have:

Proposition 2.3. The algebra homomorphism Ψ of (2.16) is injective.

In fact, Ψ is an algebra isomorphism, cf. (1.7). This was first conjectured by Enriquez, established for A_n-type in [25] (see also [30]), and finally proved in the full generality in [26].

Theorem 2.4. $\Psi : U_v^>(L\mathfrak{g}) \xrightarrow{\sim} S$ of (2.16) is a $\mathbb{Q}(v)$-algebra isomorphism.

The key to the proof of Theorem 2.4 is to obtain an upper bound estimate for the dimensions of the graded components of the shuffle algebra S. To this end, we note that [26] instrumentally used a loop version of the formal quantum shuffle algebra due to Green, Rosso, and Schauenburg. In contrast, the arguments of [25] and [30] for type A_n were quite different: an estimate of the graded dimensions was achieved by using certain specialization maps. One benefit of the latter approach is that it allows for the shuffle realization of various integral $\mathbb{Z}[v, v^{-1}]$-forms.

The key objective of the present paper is to extend the method used in [25, 30] to types B_n and G_2. This will provide a new proof of Theorem 2.4 in these types, different from [26].

2.2 Root vectors and PBWD bases

Our construction of the specialization maps and PBWD bases is based on the specific choice of a convex order on Δ^+. The one that is best suited for our purposes is arising through the lexicographical order on standard Lyndon words, see [18, 20], as we recall next.

Recall that I is the indexing set of the simple roots of $\mathfrak{g}$. The labeling of the simple roots in the corresponding Dynkin diagrams (2.2)–(2.4) provides a total order on the set I, and hence the lexicographical order on the set of words in the alphabet I. According to [20, Proposition 3.2], there is a natural bijection between the sets of positive roots Δ^+ and so-called *standard Lyndon words*, cf. (1.5). Thus, the lexicographical order on the latter gives rise to an order $<$ on Δ^+, which is convex by [18, Proposition 26], cf. [26, Proposition 2.34]. In what follows, we fix this specific convex order on Δ^+ and use standard Lyndon words to parametrize the positive roots.

Let us work this out explicitly for types A_n, B_n, G_2 with the specific order on I as in (2.2)–(2.4). Applying [18, Proposition 25], we find the set of all standard Lyndon words:

$$
\begin{aligned}
A_n\text{-type } (n \geq 1): \quad & \Delta^+ = \{[i, i+1, \ldots, j] \mid 1 \leq i \leq j \leq n\}, \\
B_n\text{-type } (n \geq 2): \quad & \Delta^+ = \{[i, i+1, \ldots, j] \mid 1 \leq i \leq j \leq n\} \\
& \qquad \cup \{[i, \ldots, n, n, n-1, \ldots, j] \mid 1 \leq i < j \leq n\}, \\
G_2\text{-type}: \quad & \Delta^+ = \{[1], [1,2], [1,2,1,2,2], [1,2,2], [1,2,2,2], [2]\}.
\end{aligned}
$$
(2.18)

For convenience, we shall use the following notations for positive roots in types A_n and B_n:

$$
\begin{aligned}
[i, j] &:= [i, i+1, \ldots, j] \quad \text{for} \quad 1 \leq i \leq j \leq n, \\
[i, n, j] &:= [i, \ldots, n, n, n-1, \ldots, j] \quad \text{for} \quad 1 \leq i < j \leq n.
\end{aligned}
$$
(2.19)

The aforementioned specific convex order on Δ^+ in types A_n, B_n, G_2 looks as follows:

- Type A_n $(n \geq 1)$:

$$[1] < [1,2] < \cdots < [1,n] < [2] < \cdots < [n-1] < [n-1, n] < [n].$$
(2.20)

- Type B_n $(n \geq 2)$:

$$[1] < [1,2] < \cdots < [1,n] < [1,n,n] < \cdots < [1,n,2] < [2] < \cdots < [n-1,n,n] < [n].$$
(2.21)

- Type G_2:

$$[1] < [1,2] < [1,2,1,2,2] < [1,2,2] < [1,2,2,2] < [2].$$
(2.22)

The *quantum root vectors* $\{E_{\beta,s}\}_{\beta \in \Delta^+}^{s \in \mathbb{Z}}$ of $U_v^>(L\mathfrak{g})$ in type A_n were defined in [30, (2.12)] via iterated v-commutators (they were called the *PBWD basis elements* and depended on certain extra choices). Here, for $x, y \in U_v^>(L\mathfrak{g})$ and $u \in \mathbb{Q}(v)$, the *u-commutator* $[x, y]_u$ is defined via:

$$[x, y]_u := xy - u \cdot yx.$$
(2.23)

We shall now similarly define the *quantum root vectors* of $U_v^>(L\mathfrak{g})$ for $\mathfrak{g}$ of type B_n and G_2:

- B_n-type.
  For any $\beta = [i_1, \ldots, i_\ell] \in \Delta^+$ from (2.18) and $s \in \mathbb{Z}$, choose a collection $\lambda_1, \ldots, \lambda_{\ell-1} \in v^{\mathbb{Z}}$ and a decomposition $s = s_1 + \cdots + s_\ell$ with $s_1, \ldots, s_\ell \in \mathbb{Z}$. Then, we define

$$E_{\beta,s} := [\cdots [[e_{i_1,s_1}, e_{i_2,s_2}]_{\lambda_1}, e_{i_3,s_3}]_{\lambda_2}, \cdots, e_{i_\ell,s_\ell}]_{\lambda_{\ell-1}}.$$
(2.24)

- G_2-type.
  For $\beta = [i_1, \ldots, i_\ell] \neq [1,2,1,2,2]$, $s \in \mathbb{Z}$, the elements $E_{\beta,s}$ are defined exactly as in (2.24).
  For $\beta = [1,2,1,2,2]$, $s \in \mathbb{Z}$, we choose a decomposition $s = s_1 + \cdots + s_5$ with $s_1, \ldots, s_5 \in \mathbb{Z}$, a collection $\lambda_1, \ldots, \lambda_4 \in v^{\mathbb{Z}}$, and define

$$E_{\beta,s} := [[e_{1,s_1}, e_{2,s_2}]_{\lambda_1}, [[e_{1,s_3}, e_{2,s_4}]_{\lambda_2}, e_{2,s_5}]_{\lambda_3}]_{\lambda_4}.$$
(2.25)

In particular, we have the following specific choices $\{\tilde{E}^{\pm}_{\beta,s}\}^{s\in\mathbb{Z}}_{\beta\in\Delta_+}$ which will be used to construct PBWD bases of the integral forms in Sections 3.2, 4.2, and 4.3:

- B_n-type.
  If $\beta = [i, j]$, $s \in \mathbb{Z}$, we choose any decomposition $s = s_i + \cdots + s_j$, fix a sign $\pm$, and define

$$\tilde{E}^{\pm}_{[i,j],s} := [\cdots[[e_{i,s_i}, e_{i+1,s_{i+1}}]_{v^{\pm 2}}, e_{i+2,s_{i+2}}]_{v^{\pm 2}}, \cdots, e_{j,s_j}]_{v^{\pm 2}}. \tag{2.26}$$

  If $\beta = [i, n, j]$, $s \in \mathbb{Z}$, we choose any decomposition $s = s_i + \cdots + s_{j-1} + 2s_j + \cdots + 2s_n$, fix a sign $\pm$, and define

$$\tilde{E}^{\pm}_{[i,n,j],s} := [\cdots[[[\cdots[e_{i,s_i}, e_{i+1,s_{i+1}}]_{v^{\pm 2}}, \cdots, e_{n,s_n}]_{v^{\pm 2}}, e_{n,s_n}], e_{n-1,s_{n-1}}]_{v^{\pm 2}}, \cdots, e_{j,s_j}]_{v^{\pm 2}}. \tag{2.27}$$

- G_2-type.
  If $\beta = [1]$ or $[2]$, $s \in \mathbb{Z}$, we define

$$\tilde{E}^{\pm}_{[i],s} := e_{i,s} \quad \text{for} \quad 1 \le i \le 2. \tag{2.28}$$

  If $\beta = [1, 2]$, $s \in \mathbb{Z}$, we choose any decomposition $s = s_1 + s_2$, and define

$$\tilde{E}^{\pm}_{[1,2],s} := [e_{1,s_1}, e_{2,s_2}]_{v^{\pm 3}}. \tag{2.29}$$

  If $\beta = [1, 2, 2]$, $s \in \mathbb{Z}$, we choose any decomposition $s = s_1 + 2s_2$, and define

$$\tilde{E}^{\pm}_{[1,2,2],s} := [[e_{1,s_1}, e_{2,s_2}]_{v^{\pm 3}}, e_{2,s_2}]_{v^{\pm 1}}. \tag{2.30}$$

  If $\beta = [1, 2, 2, 2]$, $s \in \mathbb{Z}$, we choose any decomposition $s = s_1 + 3s_2$, and define

$$\tilde{E}^{\pm}_{[1,2,2,2],s} := [[[e_{1,s_1}, e_{2,s_2}]_{v^{\pm 3}}, e_{2,s_2}]_{v^{\pm 1}}, e_{2,s_2}]_{v^{\mp 1}}. \tag{2.31}$$

  If $\beta = [1, 2, 1, 2, 2]$, $s \in \mathbb{Z}$, we choose any decomposition $s = 2s_1 + 3s_2$, and define

$$\tilde{E}^{\pm}_{[1,2,1,2,2],s} := [[e_{1,s_1}, e_{2,s_2}]_{v^{\pm 3}}, [[e_{1,s_1}, e_{2,s_2}]_{v^{\pm 3}}, e_{2,s_2}]_{v^{\pm 1}}]_{v^{\mp 1}}. \tag{2.32}$$

Evoking the specific convex orders $<$ on Δ^+ from (2.20)–(2.22), let us consider the following order $<$ on the set $\Delta^+ \times \mathbb{Z}$:

$$(\alpha, s) < (\beta, t) \quad \text{iff} \quad \alpha < \beta \text{ or } \alpha = \beta, s < t. \tag{2.33}$$

Let H denote the set of all functions $h \colon \Delta^+ \times \mathbb{Z} \to \mathbb{N}$ with finite support. The monomials

$$E_h := \overrightarrow{\prod_{(\beta,s)\in\Delta^+\times\mathbb{Z}}} E^{h(\beta,s)}_{\beta,s} \qquad \forall\, h \in H \tag{2.34}$$

will be called the *ordered PBWD monomials* of $U^>_v(L\mathfrak{g})$. Here, the arrow $\to$ over the product sign refers to the total order (2.33).

Our first key result generalizes [30, Theorem 2.16] from type A_n to types G_2 and B_n (the proof is presented in Sections 3 and 4, respectively, and is based on the shuffle approach):

Theorem 2.5. The ordered PBWD monomials $\{E_h\}_{h\in H}$ of (2.34) form $\mathbb{Q}(v)$-bases of $U^>_v(L\mathfrak{g})$ for $\mathfrak{g}$ of type B_n and G_2.

2.3 Specialization maps in type A_n

As already mentioned, our key technique used to prove Theorem 2.5 as well as Theorem 2.4 in types B_n and G_2 is that of specialization maps. Following [30], we shall now briefly recall the construction and the key properties of the latter in type A_n, while Sections 3–4 will generalize this technique to types G_2 and B_n, respectively.

Identifying each simple root α_i ($i \in I$) with a basis element $\mathbf{1}_i \in \mathbb{N}^I$ (having the i-th coordinate equal to 1 and the rest equal to 0), we can view $\mathbb{N}^I$ as the positive cone of the root lattice of $\mathfrak{g}$. For any $\underline{k} \in \mathbb{N}^I$, let $\mathrm{KP}(\underline{k})$ be the set of *Kostant partitions*, which consists of all unordered vector partitions of $\underline{k}$ into a sum of positive roots. Explicitly, a Kostant partition of $\underline{k}$ is the same as a tuple $\underline{d} = \{d_\beta\}_{\beta \in \Delta^+} \in \mathbb{N}^{\Delta^+}$ satisfying

$$\sum_{i \in I} k_i \alpha_i = \sum_{\beta \in \Delta^+} d_\beta \beta. \tag{2.35}$$

The convex order (2.20) on Δ^+ induces a total order on $\mathrm{KP}(\underline{k})$ (opposite to that of [30]):

$$\{d_\beta\}_{\beta \in \Delta^+} < \{d'_\beta\}_{\beta \in \Delta^+} \Longleftrightarrow \exists \, \gamma \in \Delta^+ \text{ s.t. } d_\gamma < d'_\gamma \text{ and } d_\beta = d'_\beta \text{ for all } \beta < \gamma. \tag{2.36}$$

For any $h \in H$, we define its *degree* $\deg(h) \in \mathbb{N}^{\Delta^+}$ as the Kostant partition $\underline{d} = \{d_\beta\}_{\beta \in \Delta^+}$ with $d_\beta = \sum_{s \in \mathbb{Z}} h(\beta, s) \in \mathbb{N}$ for all $\beta \in \Delta^+$, and the *grading* $\mathrm{gr}(h) \in \mathbb{N}^I$ so that $\deg(h) \in \mathrm{KP}(\mathrm{gr}(h))$. For any $\underline{k} \in \mathbb{N}^I$ and $\underline{d} \in \mathrm{KP}(\underline{k})$, we define the following subsets of H:

$$H_{\underline{k}} := \{h \in H \,|\, \mathrm{gr}(h) = \underline{k}\}, \qquad H_{\underline{k},\underline{d}} := \{h \in H \,|\, \deg(h) = \underline{d}\}. \tag{2.37}$$

For any $h \in H_{\underline{k},\underline{d}}$ and $\beta \in \Delta^+$, we consider the collection

$$\lambda_{h,\beta} = \{r_\beta(h,1) \leq \cdots \leq r_\beta(h,d_\beta)\} \tag{2.38}$$

obtained by listing all integers $r \in \mathbb{Z}$ with multiplicity $h(\beta, r) > 0$ in the nondecreasing order. Thus, E_h of (2.34) can be represented by

$$E_h = \prod_{\beta \in \Delta^+}^{\rightarrow} \left(E_{\beta, r_\beta(h,1)} \cdots E_{\beta, r_\beta(h,d_\beta)} \right) \qquad \forall \, h \in H_{\underline{k},\underline{d}}, \tag{2.39}$$

where the arrow $\rightarrow$ over the product sign refers to the convex order (2.20) on Δ^+.

Let us now recall the definition of the specialization maps in type A_n. For any $F \in S_{\underline{k}}$ and $\underline{d} \in \mathrm{KP}(\underline{k})$, we split the variables $\{x_{i,l}\}_{i \in I}^{1 \leq l \leq k_i}$ into the disjoint union of $\sum_{\beta \in \Delta^+} d_\beta$ groups

$$\bigsqcup_{\substack{\beta \in \Delta^+ \\ 1 \leq s \leq d_\beta}} \left\{ x_{i,t}^{(\beta,s)} \,\Big|\, i \in I, 1 \leq t \leq \nu_{\beta,i} \right\}, \tag{2.40}$$

where the integer $\nu_{\beta,i}$ is the coefficient of α_i in β as defined in the beginning of Subsection 2.1. In type A_n, any positive root $\beta \in \Delta^+$ is of the form $\beta = [i,j] = \sum_{s=i}^{j} \alpha_s$ for some $1 \leq i \leq j \leq n$, and so $\nu_{\beta,i} \in \{0,1\}$ for any $1 \leq i \leq n$. For $F \in S_{\underline{k}}$, let f denote its numerator from (2.12). Then, we define $\phi_{\underline{d}}(F)$ by specializing the variables in f as follows:

$$x_{i,1}^{(\beta,s)} \mapsto \upsilon^{-i} w_{\beta,s}, \ldots, x_{j,1}^{(\beta,s)} \mapsto \upsilon^{-j} w_{\beta,s} \qquad \forall \, \beta = [i,j], \ 1 \leq s \leq d_\beta. \tag{2.41}$$

We note that $\phi_{\underline{d}}(F)$ is symmetric in $\{w_{\beta,s}\}_{s=1}^{d_\beta}$ for any $\beta \in \Delta^+$, and is independent of our splitting (2.40) of the variables $\{x_{i,l}\}_{i \in I}^{1 \leq l \leq k_i}$ into groups. This gives rise to the **specialization map**

$$\phi_{\underline{d}} \colon S_{\underline{k}} \longrightarrow \mathbb{Q}(\upsilon)[\{w_{\beta,s}^{\pm 1}\}_{\beta \in \Delta^+}^{1 \leq s \leq d_\beta}]^{\mathfrak{S}_{\underline{d}}}. \tag{2.42}$$

We shall further extend it to the specialization map $\phi_{\underline{d}}$ on the entire shuffle algebra S:

$$\phi_{\underline{d}}: S \longrightarrow \mathbb{Q}(v)[\{w_{\beta,s}^{\pm 1}\}_{\beta \in \Delta^+}^{1 \leq s \leq d_\beta}]^{\mathfrak{S}_{\underline{d}}} \tag{2.43}$$

by declaring $\phi_{\underline{d}}(F') = 0$ for any $\underline{\ell} \neq \underline{k}$ and $F' \in S_{\underline{\ell}}$.

Let us now summarize the key properties of the above specialization maps $\phi_{\underline{d}}$ that were crucially used in [30] to derive the shuffle algebra realization and the PBWD bases in type A_n. Our presentation is adapted to allow for an immediate formulation of B_n and G_2 counterparts.

In what follows, we will use the notation $\doteq$ to denote an equality up to $\mathbb{Q}^\times \cdot v^{\mathbb{Z}}$:

$$A \doteq B \quad \text{if} \quad A = c \cdot B \quad \text{for some } c \in \mathbb{Q}^\times \cdot v^{\mathbb{Z}}. \tag{2.44}$$

In type A_n, we define $\kappa_\beta := |\beta| - 1$ for $\beta \in \Delta^+$, where $|\beta|$ denotes the height of a root $\beta \in \Delta^+$:

$$|\beta| := \sum_{i \in I} v_{\beta,i}. \tag{2.45}$$

We also recall the notation $\langle m \rangle_v$ from (2.11). The first two lemmas imply the linear independence of the ordered PBWD monomials $\{E_h\}_{h \in H}$ in type A_n (see [30, §3.2.2]):

Lemma 2.6. For any $h \in H_{\underline{k},\underline{d}}$, we have

$$\phi_{\underline{d}}(\Psi(E_h)) \doteq \langle 1 \rangle_v^{\sum_{\beta \in \Delta^+} d_\beta \kappa_\beta} \cdot \prod_{\beta,\beta' \in \Delta^+}^{\beta < \beta'} G_{\beta,\beta'} \cdot \prod_{\beta \in \Delta^+} G_\beta \cdot \prod_{\beta \in \Delta^+} P_{\lambda_{h,\beta}}. \tag{2.46}$$

Here, the factors $G_{\beta,\beta'}$ and G_β are products of linear factors $w_{\beta,s}$ and $w_{\beta,s} - v^{\mathbb{Z}} w_{\beta',s'}$ which are independent of $h \in H_{\underline{k},\underline{d}}$ and are symmetric with respect to $\mathfrak{S}_{\underline{d}}$, $\lambda_{h,\beta}$ are as in (2.38), and

$$P_{\lambda_{h,\beta}} = Sym_{\mathfrak{S}_{d_\beta}} \left(w_{\beta,1}^{r_\beta(h,1)} \cdots w_{\beta,d_\beta}^{r_\beta(h,d_\beta)} \prod_{1 \leq i < j \leq d_\beta} \frac{w_{\beta,i} - v_\beta^{-2} w_{\beta,j}}{w_{\beta,i} - w_{\beta,j}} \right). \tag{2.47}$$

This result (cf. [30, Lemma 3.17]) features a "**rank 1 reduction**": each $P_{\lambda_{h,\beta}}$ from (2.47) can be viewed as the shuffle product $x^{r_\beta(h,1)} \star \cdots \star x^{r_\beta(h,d_\beta)}$ in the shuffle algebra of type A_1, evaluated at $\{w_{\beta,s}\}_{s=1}^{d_\beta}$. The following is [30, Lemma 3.16] (keeping in mind that our total order (2.36) on $KP(\underline{k})$ is opposite to that of [30]):

Lemma 2.7. For any $h \in H_{\underline{k},\underline{d}}$ and $\underline{d}' < \underline{d}$, we have $\phi_{\underline{d}'}(\Psi(E_h)) = 0$.

Let $S'_{\underline{k}}$ be the $\mathbb{Q}(v)$-subspace of $S_{\underline{k}}$ spanned by $\{\Psi(E_h)\}_{h \in H_{\underline{k}}}$. The following result (which implies Theorem 2.4 and establishes PBWD bases in type A_n) is proved in [30, §3.2.3]:

Lemma 2.8. For any $F \in S_{\underline{k}}$ and $\underline{d} \in KP(\underline{k})$, if $\phi_{\underline{d}'}(F) = 0$ for all $\underline{d}' \in KP(\underline{k})$ such that $\underline{d}' < \underline{d}$, then there exists $F_{\underline{d}} \in S'_{\underline{k}}$ such that $\phi_{\underline{d}}(F) = \phi_{\underline{d}}(F_{\underline{d}})$ and $\phi_{\underline{d}'}(F_{\underline{d}}) = 0$ for all $\underline{d}' < \underline{d}$.

Combining Lemmas 2.6–2.8, we derive the A_n-type counterparts of Theorems 2.4, 2.5 using

(1) the validity of Theorems 2.4, 2.5 in type A_1, see [30, §3.2.1],
(2) the observation that $\phi_{\underline{d}}(F) = 0 \; \forall \underline{d} \in KP(\underline{k})$ implies $F = 0$ by taking the maximal element of $KP(\underline{k})$ with respect to (2.36) corresponding to a partition of $\underline{k}$ into a sum of simple roots; see [31, Proposition 1.6].

Let us emphasize that proofs of both theorems use only the properties of the specialization maps $\phi_{\underline{d}}$ stated above. With this in mind, we will now define similar specialization maps and verify the validity of

analogous lemmas in types G_2 (Section 3) and B_n (Section 4). This will yield a new proof of Theorem 2.4 and a proof of Theorem 2.5 in types G_2, B_n. As an important benefit of this approach, in contrast to [26], we also establish the counterparts of the above two results for integral $\mathbb{Z}[v, v^{-1}]$-forms as well as for the Yangian version (treated in Section 5).

3 Specialization Maps for Type G_2

In this section, we define specialization maps for the shuffle algebra of type G_2 and verify their key properties. This implies the shuffle algebra realization and PBWD-type theorems for $U_v^>(L\mathfrak{g}_2)$ as well as its Lusztig integral form $\mathbf{U}_v^>(L\mathfrak{g}_2)$ introduced in Subsection 3.2. Here, $\mathfrak{g}_2$ is the simple Lie algebra of type G_2.

3.1 $U_v^>(L\mathfrak{g}_2)$ and its shuffle algebra realization

In type G_2, for any $F \in S_{\underline{k}}$ with $\underline{k} = (k_1, k_2) \in \mathbb{N}^2$, the wheel conditions are:

$$F(\{x_{i,r}\}_{1 \le i \le 2}^{1 \le r \le k_i}) = 0 \quad \text{once} \quad x_{1,1} = v^6 x_{1,2} = v^3 x_{2,1},$$
$$\text{or} \quad x_{2,1} = v^2 x_{2,2} = v^4 x_{2,3} = v^6 x_{2,4} = v^3 x_{1,1}. \tag{3.1}$$

For any $\underline{k} \in \mathbb{N}^2$ and $\underline{d} \in \mathrm{KP}(\underline{k})$, the specialization map $\phi_{\underline{d}}$ as in (2.43) is defined by the following specialization of the $x_{*,*}^{(*,*)}$-variables (replacing (2.41) for type A_n):

$$x_{1,t}^{(\beta,s)} \mapsto v^{2t} w_{\beta,s} \quad (1 \le t \le \nu_{\beta,1}), \qquad x_{2,t}^{(\beta,s)} \mapsto v^{-3+2t} w_{\beta,s} \quad (1 \le t \le \nu_{\beta,2}) \tag{3.2}$$

for any positive root $\beta \in \Delta^+$, where $\nu_{\beta,i}$ is as in Subsection 2.1.

Lemma 3.1. Consider the particular choices (2.28)–(2.32) of quantum root vectors $\{\tilde{E}_{\beta,s}^\pm\}_{\beta\in\Delta^+}^{s\in\mathbb{Z}}$. Their images under Ψ in the shuffle algebra S of type G_2 are as follows:

- For $\{\tilde{E}_{\beta,s}^+\}_{\beta\in\Delta^+}^{s\in\mathbb{Z}}$:

$$\Psi(\tilde{E}_{[i],s}^+) \doteq x_{i,1}^s, \; i = 1, 2, \tag{3.3}$$

$$\Psi(\tilde{E}_{[1,2],s}^+) \doteq \frac{\langle 3 \rangle_v x_{1,1}^{s_1+1} x_{2,1}^{s_2}}{x_{1,1} - x_{2,1}}, \; \text{with } s = s_1 + s_2, \tag{3.4}$$

$$\Psi(\tilde{E}_{[1,2,2],s}^+) \doteq \frac{\langle 3 \rangle_v \langle 2 \rangle_v [2]_v \cdot x_{1,1}^{s_1+2}(x_{2,1}x_{2,2})^{s_2}}{(x_{1,1} - x_{2,1})(x_{1,1} - x_{2,2})}, \; \text{with } s = s_1 + 2s_2, \tag{3.5}$$

$$\Psi(\tilde{E}_{[1,2,2,2],s}^+) \doteq \frac{\langle 3 \rangle_v^2 \langle 2 \rangle_v [2]_v \cdot x_{1,1}^{s_1+3}(x_{2,1}x_{2,2}x_{2,3})^{s_2}}{(x_{1,1} - x_{2,1})(x_{1,1} - x_{2,2})(x_{1,1} - x_{2,3})}, \; \text{with } s = s_1 + 3s_2, \tag{3.6}$$

$$\Psi(\tilde{E}_{[1,2,1,2,2],s}^+) \doteq \frac{\langle 3 \rangle_v^3 \langle 2 \rangle_v [2]_v \cdot (x_{1,1}x_{1,2})^{s_1+1}(x_{2,1}x_{2,2}x_{2,3})^{s_2} \cdot g_1}{\prod_{1 \le r \le 2}^{1 \le t \le 3}(x_{1,r} - x_{2,t})}, \; \text{with } s = 2s_1 + 3s_2, \tag{3.7}$$

where

$$g_1 = (v^6 + 1)x_{1,1}^2 x_{1,2}^2 + (v^6 + 1)x_{1,1}x_{1,2}(x_{2,1}x_{2,2} + x_{2,1}x_{2,3} + x_{2,2}x_{2,3})$$
$$- v^3(x_{1,1} + x_{1,2})(x_{1,1}x_{1,2}x_{2,1} + x_{1,1}x_{1,2}x_{2,2} + x_{1,1}x_{1,2}x_{2,3} + x_{2,1}x_{2,2}x_{2,3}). \tag{3.8}$$

- For $\{\tilde{E}_{\beta,s}^-\}_{\beta\in\Delta^+}^{s\in\mathbb{Z}}$:

$$\Psi(\tilde{E}_{[i],s}^-) \doteq x_{i,1}^s, \; i = 1, 2, \tag{3.9}$$

$$\Psi(\tilde{E}_{[1,2],s}^-) \doteq \frac{\langle 3 \rangle_v x_{1,1}^{s_1} x_{2,1}^{s_2+1}}{x_{1,1} - x_{2,1}}, \; \text{with } s = s_1 + s_2, \tag{3.10}$$

$$\Psi(\tilde{E}_{[1,2,2],s}^-) \doteq \frac{\langle 3 \rangle_v \langle 2 \rangle_v [2]_v \cdot x_{1,1}^{s_1}(x_{2,1}x_{2,2})^{s_2+1}}{(x_{1,1} - x_{2,1})(x_{1,1} - x_{2,2})}, \; \text{with } s = s_1 + 2s_2, \tag{3.11}$$

$$\Psi(\tilde{E}^-_{[1,2,2,2],s}) \doteq \frac{\langle 3 \rangle^2_v \langle 2 \rangle_v [2]_v \cdot x^{s_1}_{1,1} (x_{2,1} x_{2,2} x_{2,3})^{s_2+1}}{(x_{1,1} - x_{2,1})(x_{1,1} - x_{2,2})(x_{1,1} - x_{2,3})}, \quad \text{with } s = s_1 + 3s_2, \tag{3.12}$$

$$\Psi(\tilde{E}^-_{[1,2,1,2,2],s}) \doteq \frac{\langle 3 \rangle^3_v \langle 2 \rangle_v [2]_v \cdot (x_{1,1} x_{1,2})^{s_1} (x_{2,1} x_{2,2} x_{2,3})^{s_2+1} \cdot g_2}{\prod^{1 \leq t \leq 3}_{1 \leq r \leq 2} (x_{1,r} - x_{2,t})}, \quad \text{with } s = 2s_1 + 3s_2, \tag{3.13}$$

where

$$g_2 = (v^6 + 1) x_{2,1} x_{2,2} x_{2,3} + (v^6 + 1) x_{1,1} x_{1,2} (x_{2,1} + x_{2,2} + x_{2,3})$$
$$- v^3 (x_{1,1} + x_{1,2})(x_{1,1} x_{1,2} + x_{2,1} x_{2,2} + x_{2,1} x_{2,3} + x_{2,2} x_{2,3}). \tag{3.14}$$

Proof. Straightforward computation. ∎

For more general quantum root vectors $\{E_{\beta,s}\}^{s \in \mathbb{Z}}_{\beta \in \Delta^+}$ of $U^>_v(L\mathfrak{g}_2)$ defined by (2.25), their images under Ψ in S are not so well factorized as for the particular choices above. Nevertheless, what is actually important is that they behave well under the specialization maps. For $\beta \in \Delta^+$, we shall use ϕ_β to denote the specialization map $\phi_{\underline{d}}$ with $\underline{d} = \{d_\alpha = \delta_{\beta,\alpha}\}_{\alpha \in \Delta^+}$.

Lemma 3.2. For any $s \in \mathbb{Z}$ and any choices of s_k and λ_k in (2.24, 2.25), we have:

$$\phi_{[i]}(\Psi(E_{[i],s})) \doteq w^s_{[i],1}, \quad 1 \leq i \leq 2, \tag{3.15}$$

$$\phi_{[1,2]}(\Psi(E_{[1,2],s})) \doteq \langle 3 \rangle_v \cdot w^{s+1}_{[1,2],1}, \tag{3.16}$$

$$\phi_{[1,2,2]}(\Psi(E_{[1,2,2],s})) \doteq \langle 3 \rangle_v \langle 2 \rangle_v [2]_v \cdot w^{s+2}_{[1,2,2],1}, \tag{3.17}$$

$$\phi_{[1,2,2,2]}(\Psi(E_{[1,2,2,2],s})) \doteq \langle 3 \rangle^2_v \langle 2 \rangle_v [2]_v \cdot w^{s+3}_{[1,2,2,2],1}, \tag{3.18}$$

$$\phi_{[1,2,1,2,2]}(\Psi(E_{[1,2,1,2,2],s})) \doteq \langle 4 \rangle_v \langle 3 \rangle^3_v \langle 2 \rangle^2_v [2]_v \cdot w^{s+6}_{[1,2,1,2,2],1}. \tag{3.19}$$

Thus, we have:

$$\phi_\beta(\Psi(E_{\beta,s})) \doteq c_\beta \cdot w^{s+\kappa_\beta}_{\beta,1} \qquad \forall (\beta,s) \in \Delta^+ \times \mathbb{Z}, \tag{3.20}$$

where the constant c_β is explicitly specified in (3.15)–(3.19), and κ_β is given by

$$\kappa_\beta = \begin{cases} |\beta| - 1 & \text{if } \beta \neq [1,2,1,2,2] \\ |\beta| + 1 & \text{if } \beta = [1,2,1,2,2] \end{cases}. \tag{3.21}$$

Proof. This follows from straightforward computations and the fact that for any positive roots $\alpha_1 < \alpha_2$ such that $\alpha_1 + \alpha_2$ is a root, we have

$$\phi_{\alpha_1+\alpha_2}(\Psi(E_{\alpha_1,s_1}) \star \Psi(E_{\alpha_2,s_2})) = 0 \qquad \forall s_1, s_2 \in \mathbb{Z}.$$

The latter fact is a special case of Proposition 3.5. ∎

Let us now generalize the above lemma by computing $\phi_{\underline{d}}(\Psi(E_h))$ for any $h \in H_{\underline{k},\underline{d}}$. According to (2.39), we have:

$$\Psi(E_h) = \overrightarrow{\prod_{\beta \in \Delta^+}} \left(\Psi(E_{h,r_\beta(h,1)}) \star \cdots \star \Psi(E_{h,r_\beta(h,d_\beta)}) \right).$$

Here, the product refers to the shuffle product and the arrow $\rightarrow$ over the product sign refers to the order (2.22). Thus, we can choose a special splitting such that the variables in $\Psi(E_{\beta,r_\beta(h,s)})$ are taken to be the group $\{x^{(\beta,s)}_{i,t}\}^{1 \leq t \leq v_{\beta,i}}_{1 \leq i \leq 2}$, and under $\phi_{\underline{d}}$ they are specialized as in (3.2).

For each $1 \leq i \leq 2$, let us consider an order on all the variables

$$X_i = \left\{ x^{(\beta,s)}_{i,t} \mid \beta \in \Delta^+, 1 \leq s \leq d_\beta, 1 \leq t \leq v_{\beta,i} \right\}, \tag{3.22}$$

defined by (cf. (2.33))

$$x^{(\beta,s)}_{i,t} < x^{(\beta',s')}_{i,t'} \quad \text{iff} \quad (\beta,s) < (\beta',s') \text{ or } (\beta,s) = (\beta',s'), t < t'. \tag{3.23}$$

Once we fix the above splitting of the $x_{*,*}$-variables, the group $\mathfrak{S}_{\underline{k}}$ acts on each X_i ($1 \le i \le 2$) by permuting the tuples

$$\left\{ (\beta, s, t) \in \Delta^+ \times \mathbb{N} \times \mathbb{N} \mid 1 \le s \le d_\beta, 1 \le t \le \nu_{\beta,i} \right\}. \tag{3.24}$$

For any $\underline{d} \in \mathrm{KP}(\underline{k})$, let $\mathrm{Sh}_{\underline{d}} \subset \mathfrak{S}_{\underline{k}}$ be the subset of "$\underline{d}$-shuffle permutations", defined via:

$$\mathrm{Sh}_{\underline{d}} := \left\{ \sigma \in \mathfrak{S}_{\underline{k}} \,\middle|\, \sigma(x_{i,t}^{(\beta,s)}) < \sigma(x_{i,t'}^{(\beta,s)}) \quad \forall\, t < t', \beta \in \Delta^+, 1 \le s \le d_\beta, 1 \le i \le 2 \right\}, \tag{3.25}$$

where the order $<$ on the variables is defined in (3.23). Then, by definition of the shuffle product (2.14), we have:

$$\Psi(E_h) \doteq \sum_{\sigma \in \mathrm{Sh}_{\underline{d}}} \sigma\left(F_h(\{x_{*,*}^{(*,*)}\})\right) = \sum_{\sigma \in \mathrm{Sh}_{\underline{d}}} F_h\left(\{\sigma(x_{*,*}^{(*,*)})\}\right), \tag{3.26}$$

where

$$F_h := \prod_{\substack{\beta \in \Delta^+ \\ 1 \le s \le d_\beta}} \Psi(E_{\beta, r_\beta(h,s)}) \prod_{\substack{(\beta,p) < (\beta',q) \\ \beta, \beta' \in \Delta^+ \\ 1 \le p \le d_\beta, 1 \le q \le d_{\beta'}}} \prod_{1 \le i,j \le 2} \prod_{\substack{1 \le r \le \nu_{\beta',j} \\ 1 \le \ell \le \nu_{\beta,i}}} \frac{x_{i,\ell}^{(\beta,p)} - v_i^{-a_{ij}} x_{j,r}^{(\beta',q)}}{x_{i,\ell}^{(\beta,p)} - x_{j,r}^{(\beta',q)}}. \tag{3.27}$$

Let us now evaluate the $\phi_{\underline{d}}$-specialization of each term $\sigma(F_h)$ in the symmetrization (3.26), many of which actually vanish. To this end, consider the elements $\sigma \in \mathfrak{S}_{\underline{k}}$ which satisfy the following condition: as a permutation of the tuples (3.24) σ fixes the indices β, t (as well as i) and only permutes the index s. In other words, for each $\beta \in \Delta^+$ there is $\sigma_\beta \in \mathfrak{S}_{d_\beta}$ such that:

$$\sigma(x_{i,t}^{(\beta,s)}) = x_{i,t}^{(\beta,\sigma_\beta(s))} \qquad \forall\, \beta \in \Delta^+,\ 1 \le s \le d_\beta,\ i \in \beta,\ 1 \le t \le \nu_{\beta,i}. \tag{3.28}$$

Such permutations σ form a subgroup of $\mathfrak{S}_{\underline{k}}$ isomorphic to $\mathfrak{S}_{\underline{d}} := \prod_{\beta \in \Delta^+} \mathfrak{S}_{d_\beta}$, and we shall denote this subgroup simply by $\mathfrak{S}_{\underline{d}}$.

In what follows, given a collection A of the variables $x_{i,*}^{(*,*)}$ with a fixed index i and a collection B of the variables $x_{j,*}^{(*,*)}$ with a fixed index j, we shall use the following notation:

$$\zeta_{i,j}(A/B) := \prod_{x \in A}^{y \in B} \zeta_{i,j}(x/y). \tag{3.29}$$

Then, we have:

Lemma 3.3. $\phi_{\underline{d}}(\sigma(F_h)) = 0$ for $\sigma \notin \mathfrak{S}_{\underline{d}}$.

Proof. Define the following sets of variables (cf. the notation (2.1)):

$$Z_i^\beta = \left\{ x_{i,t}^{(\beta,s)} \mid 1 \le s \le d_\beta, 1 \le t \le \nu_{\beta,i} \right\} \qquad \forall\, \beta \in \Delta^+, i \in \beta, \tag{3.30}$$

$$Z_i^{>\beta} = \left\{ x_{i,t}^{(\alpha,s)} \mid \beta < \alpha, 1 \le s \le d_\alpha, i \in \alpha, 1 \le t \le \nu_{\alpha,i} \right\} \qquad \forall\, \beta \in \Delta^+, 1 \le i \le 2. \tag{3.31}$$

It suffices to show that $\phi_{\underline{d}}(\sigma(F_h)) \ne 0$ only if (3.28) holds for every $\beta \in \Delta^+$. We shall prove this by induction on β with respect to the increasing order (2.22).

We note that F_h contains ζ-factors $\zeta_{1,2}(Z_1^{[1]}/Z_2^{>[1]})$, cf. (3.29), and clearly $\sigma(Z_2^{>[1]}) = Z_2^{>[1]}$. If $\sigma(Z_1^{[1]}) \ne Z_1^{[1]}$, then there is some $x_{1,1}^{([1],s)}$ such that $\sigma(x_{1,1}^{([1],s)}) = x_{1,t}^{(\gamma,r)}$ for some $\gamma > [1]$. In the latter case, $\sigma(F_h)$ contains the ζ-factor $\zeta_{1,2}(x_{1,t}^{(\gamma,r)}/x_{2,t}^{(\gamma,r)})$ and so $\phi_{\underline{d}}(\sigma(F_h)) = 0$, which is a contradiction. This establishes (3.28) for $\beta = [1]$, which is the base of induction.

Let us now prove (3.28) for β, assuming (3.28) holds for all $\alpha < \beta$. The only nontrivial check is for the case $\beta = [1, 2, 1, 2, 2]$. Assuming by induction that for all $s < s_0$ there is s' with $1 \le s' \le d_\beta$ such that $\sigma(x_{i,t}^{(\beta,s)}) = x_{i,t}^{(\beta,s')}$ for any $i \in \beta$, $1 \le t \le \nu_{\beta,i}$, we shall prove the same also holds for $s = s_0$. The base of induction is $s_0 = 1$, in which case the statement is vacuous.

Let $\sigma(x_{1,1}^{(\beta,s_0)}) = x_{1,t_1}^{(\gamma_1,r_1)}$. By the induction hypothesis, we have $(\beta,s_0) \le (\gamma_1,r_1)$. Suppose $\sigma(x_{2,\ell_1}^{(\beta_1,s_1)}) = x_{2,t_1}^{(\gamma_1,r_1)}$. If $\beta_1 < \beta$, then by the induction hypothesis we get $\gamma_1 = \beta_1 < \beta$, a contradiction. On the other hand, if $\beta_1 > \beta$, then we note that F_h contains the ζ-factors $\zeta_{1,2}(x_{1,1}^{(\beta,s_0)}/Z_2^{>\beta})$, and thus $\sigma(F_h)$ contains the factor $\zeta_{1,2}(x_{1,t_1}^{(\gamma_1,r_1)}/x_{2,t_1}^{(\gamma_1,r_1)})$, which specializes to zero under $\phi_{\underline{d}}$, a contradiction. Thus, $\beta_1 = \beta$. Similarly, there is some $x_{2,\ell_2}^{(\beta,s_2)}$ such that $\sigma(x_{2,\ell_2}^{(\beta,s_2)}) = x_{2,t_1+1}^{(\gamma_1,r_1)}$. Moreover, if $s_0 < s_1$ or $s_1 < s_2$, then $\sigma(F_h)$ contains the ζ-factor $\zeta_{1,2}(x_{1,t_1}^{(\gamma_1,r_1)}/x_{2,t_1}^{(\gamma_1,r_1)})$ or $\zeta_{2,2}(x_{2,t_1}^{(\gamma_1,r_1)}/x_{2,t_1+1}^{(\gamma_1,r_1)})$, which implies $\phi_{\underline{d}}(\sigma(F_h)) = 0$, hence, a contradiction. Thus, we have $s_0 \ge s_1 \ge s_2$. If $s_1 < s_0$, then by induction hypothesis, we get a contradiction. Similarly, if $s_2 < s_0$, we also get a contradiction. Thus, $s_0 = s_1 = s_2$. If $\ell_2 < \ell_1$, then $x_{2,\ell_2}^{(\beta,s_0)} < x_{2,\ell_1}^{(\beta,s_0)}$, and the condition $\sigma \in \mathrm{Sh}_{\underline{d}}$ implies

$$x_{2,t_1+1}^{(\gamma_1,r_1)} = \sigma(x_{2,\ell_2}^{(\beta,s_0)}) < \sigma(x_{2,\ell_1}^{(\beta,s_0)}) = x_{2,t_1}^{(\gamma_1,r_1)},$$

a contradiction. Thus, $\ell_1 < \ell_2$. Combining all the above, we get $s_1 = s_2 = s_0$ and $\ell_1 < \ell_2$.

Similarly, let $\sigma(x_{1,2}^{(\beta,s_0)}) = x_{1,t_2}^{(\gamma_2,r_2)}$. Then, there are $x_{2,\ell_1'}^{(\beta,s_0)}, x_{2,\ell_2'}^{(\beta,s_0)}$ with $\ell_1' < \ell_2'$ and

$$\sigma(x_{2,\ell_1'}^{(\beta,s_0)}) = x_{2,t_2}^{(\gamma_2,r_2)}, \quad \sigma(x_{2,\ell_2'}^{(\beta,s_0)}) = x_{2,t_2+1}^{(\gamma_2,r_2)}.$$

Note that $1 \le \ell_1 < \ell_2 \le 3$ and $1 \le \ell_1' < \ell_2' \le 3$. Then, we have three cases:

- if $\ell_1 = \ell_1'$, then $(\gamma_1,r_1) = (\gamma_2,r_2)$ and $t_1 = t_2$, a contradiction.
- if $\ell_1 > \ell_1'$, then $\ell_1' = 1, \ell_1 = 2, \ell_2 = 3, \ell_2' = \ell_1$ or ℓ_2. In either case, we get $(\gamma_1,r_1) = (\gamma_2,r_2)$. However, since $x_{1,1}^{(\beta,s_0)} < x_{1,2}^{(\beta,s_0)}$ we have $t_1 < t_2$; since $x_{2,1}^{(\beta,s_0)} < x_{2,2}^{(\beta,s_0)}$, we have $t_2 < t_1$, which is a contradiction.
- if $\ell_1 < \ell_2$, then $\ell_1 = 1, \ell_1' = 2, \ell_2' = 3$. Similarly to above, we must have $(\gamma_1,r_1) = (\gamma_2,r_2)$, $t_1 = 1, t_2 = 2$. Then $\gamma_1 = \gamma_2 = \beta$, which is precisely what we wanted.

This completes our proof of the induction step. ∎

Combining Lemmas 3.2–3.3, we obtain the following analogue of Lemma 2.6 for type G_2:

Proposition 3.4. For any $h \in H_{\underline{k},\underline{d}}$, we have

$$\phi_{\underline{d}}(\Psi(E_h)) \doteq \prod_{\substack{\beta,\beta'\in\Delta^+ \\ \beta<\beta'}}^{\beta<\beta'} G_{\beta,\beta'} \cdot \prod_{\beta\in\Delta^+} (c_\beta^{d_\beta} \cdot G_\beta) \cdot \prod_{\beta\in\Delta^+} P_{\lambda_{h,\beta}}, \tag{3.32}$$

where the factors $\{P_{\lambda_{h,\beta}}\}_{\beta\in\Delta^+}$ are given by (2.47), the constants $\{c_\beta\}_{\beta\in\Delta^+}$ are as in Lemma 3.2, and the factors $G_\beta, G_{\beta,\beta'}$ are explicitly given in the proof below.

Proof. If $\sigma \in \mathfrak{S}_{\underline{d}}$, then in F_h the following factor is invariant under σ:

$$\prod_{\substack{\beta,\beta'\in\Delta^+ \\ \beta<\beta'}}^{\beta,\beta'\in\Delta^+} \prod_{\substack{1\le q\le d_{\beta'} \\ 1\le p\le d_\beta}}^{1\le q\le d_{\beta'}} \prod_{\substack{1\le i,j\le 2 \\ 1\le l\le v_{\beta,i}}}^{1\le r\le v_{\beta',j}} \frac{x_{i,l}^{(\beta,p)} - v_i^{-a_{ij}} x_{j,r}^{(\beta',q)}}{x_{i,l}^{(\beta,p)} - x_{j,r}^{(\beta',q)}}, \tag{3.33}$$

and we denote its $\phi_{\underline{d}}$-specialization by $\prod_{\beta<\beta'} G_{\beta,\beta'}$ (with matching indices β,β'). Thus, we only need to prove the following equality for any $\beta \in \Delta^+$:

$$\phi_{\underline{d}}\left(\Psi\left(E_{\beta,r_\beta(h,1)} \star \cdots \star E_{\beta,r_\beta(h,d_\beta)}\right)\right) \doteq c_\beta^{d_\beta} \cdot G_\beta \cdot P_{\lambda_{h,\beta}}, \tag{3.34}$$

with the factor G_β being independent of $\sigma \in \mathfrak{S}_{\underline{d}}$. This is proved by straightforward computation. Explicitly, we have the following formulas:
$G_\beta \doteq$

- 1, for $\beta = [1], [2]$,
- $\prod_{1\le s\ne r\le d_\beta}(w_{\beta,s} - v^6 w_{\beta,r}) \cdot \prod_{\ell=1}^{d_\beta} w_{\beta,\ell}$, for $\beta = [1,2]$,
- $\prod_{1\le s\ne r\le d_\beta}\{(w_{\beta,s} - v^6 w_{\beta,r})(w_{\beta,s} - v^4 w_{\beta,r})\} \cdot \prod_{\ell=1}^{d_\beta} w_{\beta,\ell}^2$, for $\beta = [1,2,2]$,
- $\prod_{1\le s\ne r\le d_\beta}\{(w_{\beta,s} - v^6 w_{\beta,r})(w_{\beta,s} - v^4 w_{\beta,r})(w_{\beta,s} - v^2 w_{\beta,r})\} \cdot \prod_{\ell=1}^{d_\beta} w_{\beta,\ell}^3$, for $\beta = [1,2,2,2]$,

- $\prod_{1 \le s \neq r \le d_\beta} \left\{ (w_{\beta,s} - v^8 w_{\beta,r})(w_{\beta,s} - v^6 w_{\beta,r})^2 (w_{\beta,s} - v^4 w_{\beta,r})^2 (w_{\beta,s} - v^2 w_{\beta,r}) \right\} \cdot \prod_{\ell=1}^{d_\beta} w_{\beta,\ell}^6$, for $\beta = [1, 2, 1, 2, 2]$.

$G_{\beta,\beta'} \doteq$

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} (w_{\beta,s} - v^{-6} w_{\beta',r})$, for $\beta = [1]$ and $\beta' = [1, 2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} \left\{ (w_{\beta,s} - v^{-6} w_{\beta',r})(w_{\beta,s} - v^{-4} w_{\beta',r})(w_{\beta,s} - v^4 w_{\beta',r}) \right\}$, for $\beta = [1]$ and $\beta' = [1, 2, 1, 2, 2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} \left\{ (w_{\beta,s} - v^{-6} w_{\beta',r})(w_{\beta,s} - v^2 w_{\beta',r}) \right\}$, for $\beta = [1]$ and $\beta' = [1, 2, 2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} \left\{ (w_{\beta,s} - v^{-6} w_{\beta',r})(w_{\beta,s} - v^2 w_{\beta',r})(w_{\beta,s} - v^4 w_{\beta',r}) \right\}$, for $\beta = [1]$ and $\beta' = [1, 2, 2, 2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} (w_{\beta,s} - w_{\beta',r})$, for $\beta = [1]$ and $\beta' = [2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} \left\{ (w_{\beta,s} - v^8 w_{\beta',r})(w_{\beta,s} - v^{-6} w_{\beta',r})(w_{\beta,s} - v^6 w_{\beta',r})(w_{\beta,s} - v^{-4} w_{\beta',r})(w_{\beta,s} - v^{-2} w_{\beta',r}) \right\}$, for $\beta = [1, 2]$ and $\beta' = [1, 2, 1, 2, 2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} \left\{ (w_{\beta,s} - v^{-6} w_{\beta',r})(w_{\beta,s} - v^6 w_{\beta',r})(w_{\beta,s} - v^{-2} w_{\beta',r}) \right\}$, for $\beta = [1, 2]$ and $\beta' = [1, 2, 2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} \left\{ (w_{\beta,s} - v^{-6} w_{\beta',r})(w_{\beta,s} - v^6 w_{\beta',r})(w_{\beta,s} - v^{-2} w_{\beta',r})(w_{\beta,s} - v^2 w_{\beta',r}) \right\}$, for $\beta = [1, 2]$ and $\beta' = [1, 2, 2, 2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} (w_{\beta,s} - v^{-2} w_{\beta',r})$, for $\beta = [1, 2]$ and $\beta' = [2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} \left\{ (w_{\beta,s} - v^{-8} w_{\beta',r})(w_{\beta,s} - v^{-6} w_{\beta',r})^2 (w_{\beta,s} - v^6 w_{\beta',r})(w_{\beta,s} - v^{-4} w_{\beta',r})(w_{\beta,s} - v^4 w_{\beta',r})(w_{\beta,s} - v^2 w_{\beta',r}) \right\}$, for $\beta = [1, 2, 1, 2, 2]$ and $\beta' = [1, 2, 2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} \left\{ (w_{\beta,s} - v^{-8} w_{\beta',r})(w_{\beta,s} - v^{-6} w_{\beta',r})^2 (w_{\beta,s} - v^6 w_{\beta',r})(w_{\beta,s} - v^{-4} w_{\beta',r})(w_{\beta,s} - v^4 w_{\beta',r})(w_{\beta,s} - v^2 w_{\beta',r})^2 (w_{\beta,s} - v^{-2} w_{\beta',r}) \right\}$, for $\beta = [1, 2, 1, 2, 2]$ and $\beta' = [1, 2, 2, 2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} \left\{ (w_{\beta,s} - v^{-6} w_{\beta',r})(w_{\beta,s} - v^{-2} w_{\beta',r}) \right\}$, for $\beta = [1, 2, 1, 2, 2]$ and $\beta' = [2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} \left\{ (w_{\beta,s} - v^{-6} w_{\beta',r})(w_{\beta,s} - v^6 w_{\beta',r})(w_{\beta,s} - v^{-4} w_{\beta',r})(w_{\beta,s} - v^4 w_{\beta',r})(w_{\beta,s} - v^{-2} w_{\beta',r}) \right\}$, for $\beta = [1, 2, 2]$ and $\beta' = [1, 2, 2, 2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} (w_{\beta,s} - v^{-4} w_{\beta',r})$, for $\beta = [1, 2, 2]$ and $\beta' = [2]$,

- $\prod_{1 \le s \le d_\beta}^{1 \le r \le d_{\beta'}} (w_{\beta,s} - v^{-6} w_{\beta',r})$, for $\beta = [1, 2, 2, 2]$ and $\beta' = [2]$.

This completes our proof. ■

Proposition 3.5. Lemma 2.7 is valid for type G_2 and specialization maps $\phi_{\underline{d}}$ as in (3.2).

Proof. Given $\underline{d}, \underline{d}' \in KP(\underline{k})$ with $\underline{d}' < \underline{d}$, and any $\sigma \in \mathfrak{S}_{\underline{k}}$, we will show that $\phi_{\underline{d}'}(\sigma(F_h)) = 0$. Let $x_{*,*}^{(*,*)}$ be the above special splitting of the variables $\{x_{i,l}\}_{i=1,2}^{1 \le l \le k_i}$ for $\phi_{\underline{d}}$ (see (3.26, 3.27)) and let $x_{*,*}'^{(*,*)}$ be any splitting of the variables $\{x_{i,l}\}_{i=1,2}^{1 \le l \le k_i}$ for $\phi_{\underline{d}'}$ (see (2.40)). We consider the following sets of $x_{*,*}'$-variables:

$$Z_i^{>(\beta,s)} = \left\{ x_{i,*}'^{(\alpha,r)} \,\middle|\, (\alpha, r) > (\beta, s) \right\} \qquad \forall\, (\beta, s) \in \Delta^+ \times \mathbb{N}, 1 \le i \le 2. \tag{3.35}$$

Let β be the smallest positive root such that $d_\beta' < d_\beta$. Following Lemma 3.3, we can assume that for any $\alpha \le \beta$ the property (3.28) holds for the variables $x_{*,*}'^{(\alpha,*)}$, since otherwise the $\phi_{\underline{d}'}$-specialization is zero. Thus, without loss of generality, we can assume $d_\alpha' = 0$ for all $\alpha \le \beta$. Then, we have the following case-by-case analysis:

- Case $\beta = [1]$. Let $\sigma(x_{1,1}^{(\beta,1)}) = x_{1,t}'^{(\gamma,r)}$ for some $\gamma > \beta$. Since F_h contains the factor $\zeta_{1,2}(x_{1,1}^{(\beta,1)} / Z_2^{>(\beta,1)})$, and $x_{2,t}'^{(\gamma,r)} \in \sigma(Z_2^{>(\beta,1)})$, we get $\phi_{\underline{d}'}(\sigma(F_h)) = 0$.

- Case $\beta = [1, 2]$. Let $\sigma(x_{1,1}^{(\beta,1)}) = x_{1,t}'^{(\gamma,r)}$ for some $\gamma > \beta$. Since F_h contains the factor $\zeta_{1,2}(x_{1,1}^{(\beta,1)} / Z_2^{>(\beta,1)})$, we have $\phi_{\underline{d}'}(\sigma(F_h)) = 0$ unless $\sigma(x_{2,1}^{(\beta,1)}) = x_{2,t}'^{(\gamma,r)}$. In this case, we again get $\phi_{\underline{d}'}(\sigma(F_h)) = 0$ as F_h contains the factor $\zeta_{2,2}(x_{2,1}^{(\beta,1)} / Z_2^{>(\beta,1)})$ and $x_{2,t+1}'^{(\gamma,r)} \in \sigma(Z_2^{>(\beta,1)})$.

- Case $\beta = [1, 2, 1, 2, 2]$. Let $\sigma(x_{1,1}^{(\beta,1)}) = x_{1,t_1}'^{(\gamma_1,r_1)}$ for some $\gamma_1 > \beta$. Then, we get $\phi_{\underline{d}'}(\sigma(F_h)) = 0$ unless there is some $1 \le \ell_1 < \ell_2 \le 3$ such that $\sigma(x_{2,\ell_1}^{(\beta,1)}) = x_{2,t_1}'^{(\gamma_1,r_1)}$, $\sigma(x_{2,\ell_2}^{(\beta,1)}) = x_{2,t_1+1}'^{(\gamma_1,r_1)}$. Similarly, let $\sigma(x_{1,2}^{(\beta,1)}) = x_{1,t_2}'^{(\gamma_2,r_2)}$ for some $\gamma_2 > \beta$, then $\phi_{\underline{d}'}(\sigma(F_h)) = 0$ unless there is some $1 \le \ell_3 < \ell_4 \le 3$ such that $\sigma(x_{2,\ell_3}^{(\beta,1)}) = x_{2,t_2}'^{(\gamma_2,r_2)}$, $\sigma(x_{2,\ell_4}^{(\beta,1)}) = x_{2,t_2+1}'^{(\gamma_2,r_2)}$. This implies that $(\gamma_1, r_1) = (\gamma_2, r_2)$, $t_1 = 1$, $t_2 = 2$, thus contradicting $\gamma_1 > \beta$. Hence, $\phi_{\underline{d}'}(\sigma(F_h)) = 0$.

- Case $\beta = [1, 2, 2]$. Let $\sigma(x_{1,1}^{(\beta,1)}) = x_{1,1}'^{(\gamma,r)}$ for some $\gamma > \beta$. Then, we have $\phi_{\underline{d}'}(\sigma(F_h)) = 0$ unless $\sigma(x_{2,1}^{(\beta,1)}) = x_{2,1}'^{(\gamma,r)}$ and $\sigma(x_{2,2}^{(\beta,1)}) = x_{2,2}'^{(\gamma,r)}$. As F_h contains the factor $\zeta_{2,2}(x_{2,2}^{(\beta,1)} / Z_2^{>(\beta,1)})$ and $x_{2,3}'^{(\gamma,r)} \in \sigma(Z_2^{>(\beta,1)})$, we thus again obtain $\phi_{\underline{d}'}(\sigma(F_h)) = 0$.

- Case $\beta = [1, 2, 2, 2]$. This case is impossible with our assumptions $d_{\le \beta}' = 0$ and $d_\beta \neq 0$, as we have $d_{[2]}' \cdot [2] = \sum_{\alpha \in \Delta^+} d_\alpha' \cdot \alpha = \sum_{i=1}^2 k_i \alpha_i = \sum_{\alpha \in \Delta^+} d_\alpha \cdot \alpha = d_{[1,2,2,2]} \cdot [1, 2, 2, 2] + d_{[2]} \cdot [2]$.

This completes our proof. ∎

Proposition 3.6. Lemma 2.8 is valid for type G_2 and specialization maps $\phi_{\underline{d}}$ as in (3.2).

Proof. The wheel conditions (3.1) for $F \in S_{\underline{k}}$ guarantee that $\phi_{\underline{d}}(F)$ (which is a Laurent polynomial in the variables $\{w_{\beta,s}\}$) vanishes under specific specializations $w_{\beta,s} = v^\# \cdot w_{\beta',s'}$. To evaluate the aforementioned powers $\#$ of v and the orders of vanishing, let us view $\phi_{\underline{d}}$ as a step-by-step specialization in each interval $[\beta]$, ordered in the decreasing order with respect to

$$(\beta, s) < (\beta', s') \quad \text{iff} \quad \beta < \beta' \text{ or } \beta = \beta' \text{ and } s < s'.$$

We note that this computation is local with respect to any fixed pair $(\beta, s) \leq (\beta', s')$. Consider

$$\underline{d}_1 = \{d_\beta = 2, d_\alpha = 0, \alpha \neq \beta\} \in \mathrm{KP}(\underline{k}_1), \tag{3.36}$$

$$\underline{d}_2 = \{d_\beta = d_{\beta'} = 1, d_\alpha = 0, \alpha \neq \beta, \beta'\} \in \mathrm{KP}(\underline{k}_2). \tag{3.37}$$

For any $F_1 \in S_{\underline{k}_1}, F_2 \in S_{\underline{k}_2}$, it thus suffices to show that $\phi_{\underline{d}_1}(F_1)$ is divisible by G_β if $\phi_{\underline{d}}(F_1) = 0$ for any $\underline{d} < \underline{d}_1$ and $\phi_{\underline{d}_2}(F_2)$ is divisible by $G_{\beta,\beta'}$ if $\phi_{\underline{d}}(F_2) = 0$ for any $\underline{d} < \underline{d}_2$, cf. (3.32).

For $\underline{d}_1 = \{d_\beta = 2, d_{\alpha \neq \beta} = 0\}$, the nontrivial cases are $\beta = [1,2,2], [1,2,2,2], [1,2,1,2,2]$.

- $\beta = [1,2,2]$. For F_1, under $\phi_{\underline{d}_1}$ the wheel condition $F_1 = 0$ once $x_{1,1}^{(\beta,1)} = v^6 x_{1,1}^{(\beta,2)} = v^3 x_{2,1}^{(\beta,1)}$ becomes $\phi_{\underline{d}_1}(F_1) = 0$ once $w_{\beta,1} = v^6 w_{\beta,2}$, thus we get the vanishing factor $(w_{\beta,1} - v^6 w_{\beta,2})$; the wheel condition $F_1 = 0$ once $x_{2,2}^{(\beta,1)} = v^2 x_{2,1}^{(\beta,1)} = v^4 x_{2,2}^{(\beta,2)} = v^6 x_{2,1}^{(\beta,2)} = v^3 x_{1,1}^{(\beta,2)}$ becomes $\phi_{\underline{d}_1}(F_1) = 0$ once $w_{\beta,1} = v^4 w_{\beta,2}$, thus we get the vanishing factor $(w_{\beta,1} - v^4 w_{\beta,2})$. Since $\phi_{\underline{d}_1}(F_1)$ is symmetric with respect to $w_{\beta,1}$ and $w_{\beta,2}$, we also have the vanishing factor $(w_{\beta,2} - v^6 w_{\beta,1})$ and $(w_{\beta,2} - v^4 w_{\beta,1})$, thus we get the vanishing factor G_β.

- $\beta = [1,2,2,2]$. Besides the vanishing factors appearing in $G_{[1,2,2]}$, the wheel condition $F_1 = 0$ once $x_{2,3}^{(\beta,1)} = v^2 x_{2,2}^{(\beta,1)} = v^4 x_{2,1}^{(\beta,1)} = v^6 x_{2,1}^{(\beta,2)} = v^3 x_{1,1}^{(\beta,2)}$ becomes $\phi_{\underline{d}_1}(F_1) = 0$ once $w_{\beta,1} = v^2 w_{\beta,2}$, thus we get the vanishing factor $(w_{\beta,1} - v^2 w_{\beta,2})(w_{\beta,2} - v^2 w_{\beta,1})$, and we get all the vanishing factors in G_β.

- $\beta = [1,2,1,2,2]$. Besides the vanishing factors in $G_{[1,2,2,2]}$, the wheel conditions $F_1 = 0$ once $x_{1,1}^{(\beta,1)} = v^6 x_{1,2}^{(\beta,2)} = v^3 x_{2,1}^{(\beta,1)}$ or $x_{1,2}^{(\beta,1)} = v^6 x_{1,2}^{(\beta,2)} = v^3 x_{2,2}^{(\beta,1)}$ or $x_{1,2}^{(\beta,1)} = v^6 x_{1,1}^{(\beta,2)} = v^3 x_{2,2}^{(\beta,1)}$ give us the rest vanishing factors in G_β.

For $\underline{d}_2 = \{d_\beta = d_{\beta'} = 1, d_{\alpha \neq \beta, \beta'} = 0\}$, take $F_2 \in S_{\underline{k}}$, then

- $(\beta, \beta') = ([1], [1,2])$. Under $\phi_{\underline{d}_2}$ the wheel condition $F_2 = 0$ once $x_{1,1}^{(\beta',1)} = v^6 x_{1,1}^{(\beta,1)} = v^3 x_{2,1}^{(\beta',1)}$ becomes $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^{-6} w_{\beta',1}$, thus we have the factor $G_{\beta,\beta'}$.

- $(\beta, \beta') = ([1], [1,2,1,2,2])$. Besides the factor $G_{[1],[1,2]}$, the wheel condition $x_{1,2}^{(\beta',1)} = v^6 x_{1,1}^{(\beta,1)} = v^3 x_{2,2}^{(\beta',1)}$ gives the vanishing factor $w_{\beta,1} = v^{-4} w_{\beta',1}$. Let $\underline{d}_3 = \{d_{[1,2]} = 3, d_{\alpha \neq [1,2]} = 0\}$, then $\underline{d}_3 < \underline{d}_2$, by $\phi_{\underline{d}_3}(F_2) = 0$ we know $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^4 w_{\beta',1}$.

- $(\beta, \beta') = ([1], [1,2,2])$. Besides the factor $G_{[1],[1,2]}$, let $\underline{d}_3 = \{d_{[1,2]} = 2, d_{\alpha \neq [1,2]} = 0\}$, then $\underline{d}_3 < \underline{d}_2$, by $\phi_{\underline{d}_3}(F_2) = 0$ we know $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^2 w_{\beta',1}$.

- $(\beta, \beta') = ([1], [1,2,2,2])$. Besides the factor $G_{[1],[1,2]}$, let $\underline{d}_3 = \{d_{[1,2,1,2,2]} = 1, d_{\alpha \neq [1,2,1,2,2]} = 0\}$, then $\underline{d}_3 < \underline{d}_2$, by $\phi_{\underline{d}_3}(F_2) = 0$ we know $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^2 w_{\beta',1}$. Let $\underline{d}_4 = \{d_{[1,2]} = 1, d_{[1,2,2]} = 1, d_{\alpha \neq [1,2],[1,2,2]} = 0\}$, then $\underline{d}_4 < \underline{d}_2$, by $\phi_{\underline{d}_4}(F_2) = 0$ we know $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^4 w_{\beta',1}$.

- $(\beta, \beta') = ([1], [2])$. Let $\underline{d}_3 = \{d_{[1,2]} = 1, d_{\alpha \neq [1,2]} = 0\}$, then $\underline{d}_3 < \underline{d}_2$, by $\phi_{\underline{d}_3}(F_2) = 0$ we know $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = w_{\beta',1}$.

- $(\beta, \beta') = ([1,2], [1,2,1,2,2])$. The factors in $G_{[1,2],[1,2,1,2,2]}$ all appear in $G_{[1,2,1,2,2]}$, and they appear due to similar wheel conditions.

- $(\beta, \beta') = ([1,2], [1,2,2])$. Besides the factors in $G_{[1,2]}$, let $\underline{d}_3 = \{d_{[1,2,1,2,2]} = 1, d_{\alpha \neq [1,2,1,2,2]} = 0\}$, then $\underline{d}_3 < \underline{d}_2$, by $\phi_{\underline{d}_3}(F_2) = 0$ we know $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^{-2} w_{\beta',1}$.

- $(\beta, \beta') = ([1,2], [1,2,2,2])$. Besides the factors in $G_{[1,2]}$, let $\underline{d}_3 = \{d_{[1,2,1,2,2]} = 1, d_{[2]} = 1, d_{\alpha \neq [1,2,1,2,2],[2]} = 0\}$, $\underline{d}_4 = \{d_{[1,2,2]} = 2, d_{\alpha \neq [1,2,2]} = 0\}$, then $\underline{d}_3, \underline{d}_4 < \underline{d}_2$. By $\phi_{\underline{d}_3}(F_2) = 0$ we know $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^{-2} w_{\beta',1}$, and by $\phi_{\underline{d}_4}(F_2) = 0$ we know $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^2 w_{\beta',1}$, thus we get the vanishing factor $G_{[1,2],[1,2,2,2]}$.

- $(\beta, \beta') = ([1,2], [2])$. Let $\underline{d}_3 = \{d_{[1,2,2]} = 1, d_{\alpha \neq [1,2,2]} = 0\}$, then $\underline{d}_3 < \underline{d}_2$, by $\phi_{\underline{d}_3}(F_2) = 0$ we know $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^{-2} w_{\beta',1}$.

- $(\beta, \beta') = ([1, 2, 1, 2, 2], [1, 2, 2])$. The factors in $G_{[1,2,1,2,2],[1,2,2]}$ all appear in $G_{[1,2,1,2,2]}$, and they appear due to similar wheel conditions.
- $(\beta, \beta') = ([1, 2, 1, 2, 2], [1, 2, 2, 2])$. Besides the factors appearing in $G_{[1,2,1,2,2]}$, let $\underline{d}_3 = \{d_{[1,2,2]} = 3, d_{\alpha \neq [1,2,2]} = 0\}$, then $\underline{d}_3 < \underline{d}_2$, by $\phi_{\underline{d}_3}(F_2) = 0$ we know $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^2 w_{\beta',1}$.
- $(\beta, \beta') = ([1, 2, 1, 2, 2], [2])$. Besides the factors appearing in $G_{[1,2,1,2,2]}$, let $\underline{d}_3 = \{d_{[1,2,2]} = 2, d_{\alpha \neq [1,2,2]} = 0\}$, then $\underline{d}_3 < \underline{d}_2$, by $\phi_{\underline{d}_3}(F_2) = 0$ we know $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^{-2} w_{\beta',1}$.
- $(\beta, \beta') = ([1, 2, 2], [1, 2, 2, 2])$. The factors in $G_{[1,2,2],[1,2,2,2]}$ all appear in $G_{[1,2,2,2]}$, and they appear due to similar wheel conditions.
- $(\beta, \beta') = ([1, 2, 2], [2])$. Let $\underline{d}_3 = \{d_{[1,2,2,2]} = 1, d_{\alpha \neq [1,2,2,2]} = 0\}$, then $\underline{d}_3 < \underline{d}_2$, by $\phi_{\underline{d}_3}(F_2) = 0$ we know $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^{-4} w_{\beta',1}$.
- $(\beta, \beta') = ([1, 2, 2], [2])$. The wheel condition $F_2 = 0$ once $x_{2,1}^{(\beta',1)} = v^2 x_{2,3}^{(\beta,1)} = v^4 x_{2,2}^{(\beta,1)} = v^6 x_{2,1}^{(\beta,1)} = v^3 x_{1,1}^{(\beta,1)}$ becomes $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^{-6} w_{\beta',1}$.

This completes our proof. ∎

Combining Propositions 3.4–3.6, we immediately obtain the shuffle algebra realization and the PBWD theorem for $U_v^>(L\mathfrak{g}_2)$:

Theorem 3.7. (a) $\Psi : U_v^>(L\mathfrak{g}_2) \xrightarrow{\sim} S$ of (2.16) is a $\mathbb{Q}(v)$-algebra isomorphism.

(b) For any choices of s_k and λ_k in the definition (2.24, 2.25) of quantum root vectors $E_{\beta,s}$, the ordered PBWD monomials $\{E_h\}_{h \in H}$ from (2.34) form a $\mathbb{Q}(v)$-basis of $U_v^>(L\mathfrak{g}_2)$.

3.2 Integral form $\mathbf{U}_v^>(L\mathfrak{g}_2)$ and its shuffle algebra realization

Let us consider the *divided powers*

$$\mathbf{E}_{i,r}^{(k)} := \frac{e_{i,r}^k}{[k]_{v_i}!} \quad \forall\, 1 \leq i \leq 2,\ r \in \mathbb{Z},\ k \in \mathbb{N}. \tag{3.38}$$

Following [16, §7.7], we define the integral form $\mathbf{U}_v^>(L\mathfrak{g}_2)$ as the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $U_v^>(L\mathfrak{g}_2)$ generated by $\{\mathbf{E}_{i,r}^{(k)}\}_{1 \leq i \leq 2, r \in \mathbb{Z}}^{k \in \mathbb{N}}$. For any $(\beta, s) \in \Delta^+ \times \mathbb{Z}$, we define the *normalized divided powers* of the quantum root vectors from (2.28)–(2.32) via:

$$\tilde{\mathbf{E}}_{\beta,s}^{\pm,(k)} := \begin{cases} \frac{(\tilde{E}_{\beta,s}^\pm)^k}{[k]_{v_\beta}!} & \text{if } \beta = [1], [2], [1, 2] \\[2mm] \frac{(\tilde{E}_{\beta,s}^\pm)^k}{([2]_v!)^k [k]_{v_\beta}!} & \text{if } \beta = [1, 2, 2] \\[2mm] \frac{(\tilde{E}_{\beta,s}^\pm)^k}{([3]_v!)^k [k]_{v_\beta}!} & \text{if } \beta = [1, 2, 2, 2], [1, 2, 1, 2, 2] \end{cases}. \tag{3.39}$$

Similarly to [31, Proposition 1.2], we have:

Proposition 3.8. For any $\beta \in \Delta^+$, $s \in \mathbb{Z}$, $k \in \mathbb{N}$, we have $\tilde{\mathbf{E}}_{\beta,s}^{\pm,(k)} \in \mathbf{U}_v^>(L\mathfrak{g}_2)$.

Proof. Let $U_v^>(\mathfrak{g}_2)$ be the "positive subalgebra" of the Drinfeld-Jimbo quantum group of $\mathfrak{g}_2$. Thus, $U_v^>(\mathfrak{g}_2)$ is the $\mathbb{Q}(v)$-algebra generated by $\{E_1, E_2\}$ subject to the v-Serre relations:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{v_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0, \quad i \neq j. \tag{3.40}$$

Let $\mathbf{U}_v^>(\mathfrak{g}_2)$ be the *Lusztig integral form* defined as the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $U_v^>(\mathfrak{g}_2)$ generated by the divided powers

$$E_i^{(k)} := \frac{E_i^k}{[k]_{v_i}!} \quad \forall\, 1 \leq i \leq 2,\ k \in \mathbb{N}.$$

Recall our specific convex order (2.22) on Δ^+. Let $\{\hat{E}_\beta^-\}_{\beta \in \Delta^+}$ denote Lusztig's quantum root vectors of $U_v^>(\mathfrak{g}_2)$ associated to this convex order (defined through the use of Lusztig's braid group action; see [23, §37.1.3]). According to [22, Theorem 6.6], we have:

$$\hat{\mathbf{E}}_\beta^{-,(k)} := \frac{(\hat{E}_\beta^-)^k}{[k]_{v_\beta}!} \in \mathbf{U}_v^>(\mathfrak{g}_2) \quad \forall\, \beta \in \Delta^+,\ k \in \mathbb{N}. \tag{3.41}$$

On the other hand, let us define another set of quantum root vectors $\{\tilde{E}^-_\beta\}_{\beta \in \Delta^+}$ in $U^>_v(\mathfrak{g}_2)$ using v-commutators similar to (2.28)–(2.32):

$$\tilde{E}^-_{[i]} := E_i, \ 1 \le i \le 2; \quad \tilde{E}^-_{[1,2]} := [E_1, E_2]_{v^{-3}}; \quad \tilde{E}^-_{[1,2,2]} := [[E_1, E_2]_{v^{-3}}, E_2]_{v^{-1}};$$

$$\tilde{E}^-_{[1,2,2,2]} := [[[E_1, E_2]_{v^{-3}}, E_2]_{v^{-1}}, E_2]_v; \quad \tilde{E}^-_{[1,2,1,2,2]} := [[E_1, E_2]_{v^{-3}}, [[E_1, E_2]_{v^{-3}}, E_2]_{v^{-1}}]_v.$$

(3.42)

Due to [21, Proposition 5.5.2], the quantum root vectors $\tilde{E}^-_\beta$ and $\hat{E}^-_\beta$ differ only by a scalar multiple for any $\beta \in \Delta^+$. These scalars are determined explicitly for Drinfeld-Jimbo quantum groups of any simple Lie algebra $\mathfrak{g}$ in [2, Theorem 4.2] (note that the parameter q of [2] equals our v^{-1}). Specifically, in our case we have:

$$\tilde{E}^-_\beta = \begin{cases} \hat{E}^-_\beta & \text{if } \beta = [1], [2], [1,2] \\ [2]_v! \, \hat{E}^-_\beta & \text{if } \beta = [1,2,2] \\ [3]_v! \, \hat{E}^-_\beta & \text{if } \beta = [1,2,2,2], [1,2,1,2,2] \end{cases}.$$

(3.43)

Let us now pass from the finite to the loop setup. First, we note that comparing the coefficients of $z_1^{-s_1} \cdots z_{1-a_{ij}}^{-s_1} w^{-s_2}$ in (2.9) for any $s_1, s_2 \in \mathbb{Z}$, we obtain:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_{v_i} e_{i,s_1}^k e_{j,s_2} e_{i,s_1}^{1-a_{ij}-k} = 0, \quad i \ne j.$$

Thus, the assignment $E_1 \mapsto e_{1,s_1}, E_2 \mapsto e_{2,s_2}$ gives rise to an algebra homomorphism

$$\eta_{s_1,s_2} : U^>_v(\mathfrak{g}_2) \longrightarrow U^>_v(L\mathfrak{g}_2).$$

Clearly, we have $\eta_{s_1,s_2}(\mathbf{U}^>_v(\mathfrak{g}_2)) \subset \mathbf{U}^>_v(L\mathfrak{g}_2)$. Combining (3.41)–(3.43) with (3.39), we thus get:

$$\tilde{\mathbf{E}}^{-,(k)}_{\beta,s} = \eta_{s_1,s_2}(\hat{\mathbf{E}}^{-,(k)}_\beta) \in \mathbf{U}^>_v(L\mathfrak{g}_2).$$

To prove the other inclusions $\tilde{\mathbf{E}}^{+,(k)}_{\beta,s} \in \mathbf{U}^>_v(L\mathfrak{g}_2)$, let us consider the convex order on Δ^+ opposite to (2.22):

$$[2] < [1,2,2,2] < [1,2,2] < [1,2,1,2,2] < [1,2] < [1].$$

(3.44)

Let $\{\hat{E}^+_\beta\}_{\beta \in \Delta^+}$ denote the set of Lusztig's quantum root vectors associated to that convex order, and define another set of quantum root vectors $\{\tilde{E}^+_\beta\}_{\beta \in \Delta^+}$ via v-commutators:

$$\tilde{E}^+_{[i]} := E_i, \ 1 \le i \le 2; \quad \tilde{E}^+_{[1,2]} := [E_2, E_1]_{v^{-3}} \doteq [E_1, E_2]_{v^3};$$

$$\tilde{E}^+_{[1,2,2]} := [E_2, [E_2, E_1]_{v^{-3}}]_{v^{-1}} \doteq [[E_1, E_2]_{v^3}, E_2]_v;$$

$$\tilde{E}^+_{[1,2,2,2]} := [E_2, [E_2, [E_2, E_1]_{v^{-3}}]_{v^{-1}}]_v \doteq [[[E_1, E_2]_{v^3}, E_2]_v, E_2]_{v^{-1}};$$

$$\tilde{E}^+_{[1,2,1,2,2]} := [[E_2, [E_2, E_1]_{v^{-3}}]_{v^{-1}}, [E_2, E_1]_{v^{-3}}]_v \doteq [[E_1, E_2]_{v^3}, [[E_1, E_2]_{v^3}, E_2]_v]_{v^{-1}}.$$

(3.45)

Then, due to [22, Theorem 6.6] and [2, Theorem 4.2] the analogues of (3.41) and (3.43) with the superscript $-$ replaced by $+$ hold. Therefore, we likewise obtain:

$$\tilde{\mathbf{E}}^{+,(k)}_{\beta,s} \doteq \eta_{s_1,s_2}(\hat{\mathbf{E}}^{+,(k)}_\beta) \in \mathbf{U}^>_v(L\mathfrak{g}_2).$$

This completes our proof. ∎

For any $\underline{k} \in \mathbb{N}^2$, consider the $\mathbb{Z}[v, v^{-1}]$-submodule $\mathbf{S}_{\underline{k}}$ of $S_{\underline{k}}$ consisting of rational functions F satisfying the following two conditions:

(1) If f denotes the numerator of F from (2.12), then

$$f \in \mathbb{Z}[v, v^{-1}][\{x_{i,r}^{\pm 1}\}_{1 \le i \le 2}^{1 \le r \le k_i}]^{\mathfrak{S}_{\underline{k}}}.$$

(3.46)

(2) For any $\underline{d} \in \mathrm{KP}(\underline{k})$, the specialization $\phi_{\underline{d}}(F)$ is divisible by the product

$$\prod_{\beta \in \Delta^+} \tilde{c}_\beta^{d_\beta}, \tag{3.47}$$

where we define $\{\tilde{c}_\beta\}_{\beta \in \Delta^+}$ via $\{c_\beta\}_{\beta \in \Delta^+}$ of Lemma 3.2:

$$\tilde{c}_\beta = \begin{cases} c_\beta & \text{if } \beta = [1], [2], [1, 2] \\ \frac{c_\beta}{[2]_\upsilon!} & \text{if } \beta = [1, 2, 2] \\ \frac{c_\beta}{[3]_\upsilon!} & \text{if } \beta = [1, 2, 2, 2], [1, 2, 1, 2, 2] \end{cases}. \tag{3.48}$$

We define $\mathbf{S} := \bigoplus_{\underline{k} \in \mathbb{N}^2} \mathbf{S}_{\underline{k}}$. Then, we have:

Proposition 3.9. $\Psi(\mathbf{U}_\upsilon^>(L\mathfrak{g}_2)) \subset \mathbf{S}$.

The proof is based on the following simple "rank 1" computation from [31, Lemma 1.3]:

Lemma 3.10. For any $\ell \geq 1, r \in \mathbb{Z}$, and $1 \leq i \leq 2$, we have

$$\underbrace{x_{i,1}^r \star \cdots \star x_{i,1}^r}_{\ell \text{ times}} = \upsilon_i^{-\frac{\ell(\ell-1)}{2}} [\ell]_{\upsilon_i}! \cdot (x_{i,1} \cdots x_{i,\ell})^r. \tag{3.49}$$

Proof of Proposition 3.9. For any $m \in \mathbb{N}, 1 \leq i_1, \ldots, i_m \leq 2, r_1, \ldots, r_m \in \mathbb{Z}, \ell_1, \ldots, \ell_m \in \mathbb{N}$, let

$$F := \Psi\big(\mathbf{E}_{i_1,r_1}^{(\ell_1)} \cdots \mathbf{E}_{i_m,r_m}^{(\ell_m)}\big),$$

and f be the numerator of F from (2.12). If a variable $x_{*,*}^{(*,*)}$ is plugged into $\Psi(\mathbf{E}_{i_q,r_q}^{(\ell_q)})$, then we shall use the following notation:

$$o(x_{*,*}^{(*,*)}) = q. \tag{3.50}$$

Thus, $o(x_{i,t}^{(\beta,s)}) = q$ means that in the corresponding summand from the symmetrization of $\Psi(\mathbf{E}_{i_1,r_1}^{(\ell_1)}) \cdots \Psi(\mathbf{E}_{i_m,r_m}^{(\ell_m)}) \cdot$ (rational factor), this x-variable is placed as an argument of $\Psi(\mathbf{E}_{i_q,r_q}^{(\ell_q)})$.
According to Lemma 3.10:

$$\Psi(\mathbf{E}_{i_q,r_q}^{(\ell_q)}) = \upsilon_{i_q}^{-\frac{\ell_q(\ell_q-1)}{2}} (x_{i_q,1} \cdots x_{i_q,\ell_q})^{r_q} \qquad \forall\, 1 \leq q \leq m,$$

hence, the condition (3.46) holds. To verify the validity of the divisibility (3.47), it suffices to show that for any $\beta \in \Delta^+$ and $1 \leq s \leq d_\beta$, the total contribution of $\phi_{\underline{d}}$-specializations of the ζ-factors between the variables $\{x_{i,t}^{(\beta,s)}\}_{i \in \beta}^{1 \leq t \leq \upsilon_{\beta,i}}$ is a multiple of $\tilde{c}_\beta$. This is obvious for $\beta = [1], [2], [1, 2]$. Let us now treat the remaining three cases:

- $\beta = [1, 2, 2]$. In this case, the $\phi_{\underline{d}}$-specialization of the corresponding product of the ζ-factors vanishes unless

$$o(x_{1,1}^{(\beta,s)}) \geq o(x_{2,1}^{(\beta,s)}) \geq o(x_{2,2}^{(\beta,s)}).$$

On the other hand, $o(x_{i,t}^{(\beta,s)}) \neq o(x_{i',t'}^{(\beta,s)})$ for $i \neq i'$. Thus, we only need to deal with the case

$$o(x_{1,1}^{(\beta,s)}) > o(x_{2,1}^{(\beta,s)}) \geq o(x_{2,2}^{(\beta,s)}).$$

In this case, the product of the ζ-factors

$$\zeta_{2,1}(x_{2,1}^{(\beta,s)}/x_{1,1}^{(\beta,s)}) \cdot \zeta_{2,1}(x_{2,2}^{(\beta,s)}/x_{1,1}^{(\beta,s)})$$

has a numerator

$$(x_{2,1}^{(\beta,s)} - v^3 x_{1,1}^{(\beta,s)})(x_{2,2}^{(\beta,s)} - v^3 x_{1,1}^{(\beta,s)}),$$

which contributes precisely the factor $\langle 3 \rangle_v \langle 2 \rangle_v = \tilde{c}_{[1,2,2]}$ under $\phi_{\underline{d}}$.

- $\beta = [1,2,2,2]$. In this case, the $\phi_{\underline{d}}$-specialization of the corresponding product of the ζ-factors vanishes unless

$$o(x_{1,1}^{(\beta,s)}) > o(x_{2,1}^{(\beta,s)}) \geq o(x_{2,2}^{(\beta,s)}) \geq o(x_{2,3}^{(\beta,s)}).$$

In the latter case, the product of the ζ-factors

$$\zeta_{2,1}(x_{2,1}^{(\beta,s)}/x_{1,1}^{(\beta,s)}) \cdot \zeta_{2,1}(x_{2,2}^{(\beta,s)}/x_{1,1}^{(\beta,s)}) \cdot \zeta_{2,1}(x_{2,3}^{(\beta,s)}/x_{1,1}^{(\beta,s)})$$

has a numerator

$$(x_{2,1}^{(\beta,s)} - v^3 x_{1,1}^{(\beta,s)})(x_{2,2}^{(\beta,s)} - v^3 x_{1,1}^{(\beta,s)})(x_{2,3}^{(\beta,s)} - v^3 x_{1,1}^{(\beta,s)}),$$

which contributes precisely the factor $\langle 3 \rangle_v \langle 2 \rangle_v \langle 1 \rangle_v = \tilde{c}_{[1,2,2,2]}$ under $\phi_{\underline{d}}$.

- $\beta = [1,2,1,2,2]$. In this case, the $\phi_{\underline{d}}$-specialization of the corresponding product of the ζ-factors vanishes unless

$$o(x_{1,1}^{(\beta,s)}) > o(x_{2,1}^{(\beta,s)}) \geq o(x_{2,2}^{(\beta,s)}) \geq o(x_{2,3}^{(\beta,s)}) \quad \text{and} \quad o(x_{1,2}^{(\beta,s)}) > o(x_{2,2}^{(\beta,s)}), \tag{3.51}$$

If $o(x_{1,2}^{(\beta,s)}) > o(x_{2,1}^{(\beta,s)})$, then the product $\prod_{1 \leq k \leq 2}^{1 \leq l \leq 3} \zeta_{2,1}(x_{2,l}^{(\beta,s)}/x_{1,k}^{(\beta,s)})$ has a numerator

$$\prod_{1 \leq k \leq 2}^{1 \leq l \leq 3} (x_{2,l}^{(\beta,s)} - v^3 x_{1,k}^{(\beta,s)}),$$

which contributes precisely the factor $\langle 4 \rangle_v \langle 3 \rangle_v^2 \langle 2 \rangle_v^2 \langle 1 \rangle_v = \tilde{c}_{[1,2,1,2,2]}$ under $\phi_{\underline{d}}$. On the other hand, if $o(x_{1,2}^{(\beta,s)}) < o(x_{2,1}^{(\beta,s)})$, then we have

$$o(x_{1,1}^{(\beta,s)}) > o(x_{2,1}^{(\beta,s)}) > o(x_{1,2}^{(\beta,s)}) > o(x_{2,2}^{(\beta,s)}) \geq o(x_{2,3}^{(\beta,s)}),$$

in which case the $\phi_{\underline{d}}$-specialization of the ζ-factors between the $x_{*,*}^{(\beta,s)}$-variables contributes a total of $[3]_v \cdot \tilde{c}_{[1,2,1,2,2]}$.

This completes the verification of the divisibility (3.47), thus concluding the proof. ∎

For any $h \in H$, define the ordered monomials (cf. (2.34))

$$\tilde{\mathbf{E}}_h^+ = \prod_{(\beta,s) \in \Delta^+ \times \mathbb{Z}}^{\longrightarrow} \tilde{\mathbf{E}}_{\beta,s}^{+,(h(\beta,s))}, \qquad \tilde{\mathbf{E}}_h^- = \prod_{(\beta,s) \in \Delta^+ \times \mathbb{Z}}^{\longrightarrow} \tilde{\mathbf{E}}_{\beta,s}^{-,(h(\beta,s))}. \tag{3.52}$$

According to Proposition 3.9, we have $\tilde{\mathbf{E}}_h^\pm \in \mathbf{S}$. For any $\epsilon \in \{\pm\}$, let $\mathbf{S}_{\underline{k}}^\epsilon$ be the $\mathbb{Z}[v,v^{-1}]$-submodule of $\mathbf{S}_{\underline{k}}$ spanned by $\{\Psi(\tilde{\mathbf{E}}_h^\epsilon)\}_{h \in H_{\underline{k}}}$. Then, the following analogue of Lemma 2.8 holds:

Proposition 3.11. For any $F \in \mathbf{S}_{\underline{k}}$ and $\underline{d} \in KP(\underline{k})$, if $\phi_{\underline{d}'}(F) = 0$ for all $\underline{d}' \in KP(\underline{k})$ such that $\underline{d}' < \underline{d}$, then there exists $F_{\underline{d}} \in \mathbf{S}_{\underline{k}}^\epsilon$ such that $\phi_{\underline{d}}(F) = \phi_{\underline{d}}(F_{\underline{d}})$ and $\phi_{\underline{d}'}(F_{\underline{d}}) = 0$ for all $\underline{d}' < \underline{d}$.

Proof. Combining (3.39), (3.48), and Proposition 3.4, we obtain:

$$\phi_{\underline{d}}(\Psi(\tilde{\mathbf{E}}_h^\epsilon)) \doteq \prod_{\beta,\beta' \in \Delta^+}^{\beta < \beta'} G_{\beta,\beta'} \cdot \prod_{\beta \in \Delta^+} (\tilde{c}_\beta^{d_\beta} \cdot G_\beta) \cdot \prod_{\beta \in \Delta^+} \mathbf{P}_{\lambda_{h,\beta}}, \tag{3.53}$$

where

$$\mathbf{P}_{\lambda_{h,\beta}} := \frac{1}{\prod_{r\in\mathbb{Z}}[h(\beta,r)]_{v_\beta}!} \cdot Sym_{\mathfrak{S}_{d_\beta}}\left(w_{\beta,1}^{r_\beta(h,1)}\cdots w_{\beta,d_\beta}^{r_\beta(h,d_\beta)}\prod_{1\le i<j\le d_\beta}\frac{w_{\beta,i}-v_\beta^{-2}w_{\beta,j}}{w_{\beta,i}-w_{\beta,j}}\right),\tag{3.54}$$

cf. (2.38). As $\{\mathbf{P}_{\lambda_{h,\beta}}\}_{h\in H_{\underline{k},\underline{d}}}$ form a $\mathbb{Z}[v,v^{-1}]$-basis of $\mathbb{Z}[v,v^{-1}][\{w_{\beta,s}^{\pm 1}\}_{s=1}^{d_\beta}]^{\mathfrak{S}_{d_\beta}}$ (this is a "rank 1" computation, cf. [31, Proposition 1.4, Lemma 1.14]), the claimed result follows from Proposition 3.6 and conditions (3.46, 3.47). ∎

Combining Propositions 3.9 and 3.11, we obtain the following upgrade of Theorem 3.7:

Theorem 3.12. (a) The $\mathbb{Q}(v)$-algebra isomorphism $\Psi: U_v^>(L\mathfrak{g}_2) \xrightarrow{\sim} S$ of Theorem 3.7(a) gives rise to a $\mathbb{Z}[v,v^{-1}]$-algebra isomorphism $\Psi: \mathbf{U}_v^>(L\mathfrak{g}_2) \xrightarrow{\sim} \mathbf{S}$.

(b) For any choices of s_1,s_2 in (2.29)–(2.32) and $\epsilon\in\{\pm\}$, the ordered monomials $\{\tilde{\mathbf{E}}_h^\epsilon\}_{h\in H}$ of (3.52) form a basis of the free $\mathbb{Z}[v,v^{-1}]$-module $\mathbf{U}_v^>(L\mathfrak{g}_2)$.

4 Specialization Maps for Type B_n

In this section, we define specialization maps for the shuffle algebras of type B_n and verify their key properties. This implies the shuffle algebra realizations and PBWD-type theorems for $U_v^>(L\mathfrak{o}_{2n+1})$, as well as for its two integral forms $\mathcal{U}_v^>(L\mathfrak{o}_{2n+1})$ and $\mathbf{U}_v^>(L\mathfrak{o}_{2n+1})$. Using arguments similar to those from Section 3 we establish the counterparts of Lemmas 2.6, 2.7, 2.8 for B_2 case, and then use induction to treat the general B_n case.

4.1 $U_v^>(L\mathfrak{o}_{2n+1})$ and its shuffle algebra realization

In type B_n, for any $F\in S_{\underline{k}}$ with $\underline{k}\in\mathbb{N}^n$, the wheel conditions are:

$$\begin{aligned}
F(\{x_{i,r}\}_{1\le i\le n}^{1\le r\le k_i}) = 0 \quad &\text{once} \quad x_{i,1}=v^4x_{i,2}=v^2x_{i+1,1} \quad \text{for some} \quad 1\le i\le n-1,\\
&\text{or} \quad x_{i,1}=v^4x_{i,2}=v^2x_{i-1,1} \quad \text{for some} \quad 2\le i\le n-1,\\
&\text{or} \quad x_{n,1}=v^2x_{n,2}=v^4x_{n,3}=v^2x_{n-1,1}.
\end{aligned}\tag{4.1}$$

For any $\underline{k}\in\mathbb{N}^n$ and $\underline{d}\in\mathrm{KP}(\underline{k})$, the specialization map $\phi_{\underline{d}}$ as in (2.43) is defined by the following specialization of the $x_{*,*}^{(*,*)}$-variables (replacing (2.41) for type A_n):

$$x_{i,1}^{(\beta,s)}\mapsto v^{-2i}w_{\beta,s},\quad x_{i,2}^{(\beta,s)}\mapsto v^{-4n+2i+2}w_{\beta,s}\qquad\forall\,\beta\in\Delta^+,\,1\le s\le d_\beta,\,i\in\beta.\tag{4.2}$$

Lemma 4.1. Consider the particular choices (2.26)–(2.27) of quantum root vectors $\{\tilde{E}_{\beta,s}^\pm\}_{\beta\in\Delta^+}^{s\in\mathbb{Z}}$. Their images under Ψ in the shuffle algebra S of type B_n are as follows:

- If $\beta=[i,j]$, then for any decomposition $s=s_i+\cdots+s_j$ used in (2.26), we have:

$$\Psi(\tilde{E}_{[i,j],s}^+) \doteq \frac{\langle 2\rangle_v^{|\beta|-1}\cdot x_{i,1}^{s_i+1}\cdots x_{j-1,1}^{s_{j-1}+1}x_{j,1}^{s_j}}{\prod_{\ell=i}^{j-1}(x_{\ell,1}-x_{\ell+1,1})},\tag{4.3}$$

$$\Psi(\tilde{E}_{[i,j],s}^-) \doteq \frac{\langle 2\rangle_v^{|\beta|-1}\cdot x_{i,1}^{s_i}x_{i+1,1}^{s_{i+1}+1}\cdots x_{j,1}^{s_j+1}}{\prod_{\ell=i}^{j-1}(x_{\ell,1}-x_{\ell+1,1})}.\tag{4.4}$$

- If $\beta=[i,n,j]$, then for any decomposition $s=s_i+\cdots+s_{j-1}+2s_j+\cdots+2s_n$ used in (2.27), we have:

$$\Psi(\tilde{E}_{[i,n,j],s}^+) \doteq \frac{\langle 2\rangle_v^{|\beta|-1}\cdot g_1\cdot\prod_{\ell=j}^{n-1}(v^4x_{\ell,1}-x_{\ell,2})(v^4x_{\ell,2}-x_{\ell,1})}{(x_{i,1}-x_{i+1,1})\cdots(x_{j-1,1}-x_{j,1})(x_{j-1,1}-x_{j,2})\prod_{\ell=j}^{n-1}\prod_{1\le r,t\le 2}(x_{\ell,r}-x_{\ell+1,t})},\tag{4.5}$$

$$\Psi(\tilde{E}_{[i,n,j],s}^-) \doteq \frac{\langle 2\rangle_v^{|\beta|-1}\cdot g_2\cdot\prod_{\ell=j}^{n-1}(v^4x_{\ell,1}-x_{\ell,2})(v^4x_{\ell,2}-x_{\ell,1})}{(x_{i,1}-x_{i+1,1})\cdots(x_{j-1,1}-x_{j,1})(x_{j-1,1}-x_{j,2})\prod_{\ell=j}^{n-1}\prod_{1\le r,t\le 2}(x_{\ell,r}-x_{\ell+1,t})},\tag{4.6}$$

where

$$g_1 = \prod_{\ell=i}^{j-2} x_{\ell,1}^{s_\ell+1} \cdot x_{j-1,1}^{s_{j-1}+2} (x_{j,1}x_{j,2})^{s_j} \cdot \prod_{\ell=j+1}^{n} (x_{\ell,1}x_{\ell,2})^{s_\ell+1},$$

(4.7)

$$g_2 = x_{i,1}^{s_i} \prod_{\ell=i+1}^{j-1} x_{\ell,1}^{s_\ell+1} \prod_{\ell=j}^{n} (x_{\ell,1}x_{\ell,2})^{s_\ell+1}.$$

Proof. Straightforward computation. ∎

For more general quantum root vectors $\{E_{\beta,s}\}_{\beta\in\Delta^+}^{s\in\mathbb{Z}}$ defined by (2.24), we have the following counterpart of Lemma 3.2:

Lemma 4.2. For any choices of s_k and λ_k in (2.24), we have:

$$\phi_\beta(\Psi(E_{\beta,s})) \doteq c_\beta \cdot w_{\beta,1}^{s+\kappa_\beta} \qquad \forall\, (\beta,s) \in \Delta^+ \times \mathbb{Z},$$

(4.8)

where $\{\kappa_\beta\}_{\beta\in\Delta^+}$ are explicitly given by

$$\kappa_\beta = \begin{cases} |\beta| - 1 & \text{if } \beta = [i,j] \\ |\beta| + 2(n-j) - 1 & \text{if } \beta = [i,n,j] \end{cases}$$

(4.9)

and the constants $\{c_\beta\}_{\beta\in\Delta^+}$ are explicitly given by

$$c_\beta = \begin{cases} \langle 2 \rangle_v^{|\beta|-1} & \text{if } \beta = [i,j] \\ \langle 2 \rangle_v^{|\beta|-1} \cdot \prod_{\ell=j}^{n-1} \{ (v^{-4n+4\ell-2} - 1)(v^{-4n+4\ell+6} - 1) \} & \text{if } \beta = [i,n,j] \end{cases}.$$

(4.10)

Proof. The proof is by induction on n, the base case $n = 2$ being obvious. Let us now assume that the result holds for types B_m ($m < n$). In what follows, we shall use the following special case of Proposition 4.5: for any positive roots $\alpha_1 < \alpha_2$ such that $\alpha_1 + \alpha_2$ is a root, we have

$$\phi_{\alpha_1+\alpha_2}(\Psi(E_{\alpha_1,s_1}) \star \Psi(E_{\alpha_2,s_2})) = 0 \qquad \forall\, s_1, s_2 \in \mathbb{Z}.$$

If $\beta = [i,j]$ with $1 \le i \le j \le n$, then (4.8) follows from the A_n type; see [30, Lemma 3.14]. If $\beta = [i,n,j]$ with $1 < i < j \le n$, then (4.8) likewise follows from the B_{n-1} case. Therefore, it remains to treat the cases $\beta = [1,n,j]$ with $2 \le j \le n$.

If $\beta = [1,n,n]$, then

$$E_{\beta,s} = [[[\cdots [[e_{1,s_1}, e_{2,s_2}]_{\lambda_1}, e_{3,s_3}]_{\lambda_2}, \cdots, e_{n-1,s_{n-1}}]_{\lambda_{n-2}}, e_{n,s_n}]_{\lambda_{n-1}}, e_{n,s_{n+1}}]_{\lambda_n}$$

with $s = s_1 + \cdots + s_{n+1}$. Consider $\alpha = [1,n]$, $r = s_1 + \cdots + s_{n-1} + s_n$, and

$$E_{\alpha,r} = [[\cdots [[e_{1,s_1}, e_{2,s_2}]_{\lambda_1}, e_{3,s_3}]_{\lambda_2}, \cdots, e_{n-1,s_{n-1}}]_{\lambda_{n-2}}, e_{n,s_n}]_{\lambda_{n-1}}.$$

Then, we have:

$$\begin{aligned}
\phi_\beta(\Psi(E_{\beta,s})) &= \phi_\beta(\Psi(E_{\alpha,r}) \star \Psi(e_{n,s_{n+1}}) - \lambda_n \Psi(e_{n,s_{n+1}}) \star \Psi(E_{\alpha,r})) \\
&\doteq \phi_\beta(\Psi(e_{n,s_{n+1}}) \star \Psi(E_{\alpha,r})) \\
&\doteq \phi_\beta\left(\mathrm{Sym}_{\mathfrak{S}_2}\left(\Psi(E_{\alpha,r})(x_{n,2}^{(\beta,1)})^{s_{n+1}} \zeta_{n,n-1}(x_{n,2}^{(\beta,1)}/x_{n-1,1}^{(\beta,1)}) \zeta_{n,n}(x_{n,2}^{(\beta,1)}/x_{n,1}^{(\beta,1)}) \right) \right) \\
&\doteq \phi_\beta\left(\Psi(E_{\alpha,r})(x_{n,2}^{(\beta,1)})^{s_{n+1}} \frac{(x_{n,2}^{(\beta,1)} - v^2 x_{n-1,1}^{(\beta,1)})(x_{n,2}^{(\beta,1)} - v^{-2} x_{n,1}^{(\beta,1)})}{(x_{n,2}^{(\beta,1)} - x_{n-1,1}^{(\beta,1)})(x_{n,1}^{(\beta,1)} - x_{n,2}^{(\beta,1)})} \right) \\
&\doteq \langle 2 \rangle_v^{|\beta|-1} \cdot w_{\beta,1}^{s+|\beta|-1},
\end{aligned}$$

(4.11)

where we plug the variables $\{x_{i,1}^{(\beta,1)}\}_{1\le i\le n}$ into $\Psi(E_{\alpha,r})$, the variable $x_{n,2}^{(\beta,1)}$ into $\Psi(e_{n,s_{n+1}})$, $Sym_{\mathfrak{S}_2}$ denotes the symmetrization with respect to the variables $\{x_{n,1}^{(\beta,1)}, x_{n,2}^{(\beta,1)}\}$, and the last line follows by applying the validity of (4.8) for (α, r) established above. We also note that the second equality in (4.11) used the vanishing $\phi_\beta(\Psi(E_{\alpha,r}) \star \Psi(e_{n,s_{n+1}})) = 0$, due to

$$\phi_\beta(\zeta_{n,n}(x_{n,1}^{(\beta,1)}/x_{n,2}^{(\beta,1)})) = 0 = \phi_\beta(\zeta_{n-1,n}(x_{n-1,1}^{(\beta,1)}/x_{n,1}^{(\beta,1)})). \tag{4.12}$$

If $\beta = [1, n, j]$ with $2 \le j \le n - 1$, then

$$E_{\beta,s} = [[\cdots[[\cdots[e_{1,s_1}, e_{2,s_2}]_{\lambda_1}, \cdots, e_{n,s_n}]_{\lambda_{n-1}}, e_{n,s_{n+1}}]_{\lambda_n}, \cdots, e_{j+1,s_{2n-j}}]_{\lambda_{2n-j-1}}, e_{j,s_{2n-j+1}}]_{\lambda_{2n-j}}$$

with $s = s_1 + \cdots + s_{2n-j+1}$. Consider $\alpha = [1, n, j+1]$, $r = s_1 + \cdots + s_{2n-j}$, and

$$E_{\alpha,r} = [\cdots[[\cdots[e_{1,s_1}, e_{2,s_2}]_{\lambda_1}, \cdots, e_{n,s_n}]_{\lambda_{n-1}}, e_{n,s_{n+1}}]_{\lambda_n}, \cdots, e_{j+1,s_{2n-j}}]_{\lambda_{2n-j-1}}.$$

Note that $\phi_\beta(\zeta_{j+1,j}(x_{j+1,2}^{(\beta,1)}/x_{j,2}^{(\beta,1)})) = 0 = \phi_\beta(\zeta_{j,j+1}(x_{j,1}^{(\beta,1)}/x_{j+1,1}^{(\beta,1)}))$, cf. (4.12). Thus, we have:

$$\phi_\beta(\Psi(E_{\beta,s})) = \phi_\beta(\Psi(E_{\alpha,r}) \star \Psi(e_{j,s_{2n-j+1}}) - \lambda_{2n-j}\Psi(e_{j,s_{2n-j+1}}) \star \Psi(E_{\alpha,r}))$$

$$\doteq \phi_\beta(\Psi(e_{j,s_{2n-j+1}}) \star \Psi(E_{\alpha,r}))$$

$$\doteq \phi_\beta\Big(Sym_{\mathfrak{S}_2}\Big(\Psi(E_{\alpha,r})(x_{j,2}^{(\beta,1)})^{s_{2n-j+1}}\zeta_{j,j}(x_{j,2}^{(\beta,1)}/x_{j,1}^{(\beta,1)})\times$$

$$\zeta_{j,j-1}(x_{j,2}^{(\beta,1)}/x_{j-1,1}^{(\beta,1)})\zeta_{j,j+1}(x_{j,2}^{(\beta,1)}/x_{j+1,1}^{(\beta,1)})\zeta_{j,j+1}(x_{j,2}^{(\beta,1)}/x_{j+1,2}^{(\beta,1)})\Big)\Big)$$

$$\doteq \phi_\beta\left(\frac{\Psi(E_{\alpha,r})(x_{j,2}^{(\beta,1)})^{s_{2n-j+1}}(x_{j,2}^{(\beta,1)} - \upsilon^{-4}x_{j,1}^{(\beta,1)})}{x_{j,1}^{(\beta,1)} - x_{j,2}^{(\beta,1)}}\times\right. \tag{4.13}$$

$$\left.\frac{(x_{j,2}^{(\beta,1)} - \upsilon^2 x_{j-1,1}^{(\beta,1)})(x_{j,2}^{(\beta,1)} - \upsilon^2 x_{j+1,1}^{(\beta,1)})(x_{j,2}^{(\beta,1)} - \upsilon^2 x_{j+1,2}^{(\beta,1)})}{(x_{j,2}^{(\beta,1)} - x_{j-1,1}^{(\beta,1)})(x_{j,2}^{(\beta,1)} - x_{j+1,1}^{(\beta,1)})(x_{j,2}^{(\beta,1)} - x_{j+1,2}^{(\beta,1)})}\right)$$

$$\doteq \langle 2\rangle_\upsilon^{|\beta|-1} \cdot \prod_{\ell=j}^{n-1}\{(\upsilon^{-4n+4\ell-2} - 1)(\upsilon^{-4n+4\ell+6} - 1)\} \cdot w_{\beta,1}^{s+|\beta|+2(n-j)-1},$$

where we plug the variables $\{x_{i,t}^{(\beta,1)}, x_{j,1}^{(\beta,1)}\}_{1\le i\le n, i\ne j}^{1\le t\le v_{\beta,i}}$ into $\Psi(E_{\alpha,r})$, the variable $x_{j,2}^{(\beta,1)}$ into $\Psi(e_{j,s_{2n-j+1}})$, $Sym_{\mathfrak{S}_2}$ denotes the symmetrization with respect to the variables $\{x_{j,1}^{(\beta,1)}, x_{j,2}^{(\beta,1)}\}$, and the last line follows by applying the induction hypothesis for $\phi_\alpha(\Psi(E_{\alpha,r}))$. ∎

Let us now generalize the above lemma by computing $\phi_{\underline{d}}(\Psi(E_h))$ for any $h \in H_{k,\underline{d}}$. Similarly to type G_2, we choose a special splitting such that the variables in $\Psi(E_{\beta,r_\beta(h,s)})$ are taken to be the group $\{x_{i,t}^{(\beta,s)}\}_{1\le i\le n}^{1\le t\le v_{\beta,i}}$, and under $\phi_{\underline{d}}$ they are specialized as in (4.2). For each $1 \le i \le n$, we define the set X_i as in (3.22), and the total order on X_i as in (3.23).

For any $\underline{d} \in KP(\underline{k})$, we define the subset $Sh_{\underline{d}} \subset \mathfrak{S}_{\underline{k}}$ of "$\underline{d}$-shuffle permutations" as in (3.25):

$$Sh_{\underline{d}} = \left\{\sigma \in \mathfrak{S}_{\underline{k}} \,\middle|\, \sigma(x_{i,1}^{(\beta,s)}) < \sigma(x_{i,2}^{(\beta,s)}) \quad \forall \beta \in \Delta^+, 1 \le s \le d_\beta, 2 \le i \le n\right\}. \tag{4.14}$$

Then from (2.14), we have:

$$\Psi(E_h) \doteq \sum_{\sigma \in Sh_{\underline{d}}} \sigma\left(F_h(\{x_{*,*}^{(*,*)}\})\right) = \sum_{\sigma \in Sh_{\underline{d}}} F_h\left(\{\sigma(x_{*,*}^{(*,*)})\}\right), \tag{4.15}$$

where

$$F_h := \prod_{\substack{\beta \in \Delta^+ \\ 1 \le s \le d_\beta}} \Psi(E_{\beta,r_\beta(h,s)}) \prod_{\substack{\alpha,\beta \in \Delta^+ \\ 1 \le p \le d_\alpha, 1 \le q \le d_\beta}}^{(\alpha,p)<(\beta,q)} \prod_{1\le i,j\le n}^{a_{ij}\ne 0} \prod_{1\le \ell\le v_{\alpha,i}}^{1\le r\le v_{\beta,j}} \frac{x_{i,\ell}^{(\alpha,p)} - \upsilon_i^{-a_{ij}}x_{j,r}^{(\beta,q)}}{x_{i,\ell}^{(\alpha,p)} - x_{j,r}^{(\beta,q)}}. \tag{4.16}$$

Let us consider the elements of $\mathfrak{S}_{\underline{k}}$ satisfying (3.28), which form a subgroup isomorphic to $\mathfrak{S}_{\underline{d}}$ (we shall denote this subgroup by $\mathfrak{S}_{\underline{d}}$). Then, similarly to Lemma 3.3, we have:

Lemma 4.3. $\phi_{\underline{d}}(\sigma(F_h)) = 0$ for $\sigma \notin \mathfrak{S}_{\underline{d}}$.

Proof. The proof is by induction on n. Arguing alike in the proof of Lemma 3.3, the result is clear for B_2-type. Let $\Delta_1^+ = \{[1,j]\}_{1 \le j \le n} \cup \{[1,n,j]\}_{1 < j \le n}$. It suffices to show that $\phi_{\underline{d}}(\sigma(F_h)) \ne 0$ only if (3.28) holds for every $\beta \in \Delta_1^+$, as for $\beta > [1,n,2]$ we can apply the induction hypothesis for B_{n-1}-type. We shall now prove that (3.28) holds for any $x_{*,*}^{(\beta,s_1)}$, assuming it holds for any $x_{*,*}^{(\beta',s')}$ with $(\beta',s') < (\beta,s_1)$. Similar to (3.35), we define the following sets of $x_{*,*}^{(*,*)}$-variables:

$$Z_i^{>(\beta,s)} = \left\{ x_{i,*}^{(\alpha,r)} \,\middle|\, (\alpha,r) > (\beta,s) \right\} \qquad \forall\, (\beta,s) \in \Delta^+ \times \mathbb{N}, 1 \le i \le n. \tag{4.17}$$

- Case 1: $\beta = [1,j]$ with $1 \le j \le n-1$. Let $\sigma(x_{1,1}^{(\beta,s_1)}) = x_{1,1}^{(\gamma,r)}$ with $\gamma \ge \beta$. Since F_h contains the factor $\zeta_{1,2}(x_{1,1}^{(\beta,s_1)}/Z_2^{>(\beta,s_1)})$, we have $\phi_{\underline{d}}(\sigma(F_h)) = 0$ unless $\sigma(x_{2,1}^{(\beta,s_1)}) = x_{2,1}^{(\gamma,r)}$. Proceeding as this, we get $\phi_{\underline{d}}(\sigma(F_h)) = 0$ unless $\sigma(x_{\ell,1}^{(\beta,s_1)}) = x_{\ell,1}^{(\gamma,r)}$ for any $\ell \le j$. If $\gamma > \beta$, then we get $\phi_{\underline{d}}(\sigma(F_h)) = 0$, since F_h contains the factor $\zeta_{j,j+1}(x_{j,1}^{(\beta,s_1)}/Z_{j+1}^{>(\beta,s_1)})$ and $x_{j+1,1}^{(\gamma,r)} \in \sigma(Z_{j+1}^{>(\beta,s_1)})$, hence, a contradiction. Thus, $\gamma = \beta$, and (3.28) holds for any $x_{*,1}^{(\beta,s_1)}$.

- Case 2: $\beta = [1,n]$. Let $\sigma(x_{1,1}^{(\beta,s_1)}) = x_{1,1}^{(\gamma,r)}$ with $\gamma \ge \beta$. Then, similar to above, we get $\phi_{\underline{d}}(\sigma(F_h)) = 0$ unless $\sigma(x_{\ell,1}^{(\beta,s_1)}) = x_{\ell,1}^{(\gamma,r)}$ for any $\ell \le n$. If $\gamma > \beta$, then we get $\phi_{\underline{d}}(\sigma(F_h)) = 0$, since F_h contains the factor $\zeta_{n,n}(x_{n,1}^{(\beta,s_1)}/Z_n^{>(\beta,s_1)})$ and $x_{n,2}^{(\gamma,r)} \in \sigma(Z_n^{>(\beta,s_1)})$, a contradiction. Thus, $\gamma = \beta$, and (3.28) holds for any $x_{*,*}^{(\beta,s_1)}$.

- Case 3: $\beta = [1,n,j]$ with $2 \le j \le n$. Let $\sigma(x_{1,1}^{(\beta,s_1)}) = x_{1,1}^{(\gamma,r)}$ with $\gamma \ge \beta$. Then, $\phi_{\underline{d}}(\sigma(F_h)) = 0$ unless there exist $\{t_{\ell,1}, t_{\ell,2}\}_{\ell=j}^n$ such that

$$\begin{aligned} \sigma(x_{\ell,1}^{(\beta,s_1)}) &= x_{\ell,1}^{(\gamma,r)} && \text{if } 1 \le \ell \le j-1, \\ \sigma(x_{\ell,t_{\ell,1}}^{(\beta,s_1)}) &= x_{\ell,1}^{(\gamma,r)}, \quad \sigma(x_{\ell,t_{\ell,2}}^{(\beta,s_1)}) = x_{\ell,2}^{(\gamma,r)} && \text{if } j \le \ell \le n. \end{aligned} \tag{4.18}$$

If $t_{\ell,2} < t_{\ell,1}$ for some $\ell \in \{j, j+1, \dots, n\}$, then $x_{\ell,t_{\ell,2}}^{(\beta,s_1)} < x_{\ell,t_{\ell,1}}^{(\beta,s_1)}$, and the condition $\sigma \in \mathrm{Sh}_{\underline{d}}$ implies

$$x_{\ell,1}^{(\gamma,r)} = \sigma(x_{\ell,t_{\ell,1}}^{(\beta,s_1)}) < \sigma(x_{\ell,t_{\ell,2}}^{(\beta,s_1)}) = x_{\ell,2}^{(\gamma,r)},$$

a contradiction. Hence, $t_{\ell,1} < t_{\ell,2}$ for any $j \le \ell \le n$, so that $t_{\ell,1} = 1, t_{\ell,2} = 2$ for any $j \le \ell \le n$. If $\gamma > \beta$, then we get $\phi_{\underline{d}}(\sigma(F_h)) = 0$, since F_h contains the factor $\zeta_{j,j-1}(x_{j,2}^{(\beta,s_1)}/Z_{j-1}^{>(\beta,s_1)})$ and $x_{j-1,2}^{(\gamma,r)} \in \sigma(Z_{j-1}^{>(\beta,s_1)})$, a contradiction. Thus, $\gamma = \beta$, and (3.28) holds for any $x_{*,*}^{(\beta,s_1)}$.

This completes our proof. ∎

Combining Lemmas 4.2 and 4.3, we obtain the following analogue of Lemma 2.6 for type B_n:

Proposition 4.4. For any $h \in H_{\underline{k},\underline{d}}$, we have

$$\phi_{\underline{d}}(\Psi(E_h)) \doteq \prod_{\beta,\beta' \in \Delta^+}^{\beta < \beta'} G_{\beta,\beta'} \cdot \prod_{\beta \in \Delta^+} (c_\beta^{d_\beta} \cdot G_\beta) \cdot \prod_{\beta \in \Delta^+} P_{\lambda_{h,\beta}}, \tag{4.19}$$

where the factors $\{P_{\lambda_{h,\beta}}\}_{\beta \in \Delta^+}$ are given by (2.47), the constants $\{c_\beta\}_{\beta \in \Delta^+}$ are as in Lemma 4.2, and the terms $G_{\beta,\beta'}, G_\beta$ are products of linear factors $w_{\beta,s}$ and $w_{\beta,s} - v^{\mathbb{Z}} w_{\beta',s'}$ which are independent of $h \in H_{\underline{k},\underline{d}}$ and are $\mathfrak{S}_{\underline{d}}$-symmetric (the factors G_β are explicitly computed in (4.36)).

Proposition 4.5. Lemma 2.7 is valid for type B_n and specialization maps $\phi_{\underline{d}}$ as in (4.2).

Proof. The proof is by induction on n with the base case of B_2-type being clear, cf. the proof of Lemma 4.3. Given $\underline{d}, \underline{d}' \in \mathrm{KP}(\underline{k})$ with $\underline{d}' < \underline{d}$, let $\beta \in \Delta^+$ be the smallest root such that $d'_\beta < d_\beta$. We can assume that $\beta \in \Delta_1^+$, as otherwise the induction hypothesis for B_{n-1} type will apply. Similarly to the proof of Proposition 3.5, without loss of generality, we can assume $d'_\alpha = 0$ for all $\alpha \le \beta$. Let $x_{*,*}^{(*,*)}$ be the above special splitting of the variables for $\phi_{\underline{d}}$, and $x'^{(*,*)}_{*,*}$ be any splitting of the variables for $\phi_{\underline{d}'}$. Then, we have the following case-by-case analysis:

- Case 1: $\beta = [1,j]$ with $1 \le j \le n-1$. Since $0 = d'_\beta < d_\beta$, we have $\sigma(x_{1,1}^{(\beta,1)}) = x_{1,1}'^{(\gamma,r)}$ for some $\gamma > \beta$. Since F_h contains the ζ-factor $\zeta_{1,2}(x_{1,1}^{(\beta,1)}/x_{2,t}^{(\alpha,s)})$ for any $(\alpha,s) > (\beta,1)$ and $1 \le t \le 2$, we have $\phi_{\underline{d}'}(\sigma(F_h)) = 0$ unless $\sigma(x_{2,1}^{(\beta,1)}) = x_{2,1}'^{(\gamma,r)}$. Proceeding as this, we get $\phi_{\underline{d}'}(\sigma(F_h)) = 0$ unless $\sigma(x_{i,1}^{(\beta,1)}) = x_{i,1}'^{(\gamma,r)}$ for any $1 \le i \le j$. In the latter case, we still obtain $\phi_{\underline{d}'}(\sigma(F_h)) = 0$, since F_h contains the factor $\zeta_{j,j+1}(x_{j,1}^{(\beta,1)}/x_{j+1,1}'^{(\gamma,r)})$.

- Case 2: $\beta = [1,n]$. Let $\sigma(x_{1,1}^{(\beta,1)}) = x_{1,1}'^{(\gamma,r)}$ for some $\gamma > \beta$. Arguing as above, we have $\phi_{\underline{d}'}(\sigma(F_h)) = 0$ unless $\sigma(x_{i,1}^{(\beta,1)}) = x_{i,1}'^{(\gamma,r)}$ for any $1 \le i \le n$. If (β',s') is such that $\sigma(x_{n,t}^{(\beta,1)}) = x_{n,2}'^{(\gamma,r)}$, then we have $(\beta',s') > (\beta,1)$. Therefore, we still get $\phi_{\underline{d}'}(\sigma(F_h)) = 0$, since F_h contains the factor $\zeta_{n,n}(x_{n,1}^{(\beta,1)}/x_{n,t}^{(\beta',s')})$.

- Case 3: $\beta = [1,n,j]$ with $2 < j \le n$. Similar to above, $\phi_{\underline{d}'}(\sigma(F_h)) = 0$ unless there is $\gamma > \beta$ such that $\sigma(x_{i,1}^{(\beta,1)}) = x_{i,1}^{(\gamma,r)}$ for $1 \le i \le n$, $\sigma(x_{i,2}^{(\beta,1)}) = x_{i,2}^{(\gamma,r)}$ for $j \le i \le n$. In the latter case, we again obtain $\phi_{\underline{d}'}(\sigma(F_h)) = 0$, since F_h contains the factor $\zeta_{j,j-1}(x_{j,2}^{(\beta,1)}/x_{j-1,2}'^{(\gamma,r)})$.

- Case 4: $\beta = [1,n,2]$. This case is impossible as $0 = d'_\alpha = d_\alpha$ for $\alpha < \beta$ and $0 = d'_\beta < d_\beta$ (indeed, $k_1 = \sum_{\alpha \le \beta} d'_\alpha < \sum_{\alpha \le \beta} d_\alpha = k_1$ as $\underline{d}, \underline{d}' \in \mathrm{KP}(\underline{k})$, a contradiction).

This completes our proof. ∎

Proposition 4.6. Lemma 2.8 is valid for type B_n and specialization maps $\phi_{\underline{d}}$ as in (4.2).

Proof. Similarly to the proof of Proposition 3.6, we only need to prove the lemma to be true for any pair of roots $(\beta \le \beta')$. First let $\underline{d}_1 = \{d_\beta = 2, d_\alpha = 0, \forall \alpha \ne \beta\} \in \mathrm{KP}(\underline{k}_1)$ and $F_1 \in S_{\underline{k}_1}$. We can assume $\beta = [1,n,j]$ for some $2 \le j \le n$. If $j > 2$, let $\gamma = [2,n,j]$, and $\underline{d}_3 = \{d_\gamma = 2, d_{[1]} = 2, d_\alpha = 0, \forall \alpha \ne \gamma, [1]\}$. By induction we know $\phi_{\underline{d}_3}(F_1)$ has the factor G_γ (here we change the variable $w_{\gamma,s}$ in G_γ by $w_{\beta,s}$). Moreover, we have

$$G_\beta = (w_{\beta,1} - v^{-4}w_{\beta,2})(w_{\beta,1} - v^4 w_{\beta,2}) \cdot G_\gamma, \tag{4.20}$$

The wheel condition $F_1 = 0$ once $x_{1,1}^{(\beta,1)} = v^4 x_{1,1}^{(\beta,2)} = v^2 x_{2,1}^{(\beta,1)}$ or $x_{1,1}^{(\beta,2)} = v^4 x_{1,1}^{(\beta,1)} = v^2 x_{2,1}^{(\beta,1)}$ becomes $\phi_{\underline{d}_1}(F_1) = 0$ once $w_{\beta,1} = v^4 w_{\beta,2}$ or $w_{\beta,1} = v^{-4} w_{\beta,2}$, thus giving us the required vanishing factors. If $j = 2$, let $\gamma = [1,n,3]$ (here we assume $n > 3$), then

$$G_\beta = (w_{\beta,1} - v^{\pm(4n-6)}w_{\beta,2})(w_{\beta,1} - v^{\pm(4n-14)}w_{\beta,2})(w_{\beta,1} - v^{\pm 4}w_{\beta,2}) \cdot G_\gamma. \tag{4.21}$$

The wheel condition $F_1 = 0$ once $x_{2,2}^{(\beta,1)} = v^4 x_{2,1}^{(\beta,2)} = v^2 x_{1,1}^{(\beta,2)}$, or $x_{2,1}^{(\beta,1)} = v^4 x_{2,2}^{(\beta,2)} = v^2 x_{3,1}^{(\beta,1)}$, or $x_{2,2}^{(\beta,1)} = v^4 x_{2,2}^{(\beta,2)} = v^2 x_{3,2}^{(\beta,2)}$ give us the vanishing factors above. Thus, the lemma is true for pairs (β, β).

Now for any pair $(\beta < \beta')$, let $\underline{d}_2 = \{d_\beta = d_{\beta'} = 1, d_{\alpha \ne \beta, \beta'} = 0\} \in \mathrm{KP}(\underline{k}_2)$ and $F_2 \in S_{\underline{k}_2}$, we prove $\phi_{\underline{d}_2}(F_2)$ has the vanishing factor $G_{\beta,\beta'}$ if $\phi_{\underline{d}}(F_2) = 0$ for any $\underline{d} < \underline{d}_2$. By induction we know it is true for any $(\beta < \beta')$ with $[2] \le \beta$. And by results on type A_n, we know it is true for any $(\beta < \beta')$ with $\beta = [1,j], \beta' = [i',j']$ for some $1 \le i', j, j' \le n$.

- $\beta = [1,j]$ for some $1 \le j \le n$, $\beta' = [i,n,n]$ for some $1 \le i \le n-1$.
  Let $\gamma = [i,n]$. If $j < n-1$ then $G_{\beta,\beta'} = G_{\beta,\gamma}$ and $\phi_{\underline{d}_2}(F_2)$ has the factor $G_{\beta,\beta'}$ for the same reason. If $j = n-1$, then

$$G_{\beta,\beta'} = (w_{\beta,1} - v^2 w_{\beta',1}) \cdot G_{\beta,\gamma}, \tag{4.22}$$

and by induction we know $\phi_{\underline{d}_2}(F_2)$ has the factor $G_{\beta,\gamma}$. Set $\underline{d}_3 = \{d_{[1,n]} = 2, d_{\alpha \ne [1,n]} = 0\}$ if $i = 1$, or $\underline{d}_3 = \{d_{[1,n,n]} = 1, d_{[i,n-1]} = 1, d_{\alpha \ne [1,n,n],[i,n-1]} = 0\}$ if $i \ne 1$. Then for each case, we have $\underline{d}_3 < \underline{d}_2$ and $\phi_{\underline{d}_3}(F_2) = 0$ implies that $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^2 w_{\beta',1}$. If $j = n$ and $i = 1$, then

$$G_{\beta,\beta'} = (w_{\beta,1} - v^{-2}w_{\beta',1}) \cdot G_{\beta,\beta}, \tag{4.23}$$

where $G_{\beta,\beta}$ is obtained by changing the variable $w_{\beta,2}$ in G_β by $w_{\beta',1}$. By induction we know $\phi_{\underline{d}_2}(F_2)$ has the factor $G_{\beta,\beta}$. Also the wheel condition $F_2 = 0$ once $x_{n,2}^{(\beta',1)} = v^2 x_{n,1}^{(\beta',1)} = v^4 x_{n,1}^{(\beta,1)} = v^2 x_{n-1,1}^{(\beta,1)}$ becomes $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^{-2} w_{\beta',1}$. If $j = n$ and $i > 1$, then

$$G_{\beta,\beta'} = (w_{\beta,1} - w_{\beta',1}) \cdot G_{\beta,\gamma}. \tag{4.24}$$

Let $\underline{d}_3 = \{d_{[1,n,n]} = 1, d_{[i,n]} = 1, d_{\alpha \ne [1,n,n],[i,n]} = 0\}$, then $\underline{d}_3 < \underline{d}_2$ and $\phi_{\underline{d}_3}(F_2) = 0$ implies that $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = w_{\beta',1}$.

- $\beta = [1,j]$ for some $1 \le j \le n$, $\beta' = [i,n,\ell]$ for some $1 \le i < \ell \le n$.
  We already prove $\ell = n$ case, so we can use induction. If $j \le \ell - 2$, then $G_{\beta,\beta'} = G_{\beta,[i,\ell-1]}$ and $\phi_{\underline{d}_2}(F_2)$ has the factor $G_{\beta,\beta'}$ for the same reason. If $j \ge \ell + 1$, then

$$G_{\beta,\beta'} = (w_{\beta,1} - v^{(4\ell-4n-2)}w_{\beta',1})(w_{\beta,1} - v^{(4\ell-4n+6)}w_{\beta',1}) \cdot G_{\beta,[i,n,\ell+1]}. \tag{4.25}$$

By induction we know $\phi_{\underline{d}_2}(F_2)$ has the factor $G_{\beta,[i,n,\ell+1]}$, and the wheel conditions $F_2 = 0$ once $x_{\ell,2}^{(\beta',1)} = v^4 x_{\ell,1}^{(\beta,1)} = v^2 x_{\ell-1,1}^{(\beta,1)}$ or $x_{\ell,1}^{(\beta,1)} = v^4 x_{\ell,2}^{(\beta',1)} = v^2 x_{\ell+1,1}^{(\beta,1)}$ give the additional vanishing factors. If $j = \ell - 1$, then

$$G_{\beta,\beta'} = (w_{\beta,1} - v^{(4\ell-4n+2)}w_{\beta',1}) \cdot G_{\beta,[i,n,\ell+1]}. \tag{4.26}$$

By induction we know $\phi_{\underline{d}_2}(F_2)$ has the factor $G_{\beta,[i,n,\ell+1]}$. Let $\underline{d}_3 = \{d_{[1,\ell]} = d_{[i,n,\ell+1]} = 1, d_{\alpha \ne [1,\ell],[i,n,\ell+1]} = 0\}$, then $\underline{d}_3 < \underline{d}_2$ and $\phi_{\underline{d}_3}(F_2) = 0$ implies that $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^{(4\ell-4n+2)}w_{\beta',1}$. If $j = \ell$, then

$$G_{\beta,\beta'} = (w_{\beta,1} - v^{(4\ell-4n-2)}w_{\beta',1}) \cdot G_{\beta,[i,n,\ell+1]}, \tag{4.27}$$

and the wheel condition $F_2 = 0$ once $x_{\ell,2}^{(\beta',1)} = v^4 x_{\ell,1}^{(\beta,1)} = v^2 x_{\ell-1,1}^{(\beta,1)}$ gives the additional vanishing factor.
- $\beta = [1,n,j]$ for some $2 \le j \le n$ and $\beta' = [i,\ell]$ for some $2 \le i \le \ell \le n$.
  We only need to prove $\beta = [1,n,n]$ case, other cases can be proved by induction as above. If $\ell \le n-2$, then $G_{\beta,\beta'} = G_{[1,n-1],\beta'}$ and the vanishing factors appear for the same wheel conditions. If $\ell = n-1$, then

$$G_{\beta,\beta'} = (w_{\beta,1} - v^2 w_{\beta',1}) \cdot G_{[1,n],\beta'}. \tag{4.28}$$

Let $\underline{d}_3 = \{d_{[1,n,n-1]} = d_{[i,n-2]} = 1, d_{\alpha \ne [1,n,n-1],[i,n-2]} = 0\}$, then $\underline{d}_3 < \underline{d}_2$ and $\phi_{\underline{d}_3}(F_2) = 0$ implies that $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^2 w_{\beta',1}$. If $\ell = n$, then

$$G_{\beta,\beta'} = (w_{\beta,1} - v^{-4}w_{\beta',1}) \cdot G_{\beta,[i,n-1]}, \tag{4.29}$$

and the wheel condition $F_2 = 0$ once $x_{n,1}^{(\beta',1)} = v^2 x_{n,2}^{(\beta,1)} = v^4 x_{n,1}^{(\beta,1)} = v^2 x_{n-1,1}^{(\beta,1)}$ becomes $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = v^{-4}w_{\beta',1}$.
- $\beta = [1,n,j]$ for some $2 \le j \le n$, $\beta' = [i,n,\ell]$ for some $1 \le i < \ell \le n$.
  If $j \ge 3$ and $i > 2$, then $G_{\beta,\beta'} = G_{[2,n,j],\beta'}$ and they appear for the same reason.
  Let us consider now $j \ge 3$ and $i = 2$. Then if $\ell < j$, we have

$$G_{\beta,\beta'} = (w_{\beta,1} - w_{\beta',1}) \cdot G_{[2,n,j],\beta'}, \tag{4.30}$$

Let $\underline{d}_3 = \{d_{[1,n,\ell]} = d_{[2,n,j]} = 1, d_{\alpha \ne [1,n,\ell],[2,n,j]} = 0\}$, then $\underline{d}_3 < \underline{d}_2$ and $\phi_{\underline{d}_3}(F_2) = 0$ implies that $\phi_{\underline{d}_2}(F_2) = 0$ once $w_{\beta,1} = w_{\beta',1}$. If $\ell \ge j+2$, then

$$G_{\beta,\beta'} = (w_{\beta,1} - v^{4n-4j+2}w_{\beta',1})(w_{\beta,1} - v^{4n-4j-6}w_{\beta',1}) \cdot G_{[1,n,j+1],\beta'}, \tag{4.31}$$

and the wheel condition $x_{j,2}^{(\beta,1)} = v^4 x_{j,1}^{(\beta',1)} = v^2 x_{j-1,1}^{(\beta',1)}$ or $x_{j,1}^{(\beta',1)} = v^4 x_{j,2}^{(\beta,1)} = v^2 x_{j+1,1}^{(\beta,1)}$ gives the vanishing factors. If $\ell = j+1$, then

$$G_{\beta,\beta'} = (w_{\beta,1} - v^{4n-4j+2}w_{\beta',1})(w_{\beta,1} - v^{4n-4j-6}w_{\beta',1})(w_{\beta,1} - v^4 w_{\beta',1}) \cdot G_{[1,n,j+1],\beta'}, \tag{4.32}$$

then the wheel condition for (4.31) and $x_{j+1,2}^{(\beta,1)} = v^4 x_{j+1,2}^{(\beta',1)} = v^2 x_{j,2}^{(\beta,1)}$ give the vanishing factors. If $\ell = j$, then

$$G_{\beta,\beta'} = (w_{\beta,1} - v^{-4}w_{\beta',1}) \cdot G_{\beta',\beta'}, \tag{4.33}$$

and the wheel condition $x_{2,1}^{(\beta',1)} = v^4 x_{2,1}^{(\beta,1)} = v^2 x_{1,1}^{(\beta,1)}$ give the vanishing factor.

If $j \geq 3$ and $i = 1$, then $\ell < j$. If $\ell \geq 3$, we have

$$G_{\beta,\beta'} = (w_{\beta,1} - v^{-4} w_{\beta',1})(w_{\beta,1} - v^4 w_{\beta',1}) \cdot G_{[2,n,j],[2,n,\ell]}, \tag{4.34}$$

and the wheel conditions for (4.20) give the additional factors. If $\ell = 2$, then for $j > 3$, we can use induction. If $\ell = 2, j = 3$, then

$$G_{\beta,\beta'} = (w_{\beta,1} - v^{-4} w_{\beta',1})(w_{\beta,1} - v^{-4n+6} w_{\beta',1})(w_{\beta,1} - v^{-4n+14} w_{\beta',1}) \cdot G_{\beta,\beta}. \tag{4.35}$$

The wheel conditions for (4.25) and the wheel condition $x_{3,2}^{(\beta',1)} = v^4 x_{3,2}^{(\beta,1)} = v^2 x_{2,2}^{(\beta',1)}$ give the additional factors. Now let $\beta = [1, n, 2]$, then $i \geq 2$. If $\ell > 3$ then we can use induction. If $\ell = 3$, then $\beta' = [2, n, 3]$, and similar wheel conditions for (4.35) apply.

This completes our proof. ∎

Using formulas (4.20)–(4.21), we obtain the following explicit formulas for the factors G_β (which shall be used in Subsection 4.2):

Corollary 4.7. The factors $\{G_\beta\}_{\beta \in \Delta^+}$ featuring in (4.19) are explicitly given by:

$$G_\beta = \prod_{s=1}^{d_\beta} w_{\beta,s}^{j-i} \prod_{1 \leq s \neq s' \leq d_\beta} (w_{\beta,s} - v^4 w_{\beta,s'})^{j-i} \qquad \text{if } \beta = [i,j],$$

$$G_\beta = \prod_{s=1}^{d_\beta} w_{\beta,s}^{4n-i-3j+1} \prod_{1 \leq s \neq s' \leq d_\beta} \left\{ (w_{\beta,s} - v^4 w_{\beta,s'})^{2n-i-j} (w_{\beta,s} - v^2 w_{\beta,s'}) \right\} \times \tag{4.36}$$

$$\prod_{1 \leq s \neq s' \leq d_\beta} \prod_{\ell=j}^{n-1} \left\{ (w_{\beta,s} - v^{4n-4\ell+2} w_{\beta,s'})(w_{\beta,s} - v^{4n-4\ell-6} w_{\beta,s'}) \right\} \qquad \text{if } \beta = [i,n,j].$$

Combining Propositions 4.4–4.6, we immediately obtain the shuffle algebra realization and the PBWD theorem for $U_v^>(L\mathfrak{o}_{2n+1})$:

Theorem 4.8. (a) $\Psi : U_v^>(L\mathfrak{o}_{2n+1}) \xrightarrow{\sim} S$ of (2.16) is a $\mathbb{Q}(v)$-algebra isomorphism.

(b) For any choices of s_k and λ_k in the definition (2.24) of quantum root vectors $E_{\beta,s}$, the ordered PBWD monomials $\{E_h\}_{h \in H}$ from (2.34) form a $\mathbb{Q}(v)$-basis of $U_v^>(L\mathfrak{o}_{2n+1})$.

4.2 RTT integral form $\mathcal{U}_v^>(L\mathfrak{o}_{2n+1})$ and its shuffle algebra realization

For $\epsilon \in \{\pm\}$, we define

$$\tilde{\mathcal{E}}_{\beta,s}^\epsilon := \langle 2 \rangle_v \cdot \tilde{E}_{\beta,s}^\epsilon \qquad \forall (\beta, s) \in \Delta^+ \times \mathbb{Z}, \tag{4.37}$$

cf. (2.26)–(2.27). Similarly to (2.34), we also consider the ordered monomials

$$\tilde{\mathcal{E}}_h^\epsilon = \prod_{(\beta,s) \in \Delta^+ \times \mathbb{Z}}^{\rightarrow} (\tilde{\mathcal{E}}_{\beta,s}^\epsilon)^{h(\beta,s)} \qquad \forall h \in H, \tag{4.38}$$

with the arrow → over the product sign indicating the order (2.33) on $\Delta^+ \times \mathbb{Z}$.

Remark 4.9. The reason for the extra factor $\langle 2 \rangle_v$ in the definition (4.37) is explained in the Appendix A; see Corollary A.5 and Proposition A.6.

We define the RTT integral form $\mathcal{U}_v^>(L\mathfrak{o}_{2n+1})$ as the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $U_v^>(L\mathfrak{o}_{2n+1})$ generated by $\{\tilde{\mathcal{E}}_{\beta,s}^\epsilon\}_{\beta \in \Delta^+}^{s \in \mathbb{Z}}$. We note that the above definition depends on the choices of quantum root vectors in (2.26)–(2.27) as well as of $\epsilon \in \{\pm\}$. The main goal of the present subsection is to prove the following theorem by simultaneously establishing the shuffle algebra realization of $\mathcal{U}_v^>(L\mathfrak{o}_{2n+1})$, which is of independent interest:

Theorem 4.10. (a) $\mathcal{U}_v^>(L\mathfrak{o}_{2n+1})$ is independent of the choice of quantum root vectors $\{\tilde{\mathcal{E}}_{\beta,s}^\epsilon\}_{\beta\in\Delta^+}^{s\in\mathbb{Z}}$.

(b) For any choices of s_k in (2.26)–(2.27) and $\epsilon \in \{\pm\}$, the ordered monomials $\{\tilde{\mathcal{E}}_h^\epsilon\}_{h\in H}$ of (4.38) form a basis of the free $\mathbb{Z}[v, v^{-1}]$-module $\mathcal{U}_v^>(L\mathfrak{o}_{2n+1})$.

For any $\underline{k} \in \mathbb{N}^n$, consider the $\mathbb{Z}[v, v^{-1}]$-submodule $\tilde{S}_{\underline{k}}$ of $S_{\underline{k}}$ consisting of rational functions F satisfying the following two conditions:

(1) If f denotes the numerator of F from (2.12), then

$$f \in \langle 2 \rangle_v^{|\underline{k}|} \cdot \mathbb{Z}[v, v^{-1}][\{x_{i,r}^{\pm 1}\}_{1\leq i\leq n}^{1\leq r\leq k_i}]^{\mathfrak{S}_{\underline{k}}}, \tag{4.39}$$

where $|\underline{k}| = |(k_1, \ldots, k_n)| := k_1 + \cdots + k_n$.

(2) For any $\underline{d} \in KP(\underline{k})$, the specialization $\phi_{\underline{d}}(f \cdot \langle 2 \rangle_v^{-|\underline{k}|})$ is divisible by

$$\prod_{\beta=[i,n,j]\in\Delta^+} \prod_{\ell=j}^{n-1} \left\{ (v^{-4n+4\ell-2} - 1)^{d_\beta} (v^{-4n+4\ell+6} - 1)^{d_\beta} \right\}. \tag{4.40}$$

We define $\tilde{S} := \bigoplus_{\underline{k}\in\mathbb{N}^n} \tilde{S}_{\underline{k}}$. Then, we have:

Proposition 4.11. $\Psi(\mathcal{U}_v^>(L\mathfrak{o}_{2n+1})) \subset \tilde{S}$.

Proof. For any $\epsilon \in \{\pm\}$, $m \in \mathbb{N}$, $\beta_1, \ldots, \beta_m \in \Delta^+$, $r_1, \ldots, r_m \in \mathbb{Z}$, let

$$F := \Psi\left(\tilde{\mathcal{E}}_{\beta_1,r_1}^\epsilon \cdots \tilde{\mathcal{E}}_{\beta_m,r_m}^\epsilon\right),$$

and f be the numerator of F from (2.12). We set $\underline{k} = \sum_{q=1}^m \beta_q$. Similarly to (3.50), if a variable $x_{*,*}^{(*,*)}$ is plugged into $\Psi(\tilde{\mathcal{E}}_{\beta_q,r_q}^\epsilon)$ for some $1 \leq q \leq m$, then we shall use the notation

$$o(x_{*,*}^{(*,*)}) = q.$$

Due to Lemma 4.1, f is divisible by $\langle 2 \rangle_v^{|\underline{k}|}$, hence, the condition (4.39) holds. Now for any $\underline{d} \in KP(\underline{k})$, consider each summand from the symmetrization featuring in f. Pick any $\beta = [i, n, j]$ with $1 \leq i < j \leq n$ such that $d_\beta \neq 0$. For any $1 \leq s \leq d_\beta$ and $j \leq \ell \leq n$, it suffices to show that the contribution of the $\phi_{\underline{d}}$-specializations of ζ-factors between the variables

$$\{x_{\ell-1,1}^{(\beta,s)}, x_{\ell,1}^{(\beta,s)}, \ldots, x_{n,1}^{(\beta,s)}, x_{n,2}^{(\beta,s)}, \ldots, x_{\ell+1,2}^{(\beta,s)}, x_{\ell,2}^{(\beta,s)}\}$$

is divisible by $\prod_{t=\ell}^{n-1}\{(v^{-4n+4t-2} - 1)(v^{-4n+4t+6} - 1)\}$. The proof is by induction on ℓ, where the base step $\ell = n$ is vacuous.

We first note that this $\phi_{\underline{d}}$-specialization vanishes unless

$$o(x_{\ell-1,1}^{(\beta,s)}) \geq o(x_{\ell,1}^{(\beta,s)}) \geq \cdots \geq o(x_{n,1}^{(\beta,s)}) \geq o(x_{n,2}^{(\beta,s)}) \geq \cdots \geq o(x_{\ell+1,2}^{(\beta,s)}) \geq o(x_{\ell,2}^{(\beta,s)}).$$

If $o(x_{\ell,1}^{(\beta,s)}) = o(x_{\ell,2}^{(\beta,s)})$, then due to Lemma 4.1, this $\phi_{\underline{d}}$-specialization contains the required factor

$$\prod_{t=\ell}^{n-1}\{(v^{-4n+4t-2} - 1)(v^{-4n+4t+6} - 1)\}.$$

If $o(x_{\ell,1}^{(\beta,s)}) > o(x_{\ell,2}^{(\beta,s)})$, then we have the following two cases to consider:

- Case 1: $o(x_{\ell+1,1}^{(\beta,s)}) = o(x_{\ell,2}^{(\beta,s)})$. Then we have

$$o(x_{\ell+1,1}^{(\beta,s)}) = o(x_{\ell+2,1}^{(\beta,s)}) = \cdots = o(x_{\ell+1,2}^{(\beta,s)}) = o(x_{\ell,2}^{(\beta,s)}).$$

According to Lemma 4.1, the corresponding $\phi_{\underline{d}}$-specialization is divisible by

$$\prod_{t=\ell+1}^{n-1} \{(\upsilon^{-4n+4t-2} - 1)(\upsilon^{-4n+4t+6} - 1)\}.$$

On the other hand, the $\phi_{\underline{d}}$-specialization of the product of ζ-factors

$$\zeta_{\ell,\ell-1}(x_{\ell,2}^{(\beta,s)}/x_{\ell-1,1}^{(\beta,s)})\zeta_{\ell,\ell}(x_{\ell,2}^{(\beta,s)}/x_{\ell,1}^{(\beta,s)})\zeta_{\ell+1,\ell}(x_{\ell+1,2}^{(\beta,s)}/x_{\ell,1}^{(\beta,s)})$$

contributes the remaining required factor $(\upsilon^{-4n+4\ell-2} - 1)(\upsilon^{-4n+4\ell+6} - 1)$.

- Case 2: $o(x_{\ell+1,1}^{(\beta,s)}) > o(x_{\ell,2}^{(\beta,s)})$. Then the $\phi_{\underline{d}}$-specialization of the product of ζ-factors

$$\zeta_{\ell,\ell-1}(x_{\ell,2}^{(\beta,s)}/x_{\ell-1,1}^{(\beta,s)})\zeta_{\ell,\ell}(x_{\ell,2}^{(\beta,s)}/x_{\ell,1}^{(\beta,s)})\zeta_{\ell,\ell+1}(x_{\ell,2}^{(\beta,s)}/x_{\ell+1,1}^{(\beta,s)})$$

contributes the factor $(\upsilon^{-4n+4\ell-2} - 1)(\upsilon^{-4n+4\ell+6} - 1)$. Considering the contribution of the $\phi_{\underline{d}}$-specializations of ζ-factors between the variables

$$\{x_{\ell,1}^{(\beta,s)}, x_{\ell+1,1}^{(\beta,s)}, \ldots, x_{n,1}^{(\beta,s)}, x_{n,2}^{(\beta,s)}, \ldots, x_{\ell+2,2}^{(\beta,s)}, x_{\ell+1,2}^{(\beta,s)}\}$$

and using the induction hypothesis, we get the remaining required factors

$$\prod_{t=\ell+1}^{n-1} \{(\upsilon^{-4n+4t-2} - 1)(\upsilon^{-4n+4t+6} - 1)\}.$$

This completes our proof. ∎

We shall now introduce a certain refinement of $\tilde{\mathcal{S}}$ in order to describe the image $\Psi(\mathcal{U}_\upsilon^>(L\mathfrak{o}_{2n+1}))$. Pick any $F \in \tilde{\mathcal{S}}_{\underline{k}}$ and $\underline{d} \in \mathrm{KP}(\underline{k})$. First, according to (4.39) and (4.40), $\phi_{\underline{d}}(F)$ is divisible by

$$A_{\underline{d}} := \langle 2 \rangle_\upsilon^{|\underline{k}|} \cdot \prod_{\beta=[i,n,j]\in\Delta^+} \prod_{\ell=j}^{n-1} \{(\upsilon^{-4n+4\ell-2} - 1)^{d_\beta}(\upsilon^{-4n+4\ell+6} - 1)^{d_\beta}\}. \tag{4.41}$$

Second, following Corollary 4.7 (based solely on the wheel conditions), the specialization $\phi_{\underline{d}}(F)$ is also divisible by the product

$$B_{\underline{d}} := \prod_{\beta\in\Delta^+} G_\beta,$$

with G_β computed explicitly in (4.36). Combining these two observations, we can now define the following *reduced specialization map*

$$\xi_{\underline{d}} : \tilde{\mathcal{S}} \longrightarrow \mathbb{Z}[\upsilon, \upsilon^{-1}][\{w_{\beta,s}^{\pm 1}\}_{\beta\in\Delta^+}^{1\leq s\leq d_\beta}]^{\mathfrak{S}_{\underline{d}}} \quad \text{via} \quad \xi_{\underline{d}}(F) := \frac{\phi_{\underline{d}}(F)}{A_{\underline{d}}B_{\underline{d}}}. \tag{4.42}$$

Let us introduce another type of specialization maps. Pick any collection of positive integers $\underline{t} = \{t_{\beta,r}\}_{\beta\in\Delta^+}^{1\leq r\leq \ell_\beta}$ ($\ell_\beta \in \mathbb{N}$) satisfying

$$d_\beta = \sum_{r=1}^{\ell_\beta} t_{\beta,r} \qquad \forall \beta \in \Delta^+. \tag{4.43}$$

For any $\beta \in \Delta^+$, we split the variables $\{w_{\beta,s}\}_{s=1}^{d_\beta}$ into ℓ_β groups of size $t_{\beta,r}$ each ($1 \le r \le \ell_\beta$) and specialize the variables in the r-th group to

$$\upsilon_\beta^{-2} z_{\beta,r}, \ \upsilon_\beta^{-4} z_{\beta,r}, \ \ldots \ \upsilon_\beta^{-2t_{\beta,r}} z_{\beta,r}. \tag{4.44}$$

For any $g \in \mathbb{Z}[\upsilon, \upsilon^{-1}][\{w_{\beta,s}^{\pm 1}\}_{\beta \in \Delta^+}^{1 \le s \le d_\beta}]^{\mathfrak{S}_{\underline{d}}}$, we define $\varpi_{\underline{t}}(g)$ as the corresponding specialization of g. This gives rise to the *vertical specialization map*

$$\varpi_{\underline{t}} \colon \mathbb{Z}[\upsilon, \upsilon^{-1}][\{w_{\beta,s}^{\pm 1}\}_{\beta \in \Delta^+}^{1 \le s \le d_\beta}]^{\mathfrak{S}_{\underline{d}}} \longrightarrow \mathbb{Z}[\upsilon, \upsilon^{-1}][\{z_{\beta,r}^{\pm 1}\}_{\beta \in \Delta^+}^{1 \le r \le \ell_\beta}]. \tag{4.45}$$

Finally, given any $\underline{d} \in \mathrm{KP}(\underline{k})$ and a collection of positive integers $\underline{t} = \{t_{\beta,r}\}_{\beta \in \Delta^+}^{1 \le r \le \ell_\beta}$ satisfying (4.43), we combine (4.42) and (4.45) to define the **cross specialization map**

$$\Upsilon_{\underline{d},\underline{t}} \colon \tilde{\mathcal{S}} \longrightarrow \mathbb{Z}[\upsilon, \upsilon^{-1}][\{z_{\beta,r}^{\pm 1}\}_{\beta \in \Delta^+}^{1 \le r \le \ell_\beta}] \quad \text{via} \quad \Upsilon_{\underline{d},\underline{t}}(F) := \varpi_{\underline{t}}(\xi_{\underline{d}}(F)). \tag{4.46}$$

Similarly to [29, Definition 3.37], we introduce:

Definition 4.12. $F \in \tilde{\mathcal{S}}_{\underline{k}}$ is **integral** if $\Upsilon_{\underline{d},\underline{t}}(F)$ is divisible by $\prod_{\beta \in \Delta^+}^{1 \le r \le \ell_\beta} [t_{\beta,r}]_{\upsilon_\beta}!$ for any $\underline{d} \in \mathrm{KP}(\underline{k})$ and $\underline{t} = \{t_{\beta,r}\}_{\beta \in \Delta^+}^{1 \le r \le \ell_\beta}$ satisfying (4.43).

Let $\mathcal{S} \subset \tilde{\mathcal{S}}$ denote the $\mathbb{Z}[\upsilon, \upsilon^{-1}]$-submodule of all integral elements. Then, we have:

Proposition 4.13. $\Psi(\mathcal{U}_\upsilon^>(L\mathfrak{o}_{2n+1})) \subset \mathcal{S}$.

Proof. For any $\epsilon \in \{\pm\}$, $m \in \mathbb{N}$, $\beta_1, \ldots, \beta_m \in \Delta^+$, $r_1, \ldots, r_m \in \mathbb{Z}$, let

$$F := \Psi(\tilde{\mathcal{E}}_{\beta_1,r_1}^\epsilon \cdots \tilde{\mathcal{E}}_{\beta_m,r_m}^\epsilon).$$

For any $\beta \in \Delta^+$ and $1 \le r \le \ell_\beta$, we need to show that under $\Upsilon_{\underline{d},\underline{t}}$, the contribution of the ζ-factors between the variables $x_{*,*}^{(*,*)}$ that got specialized to $\upsilon^? z_{\beta,r}$ is divisible by $[t_{\beta,r}]_{\upsilon_\beta}!$. For $\beta = [i,j]$ with $1 \le i \le j \le n$, this follows from [30, Lemma 3.51] (note that $\upsilon_{[i,j]} = \upsilon_j$).

It remains to treat the case $\beta = [i,n,j]$ with $1 \le i < j \le n$. We note that $\upsilon_\beta = \upsilon^2$ for this β. Without loss of generality, we may assume that under the $\phi_{\underline{d}}$-specialization:

$$x_{i,1}^{(\beta,s)} \mapsto \upsilon^{-2i} w_{\beta,s}, \ \ldots, \ x_{n,1}^{(\beta,s)} \mapsto \upsilon^{-2n} w_{\beta,s}, \ x_{n,2}^{(\beta,s)} \mapsto \upsilon^{-2n+2} w_{\beta,s}, \ \ldots, \ x_{j,2}^{(\beta,s)} \mapsto \upsilon^{-4n+2j+2} w_{\beta,s}$$

for $1 \le s \le t_{\beta,r}$, while under the $\varpi_{\underline{t}}$-specialization:

$$w_{\beta,1} \mapsto \upsilon^{-4} z_{\beta,r}, \ \ldots, \ w_{\beta,t_{\beta,r}} \mapsto \upsilon^{-4t_{\beta,r}} z_{\beta,r}.$$

Fix any $1 \le s \ne s' \le t_{\beta,r}$. First, let us consider the relative position of the variables

$$\{x_{j+1,2}^{(\beta,s)}, x_{j,2}^{(\beta,s)}, x_{j+1,2}^{(\beta,s')}, x_{j,2}^{(\beta,s')}\}, \tag{4.47}$$

that is, compare their o-values with $o(x_{*,*}^{(*,*)})$ defined now as in the proof of Proposition 4.11. We can assume that

$$o(x_{j+1,2}^{(\beta,s)}) \ge o(x_{j,2}^{(\beta,s)}) \quad \text{and} \quad o(x_{j+1,2}^{(\beta,s')}) \ge o(x_{j,2}^{(\beta,s')}),$$

as otherwise the corresponding term is specialized to zero under $\phi_{\underline{d}}$. If $o(x_{j,2}^{(\beta,s)}) = o(x_{j,2}^{(\beta,s')})$ then, according to Lemma 4.1, the corresponding term is divisible by

$$(\upsilon^4 x_{j,2}^{(\beta,s)} - x_{j,2}^{(\beta,s')})(\upsilon^4 x_{j,2}^{(\beta,s')} - x_{j,2}^{(\beta,s)}),$$

whose $\phi_{\underline{d}}$-specialization contributes the factor $(w_{\beta,s} - v^4 w_{\beta,s'})(w_{\beta,s'} - v^4 w_{\beta,s})$. If $o(x_{j,2}^{(\beta,s)}) \neq o(x_{j,2}^{(\beta,s')})$, then without loss of generality, we can assume $o(x_{j,2}^{(\beta,s)}) > o(x_{j,2}^{(\beta,s')})$. Then:

- If $o(x_{j+1,2}^{(\beta,s')}) > o(x_{j,2}^{(\beta,s)})$, then the $\phi_{\underline{d}}$-specialization of the product of ζ-factors

$$\zeta_{j,j+1}(x_{j,2}^{(\beta,s)}/x_{j+1,2}^{(\beta,s')}) \cdot \zeta_{j,j+1}(x_{j,2}^{(\beta,s')}/x_{j+1,2}^{(\beta,s)})$$

  contributes the factor $(w_{\beta,s} - v^4 w_{\beta,s'})(w_{\beta,s'} - v^4 w_{\beta,s})$.

- If $o(x_{j,2}^{(\beta,s)}) \geq o(x_{j+1,2}^{(\beta,s')})$, then the $\phi_{\underline{d}}$-specialization of the product of ζ-factors

$$\zeta_{j,j}(x_{j,2}^{(\beta,s')}/x_{j,2}^{(\beta,s)}) \cdot \zeta_{j,j+1}(x_{j,2}^{(\beta,s')}/x_{j+1,2}^{(\beta,s)})$$

  contributes the factor $(w_{\beta,s} - v^4 w_{\beta,s'})(w_{\beta,s'} - v^4 w_{\beta,s})$.

Similarly, considering the $\phi_{\underline{d}}$-specialization of the ζ-factors arising from the following quadruples

$$\{x_{j+2,2}^{(\beta,s)}, x_{j+1,2}^{(\beta,s)}, x_{j+2,2}^{(\beta,s')}, x_{j+1,2}^{(\beta,s')}\}, \ldots, \{x_{n,2}^{(\beta,s)}, x_{n-1,2}^{(\beta,s)}, x_{n,2}^{(\beta,s')}, x_{n-1,2}^{(\beta,s')}\},$$

$$\{x_{n-2,1}^{(\beta,s)}, x_{n-1,1}^{(\beta,s)}, x_{n-2,1}^{(\beta,s')}, x_{n-1,1}^{(\beta,s')}\}, \ldots, \{x_{i,1}^{(\beta,s)}, x_{i+1,1}^{(\beta,s)}, x_{i,1}^{(\beta,s')}, x_{i+1,1}^{(\beta,s')}\},$$

along with the contribution of the tuple (4.47) considered above, they produce a total contribution of the factor

$$(w_{\beta,s} - v^4 w_{\beta,s'})^{2n-i-j-1}(w_{\beta,s'} - v^4 w_{\beta,s})^{2n-i-j-1}.$$

Second, let us consider the relative position of the variables

$$\{x_{n-1,1}^{(\beta,s)}, x_{n,1}^{(\beta,s)}, x_{n,2}^{(\beta,s)}, x_{n-1,1}^{(\beta,s')}, x_{n,1}^{(\beta,s')}, x_{n,2}^{(\beta,s')}\}. \tag{4.48}$$

We can assume that

$$o(x_{n-1,1}^{(\beta,s)}) \geq o(x_{n,1}^{(\beta,s)}) \geq o(x_{n,2}^{(\beta,s)}) \quad \text{and} \quad o(x_{n-1,1}^{(\beta,s')}) \geq o(x_{n,1}^{(\beta,s')}) \geq o(x_{n,2}^{(\beta,s')}), \tag{4.49}$$

as otherwise the corresponding term is specialized to zero under $\phi_{\underline{d}}$. Then:

- If $o(x_{n,2}^{(\beta,s)}) < o(x_{n,1}^{(\beta,s')})$ or $o(x_{n,1}^{(\beta,s')}) < o(x_{n-1,1}^{(\beta,s)})$, then the $\phi_{\underline{d}}$-specialization of ζ-factor

$$\zeta_{n,n}(x_{n,2}^{(\beta,s)}/x_{n,1}^{(\beta,s')}) \quad \text{or} \quad \zeta_{n,n-1}(x_{n,1}^{(\beta,s')}/x_{n-1,1}^{(\beta,s)})$$

  respectively, contributes the factor $w_{\beta,s} - v^{-4} w_{\beta,s'}$. Otherwise, $o(x_{n,2}^{(\beta,s)}) \geq o(x_{n,1}^{(\beta,s')}) \geq o(x_{n-1,1}^{(\beta,s)})$, which together with (4.49) implies:

$$o(x_{n,1}^{(\beta,s')}) = o(x_{n-1,1}^{(\beta,s)}) = o(x_{n,1}^{(\beta,s)}) = o(x_{n,2}^{(\beta,s)}).$$

  But there are at most two variables $x_{n,*}^{(*,*)}$ plugged into each $\Psi(\tilde{\mathcal{E}}_{\beta_q,r_q}^\epsilon)$, a contradiction.

- If $o(x_{n,2}^{(\beta,s)}) < o(x_{n,2}^{(\beta,s')})$, or $o(x_{n,1}^{(\beta,s)}) < o(x_{n,1}^{(\beta,s')})$, or $o(x_{n,2}^{(\beta,s')}) < o(x_{n-1,1}^{(\beta,s)})$, then the $\phi_{\underline{d}}$-specialization of ζ-factor

$$\zeta_{n,n}(x_{n,2}^{(\beta,s)}/x_{n,2}^{(\beta,s')}) \quad \text{or} \quad \zeta_{n,n}(x_{n,1}^{(\beta,s)}/x_{n,1}^{(\beta,s')}) \quad \text{or} \quad \zeta_{n,n-1}(x_{n,2}^{(\beta,s')}/x_{n-1,1}^{(\beta,s)})$$

  respectively, contributes the factor $w_{\beta,s} - v^{-2} w_{\beta,s'}$. Otherwise, if $o(x_{n,2}^{(\beta,s)}) \geq o(x_{n,2}^{(\beta,s')})$, $o(x_{n,1}^{(\beta,s)}) \geq o(x_{n,1}^{(\beta,s')})$, and $o(x_{n,2}^{(\beta,s')}) \geq o(x_{n-1,1}^{(\beta,s)})$, then according to (4.49) we have

$$o(x_{n-1,1}^{(\beta,s)}) = o(x_{n,1}^{(\beta,s)}) = o(x_{n,2}^{(\beta,s)}) = o(x_{n,2}^{(\beta,s')}).$$

The latter can not occur for the same reason as above, hence, a contradiction.

Swapping the roles of s and s' in the above two bullets, we thus conclude that the $\phi_{\underline{d}}$-specialization of the ζ-factors arising from (4.48) contributes the factor

$$(w_{\beta,s} - v^4 w_{\beta,s'})(w_{\beta,s'} - v^4 w_{\beta,s})(w_{\beta,s} - v^2 w_{\beta,s'})(w_{\beta,s'} - v^2 w_{\beta,s}).$$

Finally, for any $j \le \ell < n$, let us consider the relative position of the variables

$$\left\{ x_{\ell,2}^{(\beta,s)}, x_{\ell-1,1}^{(\beta,s')}, x_{\ell,1}^{(\beta,s')}, x_{\ell+1,1}^{(\beta,s')} \right\}.$$

We can assume that $o(x_{\ell-1,1}^{(\beta,s')}) \ge o(x_{\ell,1}^{(\beta,s')}) \ge o(x_{\ell+1,1}^{(\beta,s')})$, as otherwise the corresponding term is specialized to zero under $\phi_{\underline{d}}$. If $o(x_{\ell,2}^{(\beta,s)}) = o(x_{\ell,1}^{(\beta,s')})$, then the corresponding term has the factor

$$(x_{\ell,2}^{(\beta,s)} - v^4 x_{\ell,1}^{(\beta,s')})(x_{\ell,2}^{(\beta,s)} - v^{-4} x_{\ell,1}^{(\beta,s')})$$

as in Lemma 4.1, and so its $\phi_{\underline{d}}$-specialization produces the factor

$$(w_{\beta,s} - v^{4n-4\ell+2} w_{\beta,s'})(w_{\beta,s} - v^{4n-4\ell-6} w_{\beta,s'}).$$

If $o(x_{\ell,2}^{(\beta,s)}) < o(x_{\ell,1}^{(\beta,s')})$ or $o(x_{\ell,2}^{(\beta,s)}) > o(x_{\ell,1}^{(\beta,s')})$, then the $\phi_{\underline{d}}$-specialization of the products

$$\zeta_{\ell,\ell}(x_{\ell,2}^{(\beta,s)}/x_{\ell,1}^{(\beta,s')}) \cdot \zeta_{\ell,\ell-1}(x_{\ell,2}^{(\beta,s)}/x_{\ell-1,1}^{(\beta,s')}) \quad \text{or} \quad \zeta_{\ell+1,\ell}(x_{\ell+1,1}^{(\beta,s')}/x_{\ell,2}^{(\beta,s)}) \cdot \zeta_{\ell,\ell}(x_{\ell,1}^{(\beta,s')}/x_{\ell,2}^{(\beta,s)})$$

respectively, contributes the required factor

$$(w_{\beta,s} - v^{4n-4\ell+2} w_{\beta,s'})(w_{\beta,s} - v^{4n-4\ell-6} w_{\beta,s'}).$$

All these contributions overall produce exactly the factor G_β from Corollary 4.7. However, we have not used yet the factors $\zeta_{ii}(x_{i,1}^{(\beta,s)}/x_{i,1}^{(\beta,s')})$. We can now appeal to the "rank 1" computation of [30, Lemma 3.46] to deduce the required divisibility by $[t_{\beta,r}]_{v^2}!$. ∎

Finally, combining Propositions 4.5, 4.6, and 4.13, we obtain the following upgrade of Theorem 4.8 (we note that divisibility (4.39, 4.40) is precisely matching the constants of (4.10), while the divisibility condition from Definition 4.12 is precisely matching the "rank 1" formula (3.49)):

Theorem 4.14. (a) The $\mathbb{Q}(v)$-algebra isomorphism $\Psi \colon U_v^>(L\mathfrak{o}_{2n+1}) \xrightarrow{\sim} S$ of Theorem 4.8(a) gives rise to a $\mathbb{Z}[v, v^{-1}]$-algebra isomorphism $\Psi \colon \mathcal{U}_v^>(L\mathfrak{o}_{2n+1}) \xrightarrow{\sim} \mathcal{S}$.
(b) Theorem 4.10 holds.

4.3 Integral form $\mathbf{U}_v^>(L\mathfrak{o}_{2n+1})$ and its shuffle algebra realization

Similarly to (3.38), we consider the *divided powers*

$$\mathbf{E}_{i,r}^{(k)} := \frac{e_{i,r}^k}{[k]_{v_i}!} \quad \forall \, 1 \le i \le n, \ r \in \mathbb{Z}, \ k \in \mathbb{N}. \tag{4.50}$$

Likewise, we define the integral form $\mathbf{U}_v^>(L\mathfrak{o}_{2n+1})$ as the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $U_v^>(L\mathfrak{o}_{2n+1})$ generated by $\{\mathbf{E}_{i,r}^{(k)}\}_{1 \le i \le n, r \in \mathbb{Z}}^{k \in \mathbb{N}}$. For any $(\beta, s) \in \Delta^+ \times \mathbb{Z}$, we define the *normalized divided powers* of the quantum root vectors from (2.26)–(2.27) via:

$$\tilde{\mathbf{E}}_{\beta,s}^{\pm,(k)} := \begin{cases} \frac{(\tilde{E}_{\beta,s}^\pm)^k}{[k]_{v_\beta}!} & \text{if } \beta = [i,j] \text{ with } 1 \le j \le n \\[2mm] \frac{(\tilde{E}_{\beta,s}^\pm)^k}{([2]_v!)^k [k]_{v_\beta}!} & \text{if } \beta = [i,n,j] \text{ with } 1 \le i < j \le n \end{cases}. \tag{4.51}$$

Similarly to Proposition 3.8, we obtain:

Proposition 4.15. For any $\beta \in \Delta^+$, $s \in \mathbb{Z}$, $k \in \mathbb{N}$, we have $\tilde{\mathbf{E}}_{\beta,s}^{\pm,(k)} \in \mathbf{U}_v^>(L\mathfrak{o}_{2n+1})$.

Proof. The proof is similar to that of Proposition 3.8. Let $U_v^>(\mathfrak{o}_{2n+1})$ be the "positive subalgebra" of the Drinfeld-Jimbo quantum group of type B_n. Thus, $U_v^>(\mathfrak{o}_{2n+1})$ is the $\mathbb{Q}(v)$-algebra generated by $\{E_i\}_{i=1}^n$ subject to the v-Serre relations. Let $\mathbf{U}_v^>(\mathfrak{o}_{2n+1})$ be its *Lusztig integral form*, defined as the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $U_v^>(\mathfrak{o}_{2n+1})$ generated by the divided powers

$$E_i^{(k)} := \frac{E_i^k}{[k]_{v_i}!} \qquad \forall\, 1 \le i \le n, k \in \mathbb{N}.$$

Recall our specific convex order (2.21) on Δ^+. Let $\{\hat{E}_\beta^-\}_{\beta \in \Delta^+}$ denote Lusztig's quantum root vectors of $U_v^>(\mathfrak{o}_{2n+1})$ associated to this convex order. We also define $\{\tilde{E}_\beta^-\}_{\beta \in \Delta^+}$ as the following iterated v-commutators similar to (2.26)–(2.27):

$$\begin{aligned}
\tilde{E}_{[i,j]}^- &:= [\cdots[[E_i, E_{i+1}]_{v^{-2}}, E_{i+2}]_{v^{-2}}, \cdots, E_j]_{v^{-2}}, \\
\tilde{E}_{[i,n,j]}^- &:= [\cdots[[[\cdots[E_i, E_{i+1}]_{v^{-2}}, \cdots, E_n]_{v^{-2}}, E_n], E_{n-1}]_{v^{-2}}, \cdots, E_j]_{v^{-2}}.
\end{aligned} \tag{4.52}$$

Then, according to [21, Proposition 5.5.2] and [2, Theorem 4.2], we have:

$$\tilde{E}_\beta^- = \begin{cases} \hat{E}_\beta^- & \text{if } \beta = [i,j] \text{ with } 1 \le i \le j \le n \\ [2]_v!\, \hat{E}_\beta^- & \text{if } \beta = [i,n,j] \text{ with } 1 \le i < j \le n \end{cases}. \tag{4.53}$$

To pass from the finite to the loop setup, we note that for any $\underline{s} = (s_1, \ldots, s_n) \in \mathbb{Z}^n$, the assignment $E_1 \mapsto e_{1,s_1}, \ldots, E_n \mapsto e_{n,s_n}$ gives rise to an algebra homomorphism

$$\eta_{\underline{s}}: U_v^>(\mathfrak{o}_{2n+1}) \longrightarrow U_v^>(L\mathfrak{o}_{2n+1}),$$

such that $\eta_{\underline{s}}(\mathbf{U}_v^>(\mathfrak{o}_{2n+1})) \subset \mathbf{U}_v^>(L\mathfrak{o}_{2n+1})$. As $\tilde{\mathbf{E}}_{\beta,s}^{-,(k)} = \eta_{\underline{s}}(\hat{\mathbf{E}}_\beta^{-,(k)})$ and $\hat{\mathbf{E}}_\beta^{-,(k)} \in \mathbf{U}_v^>(\mathfrak{o}_{2n+1})$ by [22, Theorem 6.6], we get $\tilde{\mathbf{E}}_{\beta,s}^{-,(k)} \in \mathbf{U}_v^>(L\mathfrak{o}_{2n+1})$. Using similar arguments and the convex order on Δ^+ opposite to (2.21), we also obtain $\tilde{\mathbf{E}}_{\beta,s}^{+,(k)} \in \mathbf{U}_v^>(L\mathfrak{o}_{2n+1})$. This completes our proof. ∎

For any $\underline{k} \in \mathbb{N}^n$, consider the $\mathbb{Z}[v, v^{-1}]$-submodule $\mathbf{S}_{\underline{k}}$ of $S_{\underline{k}}$ consisting of rational functions F satisfying the following two conditions:

(1) If f denotes the numerator of F from (2.12), then

$$f \in \mathbb{Z}[v, v^{-1}][\{x_{i,r}^{\pm 1}\}_{1 \le i \le n}^{1 \le r \le k_i}]^{\mathfrak{S}_{\underline{k}}}. \tag{4.54}$$

(2) For any $\underline{d} \in \mathrm{KP}(\underline{k})$, the specialization $\phi_{\underline{d}}(F)$ is divisible by the product

$$\prod_{\beta=[i,j]\in\Delta^+} \langle 2 \rangle_v^{d_\beta(|\beta|-1)} \cdot \prod_{\beta=[i,n,j]\in\Delta^+} \left\{ \langle 2 \rangle_v^{d_\beta(|\beta|-2)} \langle 1 \rangle_v \right\} \times$$
$$\prod_{\beta=[i,n,j]\in\Delta^+} \prod_{\ell=j}^{n-1} \left\{ (v^{-4(n-\ell)-2} - 1)^{d_\beta} (v^{-4(n-\ell)+6} - 1)^{d_\beta} \right\}. \tag{4.55}$$

Remark 4.16. We note that this definition is much simpler than that of $\mathcal{S}_{\underline{k}}$ in Definition 4.12.

We define $\mathbf{S} := \bigoplus_{\underline{k}\in\mathbb{N}^n} \mathbf{S}_{\underline{k}}$. Then, similarly to Proposition 3.9, we have:

Proposition 4.17. $\Psi(\mathbf{U}_v^>(L\mathfrak{o}_{2n+1})) \subset \mathbf{S}$.

Proof. For any $m \in \mathbb{N}$, $1 \le i_1, \ldots, i_m \le n$, $r_1, \ldots, r_m \in \mathbb{Z}$, $\ell_1, \ldots, \ell_m \in \mathbb{N}$, let

$$F := \Psi\left(\mathbf{E}_{i_1, r_1}^{(\ell_1)} \cdots \mathbf{E}_{i_m, r_m}^{(\ell_m)}\right),$$

and f be the numerator of F from (2.12). According to Lemma 3.10:

$$\Psi\left(\mathbf{E}_{i_q, r_q}^{(\ell_q)}\right) = v_{i_q}^{-\frac{\ell_q(\ell_q-1)}{2}} (x_{i_q, 1} \cdots x_{i_q, \ell_q})^{r_q} \qquad \forall\, 1 \le q \le m,$$

hence, the condition (4.54) holds. To verify the validity of the divisibility (4.55), it suffices to show (see Lemma 3.2 for $\beta = [i, j]$) that for any $\beta = [i, n, j]$ ($1 \le i < j \le n$), and $1 \le s \le d_\beta$, the total contribution of $\phi_{\underline{d}}$-specializations of the ζ-factors between the variables $\{x_{i,t}^{(\beta,s)}\}_{i \in \beta}^{1 \le t \le v_{\beta,i}}$ is divisible by

$$\langle 2 \rangle_v^{|\beta|-2} \langle 1 \rangle_v \cdot \prod_{\ell=j}^{n-1} \left\{ (v^{-4(n-\ell)-2} - 1)(v^{-4(n-\ell)+6} - 1) \right\}. \tag{4.56}$$

We shall now use the notation $o(x_{*,*}^{(*,*)})$ defined as in (3.50). The $\phi_{\underline{d}}$-specialization of the corresponding product of ζ-factors vanishes unless

$$o(x_{i,1}^{(\beta,s)}) \ge o(x_{i+1,1}^{(\beta,s)}) \ge \cdots \ge o(x_{n,1}^{(\beta,s)}) \ge o(x_{n,2}^{(\beta,s)}) \ge \cdots \ge o(x_{j+1,2}^{(\beta,s)}) \ge o(x_{j,2}^{(\beta,s)}),$$

Since the equality above can occur only in a single spot, due to $o(x_{i,t}^{(\beta,s)}) \ne o(x_{i',t'}^{(\beta,s)})$ for $i \ne i'$, we need to treat the following two cases:

(1) $o(x_{i,1}^{(\beta,s)}) > o(x_{i+1,1}^{(\beta,s)}) > \cdots > o(x_{n,1}^{(\beta,s)}) > o(x_{n,2}^{(\beta,s)}) > \cdots > o(x_{j+1,2}^{(\beta,s)}) > o(x_{j,2}^{(\beta,s)})$.

(2) $o(x_{i,1}^{(\beta,s)}) > o(x_{i+1,1}^{(\beta,s)}) > \cdots > o(x_{n,1}^{(\beta,s)}) = o(x_{n,2}^{(\beta,s)}) > \cdots > o(x_{j+1,2}^{(\beta,s)}) > o(x_{j,2}^{(\beta,s)})$.

In the first case, the $\phi_{\underline{d}}$-specialization of each ζ-factor

$$\zeta_{i+1,i}(x_{i+1,1}^{(\beta,s)}/x_{i,1}^{(\beta,s)}), \ldots, \zeta_{n,n-1}(x_{n,1}^{(\beta,s)}/x_{n-1,1}^{(\beta,s)}), \zeta_{n-1,n}(x_{n-1,2}^{(\beta,s)}/x_{n,2}^{(\beta,s)}), \ldots, \zeta_{j,j+1}(x_{j,2}^{(\beta,s)}/x_{j+1,2}^{(\beta,s)})$$

as well as of the product

$$\zeta_{n,n}(x_{n,2}^{(\beta,s)}/x_{n,1}^{(\beta,s)}) \cdot \zeta_{n,n-1}(x_{n,2}^{(\beta,s)}/x_{n-1,1}^{(\beta,s)})$$

contributes a multiple of $\langle 2 \rangle_v$, thus totalling $\langle 2 \rangle_v^{|\beta|-1}$. On the other hand, for any $j \le \ell \le n-1$, the $\phi_{\underline{d}}$-specialization of the product of ζ-factors

$$\zeta_{\ell,\ell-1}(x_{\ell,2}^{(\beta,s)}/x_{\ell-1,1}^{(\beta,s)}) \cdot \zeta_{\ell,\ell}(x_{\ell,2}^{(\beta,s)}/x_{\ell,1}^{(\beta,s)}) \cdot \zeta_{\ell,\ell+1}(x_{\ell,2}^{(\beta,s)}/x_{\ell+1,1}^{(\beta,s)})$$

contributes precisely the required factor $(v^{-4(n-\ell)-2} - 1)(v^{-4(n-\ell)+6} - 1)$.

For the second case, the only difference is that we replace the product of two ζ-factors $\zeta_{n,n}(x_{n,2}^{(\beta,s)}/x_{n,1}^{(\beta,s)}) \cdot \zeta_{n,n-1}(x_{n,2}^{(\beta,s)}/x_{n-1,1}^{(\beta,s)})$ with a single $\zeta_{n,n-1}(x_{n,2}^{(\beta,s)}/x_{n-1,1}^{(\beta,s)})$, so that the first contribution of $\langle 2 \rangle_v^{|\beta|-1}$ is now getting replaced with $\langle 2 \rangle_v^{|\beta|-2} \cdot \langle 1 \rangle_v$.

This completes our verification of the divisibility (4.55), thus concluding the proof. ∎

For any $h \in H$, define the ordered monomials (cf. (2.34))

$$\tilde{\mathbf{E}}_h^+ = \overrightarrow{\prod_{(\beta,s) \in \Delta^+ \times \mathbb{Z}}} \tilde{\mathbf{E}}_{\beta,s}^{+,(h(\beta,s))}, \qquad \tilde{\mathbf{E}}_h^- = \overrightarrow{\prod_{(\beta,s) \in \Delta^+ \times \mathbb{Z}}} \tilde{\mathbf{E}}_{\beta,s}^{-,(h(\beta,s))}. \tag{4.57}$$

For any $\epsilon \in \{\pm\}$, let $\mathbf{S}_{\underline{k}}^\epsilon$ be the $\mathbb{Z}[v, v^{-1}]$-submodule of $\mathbf{S}_{\underline{k}}$ spanned by $\{\Psi(\tilde{\mathbf{E}}_h^\epsilon)\}_{h \in H_{\underline{k}}}$. Then, the following analogue of Lemma 2.8 holds:

Proposition 4.18. For any $F \in \mathbf{S}_{\underline{k}}$ and $\underline{d} \in \mathrm{KP}(\underline{k})$, if $\phi_{\underline{d}'}(F) = 0$ for all $\underline{d}' \in \mathrm{KP}(\underline{k})$ such that $\underline{d}' < \underline{d}$, then there exists $F_{\underline{d}} \in \mathbf{S}_{\underline{k}}^\epsilon$ such that $\phi_{\underline{d}}(F) = \phi_{\underline{d}}(F_{\underline{d}})$ and $\phi_{\underline{d}'}(F_{\underline{d}}) = 0$ for all $\underline{d}' < \underline{d}$.

The proof of this result is completely analogous to that of Proposition 3.11. Combining Propositions 4.17 and 4.18, we obtain the following upgrade of Theorem 4.8:

Theorem 4.19. (a) The $\mathbb{Q}(v)$-algebra isomorphism $\Psi \colon U_v^>(L\mathfrak{o}_{2n+1}) \xrightarrow{\sim} S$ of Theorem 4.8(a) gives rise to a $\mathbb{Z}[v, v^{-1}]$-algebra isomorphism $\Psi \colon \mathbf{U}_v^>(L\mathfrak{o}_{2n+1}) \xrightarrow{\sim} \mathbf{S}$.
(b) For any choices of s_k in (2.26)–(2.27) and $\epsilon \in \{\pm\}$, the ordered monomials $\{\tilde{\mathbf{E}}_h^\epsilon\}_{h \in H}$ of (4.57) form a basis of the free $\mathbb{Z}[v, v^{-1}]$-module $\mathbf{U}_v^>(L\mathfrak{o}_{2n+1})$.

5 Yangian Counterpart

In this section, we generalize the results of Sections 3–4 to the Yangian case, thus establishing shuffle algebra realizations of Yangians and their Drinfeld-Gavarini duals in types B_n and G_2. This should be viewed as the "rational vs trigonometric" counterpart, where we replace factors $\frac{z}{w} - v^k$ by $z - w - \frac{k}{2}\hbar$. In particular, $\zeta_{i,j}(z)$ of (2.13) will be replaced by $\hat{\zeta}_{i,j}(z) = 1 + \frac{(\alpha_i, \alpha_j) \cdot \hbar}{2z}$.

5.1 The Yangian $Y_\hbar^>(\mathfrak{g})$ and its shuffle algebra realization

We still use the notations from Section 2. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of type B_n or G_2. Following [3], the **"positive subalgebra" of the Yangian of $\mathfrak{g}$** in the new Drinfeld realization, denoted by $Y_\hbar^>(\mathfrak{g})$, is the $\mathbb{Q}[\hbar]$-algebra generated by $\{x_{i,r}\}_{i \in I}^{r \in \mathbb{N}}$ subject to the following defining relations:

$$[x_{i,r+1}, x_{j,s}] - [x_{i,r}, x_{j,s+1}] = \frac{d_i a_{ij} \hbar}{2}(x_{i,r} x_{j,s} + x_{j,s} x_{i,r}) \qquad \forall\, i, j \in I, r, s \in \mathbb{N}, \tag{5.1}$$

$$\underset{s_1, \ldots, s_{1-a_{ij}}}{Sym} [x_{i,s_1}, [x_{i,s_2}, \cdots, [x_{i,s_{1-a_{ij}}}, x_{j,r}] \cdots]] = 0 \qquad \forall\, i \neq j, s_1, \ldots, s_{1-a_{ij}}, r \in \mathbb{N}. \tag{5.2}$$

Analogously to (2.24)–(2.25), let us now define the root vectors $\{X_{\beta,s}\}_{\beta \in \Delta^+}^{s \in \mathbb{N}}$ of $Y_\hbar^>(\mathfrak{g})$:

- B_n-type.
  For any $\beta = [i_1, \ldots, i_\ell] \in \Delta^+$ from (2.18) and $s \in \mathbb{N}$, choose a decomposition $s = s_1 + \cdots + s_\ell$ with $s_1, \ldots, s_\ell \in \mathbb{N}$. Then, we define

$$X_{\beta,s} := [\cdots [[x_{i_1,s_1}, x_{i_2,s_2}], x_{i_3,s_3}], \cdots, x_{i_\ell,s_\ell}]. \tag{5.3}$$

- G_2-type.
  For $\beta = [i_1, \ldots, i_\ell] \neq [1, 2, 1, 2, 2]$, $s \in \mathbb{N}$, the elements $X_{\beta,s}$ are defined exactly as in (5.3).
  For $\beta = [1, 2, 1, 2, 2]$ and $s \in \mathbb{N}$, we choose a decomposition $s = s_1 + \cdots + s_5$ with $s_1, \ldots, s_5 \in \mathbb{N}$, and define

$$X_{\beta,s} := [[x_{1,s_1}, x_{2,s_2}], [[x_{1,s_3}, x_{2,s_4}], x_{2,s_5}]]. \tag{5.4}$$

We will also need the following specific choices $\{\tilde{X}_{\beta,s}\}_{\beta \in \Delta^+}^{s \in \mathbb{N}}$ of (5.3)–(5.4):

- B_n-type.

$$\tilde{X}_{[i,j],s} := [\cdots [[x_{i,s}, x_{i+1,0}], x_{i+2,0}], \cdots, x_{j,0}], \quad 1 \leq i \leq j \leq n, \tag{5.5}$$

$$\tilde{X}_{[i,n,j],s} := [\cdots [[[\cdots [x_{i,s}, x_{i+1,0}], \cdots, x_{n,0}], x_{n,0}], x_{n-1,0}], \cdots, x_{j,0}], \quad 1 \leq i < j \leq n. \tag{5.6}$$

- G_2-type.

$$\tilde{X}_{[i],s} := x_{i,s}, \quad 1 \leq i \leq 2, \tag{5.7}$$

$$\tilde{X}_{[1,2],s} := [x_{1,s}, x_{2,0}], \tag{5.8}$$

$$\tilde{X}_{[1,2,2],s} := [[x_{1,s}, x_{2,0}], x_{2,0}], \tag{5.9}$$

$$\tilde{X}_{[1,2,2,2],s} := [[[x_{1,s}, x_{2,0}], x_{2,0}], x_{2,0}], \tag{5.10}$$

$$\tilde{X}_{[1,2,1,2,2],s} := [[x_{1,s}, x_{2,0}], [[x_{1,0}, x_{2,0}], x_{2,0}]]. \tag{5.11}$$

Let H denote the set of all functions $h\colon \Delta^+ \times \mathbb{N} \to \mathbb{N}$ with finite support. For any $h \in$ H, we consider the ordered monomials

$$X_h = \overrightarrow{\prod_{(\beta,s)\in\Delta^+\times\mathbb{N}}} X_{\beta,s}^{h(\beta,s)} \qquad \text{and} \qquad \tilde{X}_h = \overrightarrow{\prod_{(\beta,s)\in\Delta^+\times\mathbb{N}}} \tilde{X}_{\beta,s}^{h(\beta,s)}. \tag{5.12}$$

Then, similarly to [19] (cf. [14, Theorem B.3]), we have:

Theorem 5.1. The elements $\{X_h\}_{h\in H}$ form a basis of the free $\mathbb{Q}[\hbar]$-module $Y_\hbar^>(\mathfrak{g})$.

Proof. Comparing $\tilde{X}_{\beta,s}$ to the root vectors $e_\beta^{(s)}$ used in [14, (A.11)], we see that the only difference is in the root vectors $\tilde{X}_{[1,2,1,2,2],s}$ in G_2-type. However, the two key properties (B.1) and (B.2) of [14, Appendix B] still hold for the new root vectors, cf. Remark 5.15. Hence, the proof of [14, Theorem B.2] and thus of [14, Theorem B.3] still goes through. This proves that $\{\tilde{X}_h\}_{h\in H}$ form a basis of the free $\mathbb{Q}[\hbar]$-module $Y_\hbar^>(\mathfrak{g})$. The proof of the fact that the ordered monomials $\{X_h\}_{h\in H}$ in more general root vectors of (5.3)–(5.4) also provide a basis will be derived from the shuffle algebra realization of $Y_\hbar^>(\mathfrak{g})$; see Theorem 5.14. ∎

We define the shuffle algebra $(\bar{\mathbb{W}}, \star)$ analogously to the shuffle algebra $(S, \star)$ of Section 2 with the following modifications:

(1) All rational functions $F \in \bar{\mathbb{W}}$ are defined over $\mathbb{Q}[\hbar]$.
(2) The matrix $(\hat{\zeta}_{i,j}(z))_{i,j\in I}$ is defined via

$$\hat{\zeta}_{i,j}(z) = 1 + \frac{(\alpha_i, \alpha_j) \cdot \hbar}{2z}. \tag{5.13}$$

(3) (*pole conditions*) $F \in \bar{\mathbb{W}}_{\underline{k}}$ has the form

$$F = \frac{f(\{x_{i,r}\}_{i\in I}^{1\leq r\leq k_i})}{\prod_{i<j}^{a_{ij}\neq 0} \prod_{1\leq s\leq k_j}^{1\leq r\leq k_i}(x_{i,r}-x_{j,s})}, \tag{5.14}$$

where $f \in \mathbb{Q}[\hbar][\{x_{i,r}\}_{i\in I}^{1\leq r\leq k_i}]^{\mathfrak{S}_{\underline{k}}}$ and $<$ is an arbitrary order on I.
(4) (*wheel conditions*) Let f be the numerator of $F \in \bar{\mathbb{W}}_{\underline{k}}$ from (5.14), then

$$f(\{x_{i,r}\}_{i\in I}^{1\leq r\leq k_i}) = 0 \text{ once } x_{i,s_1} = x_{i,s_2} + d_i\hbar = \cdots = x_{i,s_{1-a_{ij}}} - d_i a_{ij}\hbar = x_{j,r} - \frac{d_i a_{ij}}{2}\hbar \tag{5.15}$$

for any $i \neq j$ such that $a_{ij} \neq 0$, pairwise distinct $1 \leq s_1, \ldots, s_{1-a_{ij}} \leq k_i$, and $1 \leq r \leq k_j$.
(5) The shuffle product is defined like (2.14), but $\zeta_{i,j}(\frac{x_{i,r}}{x_{j,s}})$ are replaced by $\hat{\zeta}_{i,j}(x_{i,r}-x_{j,s})$.

Similarly to Proposition 2.1, we have:

Proposition 5.2. The assignment $x_{i,r} \mapsto x_{i,1}^r \in \bar{\mathbb{W}}_{\mathbf{1}_i}$ ($i \in I, r \in \mathbb{N}$) gives rise to a $\mathbb{Q}[\hbar]$-algebra homomorphism

$$\Psi\colon Y_\hbar^>(\mathfrak{g}) \longrightarrow \bar{\mathbb{W}}. \tag{5.16}$$

Let us adapt our key tool of *specialization maps* to the Yangian setup. In type G_2, the specialization maps $\phi_{\underline{d}}$ are defined via the following specialization of the $x_{*,*}^{(*,*)}$-variables, cf. (3.2):

$$\begin{aligned} x_{1,t}^{(\beta,s)} &\mapsto w_{\beta,s} + t\hbar, \quad 1 \leq t \leq 2, \\ x_{2,t}^{(\beta,s)} &\mapsto w_{\beta,s} - \tfrac{3}{2}\hbar + t\hbar, \quad 1 \leq t \leq 3. \end{aligned} \tag{5.17}$$

In type B_n, the specialization maps $\phi_{\underline{d}}$ are defined via the following specialization of the $x_{*,*}^{(*,*)}$-variables, cf. (4.2):

$$x_{i,1}^{(\beta,s)} \mapsto w_{\beta,s} - i\hbar, \quad x_{i,2}^{(\beta,s)} \mapsto w_{\beta,s} - (2n-i-1)\hbar \qquad \forall \beta \in \Delta^+, \ 1 \leq s \leq d_\beta, \ i \in \beta. \tag{5.18}$$

In what follows, we will use the notation $\doteq$ to denote an equality up to $\mathbb{Q}^\times$ (cf. (2.44)):

$$A \doteq B \quad \text{if} \quad A = c \cdot B \quad \text{for some } c \in \mathbb{Q}^\times. \tag{5.19}$$

Remark 5.3. We emphasize that c in (5.19) is a nonzero element of $\mathbb{Q}$ rather than $\mathbb{Q}[\hbar]$. Most importantly, the appearance of such constants occurs from the "rank 1" computations. Explicitly, the formula (3.49) in the shuffle algebra S is now replaced by the equality

$$\underbrace{x_{i,1}^r \star \cdots \star x_{i,1}^r}_{\ell \text{ times}} = \ell! \cdot (x_{i,1} \cdots x_{i,\ell})^r \tag{5.20}$$

in the shuffle algebra $\bar{\mathbb{W}}$ for any $i \in I, r \geq 0, \ell \geq 1$; see [30, Lemma 6.22]. Thus, the product of quantum integers in the trigonometric case is now replaced by the product of integers.

We have the following straightforward analogues of Lemmas 3.1 and 4.1:

Lemma 5.4. For type G_2, we have:

$$\Psi(\tilde{X}_{[i],s}) \doteq x_{i,1}^s, \quad 1 \leq i \leq 2, \tag{5.21}$$

$$\Psi(\tilde{X}_{[1,2],s}) \doteq \frac{\hbar x_{1,1}^s}{x_{1,1} - x_{2,1}}, \tag{5.22}$$

$$\Psi(\tilde{X}_{[1,2,2],s}) \doteq \frac{\hbar^2 x_{1,1}^s}{(x_{1,1} - x_{2,1})(x_{1,1} - x_{2,2})}, \tag{5.23}$$

$$\Psi(\tilde{X}_{[1,2,2,2],s}) \doteq \frac{\hbar^3 x_{1,1}^s}{(x_{1,1} - x_{2,1})(x_{1,1} - x_{2,2})(x_{1,1} - x_{2,3})}, \tag{5.24}$$

$$\Psi(\tilde{X}_{[1,2,1,2,2],s}) \doteq \frac{\hbar^4 g(\{x_{1,r}, x_{2,t}\}_{1 \leq r \leq 2}^{1 \leq t \leq 3})}{\prod_{1 \leq r \leq 2}^{1 \leq t \leq 3} (x_{1,r} - x_{2,t})}, \tag{5.25}$$

where $g \in \mathbb{Q}[\hbar][\{x_{1,r}, x_{2,t}\}_{1 \leq r \leq 2}^{1 \leq t \leq 3}]^{\mathfrak{S}_2 \times \mathfrak{S}_3}$ (we do not really need an explicit formula for this g).

Lemma 5.5. For type B_n, we have:

$$\Psi(\tilde{X}_{[i,j],s}) \doteq \frac{\hbar^{j-i} x_{i,1}^s}{(x_{i,1} - x_{i+1,1}) \cdots (x_{j-1,1} - x_{j,1})}, \tag{5.26}$$

$$\Psi(\tilde{X}_{[i,n,j],s}) \doteq \frac{\hbar^{2n-i-j+1} x_{i,1}^s \prod_{\ell=j}^{n-1} (2\hbar + x_{\ell,1} - x_{\ell,2})(2\hbar - x_{\ell,1} + x_{\ell,2})}{(x_{i,1} - x_{i+1,1}) \cdots (x_{j-1,1} - x_{j,1})(x_{j-1,1} - x_{j,2}) \prod_{\ell=j}^{n-1} \prod_{1 \leq r,t \leq 2} (x_{\ell,r} - x_{\ell+1,t})}. \tag{5.27}$$

As follows from the above two lemmas, the images $\Psi(\tilde{X}_{\beta,s})$ are divisible by $\hbar^{|\beta|-1}$. This is actually true for any root vectors $X_{\beta,s}$ defined in (5.3)–(5.4):

Lemma 5.6. For any $\beta \in \Delta^+$ and $s \in \mathbb{N}$, $\Psi(X_{\beta,s})$ is divisible by $\hbar^{|\beta|-1}$.

Proof. The proof is similar to that of Lemma 4.2, and follows immediately from the equality

$$\hat{\zeta}_{i,j}(z) - \hat{\zeta}_{j,i}(-z) = \frac{(\alpha_i, \alpha_j)}{z} \hbar. \qquad \blacksquare$$

Furthermore, invoking the constants κ_β given by (3.21) in type G_2 and by (4.9) in type B_n, we have the following straightforward analogue of Lemmas 3.2 and 4.2:

Lemma 5.7. In both types G_2 and B_n, we have:

$$\phi_\beta(\Psi(X_{\beta,s})) \doteq \hbar^{\kappa_\beta} \cdot p_{\beta,s}(w_{\beta,1}) \qquad \forall (\beta, s) \in \Delta^+ \times \mathbb{N}, \tag{5.28}$$

where $p_{\beta,s}(w) \in \mathbb{Q}[\hbar][w]$ is a monic degree s polynomial in w over $\mathbb{Q}[\hbar]$.

Recall that H denotes the set of all functions $h: \Delta^+ \times \mathbb{N} \to \mathbb{N}$ with finite support. For any $\underline{k} \in \mathbb{N}^I$ and $\underline{d} \in KP(\underline{k})$, we define the subsets $H_{\underline{k}}$, $H_{\underline{k},\underline{d}}$ of H similarly to (2.37), but with $h \in H$ been replaced by $h \in H$. Using Lemma 5.7 and arguing exactly as in Sections 3–4, we obtain the following analogues of Lemmas 2.6 and 2.7 for the Yangians of types G_2 and B_n:

Lemma 5.8. For any $h \in H_{\underline{k},\underline{d}}$, we have

$$\phi_{\underline{d}}(\Psi(X_h)) \stackrel{\circ}{=} \hbar^{\sum_{\beta \in \Delta^+} d_\beta \kappa_\beta} \cdot \prod_{\beta, \beta' \in \Delta^+}^{\beta < \beta'} \hat{G}_{\beta,\beta'} \cdot \prod_{\beta \in \Delta^+} \hat{G}_\beta \cdot \prod_{\beta \in \Delta^+} \hat{P}_{\lambda_{h,\beta}}, \tag{5.29}$$

where $\hat{G}_{\beta,\beta'}, \hat{G}_\beta$ are <u>independent of $h \in H_{\underline{k},\underline{d}}$</u> and are rational counterparts of $G_{\beta,\beta'}, G_\beta$ from Lemma 2.6 (obtained by replacing factors $(x - v^t y)$ with $(x - y - \frac{t}{2}\hbar)$), while

$$\hat{P}_{\lambda_{h,\beta}} = \mathrm{Sym}_{\mathfrak{S}_{d_\beta}} \left(\prod_{s=1}^{d_\beta} p_{\beta, r_\beta(h,s)}(w_{\beta,s}) \prod_{1 \le s < r \le d_\beta} \left(1 + \frac{(\beta, \beta) \cdot \hbar}{2(w_{\beta,s} - w_{\beta,r})} \right) \right). \tag{5.30}$$

This again features a "rank 1 reduction": each $\hat{P}_{\lambda_{h,\beta}}$ from (5.30) can be viewed as the shuffle product $p_{\beta, r_\beta(h,1)}(x) \star \cdots \star p_{\beta, r_\beta(h,d_\beta)}(x)$ in the A_1-type shuffle algebra $\bar{W}$, evaluated at $\{w_{\beta,s}\}_{s=1}^{d_\beta}$. The following result is a Yangian counterpart of Lemma 2.7 for types G_2 and B_n:

Lemma 5.9. For any $h \in H_{\underline{k},\underline{d}}$ and $\underline{d}' < \underline{d}$, we have $\phi_{\underline{d}'}(\Psi(X_h)) = 0$.

Combining Theorem 5.1 and Lemmas 5.8–5.9, we obtain (cf. [30, Proposition 6.16]):

Proposition 5.10. The homomorphism Ψ of (5.16) is injective.

Following [29, Definition 3.27], we introduce:

Definition 5.11. $F \in \bar{W}_{\underline{k}}$ is **good** if $\phi_{\underline{d}}(F)$ is divisible by $\hbar^{\sum_{\beta \in \Delta^+} d_\beta \kappa_\beta}$ for any $\underline{d} \in KP(\underline{k})$.

Let $W_{\underline{k}}$ be the $\mathbb{Q}[\hbar]$-submodule of all good elements in $\bar{W}_{\underline{k}}$, and set $W := \bigoplus_{\underline{k} \in \mathbb{N}^I} W_{\underline{k}}$.

Proposition 5.12. $\Psi(Y_\hbar^>(\mathfrak{g})) \subset W$.

Proof. For any $m \in \mathbb{N}$, $i_1, \ldots, i_m \in I$, $r_1, \ldots, r_m \in \mathbb{N}$, set $F := \Psi(x_{i_1, r_1} \cdots x_{i_m, r_m})$, and let f be the numerator of F from (5.14). Set $\underline{k} = \sum_{\ell=1}^m \alpha_{i_\ell} \in \mathbb{N}^I$, and choose any $\underline{d} \in KP(\underline{k})$. It suffices to show that the $\phi_{\underline{d}}$-specialization of each summand in the symmetrization from f is divisible by $\hbar^{\sum_{\beta \in \Delta^+} d_\beta \kappa_\beta}$. Similarly to (3.50), if a variable $x_{*,*}^{(*,*)}$ is plugged into $\Psi(x_{i_q, r_q})$ for some $1 \le q \le m$, then we shall use the notation $o(x_{*,*}^{(*,*)}) = q$.

In type B_n, the case of $\beta = [i, j]$ is analogous to A_n-type. Thus, it remains to treat the case of $\beta = [i, n, j]$ with $d_\beta \ne 0$. For any $1 \le s \le d_\beta$, the $\phi_{\underline{d}}$-specialization of the corresponding summand vanishes unless

$$o(x_{i,1}^{(\beta,s)}) > o(x_{i+1,1}^{(\beta,s)}) > \cdots > o(x_{n,1}^{(\beta,s)}) > o(x_{n,2}^{(\beta,s)}) > \cdots > o(x_{j+1,2}^{(\beta,s)}) > o(x_{j,2}^{(\beta,s)}).$$

In the latter case, the $\phi_{\underline{d}}$-specialization of the $\hat{\zeta}$-factors between pairs of the $x_{*,*}^{(\beta,s)}$-variables contributes precisely the required factor $\hbar^{\kappa_\beta}$.

In type G_2, the only nontrivial check is for $\beta = [1, 2, 1, 2, 2]$ case. Suppose $d_\beta \ne 0$. For any $1 \le s \le d_\beta$, the $\phi_{\underline{d}}$-specialization of the corresponding summand vanishes unless

$$o(x_{1,1}^{(\beta,s)}) > o(x_{2,1}^{(\beta,s)}) > o(x_{2,2}^{(\beta,s)}) > o(x_{2,3}^{(\beta,s)}) \quad \text{and} \quad o(x_{1,2}^{(\beta,s)}) > o(x_{2,2}^{(\beta,s)}).$$

In the latter case, the $\phi_{\underline{d}}$-specialization of the $\hat{\zeta}$-factors between pairs of the $x_{*,*}^{(\beta,s)}$-variables contributes precisely the required factor $\hbar^6 = \hbar^{\kappa_\beta}$ as well. ∎

Let $W'_{\underline{k}}$ be the $\mathbb{Q}[\hbar]$-submodule of $W_{\underline{k}}$ spanned by $\{\Psi(X_h)\}_{h \in H_{\underline{k}}}$. Then, the following Yangian counterpart of Lemma 2.8 holds true in types G_2 and B_n (cf. Propositions 3.6 and 4.6):

Proposition 5.13. For any $F \in \mathbb{W}_{\underline{k}}$ and $\underline{d} \in KP(\underline{k})$, if $\phi_{\underline{d}'}(F) = 0$ for all $\underline{d}' \in KP(\underline{k})$ such that $\underline{d}' < \underline{d}$, then there exists $F_{\underline{d}} \in \mathbb{W}'_{\underline{k}}$ such that $\phi_{\underline{d}}(F) = \phi_{\underline{d}}(F_{\underline{d}})$ and $\phi_{\underline{d}'}(F_{\underline{d}}) = 0$ for all $\underline{d}' < \underline{d}$.

Proof. Since F is good, the specialization $\phi_{\underline{d}}(F)$ is divisible by $\hbar^{\sum_{\beta \in \Delta^+} d_\beta \kappa_\beta}$. On the other hand, arguing as in the proofs of Propositions 3.6 and 4.6 (which utilized only wheel conditions), we conclude that $\phi_{\underline{d}}(F)$ is also divisible by $\prod_{\beta < \beta'} \hat{G}_{\beta,\beta'} \cdot \prod_{\beta \in \Delta^+} \hat{G}_\beta$. Therefore, we have:

$$\phi_{\underline{d}}(F) = \hbar^{\sum_{\beta \in \Delta^+} d_\beta \kappa_\beta} \cdot \prod_{\beta, \beta' \in \Delta^+}^{\beta < \beta'} \hat{G}_{\beta,\beta'} \cdot \prod_{\beta \in \Delta^+} \hat{G}_\beta \cdot G \tag{5.31}$$

for some symmetric polynomial $G \in \mathbb{Q}[\hbar][\{w_{\beta,s}\}_{\beta \in \Delta^+}^{1 \le s \le d_\beta}]^{\mathfrak{S}_{\underline{d}}}$. Combining (5.31) with Lemma 5.8 (and the "rank 1" counterpart from [29, end of proof of Lemma 3.18]), we see that there is a linear combination $F_{\underline{d}} = \sum_{h \in H_{\underline{k},\underline{d}}} c_h X_h$ such that $\phi_{\underline{d}}(F) = \phi_{\underline{d}}(F_{\underline{d}})$. On the other hand, the equality $\phi_{\underline{d}'}(F_{\underline{d}}) = 0$ for all $\underline{d}' < \underline{d}$ follows from Lemma 5.9. ∎

Combining Propositions 5.12–5.13, we immediately obtain the shuffle algebra realization and the PBW theorem for $Y_\hbar^>(\mathfrak{g})$ in types G_2 and B_n:

Theorem 5.14. (a) The $\mathbb{Q}[\hbar]$-algebra embedding $\Psi \colon Y_\hbar^>(\mathfrak{g}) \to \bar{\mathbb{W}}$ of Proposition 5.10 gives rise to a $\mathbb{Q}[\hbar]$-algebra isomorphism $\Psi \colon Y_\hbar^>(\mathfrak{g}) \xrightarrow{\sim} \mathbb{W}$.

(b) The ordered monomials $\{X_h\}_{h \in H}$ of (5.12) form a basis of the free $\mathbb{Q}[\hbar]$-module $Y_\hbar^>(\mathfrak{g})$, cf. Theorem 5.1.

Proof. According to Lemma 5.9 and Proposition 5.10, $\{\Psi(X_h)\}_{h \in H} \subset \mathbb{W}$ are linearly independent. On the other hand, by iterated application of Proposition 5.13, we also get that $\{\Psi(X_h)\}_{h \in H}$ span $\mathbb{W}$ over $\mathbb{Q}[\hbar]$, cf. [31, Proposition 1.6]. Thus, $\{\Psi(X_h)\}_{h \in H}$ form a basis of $\mathbb{W}$. Combining this with the injectivity of the homomorphism $\Psi \colon Y_\hbar^>(\mathfrak{g}) \to \mathbb{W}$ from Proposition 5.10 (which uses the validity of Theorem 5.14(b) only for the particular choices $\tilde{X}_{\beta,s}$, see Theorem 5.1), we immediately obtain both parts of Theorem 5.14. ∎

Remark 5.15. We note that Theorem 5.14(b) can be proved directly as Theorem B.3 in [14]. The proof of the latter relied only on (B.1) and (B.2) of *loc.cit.* The former of these holds true in our setup without any changes: $\Delta(X_{\beta,s}) = X_{\beta,s} \otimes 1 + 1 \otimes X_{\beta,s} +$ lower order terms, where Δ is the coproduct on the Yangian and we use the standard filtration on the Yangian. On the other hand, (B.2) fails on the nose, but what we really need is the property stated after (B.2) which does still hold: "for any PBW monomial y, the expression $\tau_a(y)$ is polynomial in the variable a, has a maximal degree of a equal to the filtered degree of y, and the coefficient of this leading power of a equals $\bar{y}$ which is obtained from y by replacing all $x_{i,r}$ with $x_{i,0}$".

5.2 The Drinfeld-Gavarini dual $\dot{Y}_\hbar^>(\mathfrak{g})$ and its shuffle algebra realization

For any $(\beta, s) \in \Delta^+ \times \mathbb{N}$, define $\bar{X}_{\beta,s} \in Y_\hbar^>(\mathfrak{g})$ via

$$\bar{X}_{\beta,s} := \hbar \cdot X_{\beta,s}. \tag{5.32}$$

We define $\dot{Y}_\hbar^>(\mathfrak{g})$, the *"positive subalgebra"* of the Drinfeld-Gavarini dual, as the $\mathbb{Q}[\hbar]$-subalgebra of $Y_\hbar^>(\mathfrak{g})$ generated by $\{\bar{X}_{\beta,s}\}_{\beta \in \Delta^+}^{s \in \mathbb{N}}$. For any $h \in H$, define the ordered monomial (cf. (5.12)):

$$\bar{X}_h := \prod_{(\beta,s) \in \Delta^+ \times \mathbb{N}}^{\longrightarrow} \bar{X}_{\beta,s}^{h(\beta,s)}. \tag{5.33}$$

Similarly to [14, Theorem A.7], we obtain:

Theorem 5.16. (a) $\dot{Y}_\hbar^>(\mathfrak{g})$ is independent of the choice of root vectors $X_{\beta,s}$ in (5.3)–(5.4).

(b) For any choices of s_k in (5.3)–(5.4), the ordered monomials $\{\bar{X}_h\}_{h \in H}$ form a basis of the free $\mathbb{Q}[\hbar]$-module $\dot{Y}_\hbar^>(\mathfrak{g})$.

Proof. The arguments of [14, Appendix A] apply directly to the particular choice $\{\tilde{X}_{\beta,s}\}_{\beta \in \Delta^+}^{s \in \mathbb{N}}$. The general case can be derived from the shuffle realization of $\dot{Y}_\hbar^>(\mathfrak{g})$; see Theorem 5.20. ∎

Remark 5.17. We note that Theorem 5.16 can be proved directly as Theorem A.7 in [14]. The proof of the latter relied only on the validity of properties (As1, As2, As3) from *loc.cit.* In the present setup: (As1) is obvious, (As2) is established in Theorem 5.14(b), (As3) is verified precisely as in [14, Lemma A.6] since all our $X_{\beta,s}$'s are still iterated commutators of $x_{i,r}$'s.

Following [29, Definition 3.8], we introduce:

Definition 5.18. $F \in \bar{\mathbb{W}}_{\underline{k}}$ is **integral** if F is divisible by $\hbar^{|\underline{k}|}$ and $\phi_{\underline{d}}(F)$ is divisible by $\hbar^{\sum_{\beta \in \Delta^+} d_\beta(\kappa_\beta+1)}$ for any $\underline{d} \in \mathrm{KP}(\underline{k})$.

Let $\mathbf{W}_{\underline{k}} \subset \bar{\mathbb{W}}_{\underline{k}}$ be the $\mathbb{Q}[\hbar]$-submodule of all integral elements, and set $\mathbf{W} := \bigoplus_{\underline{k} \in \mathbb{N}^I} \mathbf{W}_{\underline{k}}$. Then, following Lemmas 5.6–5.7 and the proof of Proposition 5.12, we obtain:

Proposition 5.19. $\Psi(\dot{\mathrm{Y}}_\hbar^>(\mathfrak{g})) \subset \mathbf{W}$.

Finally, we have the following upgrade of Theorem 5.14:

Theorem 5.20. The $\mathbb{Q}[\hbar]$-algebra isomorphism $\Psi : \mathrm{Y}_\hbar^>(\mathfrak{g}) \overset{\sim}{\longrightarrow} \mathbb{W}$ of Theorem 5.14(a) gives rise to a $\mathbb{Q}[\hbar]$-algebra isomorphism $\Psi : \dot{\mathrm{Y}}_\hbar^>(\mathfrak{g}) \overset{\sim}{\longrightarrow} \mathbf{W}$.

Proof. It remains to prove the opposite inclusion $\mathbf{W} \subseteq \Psi(\dot{\mathrm{Y}}_\hbar^>(\mathfrak{g}))$. To this end, it suffices to show that for any $F \in \mathbf{W}_{\underline{k}}$ and $\underline{d} \in \mathrm{KP}(\underline{k})$, if $\phi_{\underline{d}'}(F) = 0$ for all $\underline{d}' \in \mathrm{KP}(\underline{k})$ such that $\underline{d}' < \underline{d}$, then there exists $F_{\underline{d}} \in \mathbf{W}_{\underline{k}} \cap \Psi(\dot{\mathrm{Y}}_\hbar^>(\mathfrak{g}))$ such that $\phi_{\underline{d}}(F) = \phi_{\underline{d}}(F_{\underline{d}})$ and $\phi_{\underline{d}'}(F_{\underline{d}}) = 0$ for $\underline{d}' < \underline{d}$, cf. (2) after Lemma 2.8. The proof of this result is analogous to the proof of Proposition 5.13, except that the factor $\hbar^{\sum_{\beta \in \Delta^+} d_\beta \kappa_\beta}$ in (5.31) is getting replaced by $\hbar^{\sum_{\beta \in \Delta^+} d_\beta(\kappa_\beta+1)}$. ∎

A The RTT Realization in Type B_n

In this section, we recall the RTT realization of $U_v(L\mathfrak{o}_{2n+1})$, established in [17], and use it to explain the natural origin and the name of the integral form $\mathcal{U}_v^>(L\mathfrak{o}_{2n+1})$ from Subsection 4.2.

RTT realization of $U_{\mathbf{q}}(L\mathfrak{o}_{2n+1})$

Set $N = 2n + 1$. For $1 \leq i \leq N$, we define i' and $\bar{i}$ via:

$$i' := N + 1 - i, \tag{A.1}$$

$$(\bar{1}, \ldots, \bar{N}) := \left(n - \tfrac{1}{2}, \ldots, \tfrac{1}{2}, 0, -\tfrac{1}{2}, \ldots, -n + \tfrac{1}{2}\right). \tag{A.2}$$

To follow the notations of [17], we also define:

$$q := v^2 \ \text{(so that } v = q^{1/2}), \qquad \xi := q^{2-N}. \tag{A.3}$$

Consider the trigonometric R-matrix with a spectral parameter $\bar{R}_{\mathrm{trig}}(u)$ given by

$$\bar{R}_{\mathrm{trig}}(u) := \frac{u-1}{uq - q^{-1}} R + \frac{q - q^{-1}}{uq - q^{-1}} P - \frac{(q - q^{-1})(u-1)\xi}{(uq - q^{-1})(u - \xi)} Q, \tag{A.4}$$

see [17, (3.1)], where $P, Q, R \in (\mathrm{End}\,\mathbb{Q}^N)^{\otimes 2}$ are defined via (with $q = v^2$ as in (A.3)):

$$P = \sum_{1 \leq i,j \leq N} e_{ij} \otimes e_{ji},$$

$$Q = \sum_{1 \leq i,j \leq N} q^{\bar{i} - \bar{j}} e_{i'j'} \otimes e_{ij},$$

$$R = \sum_{\substack{1 \leq i \leq N \\ i \neq n+1}} q^{1 - \delta_{i,n+1}} e_{ii} \otimes e_{ii} + \sum_{\substack{1 \leq i,j \leq N \\ i \neq j,j'}} e_{ii} \otimes e_{jj} + q^{-1} \sum_{i \neq n+1} e_{ii} \otimes e_{i'i'} + \tag{A.5}$$

$$(q - q^{-1}) \sum_{i<j} e_{ij} \otimes e_{ji} - (q - q^{-1}) \sum_{i>j} q^{\bar{i} - \bar{j}} e_{i'j'} \otimes e_{ij}.$$

This $\bar{R}_{\text{trig}}(u)$ satisfies the famous *Yang-Baxter equation* (with a spectral parameter):

$$\bar{R}_{\text{trig};12}(u/v)\bar{R}_{\text{trig};13}(u/w)\bar{R}_{\text{trig};23}(v/w) = \bar{R}_{\text{trig};23}(v/w)\bar{R}_{\text{trig};13}(u/w)\bar{R}_{\text{trig};12}(u/v). \tag{A.6}$$

Following [17] (with the conceptual ideology going back to [12]), we define the **RTT integral form of the quantum loop algebra of $\mathfrak{o}_N$**, denoted by $\mathcal{U}_v^{\text{rtt}}(L\mathfrak{o}_N)$, to be the associative $\mathbb{Z}[v, v^{-1}]$-algebra generated by $\{\ell_{ij}^{\pm}[\mp r]\}_{1 \le i,j \le N}^{r \in \mathbb{N}}$ with the following defining relations:

$$\ell_{ij}^{+}[0] = \ell_{ji}^{-}[0] = 0 \text{ for } 1 \le i < j \le N,$$

$$\ell_{ii}^{\pm}[0]\ell_{ii}^{\mp}[0] = 1 \text{ for } 1 \le i \le N,$$

$$\bar{R}_{\text{trig}}(z/w)\mathcal{L}_1^{\pm}(z)\mathcal{L}_2^{\pm}(w) = \mathcal{L}_2^{\pm}(w)\mathcal{L}_1^{\pm}(z)\bar{R}_{\text{trig}}(z/w), \tag{A.7}$$

$$\bar{R}_{\text{trig}}(z/w)\mathcal{L}_1^{+}(z)\mathcal{L}_2^{-}(w) = \mathcal{L}_2^{-}(w)\mathcal{L}_1^{+}(z)\bar{R}_{\text{trig}}(z/w),$$

as well as

$$\mathcal{L}^{\pm}(u)D\mathcal{L}^{\pm}(u\xi)^{\text{t}}D^{-1} = D\mathcal{L}^{\pm}(u\xi)^{\text{t}}D^{-1}\mathcal{L}^{\pm}(u) = 1, \tag{A.8}$$

where t denotes the matrix transposition with $E_{ij}^{\text{t}} = E_{j'i'}$ and D is the diagonal matrix

$$D = \text{diag}(q^{\bar{1}}, q^{\bar{2}}, \ldots, q^{\bar{N}}).$$

Here, $\mathcal{L}^{\pm}(u) \in \mathcal{U}_v^{\text{rtt}}(L\mathfrak{o}_N)[[u, u^{-1}]] \otimes \text{End } \mathbb{Q}^N$ is defined by

$$\mathcal{L}^{\pm}(u) = \sum_{1 \le i,j \le N} \ell_{ij}^{\pm}(u) \otimes E_{ij} \quad \text{with} \quad \ell_{ij}^{\pm}(u) := \sum_{r \ge 0} \ell_{ij}^{\pm}[\mp r]u^{\pm r}. \tag{A.9}$$

We also define the $\mathbb{Q}(v)$-counterpart $U_v^{\text{rtt}}(L\mathfrak{o}_N) := \mathcal{U}_v^{\text{rtt}}(L\mathfrak{o}_N) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{Q}(v)$.

> **Remark A.1.** The last two relations in (A7) are commonly referred to as the *RTT relations*. However, without imposing (A8), one actually gets an extended version of that algebra featuring an extra Heisenberg algebra factor.

Let $U_v(L\mathfrak{o}_N)$ be the quantum loop algebra of type B_n in the new Drinfeld realization. It is a $\mathbb{Q}(v)$-algebra generated by $\{x_{i,r}^{\pm}, \varphi_{i,-k}, \psi_{i,k}, k_i^{\pm 1}\}_{1 \le i \le n}^{r \in \mathbb{Z}, k \in \mathbb{N}}$ with the relations as in [17, §1]. Identifying $x_{i,r}^{+}$ with our $e_{i,r}$, the subalgebra of $U_v(L\mathfrak{o}_N)$ generated by $\{x_{i,r}^{+}\}_{1 \le i \le n}^{r \in \mathbb{Z}}$ recovers $U_v^{>}(L\mathfrak{o}_N)$ from Subsection 2.1. In what follows, we will consider the following generating series:

$$x_i^{\pm}(u) = \sum_{r \in \mathbb{Z}} x_{i,r}^{+}u^{-r}, \quad \varphi_i(u) = \sum_{k \ge 0} \varphi_{i,-k}u^{k}, \quad \psi_i(u) = \sum_{k \ge 0} \psi_{i,k}u^{-k}.$$

The relation between the algebras $U_v(L\mathfrak{o}_N)$ and $\mathcal{U}_v^{\text{rtt}}(L\mathfrak{o}_N)$ was recently established in [17]. To state the main result, we consider the Gauss decomposition of the matrices $\mathcal{L}^{\pm}(u)$ from (A.9):

$$\mathcal{L}^{\pm}(u) = F^{\pm}(u) \cdot H^{\pm}(u) \cdot E^{\pm}(u).$$

Here, $F^{\pm}(u), H^{\pm}(u), E^{\pm}(u) \in \mathcal{U}_v^{\text{rtt}}(L\mathfrak{o}_N)[[u, u^{-1}]] \otimes \text{End } \mathbb{Q}^N$ are of the form

$$F^{\pm}(u) = \sum_i E_{ii} + \sum_{i>j} f_{ij}^{\pm}(u) \otimes E_{ij}, \ H^{\pm}(u) = \sum_i h_i^{\pm}(u) \otimes E_{ii}, \ E^{\pm}(u) = \sum_i E_{ii} + \sum_{i<j} e_{ij}^{\pm}(u) \otimes E_{ij}.$$

Theorem A.2 ([17]). There is a unique $\mathbb{Q}(v)$-algebra isomorphism

$$\varrho : U_v(L\mathfrak{o}_N) \xrightarrow{\sim} \mathcal{U}_v^{\text{rtt}}(L\mathfrak{o}_N)$$

defined by

$$x_i^{+}(u) \mapsto \frac{e_{i,i+1}^{+}(uq^i) - e_{i,i+1}^{-}(uq^i)}{q - q^{-1}}, \quad x_i^{-}(u) \mapsto \frac{f_{i+1,i}^{+}(uq^i) - f_{i+1,i}^{-}(uq^i)}{q^{1-\delta_{in}/2} - q^{-1+\delta_{in}/2}},$$

$$\psi_i(u) \mapsto h_{i+1}^{-}(uq^i)h_i^{-}(uq^i)^{-1}, \quad \varphi_i(u) \mapsto h_{i+1}^{+}(uq^i)h_i^{+}(uq^i)^{-1}, \tag{A.10}$$

where $q = v^2$ as in (A3).

Remark A.3. We note right away that [17] established a similar isomorphism for centrally extended algebras, and also that they used slightly rescaled formulas for the images of $x_n^{\pm}(u)$.

The RTT realization of $\mathcal{U}_v^{>}(L\mathfrak{o}_{2n+1})$

Let $\mathcal{U}_v^{\mathrm{rtt},>}(L\mathfrak{o}_N)$ be the $\mathbb{Z}[v, v^{-1}]$-subalgebra of $\mathcal{U}_v^{\mathrm{rtt}}(L\mathfrak{o}_N)$ generated by the coefficients of $\{e_{ij}^{\pm}(u)\}_{1 \le i < j \le N}$, the matrix coefficients of $E^{\pm}(u)$. The key goal of this appendix is to highlight the natural origin of the integral form $\mathcal{U}_v^{>}(L\mathfrak{o}_{2n+1})$ introduced in Subsection 4.2 and its specific quantum root vectors (a special case of (4.37))

$$\tilde{\mathcal{E}}_{[i,j],s}^{\mathrm{rtt}} := \langle 2 \rangle_v \cdot [\cdots [[e_{i,s}, e_{i+1,0}]_{v^2}, e_{i+2,0}]_{v^2}, \cdots, e_{j,0}]_{v^2},$$

$$\tilde{\mathcal{E}}_{[i,n,j],s}^{\mathrm{rtt}} := \langle 2 \rangle_v \cdot [\cdots [[[\cdots [e_{i,s}, e_{i+1,0}]_{v^2}, \cdots, e_{n,0}]_{v^2}, e_{n,0}], e_{n-1,0}]_{v^2}, \cdots, e_{j,0}]_{v^2}. \tag{A.11}$$

Let us express the matrix coefficients of $E^{\pm}(u)$ as series in $u^{\pm 1}$ with coefficients in $\mathcal{U}_v^{\mathrm{rtt}}(L\mathfrak{o}_N)$:

$$e_{ij}^{+}(u) = \sum_{r>0} e_{ij}^{(-r)} u^r, \quad e_{ij}^{-}(u) = \sum_{r \ge 0} e_{ij}^{(r)} u^{-r} \qquad \forall\, 1 \le i < j \le N. \tag{A.12}$$

We also define $e_{ij}(u) := e_{ij}^{+}(u) - e_{ij}^{-}(u)$. The key technical result of this subsection is:

Proposition A.4. For any $1 \le i < j \le n$, we have:

$$e_{i,j+1}(u) = (1-q^2)^{i-j} \cdot [\cdots [[e_{i,i+1}(u), e_{i+1,i+2}^{(0)}]_q, e_{i+2,i+3}^{(0)}]_q, \cdots, e_{j,j+1}^{(0)}]_q \tag{A.13}$$

and

$$e_{i,j'}(u) = q(1-q^2)^{i+j-2n-1}(-1)^{j-n-1} \times$$
$$[\cdots [[[\cdots [e_{i,i+1}(u), e_{i+1,i+2}^{(0)}]_q, \cdots, e_{n,n+1}^{(0)}]_q, e_{n,n+1}^{(0)}], e_{n-1,n}^{(0)}]_q, \cdots, e_{j,j+1}^{(0)}]_q. \tag{A.14}$$

Proof. Due to the "rank reduction" embedding homomorphisms of [17, §3.2, Proposition 4.2], it suffices to prove both formulas (A.13) and (A.14) for $i = 1$ and $1 < j \le n$.

We prove (A.13) for $i = 1$ by induction on $j \ge 1$, the base case $j = 1$ being vacuous. Comparing the matrix coefficients $\langle v_1 \otimes v_j | \cdots | v_j \otimes v_{j+1} \rangle$ of both sides of the RTT relation $\bar{R}_{\mathrm{trig}}(z/w)\mathcal{L}_1^{-}(z)\mathcal{L}_2^{-}(w) = \mathcal{L}_2^{-}(w)\mathcal{L}_1^{-}(z)\bar{R}_{\mathrm{trig}}(z/w)$, we get:

$$\frac{z-w}{qz-q^{-1}w} \ell_{1j}^{-}(z)\ell_{j,j+1}^{-}(w) + \frac{(q-q^{-1})z}{qz-q^{-1}w} \ell_{jj}^{-}(z)\ell_{1,j+1}^{-}(w) =$$
$$\frac{z-w}{qz-q^{-1}w} \ell_{j,j+1}^{-}(w)\ell_{1j}^{-}(z) + \frac{(q-q^{-1})w}{qz-q^{-1}w} \ell_{jj}^{-}(w)\ell_{1,j+1}^{-}(z). \tag{A.15}$$

Expanding all rational factors as series in z/w and evaluating the $[w^0]$-coefficients, we obtain:

$$q\ell_{1j}^{-}(z)\ell_{j,j+1}^{-}[0] = q\ell_{j,j+1}^{-}[0]\ell_{1j}^{-}(z) + (1-q^2)\ell_{jj}^{-}[0]\ell_{1,j+1}^{-}(z). \tag{A.16}$$

Likewise, comparing the matrix coefficients $\langle v_1 \otimes v_j | \cdots | v_j \otimes v_j \rangle$ of both sides of the same RTT relation, we also get:

$$\frac{z-w}{qz-q^{-1}w} \ell_{1j}^{-}(z)\ell_{jj}^{-}(w) + \frac{(q-q^{-1})z}{qz-q^{-1}w} \ell_{jj}^{-}(z)\ell_{1j}^{-}(w) = \ell_{jj}^{-}(w)\ell_{1j}^{-}(z). \tag{A.17}$$

Expanding both rational factors as series in z/w and evaluating the $[w^0]$-coefficients, we obtain:

$$q\ell_{1j}^{-}(z)\ell_{jj}^{-}[0] = \ell_{jj}^{-}[0]\ell_{1j}^{-}(z), \tag{A.18}$$

or equivalently:

$$\ell_{jj}^-[0]^{-1}\ell_{1j}^-(z) = q^{-1}\ell_{1j}^-(z)\ell_{jj}^-[0]^{-1}. \tag{A.19}$$

Let $h_j^-[0]$ be the u^0-coefficient of $h_j^-(u)$. Then, $\ell_{jj}^-[0] = h_j^-[0]$ and $\ell_{j,j+1}^-[0] = h_j^-[0]e_{j,j+1}^{(0)} = \ell_{jj}^-[0]e_{j,j+1}^{(0)}$. Thus, multiplying (A.16) by $\ell_{jj}^-[0]^{-1}$ on the left and applying (A.19), we obtain:

$$(1-q^2)\ell_{1,j+1}^-(z) = [\ell_{1j}^-(z), e_{j,j+1}^{(0)}]_q. \tag{A.20}$$

Here, $\ell_{1,j+1}^-(z) = h_1^-(z)e_{1,j+1}^-(z)$ and $\ell_{1j}^-(z) = h_1^-(z)e_{1j}^-(z)$. As $h_1^-(z)$ commutes with $e_{j,j+1}^-(w)$ for $1 < j < 2'$, we derive:

$$e_{1,j+1}^-(z) = (1-q^2)^{-1} \cdot [e_{1j}^-(z), e_{j,j+1}^{(0)}]_q. \tag{A.21}$$

Arguing in the same way, but using the other RTT relation $\bar{R}_{\text{trig}}(z/w)\mathcal{L}_1^+(z)\mathcal{L}_2^-(w) = \mathcal{L}_2^-(w)\mathcal{L}_1^+(z)\bar{R}_{\text{trig}}(z/w)$, we also obtain:

$$e_{1,j+1}^+(z) = (1-q^2)^{-1} \cdot [e_{1j}^+(z), e_{j,j+1}^{(0)}]_q. \tag{A.22}$$

Subtracting (A.21) from (A.22), we finally get:

$$e_{1,j+1}(z) = (1-q^2)^{-1} \cdot [e_{1j}(z), e_{j,j+1}^{(0)}]_q. \tag{A.23}$$

Applying the induction assumption for $e_{1j}(z)$ completes our proof of (A.13) for $i = 1$.

To prove (A.14) for $i = 1$, we note first that $e_{1,k+1}^\pm(z) = \frac{[e_{1k}^\pm(z), e_{k,k+1}^{(0)}]_q}{1-q^2}$ for $n+1 < k < 2n$, similarly to (A.21)–(A.22). But according to [17, Proposition 5.4] (corrected by replacing $-e_{n-1}^\pm(u\xi q^{2n-2})$ with $-e_n^\pm(u\xi q^{2n})$ in loc.cit.), we have $e_{k,k+1}^{(0)} = -e_{k'-1,k'}^{(0)}$. Therefore, we get:

$$e_{1,k+1}^\pm(z) = -(1-q^2)^{-1} \cdot [e_{1k}^\pm(z), e_{k'-1,k'}^{(0)}]_q. \tag{A.24}$$

Finally, a special care should be taken of the $k = n+1$ case as in that case the matrix coefficient $\langle v_1 \otimes v_{n+1}|\mathcal{L}_2^-(w)\mathcal{L}_1^\pm(z)\bar{R}_{\text{trig}}(z/w)|v_{n+1} \otimes v_{n+1}\rangle$ is given by a different formula:

$$\langle v_1 \otimes v_{n+1} \,|\, \mathcal{L}_2^-(w)\mathcal{L}_1^\pm(z)\bar{R}_{\text{trig}}(z/w) \,|\, v_{n+1} \otimes v_{n+1}\rangle \;=\; \sum_{i=1}^N \frac{a_{i,n+1}(z/w)}{(z/w - q^{-2})(z/w - \xi)} \ell_{n+1,i}^-(w)\ell_{1,i'}^\pm(z), \tag{A.25}$$

where

$$a_{i,n+1}(z/w) = \begin{cases} q^{-1}(z/w - \xi)(z/w - 1) + (\xi - 1)(q^{-2} - 1)z/w & \text{if } i = n+1 \\ (q^{-2} - 1)q^{i-\overline{n+1}}\xi \cdot (z/w - 1) & \text{if } i < n+1 \\ (q^{-2} - 1)q^{\overline{i}-\overline{n+1}} \cdot (z/w - 1)z/w & \text{if } i > n+1 \end{cases}. \tag{A.26}$$

Expanding all rational factors in z/w and evaluating the w^0-coefficient, only the $i = n+1$ term will have a nontrivial contribution. Explicitly, we obtain the following analogue of (A.18):

$$q\ell_{1,n+1}^\pm(z)\ell_{n+1,n+1}^-[0] = q\ell_{n+1,n+1}^-[0]\ell_{1,n+1}^\pm(z), \tag{A.27}$$

so that (A.19) will get replaced by

$$\ell_{n+1,n+1}^-[0]^{-1}\ell_{1,n+1}^\pm(z) = \ell_{1,n+1}^\pm(z)\ell_{n+1,n+1}^-[0]^{-1}. \tag{A.28}$$

This establishes (A.14) for $i = 1$ and $j = n$, while (A.24) establishes it then for $1 < j < n$. ∎

Combining Proposition A.4 with $\varrho(x_i^+(u)) = \frac{-q}{1-q^2}e_{i,i+1}(uq^i)$ of (A.10), the definition (A.11), and identification of the present currents $x_i^+(u)$ with the currents $e_i(u)$ of (2.10), we obtain:

Corollary A.5. For any $1 \leq i < j \leq n$ and $s \in \mathbb{Z}$, we have:

$$\varrho(\tilde{\mathcal{E}}_{[i,j],s}^{\mathrm{rtt}}) \doteq e_{i,j+1}^{(s)} \qquad \text{and} \qquad \varrho(\tilde{\mathcal{E}}_{[i,n,j],s}^{\mathrm{rtt}}) \doteq e_{i,j'}^{(s)}. \tag{A.29}$$

Since the elements (A.11) are specific case of quantum root vectors (4.37), we finally obtain:

Proposition A.6. $\varrho(\mathcal{U}_v^>(L\mathfrak{o}_{2n+1})) = \mathcal{U}_v^{\mathrm{rtt},>}(L\mathfrak{o}_{2n+1})$.

This result explains why we called $\mathcal{U}_v^>(L\mathfrak{o}_{2n+1})$ the RTT integral form of $U_v^>(L\mathfrak{o}_{2n+1})$. Moreover, Theorem 4.10(b) implies the PBWD theorem for $\mathcal{U}_v^{\mathrm{rtt},>}(L\mathfrak{o}_{2n+1})$, cf. [14, Theorem 3.25]:

Corollary A.7. The ordered monomials in $\{e_{ij}^{(r)} \mid i < j \text{ such that } i+j \leq N, r \in \mathbb{Z}\}$ form a basis of the free $\mathbb{Z}[v, v^{-1}]$-module $\mathcal{U}_v^{\mathrm{rtt},>}(L\mathfrak{o}_{2n+1})$.

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