Orthosymplectic superoscillator Lax matrices

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Abstract

We construct Lax matrices of superoscillator type that are solutions of the RTT-relation for the rational orthosymplectic R-matrix, generalizing orthogonal and symplectic oscillator type Lax matrices previously constructed by the authors in Frassek (Nuclear Phys B, 2020), Frassek and Tsymbaliuk (Commun Math Phys 392 (2):545–619, 2022), Frassek et al. (Commun Math Phys 400 (1):1–82, 2023). We further establish factorisation formulas among the presented solutions.

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1 Introduction

1.1 Summary

The study of supersymmetric solutions to the Yang–Baxter equation goes back to the works of Kulish and Sklyanin in the early 80's, see e.g. [25], who introduced the R-matrix that generates the supersymmetric Yangian of $\mathfrak{gl}(n|m)$, see [30]. As common for Lie superalgebras, the underlying vector space is equipped with a $\mathbb{Z}_2$-grading to incorporate bosonic and fermionic degrees of freedom. Similarly to the purely bosonic case, there exists an evaluation map ev: $Y(\mathfrak{gl}(n|m)) \to U(\mathfrak{gl}(n|m))$ from the Yangian of $\mathfrak{gl}(n|m)$ into the universal enveloping algebra of the Lie superalgebra $\mathfrak{gl}(n|m)$, see [22, 30], which facilitates the study of the spectrum of supersymmetric spin chains. In particular, the algebraic Bethe ansatz for a large class of representations in the quantum space and the construction of the corresponding transfer matrices are well-understood, see [22]. The same is true for the functional relations (T-systems and QQ-systems) among such transfer matrices and Q-operators, see [26, 31–33] as well as [24] for an overview.

The construction of the Q-operators has been carried out more recently in [13] employing certain degenerate Lax matrices of superoscillator type (see also [23] for a different approach). The Lax matrices for Q-operators in the trigonometric case were obtained in [7] for $n + m = 3$ and in [34, 35] for arbitrary n, m. The degenerate solutions of the Yang–Baxter equation with a rational (resp. trigonometric) R-matrix arise naturally by taking certain limits of the evaluation representation of the Yangian (resp. quantum affine algebra) in the parabolic Verma modules of the underlying Lie algebra, realized in terms of superoscillator algebras. The crucial difference between the solutions that arise from the ordinary Yangian and the degenerate solutions is that the coefficients of the leading power of the spectral parameter are not of full rank in degenerate case. Therefore, this class of solutions does not arise through the ordinary Yangian, but is rather related to the so-called (RTT antidominantly) shifted Yangians, as has been recently realized in [15] (cf. [8] for an interpretation via the 4d Chern-Simons theory). The latter are usually defined in terms of Drinfeld's current realization, see e.g. [3, Appendix B], and the identification with the aforementioned RTT ones goes through the Gauss decomposition of the generating matrix $T(x)$ as in the ordinary case [5], see [15, Theorem 2.54]. For the bosonic case of $\mathfrak{gl}(n)$, the degenerate Lax matrices that are linear in the spectral parameter were constructed in [14], while the reconstruction of degenerate Lax matrices at any order of the spectral parameter has been achieved in [15] using the results of [3]. Using the $S(n)$-invariance of the rational R-matrix of $\mathfrak{gl}_n$ one can further obtain other degenerate Lax matrices which do not admit a Gauss decomposition (and thus are no longer directly related to

the *shifted Yangians*). Thus, the transfer matrices are constructed from representations of the ordinary Yangian, while the Q-operators are constructed from representations of the shifted Yangian, cf. [20] and references therein. We note that this approach allows to deduce functional relations among transfer matrices and Q-operators directly from the Yang–Baxter equation using certain factorisation properties of the Lax matrices combined with the BGG-type resolutions for the underlying Lie algebras, see [4, 6, 12].

The situation changes drastically for the Yangians of the orthosymplectic Lie superalgebras $\mathfrak{osp}(N|2m)$ which unify the bosonic cases in BCD-types, that is, Yangians of orthogonal $\mathfrak{so}_N$ and symplectic $\mathfrak{sp}_{2m}$ Lie algebras. Similarly to BCD-type, the evaluation map no longer exists and representations of the orthosymplectic Lie superalgebra cannot be always lifted to representations of the corresponding Yangian. The R-matrix in the defining vector representation was obtained in [1], generalizing the BCD-type R-matrix of [37], but other solutions of the Yang–Baxter relation or the RTT-relation (for the Lax matrix) are scarce, see [11, 21] and references therein. The algebraic Bethe ansatz has been obtained for spin chains in the defining vector representation in [19], but little is known about other representations as well as the underlying functional relations. A glimpse towards the latter appeared in the study [2] of the AdS/CFT-correspondence in relation to the quantum spectral curve for AdS_4/CFT_3, but remain to be confirmed from the first principles. Further results for more general Lie superalgebras were recently obtained in [36]. The full understanding of the QQ-system may yield new methods of solving Bethe equations similar to the method developed in [29] for $\mathfrak{gl}(n|m)$.

In this paper, we enlarge the class of representations of the orthosymplectic (shifted) Yangians by introducing several Lax matrices of superoscillator type, which can be used to construct transfer matrices and Q-operators. We anticipate the BGG-type functional relations among those, generalizing our recent BCD-type results of [12].

Finally, let us note that there has been an increased mathematical interest in the theory of quantum supergroups. In the context of orthosymplectic Yangians of $\mathfrak{osp}(N|2m)$ specifically, their Drinfeld realizations were established recently in [28] and [27] for the cases $N = 1$ and $N > 3$ with the standard parity, respectively. In the sequel paper [17], we present uniform Drinfeld realizations of orthosymplectic Yangians for any N, m and, most importantly, any parity sequence.

1.2 Outline

The structure of the present paper is the following:

• In Sect. 2, we recall the key results of [13] on $\mathfrak{gl}(n|m)$-type Lax matrices that serve as motivation and prototype for our new constructions in the orthosymplectic type.

• In Sect. 3, we set the notation for the orthosymplectic R-matrix and the corresponding Lax matrices, as well as discuss the invariance of this R-matrix that is needed for our latter results.

• In Sect. 4, we construct some linear superoscillator Lax matrices of orthosymplectic type. First, we construct a degenerate linear Lax matrix in Theorem 4.11. Fusing two

of those, we then construct a non-degenerate linear Lax matrix in Proposition 4.55, whose normalized limits recover back the degenerate Lax matrices, see Remark 4.60. In the special cases $m = 0$ or $n = 0$, we recover the corresponding orthogonal and symplectic Lax matrices of [9, 12, 16], respectively.

• In Sect. 5, we investigate some quadratic orthosymplectic Lax matrices of superoscillator type. First, fusing two degenerate linear Lax matrices from Sect. 4, we construct a Lax matrix of size $(N + 2m) \times (N + 2m)$ for even N in Theorem 5.44. We call this matrix degenerate quadratic Lax as its diagonal is $\sim (x^2, x, \ldots, x, 1)$ with respect to the spectral parameter x. A further degeneration of this matrix, depending only in $N + 2m - 2$ pairs of superoscillators, is obtained in Proposition 5.52. A similar formula provides an orthosymplectic Lax matrix for odd N, see Conjecture 5.64. Finally, fusing two degenerate quadratic Lax matrices we derive explicit non-degenerate quadratic orthosymplectic Lax matrices in Proposition 5.80 and Theorem 5.85, whose normalized limits recover back the degenerate quadratic Lax matrices, see Remark 5.90.

• In Appendix A, we present explicit formulas (A.1, A.2) and (A.4, A.5, A.6) for the twists needed to define both the transfer matrices and the Q-operators, as mentioned in the Introduction above.

While the constructions of Sects. 4–5 are presented for the specific parity sequences (3.4) or (5.58), similar Lax matrices exist for other parity sequences as well, according to Remarks 3.16, 4.62, 5.89.

2 General linear Lax matrices

The first results on superoscillator type Lax matrices for Q-operators and their factorisation formulas were presented in [13] for the rational R-matrices of $\mathfrak{gl}(n|m)$-type. The Lax matrices for the trigonometric case were obtained in [7] for $n + m = 3$ and in [34, 35] for arbitrary n, m. In this section, we briefly recall the results of [13] that are relevant to the rest of the paper.

Fix $n, m \geq 0$ and consider a superspace $V = V_{\bar{0}} \oplus V_{\bar{1}}$ with a $\mathbb{C}$-basis $v_1, \ldots, v_{n+m}$ such that each v_i is either *even* (that is, $v_i \in V_{\bar{0}}$) or *odd* (that is, $v_i \in V_{\bar{1}}$) and $\dim(V_{\bar{0}}) = n, \dim(V_{\bar{1}}) = m$. For $1 \leq i \leq n + m$, we define $|i| := |v_i| \in \mathbb{Z}_2$. We define the *parity sequence* associated to V via

$$\Upsilon_V := \big(|v_1|, \ldots, |v_{n+m}|\big) \in \big\{\bar{0}, \bar{1}\big\}^{n+m}. \tag{2.1}$$

For a superalgebra A and homogeneous elements $a, a' \in A$, their *supercommutator* is defined as

$$[a, a'] = aa' - (-1)^{|a||a'|} a'a, \tag{2.2}$$

where $|a|$ denotes the $\mathbb{Z}_2$-grading of a. Given two superspaces $A = A_{\bar{0}} \oplus A_{\bar{1}}$ and $B = B_{\bar{0}} \oplus B_{\bar{1}}$, their tensor product $A \otimes B$ is also a superspace with $(A \otimes B)_{\bar{0}} = A_{\bar{0}} \otimes B_{\bar{0}} \oplus A_{\bar{1}} \otimes B_{\bar{1}}$ and $(A \otimes B)_{\bar{1}} = A_{\bar{0}} \otimes B_{\bar{1}} \oplus A_{\bar{1}} \otimes B_{\bar{0}}$. Furthermore, if A and B are superalgebras, then $A \otimes B$ is also a superalgebra, called the *graded tensor product*

of A and B, with the multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} (aa') \otimes (bb') \tag{2.3}$$

for any homogeneous elements $a \in A_{|a|}, a' \in A_{|a'|}, b \in B_{|b|}, b' \in B_{|b'|}$.

Let $\mathsf{P} \colon \mathsf{V} \otimes \mathsf{V} \to \mathsf{V} \otimes \mathsf{V}$ be the permutation operator defined by

$$\mathsf{P} = \sum_{i,j=1}^{n+m} (-1)^{|j|} e_{ij} \otimes e_{ji}, \tag{2.4}$$

whose action is explicitly given by:

$$\mathsf{P}(\mathsf{v}_j \otimes \mathsf{v}_i) = (-1)^{|i||j|} \mathsf{v}_i \otimes \mathsf{v}_j. \tag{2.5}$$

Consider the corresponding *rational R-matrix* (of general linear type, or super A-type for short):

$$R(x) = \mathsf{R}^\mathsf{V}(x) = x\mathrm{Id} + \mathsf{P}, \tag{2.6}$$

which satisfies the famous *Yang-Baxter equation* (with a spectral parameter):

$$R_{12}(x)R_{13}(x+y)R_{23}(y) = R_{23}(y)R_{13}(x+y)R_{12}(x). \tag{2.7}$$

For any superalgebra A, an even matrix $L(x) = L^\mathsf{V}(x) = (L_{ij}(x))_{i,j=1}^{n+m} \in \mathrm{End}\,\mathsf{V} \otimes A[[x, x^{-1}]]$ is called an (A-valued) *Lax matrix* if it satisfies the corresponding *RTT-relation* with $R(x)$ of (2.6)

$$R_{12}(x-y)L_1(x)L_2(y) = L_2(y)L_1(x)R_{12}(x-y), \tag{2.8}$$

viewed as an equality in $\mathrm{End}\,\mathsf{V} \otimes \mathrm{End}\,\mathsf{V} \otimes A[[x, y, x^{-1}, y^{-1}]]$. Coefficient-wise, the equation (2.8) is equivalent to (see [18]) the well-known system of equations for all $1 \leq i, j, k, \ell \leq n+m$:

$$[L_{ij}(x), L_{k\ell}(y)] = \frac{(-1)^{|i||j|+|i||k|+|j||k|}}{x-y}\Big(L_{kj}(y)L_{i\ell}(x) - L_{kj}(x)L_{i\ell}(y)\Big). \tag{2.9}$$

Remark 2.10 (a) Here, we identify the matrix $(L_{ij}(x))_{i,j=1}^{n+m}$ with $\sum_{i,j=1}^{n+m}(-1)^{|i||j|+|j|} e_{ij} \otimes L_{ij}(x)$. Evoking (2.3), the extra sign $(-1)^{|i||j|+|j|}$ ensures that the product of matrices is calculated as usual. The above "even" condition means that $\mathbb{Z}_2$-grading of all coefficients of $L_{ij}(x)$ is $|i| + |j|$.

(b) The data of a Lax matrix $L^\mathsf{V}(x) = (L_{ij}(x))_{i,j=1}^{n+m}$ with $L_{ij}(x) \in \delta_{ij} + x^{-1}A[[x^{-1}]]$ for all i, j is equivalent to an algebra homomorphism $Y^{\mathrm{rtt}}(\mathfrak{gl}(\mathsf{V})) \to A$ from an RTT super Yangian of $\mathfrak{gl}(\mathsf{V})$.

(c) Unless we want to emphasize the dependence on Υ_V, the superscript V will be ignored.

Remark 2.11 Let $\tilde{\mathsf{V}}$ be another superspace with a $\mathbb{C}$-basis $\tilde{\mathsf{v}}_1, \ldots, \tilde{\mathsf{v}}_{n+m}$ such that each $\tilde{\mathsf{v}}_i$ is even or odd and $\dim(\mathsf{V}_{\bar{0}}) = \dim(\tilde{\mathsf{V}}_{\bar{0}})$, $\dim(\mathsf{V}_{\bar{1}}) = \dim(\tilde{\mathsf{V}}_{\bar{1}})$. Pick a permutation $\sigma \in S(n + m)$ such that $\mathsf{v}_i \in \mathsf{V}$ and $\tilde{\mathsf{v}}_{\sigma(i)} \in \tilde{\mathsf{V}}$ have the same $\mathbb{Z}_2$-grading for all $1 \le i \le n + m$, and define a superspace isomorphism $\mathsf{J}_\sigma : \mathsf{V} \xrightarrow{\sim} \tilde{\mathsf{V}}$ via $\mathsf{v}_i \mapsto \tilde{\mathsf{v}}_{\sigma(i)}$. The corresponding R-matrices (2.6) are related via

$$\mathsf{R}^{\tilde{\mathsf{V}}}(x) = (\mathsf{J}_\sigma \otimes \mathsf{J}_\sigma)\,\mathsf{R}^{\mathsf{V}}(x)\,(\mathsf{J}_\sigma \otimes \mathsf{J}_\sigma)^{-1}\,. \tag{2.12}$$

As a result, if $L^{\mathsf{V}}(x) = (L_{ij}(x))_{i,j=1}^{n+m}$ is a solution of (2.8), then $L^{\tilde{\mathsf{V}}}(x) := \mathsf{J}_\sigma L^{\mathsf{V}}(x)\mathsf{J}_\sigma^{-1}$ is a solution of (2.8) for $\tilde{\mathsf{V}}$ used instead of V. In other words, having constructed some Lax matrices, a natural $S(n+m)$-symmetry allows for analogous Lax matrices for all parity sequences (2.1).

Among other Lax matrices, the following family $\{L_a^{\mathsf{V}}(x)\}_{a=0}^{n+m}$ was constructed in [13][1]:

$$L_a(x) = L_a^{\mathsf{V}}(x) = \left(\begin{array}{c|c} x\mathrm{Id}_a - \bar{\mathbf{K}}\mathbf{K} & \bar{\mathbf{K}} \\ \hline -\mathbf{K} & \mathrm{Id}_{n+m-a} \end{array} \right) \tag{2.13}$$

with

$$\bar{\mathbf{K}} = \begin{pmatrix} \bar{\xi}_{1,a+1} & \bar{\xi}_{1,a+2} & \cdots & \bar{\xi}_{1,n+m} \\ \bar{\xi}_{2,a+1} & \bar{\xi}_{2,a+2} & \cdots & \bar{\xi}_{2,n+m} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\xi}_{a,a+1} & \bar{\xi}_{a,a+2} & \cdots & \bar{\xi}_{a,n+m} \end{pmatrix} \tag{2.14}$$

and

$$\mathbf{K} = \begin{pmatrix} (-1)^{|1|}\xi_{a+1,1} & (-1)^{|2|}\xi_{a+1,2} & \cdots & (-1)^{|a|}\xi_{a+1,a} \\ (-1)^{|1|}\xi_{a+2,1} & (-1)^{|2|}\xi_{a+2,2} & \cdots & (-1)^{|a|}\xi_{a+2,a} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{|1|}\xi_{n+m,1} & (-1)^{|2|}\xi_{n+m,2} & \cdots & (-1)^{|a|}\xi_{n+m,a} \end{pmatrix}. \tag{2.15}$$

Here, $(\xi_{ij}, \bar{\xi}_{k\ell})$ are superoscillators with $\mathbb{Z}_2$-grading $|\xi_{ij}| = |i| + |j|$ and $|\bar{\xi}_{k\ell}| = |k| + |\ell|$ that obey the following commutation relations:

$$[\xi_{ij}, \bar{\xi}_{k\ell}] = \delta_{i\ell}\delta_{jk}\,, \qquad [\xi_{ij}, \xi_{\ell k}] = 0\,, \qquad [\bar{\xi}_{k\ell}, \bar{\xi}_{ji}] = 0 \tag{2.16}$$

for any $1 \le j, k \le a < i, \ell \le n + m$, with the supercommutator $[-, -]$ defined in (2.2).

It was further noticed in [13] that for any $0 \le a \le n + m$, one has a total of $\binom{n+m}{a}$ Lax matrices analogous to (2.13), see [13, (2.12)] onwards, corresponding to the number of choices to distribute a spectral parameters x on the main diagonal of

---

[1] It is obtained from the *linear canonical* $\mathbf{L}$-operator of [13, (2.20)] for the trivial representation of the additional generators of $\mathfrak{gl}(p|q)$ in *loc.cit.*, i.e. setting $E_{ab} \mapsto 0$, and an appropriate shift of the spectral parameter x.

the Lax matrix of size $n + m$. Evoking the invariance of the R-matrix (2.6) under the symmetric group $S(n + m)$, see (2.12), these Lax matrices can be obtained by permuting rows and columns of the one in (2.13), see Remark 2.11. The following family of such Lax matrices will be relevant to us in the following:

$$\bar{L}_a(y) = \bar{L}_a^{\mathsf{V}}(y) = \left(\begin{array}{c|c} \mathrm{Id}_a & \bar{\mathbf{K}} \\ \hline \mathbf{K} & y\mathrm{Id}_{n+m-a} + \mathbf{K}\bar{\mathbf{K}} \end{array} \right). \tag{2.17}$$

Let us first explain how these Lax matrices are related to those in (2.13). To this end, let $\bar{\mathsf{V}}$ be the superspace with a basis $\{\bar{v}_i\}_{i=1}^{n+m}$ whose parity sequence (2.1) is opposite to that of V, that is

$$|\bar{v}_i| = |v_{n+m+1-i}| \qquad \forall\, 1 \le i \le n + m. \tag{2.18}$$

The corresponding rational R-matrices are related via (2.12) with $\sigma(i) = n+m+1-i$, so that

$$\mathrm{J}_\sigma = \mathrm{J}_{n+m} = \left(\begin{array}{cccc} 0 & \cdots & 0 & 1 \\ \vdots & \reflectbox{$\ddots$} & \reflectbox{$\ddots$} & 0 \\ 0 & \reflectbox{$\ddots$} & \reflectbox{$\ddots$} & \vdots \\ 1 & 0 & \cdots & 0 \end{array} \right). \tag{2.19}$$

Then, as noted in Remark 2.11, the matrix $\hat{L}_{n+m-a}^{\bar{\mathsf{V}}}(z)$ obtained from (2.17) through

$$\hat{L}_{n+m-a}^{\bar{\mathsf{V}}}(x) = \mathrm{J}_{n+m}\bar{L}_a^{\mathsf{V}}(x)\mathrm{J}_{n+m}^{-1} \tag{2.20}$$

is Lax and has opposite $\mathbb{Z}_2$-grading to that of $\bar{L}_a^{\mathsf{V}}(x)$, i.e. the underlying vector super-spaces have opposite parity sequences. Using the notation $\bar{a} = n + m - a$ and $i' = n + m + 1 - i$, we find that

$$\hat{L}_{\bar{a}}^{\bar{\mathsf{V}}}(x) = \left(\begin{array}{c|c} x\mathrm{Id}_{\bar{a}} + \mathbf{Q}\bar{\mathbf{Q}} & \mathbf{Q} \\ \hline \bar{\mathbf{Q}} & \mathrm{Id}_{n+m-\bar{a}} \end{array} \right) \tag{2.21}$$

with

$$\bar{\mathbf{Q}} = \left(\begin{array}{cccc} \bar{\bar{\xi}}_{(\bar{a}+1)',1'} & \bar{\bar{\xi}}_{(\bar{a}+1)',2'} & \cdots & \bar{\bar{\xi}}_{(\bar{a}+1)',\bar{a}'} \\ \bar{\bar{\xi}}_{(\bar{a}+2)',1'} & \bar{\bar{\xi}}_{(\bar{a}+2)',2'} & \cdots & \bar{\bar{\xi}}_{(\bar{a}+2)',\bar{a}'} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{\xi}_{(n+m)',1'} & \bar{\xi}_{(n+m)',2'} & \cdots & \bar{\xi}_{(n+m)',\bar{a}'} \end{array} \right) \tag{2.22}$$

and

$$
\mathbf{Q} = \begin{pmatrix}
(-1)^{|(\bar{a}+1)'|}\xi_{1',(\bar{a}+1)'} & (-1)^{|(\bar{a}+2)'|}\xi_{1',(\bar{a}+2)'} & \cdots & (-1)^{|(n+m)'|}\xi_{1',(n+m)'} \\
(-1)^{|(\bar{a}+1)'|}\xi_{2',(\bar{a}+1)'} & (-1)^{|(\bar{a}+2)'|}\xi_{2',(\bar{a}+2)'} & \cdots & (-1)^{|(n+m)'|}\xi_{2',(n+m)'} \\
\vdots & \vdots & \ddots & \vdots \\
(-1)^{|(\bar{a}+1)'|}\xi_{\bar{a}',(\bar{a}+1)'} & (-1)^{|(\bar{a}+2)'|}\xi_{\bar{a}',(\bar{a}+2)'} & \cdots & (-1)^{|(n+m)'|}\xi_{\bar{a}',(n+m)'}
\end{pmatrix}.
\tag{2.23}
$$

Applying further the particle-hole transformation

$$
\bar{\xi}_{i'j'} \mapsto -(-1)^{|j'|}\xi_{ij}, \qquad \xi_{j'i'} \mapsto (-1)^{|i'|}\bar{\xi}_{ji} \qquad \forall\, 1 \le j \le \bar{a} < i \le n+m, \tag{2.24}
$$

we obtain the Lax matrix of (2.13) defined on the vector space $\bar{\mathsf{V}}$ with the opposite grading, i.e.

$$
L_{\bar{a}}^{\bar{\mathsf{V}}}(x) = \hat{L}_a^{\bar{\mathsf{V}}}(x)|_{p.h.}. \tag{2.25}
$$

It thus follows that $L_a^{\mathsf{V}}(x)$ of (2.13) and $\bar{L}_a^{\mathsf{V}}(x)$ of (2.17) satisfy the same RTT-relation (2.8).

Remark 2.26 The Lax matrix (2.17) is obtained through a particle-hole transformation (2.24) from that of [13, (3.4)] for $I = \{1, \ldots, a\}$, $J = \{a+1, \ldots, n+m\}$, the trivial representation of the additional copy of $\mathfrak{gl}(p|q)$, and an appropriate shift of the spectral parameter, cf. Footnote 1.

Let us now consider two copies of mutually supercommuting superoscillators $\{(\xi_{ij}^{[r]}, \bar{\xi}_{ji}^{[r]})\}_{1 \le j \le a}^{a < i \le n+m}$, where the superscript $r = 1, 2$ indicates whether they appear in $L_a^{\mathsf{V},[1]}(x)$ or $\bar{L}_a^{\mathsf{V},[2]}(y)$, respectively. The subsequent factorisation was considered in [13, §3.1]:

$$
L_a^{\mathsf{V},[1]}(x)\bar{L}_a^{\mathsf{V},[2]}(y) = \left(\begin{array}{c|c}
x\mathrm{Id}_a - \bar{\mathbf{K}}_1'\mathbf{K}_1' & \left((y-x)\mathrm{Id}_a + \bar{\mathbf{K}}_1'\mathbf{K}_1'\right)\bar{\mathbf{K}}_1' \\
\hline
-\mathbf{K}_1' & y\mathrm{Id}_{n+m-a} + \mathbf{K}_1'\bar{\mathbf{K}}_1'
\end{array}\right)
\left(\begin{array}{c|c}
\mathrm{Id}_a & \bar{\mathbf{K}}_2' \\
\hline
0 & \mathrm{Id}_{n+m-a}
\end{array}\right)
\tag{2.27}
$$

where

$$
\begin{aligned}
\mathbf{K}_1' = \mathbf{K}_1 - \mathbf{K}_2, &\qquad \bar{\mathbf{K}}_1' = \bar{\mathbf{K}}_1, \\
\bar{\mathbf{K}}_2' = \bar{\mathbf{K}}_2 + \bar{\mathbf{K}}_1, &\qquad \mathbf{K}_2' = \mathbf{K}_2,
\end{aligned}
\tag{2.28}
$$

and the subscript $r = 1, 2$ denotes the corresponding family of oscillators. As noted in [13, §3.1], the generators (2.28) are related to those in (2.13)–(2.17) through a similarity transformation:

$$
\mathbf{K}_r' = \mathbf{S}_a \mathbf{K}_r \mathbf{S}_a^{-1}, \qquad \bar{\mathbf{K}}_r' = \mathbf{S}_a \bar{\mathbf{K}}_r \mathbf{S}_a^{-1} \qquad (r = 1, 2) \tag{2.29}
$$

with

$$
\mathbf{S}_a = \exp\left[\sum_{1 \le i \le a}^{a < j \le n+m} \bar{\xi}_{ij}^{[1]}\xi_{ji}^{[2]}\right]. \tag{2.30}
$$

We note that all the summands in the exponent above are bosonic[2] and pairwise supercommute.

Remark 2.31 It immediately follows from (2.29), but can be also directly checked from (2.28), that the entries of the matrices $\mathbf{K}'_r$, $\bar{\mathbf{K}}'_r$ encode mutually supercommuting pairs of superoscillators.

It follows that for any $x_1, x_2 \in \mathbb{C}$, the matrix

$$\mathcal{L}_{x_1, x_2}(x) = \mathcal{L}^{\mathsf{V}}_{x_1, x_2}(x) = \left(\begin{array}{c|c} (x + x_1)\mathrm{Id}_a - \bar{\mathbf{K}}_1 \mathbf{K}_1 & \left((x_2 - x_1)\mathrm{Id}_a + \bar{\mathbf{K}}_1 \mathbf{K}_1\right) \bar{\mathbf{K}}_1 \\ \hline -\mathbf{K}_1 & (x + x_2)\mathrm{Id}_{n+m-a} + \mathbf{K}_1 \bar{\mathbf{K}}_1 \end{array} \right) \tag{2.32}$$

is a solution to the RTT-relation (2.8), hence, is Lax. Moreover, it arises through the fusion

$$L^{\mathsf{V},[1]}_a(x + x_1)\bar{L}^{\mathsf{V},[2]}_a(x + x_2) = \mathbf{S}_a \, \mathcal{L}^{\mathsf{V}}_{x_1, x_2}(x) \left(\begin{array}{c|c} \mathrm{Id}_a & \bar{\mathbf{K}}_2 \\ \hline 0 & \mathrm{Id}_{n+m-a} \end{array} \right) \mathbf{S}^{-1}_a \tag{2.33}$$

with the similarity transformation $\mathbf{S}_a$ of (2.30).

Remark 2.34 Similarly to [12, §8], we note that we can vice versa obtain the degenerate linear matrices $L^{\mathsf{V}}_a(x)$ and $\bar{L}^{\mathsf{V}}_a(x)$ of (2.13, 2.17) from the non-degenerate linear Lax matrix $\mathcal{L}^{\mathsf{V}}_{x_1, x_2}(x)$ of (2.32) via the *renormalized limit* procedures (which clearly preserve the property of being Lax):

$$
\begin{aligned}
L^{\mathsf{V}}_a(x) &= \lim_{t \to \infty} \left\{ \mathcal{L}^{\mathsf{V}}_{0,t}(x) \cdot \mathrm{diag}\Big(\underbrace{1, \ldots, 1}_{a}; \underbrace{\tfrac{1}{t}, \ldots, \tfrac{1}{t}}_{n+m-a} \Big) \right\}, \\
\bar{L}^{\mathsf{V}}_a(x) &= \lim_{t \to \infty} \left\{ \mathrm{diag}\Big(\underbrace{\tfrac{1}{t}, \ldots, \tfrac{1}{t}}_{a}; \underbrace{1, \ldots, 1}_{n+m-a} \Big) \cdot \mathcal{L}^{\mathsf{V}}_{t,0}(x) \right\}\Big|_{\bar{\xi}_{ij} \mapsto -\bar{\xi}_{ij}, \, \xi_{ij} \mapsto -\xi_{ij}}.
\end{aligned} \tag{2.35}
$$

3 Orthosymplectic Lax matrices

In this section, we set the notation for the orthosymplectic R-matrix and the corresponding Lax matrices, as well as discuss the invariance of this R-matrix that will be instrumental later on.

3.1 Orthosymplectic setup

Fix $N, m \geq 0$, and consider the set $\mathbb{I} := \{1, 2, \ldots, N + 2m\}$ equipped with an involution $'$:

$$i' := N + 2m + 1 - i. \tag{3.1}$$

---

2 Elements of a superalgebra are called "bosonic" or "fermionic" if their $\mathbb{Z}_2$-degree is $\bar{0}$ or $\bar{1}$, respectively.

Let V be a superspace with a $\mathbb{Z}_2$-homogeneous basis $v_1, \ldots, v_{N+2m}$ such that

$$\dim(V_{\bar{0}}) = N, \qquad \dim(V_{\bar{1}}) = 2m, \tag{3.2}$$

and the grading is $\mathbb{Z}_2$-symmetric in the following sense:

$$|v_i| = |v_{i'}| \qquad \forall\, 1 \le i \le N + 2m. \tag{3.3}$$

For the major part of our constructions (except for Sects. 5.3–5.4), we shall assume that N is even: $N = 2n$. In this case, we pick the following specific $\mathbb{Z}_2$-grading of V:

$$|i| := |v_i| = \begin{cases} \bar{0} & \text{for } 1 \le i \le n \\ \bar{1} & \text{for } n+1 \le i \le n+2m \\ \bar{0} & \text{for } n+2m+1 \le i \le 2n+2m \end{cases} \tag{3.4}$$

which corresponds to the following parity sequence, cf. (2.1):

$$\Upsilon_V = \Big(\underbrace{\bar{0}, \ldots, \bar{0}}_{n}, \underbrace{\bar{1}, \ldots, \bar{1}}_{2m}, \underbrace{\bar{0}, \ldots, \bar{0}}_{n} \Big). \tag{3.5}$$

Similarly to (2.4), we consider the permutation operator $\mathrm{P} \colon V \otimes V \to V \otimes V$ defined by

$$\mathrm{P} = \sum_{i,j=1}^{N+2m} (-1)^{|j|} e_{ij} \otimes e_{ji}. \tag{3.6}$$

We also consider the operator $\mathrm{Q} \colon V \otimes V \to V \otimes V$ defined by

$$\mathrm{Q} = \sum_{i,j=1}^{N+2m} (-1)^{|i||j|} \theta_i \theta_j\, e_{ij} \otimes e_{i'j'}. \tag{3.7}$$

Here, the sequence $\theta = \theta_V = (\theta_1, \ldots, \theta_{N+2m})$ of ± 1's is determined uniquely by the conditions

$$\theta_{i'} = (-1)^{|i|} \theta_i, \qquad \theta_{\le (n+m)} = 1, \tag{3.8}$$

so that

$$\theta = \theta_V = \Big(\underbrace{1, \ldots, 1}_{n+m}, \underbrace{-1, \ldots, -1}_{m}, \underbrace{1, \ldots, 1}_{n} \Big) \tag{3.9}$$

for the specific $\mathbb{Z}_2$-grading (3.4). Explicitly, the action of Q is given by:

$$\mathrm{Q}(v_a \otimes v_b) = \begin{cases} 0 & \text{if } b \ne a' \\ \sum_{i=1}^{N+2m} \theta_i\, v_i \otimes v_{i'} & \text{if } b = a',\, a > n+m \\ (-1)^{|a|} \sum_{i=1}^{N+2m} \theta_i\, v_i \otimes v_{i'} & \text{if } b = a',\, a \le n+m \end{cases} \tag{3.10}$$

We also introduce a constant κ via:

$$\kappa = \frac{N}{2} - m - 1 = n - m - 1 \,. \tag{3.11}$$

Consider the *rational* R-matrix (a super-version of the one considered in [37]):

$$R(x) = R^V(x) = x(x + \kappa)\mathrm{Id} + (x + \kappa)\mathrm{P} - x\mathrm{Q}\,, \tag{3.12}$$

which satisfies the Yang–Baxter equation with a spectral parameter (2.7) according to [1].

For any superalgebra A, an even matrix $L(x) = L^V(x) = (L_{ij}(x))_{i,j=1}^{N+2m} \in$ End $V \otimes A[[x, x^{-1}]]$ will be called an (orthosymplectic) *Lax matrix* if it satisfies the RTT-relation (2.8) with $R(x)$ of (3.12). Coefficient-wise, this is equivalent to the following system of relations (see [1]):

$$
\begin{aligned}
[L_{ij}(x), L_{k\ell}(y)] = &\frac{(-1)^{|i||j|+|i||k|+|j||k|}}{x-y}\Big(L_{kj}(y)L_{i\ell}(x) - L_{kj}(x)L_{i\ell}(y)\Big) \\
&+ \frac{1}{x-y+\kappa}\Bigg(\delta_{ki'}\sum_{p=1}^{N+2m}L_{pj}(x)L_{p'\ell}(y)(-1)^{|i|+|i||j|+|j||p|}\theta_i\theta_p \\
&- \delta_{\ell j'}\sum_{p=1}^{N+2m}L_{kp'}(y)L_{ip}(x)(-1)^{|p|+|j|+|i||k|+|i||p|+|j||k|}\theta_p\theta_j\Bigg)\,.
\end{aligned}
\tag{3.13}
$$

Remark 3.14 The data of a Lax matrix $(L_{ij}(x))_{i,j=1}^{N+2m}$ with $L_{ij}(x) \in \delta_{ij}+x^{-1}A[[x^{-1}]]$ is equivalent to an algebra homomorphism $X^{\mathrm{rtt}}(\mathfrak{osp}(V)) \to A$ from an RTT extended orthosymplectic Yangian.

Remark 3.15 We recover orthogonal and symplectic types as special cases of the above setup:

- For $m = 0$, we have $\theta = \underbrace{(1, \ldots, 1)}_{2n}$ and $|i| = \bar{0}$ for all $1 \le i \le 2n$, so that $R(x)$ of (3.12) coincides with the D_n-type rational R-matrix of [12, (1.19)].
- For $n = 0$, we have $\theta = (\underbrace{1, \ldots, 1}_{m}, \underbrace{-1, \ldots, -1}_{m})$ and $|i| = \bar{1}$ for all $1 \le i \le 2m$, so that our $R(-x)$ of (3.12) coincides with the C_m-type rational R-matrix of [12, (1.19)].

Remark 3.16 Let $\tilde{V}$ be another superspace with a $\mathbb{C}$-basis $\tilde{v}_1, \ldots, \tilde{v}_{N+2m}$ satisfying (3.2, 3.3). Pick a permutation $\sigma \in S(n + m)$ such that $v_i \in V$ and $\tilde{v}_{\sigma(i)} \in \tilde{V}$ have the same $\mathbb{Z}_2$-grading for all $1 \le i \le n+m$, and extend it to $\sigma \in S(N + 2m)$ via $\sigma(i') = \sigma(i)'$. Define a superspace isomorphism $\tilde{\mathrm{J}}_\sigma : V \xrightarrow{\sim} \tilde{V}$ via $v_i \mapsto \tilde{v}_{\sigma(i)}$. The corresponding R-matrices (3.12) are related via

$$R^{\tilde{V}}(x) = (\tilde{\mathrm{J}}_\sigma \otimes \tilde{\mathrm{J}}_\sigma)\,R^V(x)\,(\tilde{\mathrm{J}}_\sigma \otimes \tilde{\mathrm{J}}_\sigma)^{-1}\,. \tag{3.17}$$

As a result, if $L^V(x) = (L_{ij}(x))_{i,j=1}^{N+2m}$ is a Lax matrix corresponding to the fixed parity Υ_V, then $L^{\tilde{V}}(x) := \tilde{\mathrm{J}}_\sigma L^V(x) \tilde{\mathrm{J}}_\sigma^{-1}$ is a Lax matrix corresponding to the parity $\Upsilon_{\tilde{V}}$, cf. Remark 2.11.

3.2 Symmetries of the orthosymplectic *R*-matrix

In this section, we establish the invariance of the *R*-matrix (3.12) under certain *graded permutation* matrices that will be used in the later constructions. The proofs are based on direct computations of the commutators of the corresponding operators with P, Q of (3.6, 3.7). To this end, let us recall the explicit formulas for the action of P and Q, cf. (2.5, 3.10):

$$
\begin{aligned}
&\mathrm{P}\colon v_i \otimes v_j \mapsto (-1)^{|i||j|}\, v_j \otimes v_i\,, \\
&\mathrm{Q}\colon v_i \otimes v_{i'} \mapsto (-1)^{|i|}\theta_i \sum_{j=1}^{N+2m} \theta_j\, v_j \otimes v_{j'}\,, \quad v_i \otimes v_{j \neq i'} \mapsto 0\,.
\end{aligned}
\tag{3.18}
$$

Henceforth, J_r will denote the $r \times r$ matrix with "1" on the antidiagonal, cf. (2.19).

Lemma 3.19 *The R-matrix* (3.12) *commutes with the tensor product of two matrices*

$$
\mathrm{J}_\theta = \left(
\begin{array}{c|c}
0 & -\mathrm{J}_n \\
\hline
\mathrm{J}_{n+2m} & 0
\end{array}
\right)
\tag{3.20}
$$

that is, $[R(x), \mathrm{J}_\theta \otimes \mathrm{J}_\theta] = 0$.

Proof The tensor product of the matrices J_θ acts explicitly via

$$
\mathrm{J}_\theta \otimes \mathrm{J}_\theta \colon v_i \otimes v_j \mapsto (-1)^{\delta_{i>n+2m}+\delta_{j>n+2m}}\, v_{i'} \otimes v_{j'}\,.
\tag{3.21}
$$

We thus have:

$$
\mathrm{P}\,(\mathrm{J}_\theta \otimes \mathrm{J}_\theta)\colon v_i \otimes v_j \mapsto (-1)^{|i'||j'|+\delta_{i>n+2m}+\delta_{j>n+2m}}\, v_{j'} \otimes v_{i'}\,,
\tag{3.22}
$$

$$
(\mathrm{J}_\theta \otimes \mathrm{J}_\theta)\,\mathrm{P}\colon v_i \otimes v_j \mapsto (-1)^{|i||j|+\delta_{j>n+2m}+\delta_{i>n+2m}}\, v_{j'} \otimes v_{i'}\,,
\tag{3.23}
$$

so that $[\mathrm{P}, \mathrm{J}_\theta \otimes \mathrm{J}_\theta] = 0$ since $|\iota| = |\iota'|$ for all ι.

To check the invariance of Q, we first note that

$$
\mathrm{Q}\,(\mathrm{J}_\theta \otimes \mathrm{J}_\theta)\,(v_i \otimes v_j) = 0 = (\mathrm{J}_\theta \otimes \mathrm{J}_\theta)\,\mathrm{Q}\,(v_i \otimes v_j) \quad \text{for } j \neq i'.
$$

It thus remains to compare the images of $v_i \otimes v_{i'}$ under both $Q\,(J_\theta \otimes J_\theta)$ and $(J_\theta \otimes J_\theta)\,Q$:

$$Q\,(J_\theta \otimes J_\theta)\colon v_i \otimes v_{i'} \mapsto (-1)^{|i'|+\delta_{i>n+2m}+\delta_{i'>n+2m}}\theta_{i'} \sum_{j=1}^{N+2m} \theta_j\, v_j \otimes v_{j'}, \tag{3.24}$$

$$(J_\theta \otimes J_\theta)\,Q\colon v_i \otimes v_{i'} \mapsto (-1)^{|i|}\theta_i \sum_{j=1}^{N+2m} (-1)^{\delta_{j>n+2m}+\delta_{j'>n+2m}}\theta_j\, v_{j'} \otimes v_j . \tag{3.25}$$

The right-hand sides of (3.24, 3.25) coincide due to the first equality in (3.8) and the identity

$$(-1)^{\delta_{i>n+2m}+\delta_{i'>n+2m}} = (-1)^{\delta_{i>n+2m}+\delta_{i\leq n}} = -(-1)^{|i|}$$

that follows immediately from (3.4). This verifies $[Q, J_\theta \otimes J_\theta] = 0$.

As $J_\theta \otimes J_\theta$ commutes with both P and Q, so it does with the R-matrix (3.12). $\square$

Lemma 3.26 *The R-matrix* (3.12) *commutes with the tensor product of two matrices*

$$\tilde{J} = \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & \mathrm{Id}_{N+2m-2} & 0 \\ \hline 1 & 0 & 0 \end{array}\right) \tag{3.27}$$

that is, $[R(x), \tilde{J} \otimes \tilde{J}] = 0.$

Proof The tensor product of the matrices $\tilde{J}$ acts explicitly via

$$\tilde{J} \otimes \tilde{J}\colon v_i \otimes v_j \mapsto v_{\tilde{i}} \otimes v_{\tilde{j}} \quad \text{with} \quad \tilde{i} = \begin{cases} i' & \text{if } i = 1, 1' \\ i & \text{if } i = 2, \ldots, 2' \end{cases}. \tag{3.28}$$

We thus have:

$$P(\tilde{J} \otimes \tilde{J})\colon v_i \otimes v_j \mapsto (-1)^{|\tilde{i}||\tilde{j}|}\, v_{\tilde{j}} \otimes v_{\tilde{i}}, \tag{3.29}$$

$$(\tilde{J} \otimes \tilde{J})P\colon v_i \otimes v_j \mapsto (-1)^{|i||j|}\, v_{\tilde{j}} \otimes v_{\tilde{i}}, \tag{3.30}$$

so that $[P, \tilde{J} \otimes \tilde{J}] = 0$ since $|\tilde{\iota}| = |\iota|$ for all ι.

To check the invariance of Q, we first note that

$$Q(\tilde{J} \otimes \tilde{J})(v_i \otimes v_J) = 0 = (\tilde{J} \otimes \tilde{J})Q\,(v_i \otimes v_J) \quad \text{for} \quad J \neq i'.$$

It thus remains to compare the images of $v_i \otimes v_{i'}$ under both $Q(\tilde{J} \otimes \tilde{J})$ and $(\tilde{J} \otimes \tilde{J})Q$:

$$Q(\tilde{J} \otimes \tilde{J}): v_i \otimes v_{i'} \mapsto (-1)^{|\tilde{i}|}\theta_{\tilde{i}} \sum_{j=1}^{N+2m} \theta_j \, v_j \otimes v_{j'}, \tag{3.31}$$

$$(\tilde{J} \otimes \tilde{J})Q: v_i \otimes v_{i'} \mapsto (-1)^{|i|}\theta_i \sum_{j=1}^{N+2m} \theta_j \, v_{\tilde{j}} \otimes v_{\tilde{j'}}, \tag{3.32}$$

and the two images coincides as $\theta_{\tilde{i}} = \theta_i$ (which uses that $|v_1| = \bar{0}$).

As $\tilde{J} \otimes \tilde{J}$ commutes with both P and Q, so it does with the R-matrix (3.12).   □

Lemma 3.33 *The R-matrix* (3.12) *commutes with the tensor product of two matrices*

$$\hat{J}_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & G_{n-1,m} & 0 \\ \hline 0 & J_{n+m-1} & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad with \quad G_{n-1,m} = \begin{pmatrix} 0 & -J_{n-1} \\ \hline J_m & 0 \end{pmatrix} \tag{3.34}$$

that is, $[R(x), \hat{J}_\theta \otimes \hat{J}_\theta] = 0.$

Proof As $\hat{J}_\theta = \tilde{J} \cdot J_\theta$, the result follows immediately from the previous two lemmas.   □

Lemma 3.35 *For* $N = 0$, *the R-matrix* (3.12) *commutes with the tensor product of two matrices*

$$Id_\theta = \begin{pmatrix} Id_m & 0 \\ \hline 0 & -Id_m \end{pmatrix} \tag{3.36}$$

that is, $[R(x), Id_\theta \otimes Id_\theta] = 0.$

Proof For $N = 0$, the θ of (3.9) are given by $\theta_i = (-1)^{\delta_{i>m}}$, so that

$$Id_\theta \otimes Id_\theta: v_i \otimes v_j \mapsto \theta_i \theta_j \, v_i \otimes v_j. \tag{3.37}$$

The commutativity $[P, Id_\theta \otimes Id_\theta] = 0$ follows immediately from

$$P(Id_\theta \otimes Id_\theta): v_i \otimes v_j \mapsto (-1)^{|i||j|}\theta_i \theta_j \, v_j \otimes v_i, \tag{3.38}$$

$$(Id_\theta \otimes Id_\theta)P: v_i \otimes v_j \mapsto (-1)^{|i||j|}\theta_i \theta_j \, v_j \otimes v_i. \tag{3.39}$$

Similarly, for the operator Q we have:

$$Q\,(\mathrm{Id}_\theta \otimes \mathrm{Id}_\theta)\colon v_i \otimes v_{i'} \mapsto (-1)^{|i|}\theta_{i'}\sum_{j=1}^{2m}\theta_j\,v_j \otimes v_{j'}\,, \tag{3.40}$$

$$(\mathrm{Id}_\theta \otimes \mathrm{Id}_\theta)\,Q\colon v_i \otimes v_{i'} \mapsto (-1)^{|i|}\theta_i\sum_{j=1}^{2m}\theta_{j'}\,v_j \otimes v_{j'}\,, \tag{3.41}$$

and the two images coincide as $\theta_\iota = -\theta_{\iota'}$ for all ι. This implies $[Q, \mathrm{Id}_\theta \otimes \mathrm{Id}_\theta] = 0$.

As $\mathrm{Id}_\theta \otimes \mathrm{Id}_\theta$ commutes with both P and Q, so it does with the R-matrix (3.12). $\square$

Remark 3.42 (a) More generally, the R-matrix (3.12) commutes with $\sum_{i=1}^{n+m}(a_i e_{ii} + b_i e_{ii'} + c_i e_{i'i} + d_i e_{i'i'})$ with either $a_i, d_i \in \{\pm 1\}$ and $b_i = c_i = 0$ (for which we set $\gamma_i := a_i d_i$) or $b_i, c_i \in \{\pm 1\}$ and $a_i = d_i = 0$ (for which we set $\gamma_i := (-1)^{|i|}b_i c_i$), and such that γ_i are the same for all $1 \le i \le n + m$.

(b) According to Remark 3.16, using some other permutation matrices ($\tilde{\mathrm{J}}_\sigma$ from *loc.cit.*) will rather produce orthosymplectic Lax matrices for other $\mathbb{Z}_2$-gradings of V, see Remarks 4.62 and 5.89.

4 Linear orthosymplectic Lax matrices

In this section, we construct some linear orthosymplectic Lax matrices of superoscillator type.

4.1 Degenerate linear orthosymplectic Lax matrices

In this subsection, we construct a degenerate linear orthosymplectic Lax matrix for the parity sequence (3.5). To this end, let us consider first bosonic pairs of superoscillators:

$$\left\{(\mathbf{a}_{ij}, \bar{\mathbf{a}}_{ji})\,\middle|\,n+2m+1 \le i \le 2n+2m-1\,,\ 1 \le j \le n-1\,,\ i+j \le 2n+2m\right\}, \tag{4.1}$$

$$\left\{(\mathbf{b}_{ij}, \bar{\mathbf{b}}_{ji})\,\middle|\,n+m+1 \le i \le n+2m\,,\ n+1 \le j \le n+m\,,\ i+j \le 2n+2m+1\right\}, \tag{4.2}$$

with $|\mathbf{a}_{ij}| = |\bar{\mathbf{a}}_{ji}| = |i| + |j| = \bar{0}$, $|\mathbf{b}_{ij}| = |\bar{\mathbf{b}}_{ji}| = |i| + |j| = \bar{0}$, and the only nonzero supercommutators

$$[\mathbf{a}_{ij}, \bar{\mathbf{a}}_{ji}] = \mathbf{a}_{ij}\bar{\mathbf{a}}_{ji} - \bar{\mathbf{a}}_{ji}\mathbf{a}_{ij} = 1\,, \qquad [\mathbf{b}_{ij}, \bar{\mathbf{b}}_{ji}] = \mathbf{b}_{ij}\bar{\mathbf{b}}_{ji} - \bar{\mathbf{b}}_{ji}\mathbf{b}_{ij} = 1\,. \tag{4.3}$$

In addition, we also consider fermionic pairs of superoscillators

$$\left\{ (\mathbf{c}_{ij}, \bar{\mathbf{c}}_{ji}) \,\Big|\, n + m + 1 \leq i \leq n + 2m \,,\; 1 \leq j \leq n \right\} \tag{4.4}$$

with $|\mathbf{c}_{ij}| = |\bar{\mathbf{c}}_{ji}| = |i| + |j| = \bar{1}$ and the only nonzero supercommutators

$$[\mathbf{c}_{ij}, \bar{\mathbf{c}}_{ji}] = \mathbf{c}_{ij} \bar{\mathbf{c}}_{ji} + \bar{\mathbf{c}}_{ji} \mathbf{c}_{ij} = 1 \,. \tag{4.5}$$

We encode the above bosonic and fermionic generators by the corresponding three pairs of matrices:

$$\bar{\mathbf{A}} = \begin{pmatrix} \bar{\mathbf{a}}_{1,n+2m+1} & \cdots & \bar{\mathbf{a}}_{1,2n+2m-1} & 0 \\ \vdots & \reflectbox{$\ddots$} & 0 & -\bar{\mathbf{a}}_{1,2n+2m-1} \\ \bar{\mathbf{a}}_{n-1,n+2m+1} & 0 & \reflectbox{$\ddots$} & \vdots \\ 0 & -\bar{\mathbf{a}}_{n-1,n+2m+1} & \cdots & -\bar{\mathbf{a}}_{1,n+2m+1} \end{pmatrix}, \tag{4.6}$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_{n+2m+1,1} & \cdots & \mathbf{a}_{n+2m+1,n-1} & 0 \\ \vdots & \reflectbox{$\ddots$} & 0 & -\mathbf{a}_{n+2m+1,n-1} \\ \mathbf{a}_{2n+2m-1,1} & 0 & \reflectbox{$\ddots$} & \vdots \\ 0 & -\mathbf{a}_{2n+2m-1,1} & \cdots & -\mathbf{a}_{n+2m+1,1} \end{pmatrix} \tag{4.7}$$

that are (skew-symmetric along the antidiagonal) $n \times n$ matrices encoding (4.1),

$$\bar{\mathbf{B}} = \begin{pmatrix} \bar{\mathbf{b}}_{n+1,n+m+1} & \cdots & \bar{\mathbf{b}}_{n+1,n+2m-1} & 2\bar{\mathbf{b}}_{n+1,n+2m} \\ \vdots & \reflectbox{$\ddots$} & 2\bar{\mathbf{b}}_{n+2,n+2m-1} & \bar{\mathbf{b}}_{n+1,n+2m-1} \\ \bar{\mathbf{b}}_{n+m-1,n+m+1} & 2\bar{\mathbf{b}}_{n+m-1,n+m+2} & \reflectbox{$\ddots$} & \vdots \\ 2\bar{\mathbf{b}}_{n+m,n+m+1} & \bar{\mathbf{b}}_{n+m-1,n+m+1} & \cdots & \bar{\mathbf{b}}_{n+1,n+m+1} \end{pmatrix}, \tag{4.8}$$

$$\mathbf{B} = \begin{pmatrix} \mathbf{b}_{n+m+1,n+1} & \cdots & \mathbf{b}_{n+m+1,n+m-1} & \mathbf{b}_{n+m+1,n+m} \\ \vdots & \reflectbox{$\ddots$} & \mathbf{b}_{n+m+2,n+m-1} & \mathbf{b}_{n+m+1,n+m-1} \\ \mathbf{b}_{n+2m-1,n+1} & \mathbf{b}_{n+2m-1,n+2} & \reflectbox{$\ddots$} & \vdots \\ \mathbf{b}_{n+2m,n+1} & \mathbf{b}_{n+2m-1,n+1} & \cdots & \mathbf{b}_{n+m+1,n+1} \end{pmatrix} \tag{4.9}$$

that are (symmetric along the antidiagonal) $m \times m$ matrices encoding (4.2),

$$\bar{\mathbf{C}} = \begin{pmatrix} \bar{\mathbf{c}}_{1,n+m+1} & \cdots & \bar{\mathbf{c}}_{1,n+2m} \\ \vdots & \ddots & \vdots \\ \bar{\mathbf{c}}_{n,n+m+1} & \cdots & \bar{\mathbf{c}}_{n,n+2m} \end{pmatrix}, \qquad \mathbf{C} = \begin{pmatrix} \mathbf{c}_{n+m+1,1} & \cdots & \mathbf{c}_{n+m+1,n} \\ \vdots & \ddots & \vdots \\ \mathbf{c}_{n+2m,1} & \cdots & \mathbf{c}_{n+2m,n} \end{pmatrix} \tag{4.10}$$

that are $n \times m$ and $m \times n$ matrices encoding (4.4).

Now we are ready to present the main result of this subsection:

Theorem 4.11 *The following is a solution to the RTT-relation* (2.8) *with the R-matrix* (3.12)*:*

$$L(x) = \left(\begin{array}{c|c} x\mathrm{Id}_{n+m} - \bar{\mathbf{K}}\mathbf{K} & \bar{\mathbf{K}} \\ \hline -\mathbf{K} & \mathrm{Id}_{n+m} \end{array} \right) \tag{4.12}$$

with

$$\bar{\mathbf{K}} = \left(\begin{array}{c|c} \bar{\mathbf{C}} & \bar{\mathbf{A}} \\ \hline \bar{\mathbf{B}} & -\mathrm{J}_m \bar{\mathbf{C}}^t \mathrm{J}_n \end{array} \right) \quad \text{and} \quad \mathbf{K} = \left(\begin{array}{c|c} \mathbf{C} & -\mathbf{B} \\ \hline \mathbf{A} & \mathrm{J}_n \mathbf{C}^t \mathrm{J}_m \end{array} \right). \tag{4.13}$$

Here, t denotes the standard (bosonic) transpose, J_n is the $n \times n$ matrix with "1" on the antidiagonal, and the matrices $\bar{\mathbf{A}}, \mathbf{A}, \bar{\mathbf{B}}, \mathbf{B}, \bar{\mathbf{C}}, \mathbf{C}$ are as in (4.6)–(4.10)*.*

Proof The proof is straightforward and is analogous to that for C- and D-types, directly verifying the commutation relations using Lemma 4.24 below. In the first step, we insert the relations (4.25) and (4.26) into (3.13) and split the resulting equations into 4×4 block structure according to the block structure of the Lax matrices (4.12). The supercommutator on the left-hand side of (3.13) indicates that the right-hand side is symmetric under the combined exchange of $i \leftrightarrow k$, $j \leftrightarrow \ell$ and $x \leftrightarrow y$ when multiplied by the factor $-(-1)^{(|i|+|j|)(|k|+|\ell|)}$. This symmetry also holds for both terms individually on the right-hand side. For the first term, proportional to $(x-y)^{-1}$, the symmetry follows from the relation

$$L_{kj}(y)L_{i\ell}(x) - L_{i\ell}(x)L_{kj}(y) = L_{kj}(x)L_{i\ell}(y) - L_{i\ell}(y)L_{kj}(x). \tag{4.14}$$

For the second term, proportional to $(x - y + \kappa)^{-1}$, the symmetry is less obvious. It follows by noting that

$$\frac{1}{x-y+\kappa} \sum_p L_{pj}(x)L_{p'\ell}(y)(-1)^{|j||p|}\theta_p + \frac{(-1)^{|\ell||j|}}{y-x+\kappa} \sum_p L_{p\ell}(y)L_{p'j}(x)(-1)^{|\ell||p|}\theta_p$$
$$= \delta_{j\ell'}\theta_{j'} \frac{\kappa(x+y+\kappa)}{(x-y+\kappa)(y-x+\kappa)} \tag{4.15}$$

and

$$\frac{1}{x-y+\kappa} \sum_p L_{kp'}(y)L_{ip}(x)(-1)^{|i||p|}\theta_{p'} + \frac{(-1)^{|i||k|}}{y-x+\kappa} \sum_p L_{ip'}(x)L_{kp}(y)(-1)^{|k||p|}\theta_{p'}$$
$$= \delta_{i'k}\theta_i \frac{\kappa(x+y+\kappa)}{(x-y+\kappa)(y-x+\kappa)}. \tag{4.16}$$

These equations are verified using Lemma 4.24. Thus, the $4 \times 4 = 16$ equations arising from the block structure are reduced to 10 equations. In the following, we verify the remaining commutation relations.

We start with the "diagonal terms" where the indices are chosen such that the commutator on the left-hand side is among the elements of the Lax matrices within

the same block. In this case, the second term on the right-hand side always vanishes because of the Kronecker deltas. Further, unless we are considering the upper left block the first term on the right-hand side has to vanish as the entries of the Lax matrix will not depend on the spectral parameter. The case with $1 \leq i, j, k, \ell \leq n + m$ follows from the following equality:

$$\left[(\bar{\mathbf{K}}\mathbf{K})_{ij}, (\bar{\mathbf{K}}\mathbf{K})_{k\ell} \right] = (-1)^{|i||j|+|i||k|+|j||k|} \left(\delta_{kj} (\bar{\mathbf{K}}\mathbf{K})_{i\ell} - \delta_{i\ell} (\bar{\mathbf{K}}\mathbf{K})_{kj} \right) . \quad (4.17)$$

This equality follows in turn from

$$\mathbf{K}_{n+m-i+1,j} \bar{\mathbf{K}}_{k,n+m-\ell+1} - (-1)^{(|i|+|j|)(|k|+|\ell|)} \bar{\mathbf{K}}_{k,n+m-\ell+1} \mathbf{K}_{n+m-i+1,j} =$$
$$(-1)^{|j|} \delta_{i\ell} \delta_{jk} - (-1)^{|i||j|+|j|} \delta_{ik} \delta_{j\ell} \quad (4.18)$$

for $1 \leq i, j, k, \ell \leq n + m$, or equivalently (written in terms of the Lax matrix)

$$\left[L(x)_{i'j}, L(y)_{k\ell'} \right] = (-1)^{|i||j|+|j|} \delta_{ik} \delta_{j\ell} - (-1)^{|j|} \delta_{i\ell} \delta_{jk} . \quad (4.19)$$

This leaves us with 6 equations to verify. The case

$$[L_{ij}(x), L_{k\ell'}(y)] = \frac{1}{x-y}(-1)^{|i||j|+|i||k|+|j||k|} \left(L_{kj}(y) L_{i\ell'}(x) - L_{kj}(x) L_{i\ell'}(y) \right)$$
$$- \frac{1}{x-y+\kappa} \theta_j (-1)^{|j|+|i||k|+|j||k|} \delta_{\ell j} \sum_p L_{kp'}(y) L_{ip}(x) (-1)^{|p|+|i||p|} \theta_p \quad (4.20)$$

with $1 \leq i, j, k, \ell \leq n + m$ is reduced to proving

$$[(\bar{\mathbf{K}}\mathbf{K})_{ij}, \bar{\mathbf{K}}_{k,n+m-\ell+1}] =$$
$$(-1)^{|i||j|+|i||k|+|j||k|} \delta_{kj} \bar{\mathbf{K}}_{i,n+m-\ell+1} + (-1)^{(|i|+|j|)|k|} \delta_{\ell j'} \bar{\mathbf{K}}_{k,n+m-i+1} . \quad (4.21)$$

Similarly, the case

$$[L_{ij}(x), L_{k'\ell}(y)] = \frac{1}{x-y}(-1)^{|i||j|+|i||k|+|j||k|} \left(L_{k'j}(y) L_{i\ell}(x) - L_{k'j}(x) L_{i\ell}(y) \right)$$
$$+ \frac{1}{x-y+\kappa}(-1)^{|i|+|i||j|} \theta_i \delta_{ki} \sum_p L_{pj}(x) L_{p'\ell}(y) (-1)^{|j||p|} \theta_p \quad (4.22)$$

with $1 \leq i, j, k, \ell \leq n + m$ is reduced to proving

$$[(\bar{\mathbf{K}}\mathbf{K})_{ij}, \mathbf{K}_{n+m-k+1,\ell}] =$$
$$- (-1)^{|i||j|+|i||k|+|j||k|} \delta_{i\ell} \mathbf{K}_{n+m-k+1,j} - (-1)^{|j|+|i|+|i||j|} \delta_{ki} \mathbf{K}_{n+m-j+1,\ell} . \quad (4.23)$$

Both equations (4.21) and (4.23) can be derived directly from (4.18).

The case with i, j', k', ℓ for $1 \leq i, j, k, \ell \leq n + m$ is verified using (4.19) and Lemma 4.24, while the remaining three equations follow directly from Lemma 4.24.

This completes our proof of the theorem.   □

Lemma 4.24 *For the matrix $L(x)$ of* (4.12)*, we have the following matrix equalities:*

$$\theta_{j'} \sum_p (-1)^{|j||p|} \theta_p \, L_{pj}(x) L_{p'\ell}(y) = \left(\begin{array}{c|c} -(x-y+\kappa)\mathrm{J}_{n+m}\mathbf{K} & (x+\kappa)\mathrm{J}_{n+m} \\ \hline y\mathrm{J}_{n+m} & 0 \end{array} \right)_{j,\ell}$$

$$\tag{4.25}$$

and

$$\theta_i \sum_p (-1)^{|p|+|i||p|} \theta_p \, L_{kp'}(y) L_{ip}(x) = \left(\begin{array}{c|c} (x-y+\kappa)\bar{\mathbf{K}}\mathrm{J}_{n+m} & y\mathrm{J}_{n+m} \\ \hline (x+\kappa)\mathrm{J}_{n+m} & 0 \end{array} \right)_{k,i} . \tag{4.26}$$

Proof Let us verify (4.25). For the lower right block, we use that

$$(-1)^{|i||j|}\theta_{i'} L_{i'j}(x) + (-1)^{|j|}\theta_j L_{j'i}(x) = 0 \quad \text{for} \quad n+m+1 \leq i,j \leq 2n+2m, \tag{4.27}$$

which can be shown using the equality

$$\bar{\mathbf{K}}_{j,n+m-i+1} = -(-1)^{|i||j|}\bar{\mathbf{K}}_{i,n+m-j+1} \quad \text{for} \quad 1 \leq i,j \leq n+m. \tag{4.28}$$

For the upper right block, we use that

$$(-1)^{|j||\ell|+|j|}\theta_j\theta_\ell \, L_{\ell j}(x) + \sum_{p=1}^{n+m} \theta_j(-1)^{|j||p|+|j|+|p|}\theta_p \, L_{p'j}(x) L_{p\ell'}(y) = (x+\kappa)\delta_{j\ell} \tag{4.29}$$

for any $1 \leq j, \ell \leq n+m$. The proof of this follows from the equality

$$-\sum_{p=1}^{n+m} (-1)^{|j||p|+|j|+|p|} \mathbf{K}_{n+m-p+1,j} \bar{\mathbf{K}}_{p,n+m-\ell+1} - (-1)^{|j||\ell|+|\ell|}(\bar{\mathbf{K}}\mathbf{K})_{\ell j} = \kappa \, \delta_{j\ell} \tag{4.30}$$

for any $1 \leq j, \ell \leq n+m$. For the lower left block, we use the relation

$$L_{j\ell}(y) + \theta_j \sum_{p=1}^{n+m} (-1)^{|j||p|}\theta_p \, L_{pj'}(x) L_{p'\ell}(y) = y\delta_{j\ell} \quad \text{for} \quad 1 \leq j, \ell \leq n+m \tag{4.31}$$

based on

$$(\bar{\mathbf{K}}\mathbf{K})_{j\ell} + \sum_{p=1}^{n+m} (-1)^{|j||p|} \bar{\mathbf{K}}_{p,n+m-j+1}\mathbf{K}_{n+m-p+1,\ell} = 0 \quad \text{for} \quad 1 \leq j, \ell \leq n+m, \tag{4.32}$$

which follows from (4.28). Finally, for the upper left block, we note that

$$\sum_{p=1}^{n+m}(-1)^{|j||p|+|j|}L_{pj}(x)L_{p'\ell}(y) + \sum_{p=1}^{n+m}(-1)^{|j||p|+|j|+|p|}L_{p'j}(x)L_{p\ell}(y) =$$
$$-(x - y + \kappa)\mathbf{K}_{n+m-j+1,\ell} \tag{4.33}$$

for any $1 \le j, \ell \le n + m$. This relation is equivalent to

$$\sum_{p=1}^{n+m}(-1)^{|j||p|+|j|}(\bar{\mathbf{K}}\mathbf{K})_{pj}\mathbf{K}_{n+m-p+1,\ell} + \sum_{p=1}^{n+m}(-1)^{|j||p|+|j|+|p|}\mathbf{K}_{n+m-p+1,j}(\bar{\mathbf{K}}\mathbf{K})_{p\ell}$$
$$= -\kappa\,\mathbf{K}_{n+m-j+1,\ell}. \tag{4.34}$$

Due to the symmetry

$$(-1)^{|j||\ell|+|j|+|\ell|}\,\mathbf{K}_{n+m-\ell+1,j} = -\mathbf{K}_{n+m-j+1,\ell} \quad \text{for} \quad 1 \le j, \ell \le n + m, \tag{4.35}$$

the relation (4.34) can be easily verified using (4.30).

The proof of (4.26) is completely analogous; we leave details to the interested reader.   □

Remark 4.36 In particular, we recover orthogonal and symplectic degenerate linear Lax matrices:

- For $m = 0$, we recover precisely the D_n-type Lax matrix of [16, (2.231)], constructed first in [9, (4.3)].
- For $n = 0$, we get

$$L(-x) = \left(\begin{array}{c|c} -x\mathrm{Id}_m + \bar{\mathbf{B}}\mathbf{B} & \bar{\mathbf{B}} \\ \hline \mathbf{B} & \mathrm{Id}_m \end{array}\right) = -L_{FKT}(x)\left(\begin{array}{c|c} \mathrm{Id}_m & 0 \\ \hline 0 & -\mathrm{Id}_m \end{array}\right), \tag{4.37}$$

where $L_{FKT}(x)$ is the C_m-type Lax matrix of [12, (8.55)], discovered first in [16, (3.50)]. We note that the constant matrix above is Id_θ of (3.36) and hence it can be dropped out of the RTT-relation due to the invariance $[R(x), \mathrm{Id}_\theta \otimes \mathrm{Id}_\theta] = 0$ established in Lemma 3.35.

4.2 Non-degenerate linear Lax matrix through the fusion of two degenerate

Similarly to the $\mathfrak{gl}(n|m)$-case of Sect. 2, one can fuse two degenerate orthosymplectic Lax matrices to obtain a non-degenerate linear one. The main result of this subsection is Proposition 4.55.

Evoking the Lax matrix $L(x)$ of (4.12), we define

$$\bar{L}_\theta(x) = \mathrm{J}_\theta L(x)\mathrm{J}_\theta^{-1} \tag{4.38}$$

with J_θ as in (3.20):

$$J_\theta = \left(\begin{array}{c|c} 0 & -J_n \\ \hline J_{n+2m} & 0 \end{array}\right), \qquad J_\theta^{-1} = \left(\begin{array}{c|c} 0 & J_{n+2m} \\ \hline -J_n & 0 \end{array}\right). \tag{4.39}$$

Due to the invariance $[R(x), J_\theta \otimes J_\theta] = 0$ of the orthosymplectic R-matrix (3.12), established in Lemma 3.19, the matrix $\bar{L}_\theta(x)$ in (4.38) is a solution to the same RTT-relation (2.8), hence, is a Lax matrix.

Remark 4.40 In contrast to (2.20), $\mathbb{Z}_2$-gradings of the Lax matrices $\bar{L}_\theta(x)$ and $L(x)$ coincide.

Explicitly, we have:

$$\bar{L}_\theta(y) = \left(\begin{array}{c|c} \mathrm{Id}_{n+m} & \mathbf{K}^\theta \\ \hline \bar{\mathbf{K}}^\theta & y\mathrm{Id}_{n+m} + \bar{\mathbf{K}}^\theta \mathbf{K}^\theta \end{array}\right) \tag{4.41}$$

with

$$\bar{\mathbf{K}}^\theta = \left(\begin{array}{c|c} \bar{\mathbf{C}}^t & J_m \bar{\mathbf{B}} J_m \\ \hline -J_n \bar{\mathbf{A}} J_n & J_n \bar{\mathbf{C}} J_m \end{array}\right) \quad \text{and} \quad \mathbf{K}^\theta = \left(\begin{array}{c|c} \mathbf{C}^t & J_n \mathbf{A} J_n \\ \hline J_m \mathbf{B} J_m & -J_m \mathbf{C} J_n \end{array}\right). \tag{4.42}$$

We further apply the fermionic particle-hole transformation

$$\mathbf{c}_{ij} \mapsto \bar{\mathbf{c}}_{ji}, \quad \bar{\mathbf{c}}_{ji} \mapsto \mathbf{c}_{ij} \quad \text{for} \quad 1 \le j \le n, \, n+m+1 \le i \le n+2m, \tag{4.43}$$

as well as the bosonic particle-hole transformation

$$\mathbf{a}_{ij} \mapsto -\bar{\mathbf{a}}_{ji}, \quad \bar{\mathbf{a}}_{ji} \mapsto \mathbf{a}_{ij} \quad \text{for} \quad 1 \le j \le n-1, \, n+2m+1 \le i \le 2n+2m-j, \tag{4.44}$$

$$\mathbf{b}_{ij} \mapsto (1+\delta_{ij'})\bar{\mathbf{b}}_{ji}, \quad \bar{\mathbf{b}}_{ji} \mapsto -\frac{1}{1+\delta_{ij'}}\mathbf{b}_{ij} \quad \text{for} \quad n+1 \le j \le n+m < i \le 2n+2m+1-j. \tag{4.45}$$

This yields the following Lax matrix:

$$\bar{L}(y) = \bar{L}_\theta(y)|_{p.h.} = \left(\begin{array}{c|c} \mathrm{Id}_{n+m} & \bar{\mathbf{K}} \\ \hline \mathbf{K} & y\mathrm{Id}_{n+m} + \mathbf{K}\bar{\mathbf{K}} \end{array}\right) \tag{4.46}$$

with $\bar{\mathbf{K}}$, $\mathbf{K}$ precisely as in (4.13).

Remark 4.47 In particular, we recover orthogonal and symplectic degenerate linear Lax matrices:

- For $m = 0$, we recover precisely the D_n-type Lax matrix $L_{(-,\ldots,-)}(x)$ of [12, (8.84)] (after swapping indices in the generators, i.e. $\mathbf{a}_{ij} \mapsto \mathbf{a}_{ji}$, $\bar{\mathbf{a}}_{ji} \mapsto \bar{\mathbf{a}}_{ij}$).
- For $n = 0$, we get

$$\bar{L}(-x) = \left(\begin{array}{c|c} \mathrm{Id}_m & \bar{\mathbf{B}} \\ \hline -\mathbf{B} & -x\mathrm{Id}_m - \mathbf{B}\bar{\mathbf{B}} \end{array} \right) = \left(\begin{array}{c|c} \mathrm{Id}_m & 0 \\ \hline 0 & -\mathrm{Id}_m \end{array} \right) L^-_{FKT}(x), \quad (4.48)$$

where $L^-_{FKT}(x)$ is the C_m-type Lax matrix equivalent to $L_{(-,\ldots,-)}(x)$ of [12, (8.56)] (after swapping indices and redistributing factor 2, i.e. $\mathbf{b}_{ij} \mapsto (1+\delta_{ij'})\mathbf{b}_{ji}$, $\bar{\mathbf{b}}_{ji} \mapsto \frac{1}{1+\delta_{ji'}}\bar{\mathbf{b}}_{ij}$). We note that the constant matrix above is Id_θ of (3.36) and hence it can be dropped out of the RTT-relation due to the invariance $[R(x), \mathrm{Id}_\theta \otimes \mathrm{Id}_\theta] = 0$ established in Lemma 3.35.

Let us now consider two copies of mutually supercommuting superoscillators labelled by the extra superscript $i = 1, 2$, which will be now encoded by the corresponding matrices $\bar{\mathbf{A}}_i$, $\mathbf{A}_i$, $\bar{\mathbf{B}}_i$, $\mathbf{B}_i$, $\bar{\mathbf{C}}_i$, $\mathbf{C}_i$, hence also, the subscript in $\bar{\mathbf{K}}_i$, $\mathbf{K}_i$. We consider the following two orthosymplectic Lax matrices:

$$L^{[1]}(x) = \left(\begin{array}{c|c} x\mathrm{Id}_{n+m} - \bar{\mathbf{K}}_1\mathbf{K}_1 & \bar{\mathbf{K}}_1 \\ \hline -\mathbf{K}_1 & \mathrm{Id}_{n+m} \end{array} \right),$$

$$\bar{L}^{[2]}(y) = \left(\begin{array}{c|c} \mathrm{Id}_{n+m} & \bar{\mathbf{K}}_2 \\ \hline \mathbf{K}_2 & y\mathrm{Id}_{n+m} + \mathbf{K}_2\bar{\mathbf{K}}_2 \end{array} \right)$$

$$(4.49)$$

of (4.12, 4.46) with

$$\bar{\mathbf{K}}_i = \left(\begin{array}{c|c} \bar{\mathbf{C}}_i & \bar{\mathbf{A}}_i \\ \hline \bar{\mathbf{B}}_i & -\mathrm{J}_m \bar{\mathbf{C}}_i^t \mathrm{J}_n \end{array} \right) \quad \text{and} \quad \mathbf{K}_i = \left(\begin{array}{c|c} \mathbf{C}_i & -\mathbf{B}_i \\ \hline \mathbf{A}_i & \mathrm{J}_n \mathbf{C}_i^t \mathrm{J}_m \end{array} \right). \quad (4.50)$$

The Lax matrices in (4.49) obey the following factorisation formula, cf. (2.27, 2.28):

$$L^{[1]}(x)\bar{L}^{[2]}(y)$$

$$= \left(\begin{array}{c|c} x\mathrm{Id}_{n+m} - \bar{\mathbf{K}}'_1\mathbf{K}'_1 & \left((y-x)\mathrm{Id}_{n+m} + \bar{\mathbf{K}}'_1\mathbf{K}'_1\right)\bar{\mathbf{K}}'_1 \\ \hline -\mathbf{K}'_1 & y\mathrm{Id}_{n+m} + \mathbf{K}'_1\bar{\mathbf{K}}'_1 \end{array} \right) \cdot \left(\begin{array}{c|c} \mathrm{Id}_{n+m} & \bar{\mathbf{K}}'_2 \\ \hline 0 & \mathrm{Id}_{n+m} \end{array} \right)$$

$$(4.51)$$

where we set

$$\mathbf{K}'_1 = \mathbf{K}_1 - \mathbf{K}_2, \qquad \bar{\mathbf{K}}'_1 = \bar{\mathbf{K}}_1,$$
$$\bar{\mathbf{K}}'_2 = \bar{\mathbf{K}}_2 + \bar{\mathbf{K}}_1, \qquad \mathbf{K}'_2 = \mathbf{K}_2. \quad (4.52)$$

Moreover, similarly to (2.29, 2.30), the transformation (4.52) can be expressed through a similarity transformation in the superoscillator space:

$$\mathbf{K}_i' = \mathbf{S}\mathbf{K}_i\mathbf{S}^{-1}, \qquad \bar{\mathbf{K}}_i' = \mathbf{S}\bar{\mathbf{K}}_i\mathbf{S}^{-1} \qquad (i = 1, 2) \tag{4.53}$$

with

$$\mathbf{S} = \exp\left[\sum_{i,j} \bar{\mathbf{a}}_{ji}^{[1]}\mathbf{a}_{ij}^{[2]} + \sum_{i,j} \bar{\mathbf{b}}_{ji}^{[1]}\mathbf{b}_{ij}^{[2]} + \sum_{i,j} \bar{\mathbf{c}}_{ji}^{[1]}\mathbf{c}_{ij}^{[2]}\right], \tag{4.54}$$

where the indices i, j take all possible values as given in (4.43)–(4.45). We note that all the summands in the exponent above are bosonic and pairwise supercommute.

We thus obtain the key result of this subsection:

Proposition 4.55 *(a) For any $x_1, x_2 \in \mathbb{C}$, the matrix*

$$\mathcal{L}_{x_1,x_2}(x) = \left(\begin{array}{c|c} (x + x_1)\mathrm{Id}_{n+m} - \bar{\mathbf{K}}_1\mathbf{K}_1 & \left((x_2 - x_1)\mathrm{Id}_{n+m} + \bar{\mathbf{K}}_1\mathbf{K}_1\right)\bar{\mathbf{K}}_1 \\ \hline -\mathbf{K}_1 & (x + x_2)\mathrm{Id}_{n+m} + \mathbf{K}_1\bar{\mathbf{K}}_1 \end{array}\right) \tag{4.56}$$

is a solution to the RTT-relation (2.8) with the R-matrix (3.12), hence, is a Lax matrix.

(b) The Lax matrix from (a) is a fusion of two degenerate Lax matrices in (4.49) through:

$$L^{[1]}(x + x_1)\bar{L}^{[2]}(x + x_2) = \mathbf{S}\,\mathcal{L}_{x_1,x_2}(x)\left(\begin{array}{c|c} \mathrm{Id}_{n+m} & \bar{\mathbf{K}}_2 \\ \hline 0 & \mathrm{Id}_{n+m} \end{array}\right)\mathbf{S}^{-1} \tag{4.57}$$

with the similarity transformation $\mathbf{S}$ of (4.54).

Remark 4.58 Similarly to Remarks 4.36 and 4.47, we note that the orthosymplectic Lax matrix $\mathcal{L}_{x_1,x_2}(x)$ of (4.56) generalizes the authors' previous work for orthogonal and symplectic types:

- For $m = 0$, we recover precisely the D_n-type Lax matrix of [12, (6.2)] when setting $x_1 = t$ and $x_2 = -t - n + 1$.
- For $n = 0$, setting $x_1 = -t$ and $x_2 = t + m + 1$, we recover the C_m-type Lax matrix $\mathcal{L}_{FKT}(-x)$ of [12, (5.3)]:

$$\mathcal{L}_{x_1=-t,x_2=t+m+1}(x) = \left(\begin{array}{c|c} (x - t)\mathrm{Id}_m + \bar{\mathbf{B}}\mathbf{B} & \bar{\mathbf{B}}\left((2t + m + 1)\mathrm{Id}_m - \mathbf{B}\bar{\mathbf{B}}\right) \\ \hline \mathbf{B} & (x + t + m + 1)\mathrm{Id}_m - \mathbf{B}\bar{\mathbf{B}} \end{array}\right)$$

$$= -\mathcal{L}_{FKT}(-x). \tag{4.59}$$

Remark 4.60 Similarly to Remark 2.34 and generalizing [12, §8], we can vice versa obtain the matrices $L(x)$ and $\bar{L}(x)$ of (4.12, 4.46) from the non-degenerate linear Lax matrix $\mathcal{L}_{x_1,x_2}(x)$ of (4.56) via the *renormalized limit* procedures (which clearly preserve the property of being Lax):

$$
\begin{aligned}
L(x) &= \lim_{t\to\infty} \left\{ \mathcal{L}_{0,t}(x) \cdot \mathrm{diag}\Big(\underbrace{1,\ldots,1}_{n+m}; \underbrace{\tfrac{1}{t},\ldots,\tfrac{1}{t}}_{n+m} \Big) \right\}, \\
\bar{L}(x) &= \lim_{t\to\infty} \left\{ \mathrm{diag}\Big(\underbrace{\tfrac{1}{t},\ldots,\tfrac{1}{t}}_{n+m}; \underbrace{1,\ldots,1}_{n+m} \Big) \cdot \mathcal{L}_{t,0}(x) \right\} \Big|_{\bar{\mathbf{K}}\mapsto-\bar{\mathbf{K}},\,\mathbf{K}\mapsto-\mathbf{K}}.
\end{aligned}
\tag{4.61}
$$

Remark 4.62 According to Remark 3.16, the Lax matrix $L(x)$ from Theorem 4.11 and the Lax matrix $\mathcal{L}_{x_1,x_2}(x)$ from Proposition 4.55(a) give rise to the corresponding degenerate and non-degenerate linear Lax matrices for all $\mathbb{Z}_2$-gradings of V.

5 Quadratic orthosymplectic Lax matrices

In this section, we investigate some quadratic superoscillator orthosymplectic Lax matrices.

5.1 From linear to quadratic Lax matrices

In this subsection, we consider a factorisation formula different from the one presented in Sect. 4.2 that allows to derive a degenerate quadratic Lax matrix, see Theorem 5.44.

Consider the Lax matrix of (4.12) written in the block form with blocks on the diagonal of size 1×1, $(n+m-1) \times (n+m-1)$, $(n+m-1) \times (n+m-1)$, and 1×1, respectively;

$$
L(x) = \left(
\begin{array}{c|c|c|c}
x - \bar{\mathbf{v}}\mathbf{v} & -\bar{\mathbf{v}}\mathbf{K}^\circ & \bar{\mathbf{v}} & 0 \\
\hline
-\bar{\mathbf{K}}^\circ\mathbf{v} & x\mathrm{Id}_{n+m-1} - \bar{\mathbf{K}}^\circ\mathbf{K}^\circ + \mathrm{J}_{n+m-1}\bar{\mathbf{v}}^t\mathbf{v}^t\mathrm{G}_{n-1,m}^{-1} & \bar{\mathbf{K}}^\circ & -\mathrm{J}_{n+m-1}\bar{\mathbf{v}}^t \\
\hline
-\mathbf{v} & -\mathbf{K}^\circ & \mathrm{Id}_{n+m-1} & 0 \\
\hline
0 & -\mathbf{v}^t\mathrm{G}_{n-1,m}^{-1} & 0 & 1
\end{array}
\right).
\tag{5.1}
$$

Here, we expressed $\bar{\mathbf{K}}$, $\mathbf{K}$ of (4.13) via

$$
\bar{\mathbf{K}} = \left(
\begin{array}{c|c}
\bar{\mathbf{v}} & 0 \\
\hline
\bar{\mathbf{K}}^\circ & -\mathrm{J}_{n+m-1}\bar{\mathbf{v}}^t
\end{array}
\right), \qquad
\mathbf{K} = \left(
\begin{array}{c|c}
\mathbf{v} & \mathbf{K}^\circ \\
\hline
0 & \mathbf{v}^t\mathrm{G}_{n-1,m}^{-1}
\end{array}
\right)
\tag{5.2}
$$

with

$$
\mathrm{G}_{n-1,m} = \left(
\begin{array}{c|c}
0 & -\mathrm{J}_{n-1} \\
\hline
\mathrm{J}_m & 0
\end{array}
\right)
\tag{5.3}
$$

as in (3.34), and used the $(n + m - 1)$-dimensional vectors

$$\bar{\mathbf{v}} = \left(\bar{\mathbf{c}}_{1,n+m+1} \cdots \bar{\mathbf{c}}_{1,n+2m} \; \bar{\mathbf{a}}_{1,n+2m+1} \cdots \bar{\mathbf{a}}_{1,2n+2m-1} \right), \tag{5.4}$$

$$\mathbf{v} = \left(\mathbf{c}_{n+m+1,1} \cdots \mathbf{c}_{n+2m,1} \; \mathbf{a}_{n+2m+1,1} \cdots \mathbf{a}_{2n+2m-1,1} \right)^t. \tag{5.5}$$

Finally, the submatrices $\mathbf{K}^\circ$, $\bar{\mathbf{K}}^\circ$ of $\mathbf{K}$, $\bar{\mathbf{K}}$ appearing in (5.2) read as follows:

$$\bar{\mathbf{K}}^\circ = \left(\begin{array}{c|c} \bar{\mathbf{C}}^\circ & \bar{\mathbf{A}}^\circ \\ \hline \mathbf{B} & -\mathbf{J}_m(\bar{\mathbf{C}}^\circ)^t \mathbf{J}_{n-1} \end{array} \right) \quad \text{and} \quad \mathbf{K}^\circ = \left(\begin{array}{c|c} \mathbf{C}^\circ & -\mathbf{B} \\ \hline \mathbf{A}^\circ & \mathbf{J}_{n-1}(\mathbf{C}^\circ)^t \mathbf{J}_m \end{array} \right), \tag{5.6}$$

where the block matrices $\bar{\mathbf{A}}^\circ$, $\mathbf{A}^\circ$ and $\bar{\mathbf{C}}^\circ$, $\mathbf{C}^\circ$ encode the bosonic and fermionic superoscillators via

$$\bar{\mathbf{A}}^\circ = \left(\begin{array}{cccc} \bar{\mathbf{a}}_{2,n+2m+1} & \cdots & \bar{\mathbf{a}}_{2,2n+2m-2} & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & 0 & -\bar{\mathbf{a}}_{2,2n+2m-2} \\ \bar{\mathbf{a}}_{n-1,n+2m+1} & 0 & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & -\bar{\mathbf{a}}_{n-1,n+2m+1} & \cdots & -\bar{\mathbf{a}}_{2,n+2m+1} \end{array} \right), \tag{5.7}$$

$$\mathbf{A}^\circ = \left(\begin{array}{cccc} \mathbf{a}_{n+2m+1,2} & \cdots & \mathbf{a}_{n+2m+1,n-1} & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & 0 & -\mathbf{a}_{n+2m+1,n-1} \\ \mathbf{a}_{2n+2m-2,2} & 0 & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & -\mathbf{a}_{2n+2m-2,2} & \cdots & -\mathbf{a}_{n+2m+1,2} \end{array} \right), \tag{5.8}$$

$$\bar{\mathbf{C}}^\circ = \left(\begin{array}{ccc} \bar{\mathbf{c}}_{2,n+m+1} & \cdots & \bar{\mathbf{c}}_{2,n+2m} \\ \vdots & \ddots & \vdots \\ \bar{\mathbf{c}}_{n,n+m+1} & \cdots & \bar{\mathbf{c}}_{n,n+2m} \end{array} \right), \quad \mathbf{C}^\circ = \left(\begin{array}{ccc} \mathbf{c}_{n+m+1,2} & \cdots & \mathbf{c}_{n+m+1,n} \\ \vdots & \ddots & \vdots \\ \mathbf{c}_{n+2m,2} & \cdots & \mathbf{c}_{n+2m,n} \end{array} \right), \tag{5.9}$$

while the block matrices $\bar{\mathbf{B}}$, $\mathbf{B}$ encoding the bosonic superoscillators are precisely as in (4.8, 4.9).

To construct the degenerate quadratic Lax matrix, we define

$$\hat{L}_\theta(x) = \hat{\mathbf{J}}_\theta L(x) \hat{\mathbf{J}}_\theta^{-1} \tag{5.10}$$

with $\hat{\mathbf{J}}_\theta$ as in (3.34):

$$\hat{\mathbf{J}}_\theta = \left(\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{G}_{n-1,m} & 0 \\ \hline 0 & \mathbf{J}_{n+m-1} & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array} \right), \quad \hat{\mathbf{J}}_\theta^{-1} = \left(\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \hline 0 & 0 & \mathbf{J}_{n+m-1} & 0 \\ \hline 0 & \mathbf{G}_{n-1,m}^t & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \end{array} \right). \tag{5.11}$$

as $G_{n-1,m}^t = G_{n-1,m}^{-1}$. The matrix $\hat{L}_\theta(x)$ in (5.10) is again a solution to the RTT-relation (2.8) with the R-matrix (3.12), hence, is a Lax matrix. This follows from the invariance $[R(x), \hat{J}_\theta \otimes \hat{J}_\theta] = 0$ of the orthosymplectic R-matrix (3.12), established in Lemma 3.33.

We rename the fermionic superoscillators

$$\mathbf{c}_{i1} \mapsto \mathbf{c}_{i'1}, \quad \bar{\mathbf{c}}_{1i} \mapsto \bar{\mathbf{c}}_{1i'} \quad \text{for} \quad n+m+1 \leq i \leq n+2m \qquad (5.12)$$

and also rename the bosonic superoscillators

$$\mathbf{a}_{i1} \mapsto -\mathbf{a}_{i'1}, \quad \bar{\mathbf{a}}_{1i} \mapsto -\bar{\mathbf{a}}_{1i'} \quad \text{for} \quad n+2m+1 \leq i \leq 2n+2m-1. \quad (5.13)$$

We further apply the following particle-hole transformation of the remaining superoscillators:

$$\mathbf{c}_{ij} \mapsto \bar{\mathbf{c}}_{ji}, \quad \bar{\mathbf{c}}_{ji} \to \mathbf{c}_{ij} \quad \text{for} \quad 2 \leq j \leq n, \, n+m+1 \leq i \leq n+2m, \qquad (5.14)$$

$$\mathbf{a}_{ij} \mapsto -\bar{\mathbf{a}}_{ji}, \quad \bar{\mathbf{a}}_{ji} \to \mathbf{a}_{ij} \quad \text{for} \quad 2 \leq j \leq n-1, \, n+2m+1 \leq i \leq 2n+2m-j, \qquad (5.15)$$

$$\mathbf{b}_{ij} \mapsto (1+\delta_{ij'})\bar{\mathbf{b}}_{ji}, \quad \bar{\mathbf{b}}_{ji} \mapsto \frac{-1}{1+\delta_{ij'}}\mathbf{b}_{ij} \quad \text{for} \quad n+1 \leq j \leq n+m < i \leq 2n+2m+1-j. \qquad (5.16)$$

Applying the above operations (5.12)–(5.16) to (5.19) yields the following Lax matrix:

$$\hat{L}(y) = \hat{L}_\theta(y)|_{p.h.} = \begin{pmatrix} y - \bar{\mathbf{w}}\mathbf{w} & \bar{\mathbf{w}} & \bar{\mathbf{w}}\bar{\mathbf{K}}^\circ & 0 \\ -\mathbf{w} & \mathrm{Id} & \bar{\mathbf{K}}^\circ & 0 \\ -\mathbf{K}^\circ\mathbf{w} & \mathbf{K}^\circ & y\mathrm{Id} + \mathbf{K}^\circ\bar{\mathbf{K}}^\circ + G^{-1}\bar{\mathbf{w}}^t\mathbf{w}^tJ & G^{-1}\bar{\mathbf{w}}^t \\ 0 & 0 & \mathbf{w}^tJ & 1 \end{pmatrix} \qquad (5.17)$$

with

$$\bar{\mathbf{w}} = \begin{pmatrix} \bar{\mathbf{a}}_{1,2} \cdots \bar{\mathbf{a}}_{1,n} \, \bar{\mathbf{c}}_{1,n+1} \cdots \bar{\mathbf{c}}_{1,n+m} \end{pmatrix}, \qquad (5.18)$$

$$\mathbf{w} = \begin{pmatrix} \mathbf{a}_{2,1} \ldots \mathbf{a}_{n,1} \, \mathbf{c}_{n+1,1} \cdots \mathbf{c}_{n+m,1} \end{pmatrix}^t, \qquad (5.19)$$

and the $(n+m-1) \times (n+m-1)$ matrices Id, J, G defined via:

$$\mathrm{Id} = \mathrm{Id}_{n+m-1}, \quad J = J_{n+m-1}, \quad G = G_{n-1,m}. \qquad (5.20)$$

Similarly to (2.27) and (4.51), let us factorise the product of the Lax matrices in (5.1) and (5.17). As before, we shall use the subscript $i = 1, 2$ to distinguish between the superoscillators and the corresponding matrices (using $i = 1$ in the context of

$L(x)$, $i = 2$ in the context of $\hat{L}(y)$). One has the following factorisation:

$$
L^{[1]}(x)\hat{L}^{[2]}(y) = \mathbf{L}'(x, y)
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \mathrm{Id} & (\bar{\mathbf{K}}_2^\circ)' & 0 \\
\hline
0 & 0 & \mathrm{Id} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\tag{5.21}
$$

where

$$
(\bar{\mathbf{K}}_2^\circ)' = \bar{\mathbf{K}}_2^\circ + \bar{\mathbf{K}}_1^\circ, \qquad (\mathbf{K}_2^\circ)' = \mathbf{K}_2^\circ,
\tag{5.22}
$$

and $\mathbf{L}'(x, y)$ is further factorised as

$$
\mathbf{L}'(x, y) =
\begin{pmatrix}
1 & \bar{\mathbf{u}}' & \frac{1}{2}\bar{\mathbf{u}}' M(\bar{\mathbf{u}}')^t \\
\hline
0 & \mathrm{Id}_{N+2m-2} & M(\bar{\mathbf{u}}')^t \\
\hline
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
x \cdot y & 0 & 0 \\
\hline
0 & \mathcal{L}'(x, y) & 0 \\
\hline
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
\hline
-\mathbf{u}' & \mathrm{Id}_{N+2m-2} & 0 \\
\hline
\frac{1}{2}(\mathbf{u}')^t M\mathbf{u}' & -(\mathbf{u}')^t M & 1
\end{pmatrix}
\tag{5.23}
$$

with

$$
M = \begin{pmatrix}
0 & -\mathrm{J} \\
\hline
\mathrm{G}^{-1} & 0
\end{pmatrix}.
\tag{5.24}
$$

Let us now describe $\bar{\mathbf{u}}'$, $\mathbf{u}'$, and $\mathcal{L}'(x, y)$ featuring in (5.23). The first two are defined via

$$
\bar{\mathbf{u}}' = \begin{pmatrix} \bar{\mathbf{w}}' & \bar{\mathbf{v}}' \end{pmatrix}, \qquad \mathbf{u}' = \begin{pmatrix} \mathbf{w}' \\ \mathbf{v}' \end{pmatrix},
\tag{5.25}
$$

where the vectors $\bar{\mathbf{v}}'$, $\mathbf{v}'$ and $\mathbf{w}'$, $\bar{\mathbf{w}}'$ are given by (cf. (5.4, 5.5, 5.6, 5.18, 5.19))

$$
\bar{\mathbf{v}}' = \bar{\mathbf{v}} - \bar{\mathbf{w}}\bar{\mathbf{K}}_1^\circ, \qquad \mathbf{v}' = \mathbf{v},
\tag{5.26}
$$

$$
\mathbf{w}' = \mathbf{w} + \bar{\mathbf{K}}_1^\circ \mathbf{v}, \qquad \bar{\mathbf{w}}' = \bar{\mathbf{w}}.
\tag{5.27}
$$

The matrix $\mathcal{L}'(x, y)$ in the middle factor of the factorisation (5.23) reads (cf. (4.56))

$$
\mathcal{L}'(x, y) =
\begin{pmatrix}
x\mathrm{Id} - (\bar{\mathbf{K}}_1^\circ)'(\mathbf{K}_1^\circ)' & ((y-x)\mathrm{Id} + (\bar{\mathbf{K}}_1^\circ)'(\mathbf{K}_1^\circ)')(\bar{\mathbf{K}}_1^\circ)' \\
\hline
-(\mathbf{K}_1^\circ)' & y\mathrm{Id} + (\mathbf{K}_1^\circ)'(\bar{\mathbf{K}}_1^\circ)'
\end{pmatrix}
\tag{5.28}
$$

where

$$
(\mathbf{K}_1^\circ)' = \mathbf{K}_1^\circ - \mathbf{K}_2^\circ + \mathbf{v}\bar{\mathbf{w}} - \mathrm{G}^{-1}\bar{\mathbf{w}}^t\mathbf{v}^t\mathrm{G}^{-1}, \qquad (\bar{\mathbf{K}}_1^\circ)' = \bar{\mathbf{K}}_1^\circ.
\tag{5.29}
$$

Remark 5.30 In the derivation of (5.23), we used the following two equalities:

$$(\bar{\mathbf{v}}')^t = \bar{\mathbf{v}}^t - J\bar{\mathbf{K}}_1^\circ G^{-1}\bar{\mathbf{w}}^t\,, \qquad (\mathbf{w}')^t = \mathbf{w}^t + \mathbf{v}^t G^{-1}\bar{\mathbf{K}}_1^\circ J\,, \tag{5.31}$$

where $\bar{\mathbf{v}}'$ and $\mathbf{w}'$ are given by (5.26) and (5.27), respectively. To verify (5.31), we use

$$\bar{\mathbf{w}}J\bar{\mathbf{v}}^t = -\bar{\mathbf{v}}G^{-1}\bar{\mathbf{w}}^t\,, \qquad \mathbf{w}^t J\mathbf{v} = -\mathbf{v}^t G^{-1}\mathbf{w}\,, \tag{5.32}$$

as well as

$$\mathbf{v}^t G^{-1}\bar{\mathbf{K}}_1^\circ \mathbf{v} = 0 = \bar{\mathbf{w}}\bar{\mathbf{K}}_1^\circ G^{-1}\bar{\mathbf{w}}^t \qquad \text{and} \qquad \bar{\mathbf{w}}J(\bar{\mathbf{w}}\bar{\mathbf{K}}_1^\circ)^t = 0 = (\bar{\mathbf{K}}_1^\circ \mathbf{v})^t J\mathbf{v}\,, \tag{5.33}$$

where we take into account the $\mathbb{Z}_2$-grading of the superoscillators. It also follows from (5.33) that

$$\bar{\mathbf{u}}'M(\bar{\mathbf{u}}')^t = \bar{\mathbf{u}}M(\bar{\mathbf{u}})^t\,, \qquad (\mathbf{u}')^t M\mathbf{u}' = (\mathbf{u})^t M\mathbf{u} \tag{5.34}$$

with

$$\bar{\mathbf{u}} = \begin{pmatrix} \bar{\mathbf{w}} & \bar{\mathbf{v}} \end{pmatrix}\,, \qquad \mathbf{u} = \begin{pmatrix} \mathbf{w} \\ \mathbf{v} \end{pmatrix}. \tag{5.35}$$

Let us also note that the first and third terms in the right-hand side of the factorisation (5.23) can be written as exponential expansions that truncate after the second term, cf. [9, (3.2)].

Finally, we note that there is a similarity transformation in the superoscillator space such that

$$(\mathbf{K}_1^\circ)' = S\mathbf{K}_1^\circ S^{-1}, \quad (\bar{\mathbf{K}}_2^\circ)' = S\bar{\mathbf{K}}_2^\circ S^{-1}, \quad (\bar{\mathbf{K}}_1^\circ)' = S\bar{\mathbf{K}}_1^\circ S^{-1}, \quad (\mathbf{K}_2^\circ)' = S\mathbf{K}_2^\circ S^{-1},$$
$$\mathbf{w}' = S\mathbf{w}S^{-1}, \quad \bar{\mathbf{v}}' = S\bar{\mathbf{v}}S^{-1}, \quad \bar{\mathbf{w}}' = S\bar{\mathbf{w}}S^{-1}, \quad \mathbf{v}' = S\mathbf{v}S^{-1}. \tag{5.36}$$

Here, $\mathbf{S}$ is given explicitly by

$$\mathbf{S} = \mathbf{S}_\circ \mathbf{S}_3 \tag{5.37}$$

with

$$\mathbf{S}_3 = \exp\left(-\bar{\mathbf{w}}\bar{\mathbf{K}}_1^\circ \mathbf{v}\right) \tag{5.38}$$

and

$$\mathbf{S}_\circ = \exp\left[\sum_{i,j} \bar{\mathbf{a}}_{ji}^{[1]}\mathbf{a}_{ij}^{[2]} + \sum_{i,j} \bar{\mathbf{b}}_{ji}^{[1]}\mathbf{b}_{ij}^{[2]} + \sum_{i,j} \bar{\mathbf{c}}_{ji}^{[1]}\mathbf{c}_{ij}^{[2]}\right], \tag{5.39}$$

where the indices i, j take all possible values as they appear in (5.6). It is obvious that the summands in the exponents of (5.38) and (5.39), respectively, are bosonic and pairwise commute (however, $\mathbf{S}_\circ$ and $\mathbf{S}_3$ do not commute!).

Remark 5.40 The equalities in (5.36) are verified by direct computations. Let us only evaluate the transformation of $\mathbf{K}_1^\circ$ under $\mathbf{S}_3$ (the rest of computations being much simpler), cf. (5.29):

$$
\begin{aligned}
\mathbf{S}_3(\mathbf{K}_1^\circ)_{ij}\mathbf{S}_3^{-1} &= (\mathbf{K}_1^\circ)_{ij} - \bar{\mathbf{w}}\bar{\mathbf{K}}_1^\circ \mathbf{v}(\mathbf{K}_1^\circ)_{ij} + (\mathbf{K}_1^\circ)_{ij}\bar{\mathbf{w}}\bar{\mathbf{K}}_1^\circ \mathbf{v} \\
&= (\mathbf{K}_1^\circ)_{ij} - \bar{\mathbf{w}}_k(\bar{\mathbf{K}}_1^\circ)_{k\ell}\mathbf{v}_\ell(\mathbf{K}_1^\circ)_{ij} + (\mathbf{K}_1^\circ)_{ij}\bar{\mathbf{w}}_k(\bar{\mathbf{K}}_1^\circ)_{k\ell}\mathbf{v}_\ell \\
&= (\mathbf{K}_1^\circ)_{ij} - \bar{\mathbf{w}}_k(\bar{\mathbf{K}}_1^\circ)_{k\ell}\mathbf{v}_\ell(\mathbf{K}_1^\circ)_{ij} + (-1)^{(|n+m-i+1|+|j+1|)|k+1|}\bar{\mathbf{w}}_k(\mathbf{K}_1^\circ)_{ij}(\bar{\mathbf{K}}_1^\circ)_{k\ell}\mathbf{v}_\ell \\
&= (\mathbf{K}_1^\circ)_{ij} + \mathbf{v}_i\bar{\mathbf{w}}_j - (\mathbf{G}^{-1}\bar{\mathbf{w}}^t)_i(\mathbf{v}^t\mathbf{G}^{-1})_j \,,
\end{aligned}
\tag{5.41}
$$

where we summed over all possible k, ℓ. The last equality above follows from (4.18) which yields

$$
\begin{aligned}
(-1)^{(|n+m-i+1|+|j+1|)|k+1|}\bar{\mathbf{w}}_k(\mathbf{K}_1^\circ)_{ij}(\bar{\mathbf{K}}_1^\circ)_{k\ell}\mathbf{v}_\ell \\
= \bar{\mathbf{w}}_k(\bar{\mathbf{K}}_1^\circ)_{k\ell}\mathbf{v}_\ell(\mathbf{K}_1^\circ)_{ij} + \mathbf{v}_i\bar{\mathbf{w}}_j - (-1)^{|n+m-i+1|+|j+1|}\bar{\mathbf{w}}_{n+m-i}\mathbf{v}_{n+m-j}
\end{aligned}
\tag{5.42}
$$

with

$$
(\mathbf{G}^{-1}\bar{\mathbf{w}}^t)_i = -(-1)^{|n+m-i+1|}\bar{\mathbf{w}}_{n+m-i}\,, \qquad (\mathbf{v}^t\mathbf{G}^{-1})_j = -(-1)^{|j+1|}\mathbf{v}_{n+m-j}\,.
\tag{5.43}
$$

We thus obtain the key result of this subsection:

Theorem 5.44 (a) For any $x_1, x_2 \in \mathbb{C}$, the matrix

$\mathbf{L}_{x_1,x_2}(x) =$

$$
\begin{pmatrix}
1 & \bar{\mathbf{u}} & \frac{1}{2}\bar{\mathbf{u}}M\bar{\mathbf{u}}^t \\
\hline
0 & \mathrm{Id}_{N+2m-2} & M\bar{\mathbf{u}}^t \\
\hline
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
(x+x_1)(x+x_2) & 0 & 0 \\
\hline
0 & \mathcal{L}_{x_1,x_2}(x) & 0 \\
\hline
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
\hline
-\mathbf{u} & \mathrm{Id}_{N+2m-2} & 0 \\
\hline
\frac{1}{2}\mathbf{u}^t M\mathbf{u} & -\mathbf{u}^t M & 1
\end{pmatrix}
\tag{5.45}
$$

with

$$
\bar{\mathbf{u}} = \begin{pmatrix} \bar{\mathbf{w}} \ \bar{\mathbf{v}} \end{pmatrix} \quad \text{and} \quad \mathbf{u} = \begin{pmatrix} \mathbf{w} \\ \mathbf{v} \end{pmatrix},
\tag{5.46}
$$

cf. (5.4, 5.5, 5.18, 5.19), and $\mathcal{L}_{x_1,x_2}(x)$ being the linear $\mathfrak{osp}(2n-2|2m)$ type Lax matrix of (4.56):

$$
\mathcal{L}_{x_1,x_2}(x) = \begin{pmatrix}
(x+x_1)\mathrm{Id} - \bar{\mathbf{K}}_1^\circ\mathbf{K}_1^\circ & \big((x_2-x_1)\mathrm{Id} + \bar{\mathbf{K}}_1^\circ\mathbf{K}_1^\circ\big)\bar{\mathbf{K}}_1^\circ \\
\hline
-\mathbf{K}_1^\circ & (x+x_2)\mathrm{Id} + \mathbf{K}_1^\circ\bar{\mathbf{K}}_1^\circ
\end{pmatrix},
\tag{5.47}
$$

cf. (5.28) is a solution to the RTT-relation (2.8) with the R-matrix (3.12), hence, is a Lax matrix.

(b) The Lax matrix from (a) is a fusion of two degenerate Lax matrices through:

$$L^{[1]}(x + x_1)\hat{L}^{[2]}(x + x_2) = \mathbf{S}\,\mathbf{L}_{x_1,x_2}(x) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{Id} & \bar{\mathbf{K}}_2^\circ & 0 \\ 0 & 0 & \mathrm{Id} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mathbf{S}^{-1} \tag{5.48}$$

with the Lax matrices $L(x)$, $\hat{L}(x)$ of (5.1, 5.17) and the similarity transformation $\mathbf{S}$ of (5.37)–(5.39).

Remark 5.49 For $m = 0$, we recover precisely the D_n-type Lax matrix of [9, (5.24, 5.25)] when setting $x_1 = s$, $x_2 = -s - n + 2$.

5.2 Degenerate quadratic Lax matrices: even N case

Dropping the index 1 in the oscillators of (5.4, 5.5, 5.18, 5.19), we consider the vectors

$$\mathbf{u} = \begin{pmatrix} \mathbf{a}_2 & \ldots & \mathbf{a}_n & \mathbf{c}_{n+1} & \cdots & \mathbf{c}_{n+m} & \mathbf{c}_{n+m+1} & \cdots & \mathbf{c}_{n+2m} & \mathbf{a}_{n+2m+1} & \cdots & \mathbf{a}_{2n+2m-1} \end{pmatrix}^t \tag{5.50}$$

and

$$\bar{\mathbf{u}} = \begin{pmatrix} \bar{\mathbf{a}}_2 & \cdots & \bar{\mathbf{a}}_n & \bar{\mathbf{c}}_{n+1} & \cdots & \bar{\mathbf{c}}_{n+m} & \bar{\mathbf{c}}_{n+m+1} & \cdots & \bar{\mathbf{c}}_{n+2m} & \bar{\mathbf{a}}_{n+2m+1} & \cdots & \bar{\mathbf{a}}_{2n+2m-1} \end{pmatrix}, \tag{5.51}$$

cf. (5.46) so that $\mathbf{a}_j = \mathbf{a}_{j,1}$, $\mathbf{c}_j = \mathbf{c}_{j,1}$, $\bar{\mathbf{a}}_j = \bar{\mathbf{a}}_{1,j}$, and $\bar{\mathbf{c}}_j = \bar{\mathbf{c}}_{1,j}$.

As an immediate corollary of Theorem 5.44, we obtain the following result:

Proposition 5.52 *The matrix*

$$\mathbf{L}(x) = \begin{pmatrix} 1 & \bar{\mathbf{u}} & \frac{1}{2}\bar{\mathbf{u}}M\bar{\mathbf{u}}^t \\ 0 & \mathrm{Id}_{N+2m-2} & M\bar{\mathbf{u}}^t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x(x - \kappa + 1) & 0 & 0 \\ 0 & x\mathrm{Id}_{N+2m-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\mathbf{u} & \mathrm{Id}_{N+2m-2} & 0 \\ \frac{1}{2}\mathbf{u}^t M\mathbf{u} & -\mathbf{u}^t M & 1 \end{pmatrix} \tag{5.53}$$

with u of (5.50), $\bar{u}$ of (5.51), M of (5.24), is a solution to the RTT-relation (2.8) with the R-matrix (3.12), hence is a Lax matrix. It can be explicitly written as

$$\mathbf{L}(x) = \left(\begin{array}{c|c|c} x(x-\kappa+1) - x\bar{\mathbf{u}}\mathbf{u} + \dfrac{1}{4}\bar{\mathbf{u}}M\bar{\mathbf{u}}^t\mathbf{u}^t M\mathbf{u} & x\bar{\mathbf{u}} - \dfrac{1}{2}\bar{\mathbf{u}}M\bar{\mathbf{u}}^t\mathbf{u}^t M & \dfrac{1}{2}\bar{\mathbf{u}}M\bar{\mathbf{u}}^t \\ \hline -x\mathbf{u} + \dfrac{1}{2}M\bar{\mathbf{u}}^t\mathbf{u}^t M\mathbf{u} & x\mathrm{Id}_{N+2m-2} - M\bar{\mathbf{u}}^t\mathbf{u}^t M & M\bar{\mathbf{u}}^t \\ \hline \dfrac{1}{2}\mathbf{u}^t M\mathbf{u} & -\mathbf{u}^t M & 1 \end{array} \right).$$

$$\tag{5.54}$$

Proof According to Theorem 5.44, the matrix $\mathbf{L}_{x_1,x_2}(x)$ of (5.45) is a solution to the RTT-relation (2.8) with the R-matrix (3.12). The superoscillators used in $\mathbf{u}$ and $\bar{\mathbf{u}}$ pairwise supercommute with the ones used in the construction of $\mathcal{L}_{x_1,x_2}(x)$. Furthermore, the coefficients of the latter generate an orthosymplectic subalgebra. To prove that the matrix $\mathbf{L}(x)$ in (5.53) is a solution to the same RTT-relation, we consider the trivial representation of the aforementioned orthosymplectic subalgebra. It is obtained from the action of the Lax matrix $\mathcal{L}_{x_1,x_2}(x)$ on the Fock vacuum $|0\rangle$ (in the Fock module for the superoscillator algebra generated by the entries of $\mathbf{K}^\circ$, $\bar{\mathbf{K}}^\circ$ from (5.6) so that $\mathbf{K}^\circ|0\rangle = 0$), and further fixing $x_2 = x_1 - \kappa + 1$ with $\kappa = n - m - 1$ as in (3.11). Using the simple equalities

$$\mathbf{K}^\circ\bar{\mathbf{K}}^\circ|0\rangle = (\kappa-1)\mathrm{Id}|0\rangle, \qquad \bar{\mathbf{K}}^\circ\mathbf{K}^\circ\bar{\mathbf{K}}^\circ|0\rangle = (\kappa-1)\bar{\mathbf{K}}^\circ|0\rangle, \tag{5.55}$$

we find

$$\mathcal{L}_{x_1,x_2=x_1-\kappa+1}(x)|0\rangle = (x+x_1)\mathrm{Id}_{N+2m-2}|0\rangle. \tag{5.56}$$

Specializing further $x_1 = 0$, we conclude that (5.54) is indeed an orthosymplectic Lax matrix. $\qquad\square$

Remark 5.57 For $m = 0$, we recover precisely the D_n-type Lax matrix of [9, (4.12)].

5.3 Degenerate quadratic Lax matrices: odd N case

In this subsection, we consider the case $N = 2n + 1$. In this setup, the operators P, Q are defined as in (3.6, 3.7), where we choose the following $\mathbb{Z}_2$-grading of the superspace V:

$$|i| := |v_i| = \begin{cases} \bar{0} & \text{for } 1 \leq i \leq n \\ \bar{1} & \text{for } n+1 \leq i \leq n+m \\ \bar{0} & \text{for } i = n+m+1 \\ \bar{1} & \text{for } n+m+2 \leq i \leq n+2m+1 \\ \bar{0} & \text{for } n+2m+2 \leq i \leq 2n+2m+1 \end{cases}. \tag{5.58}$$

This $\mathbb{Z}_2$-grading corresponds to the parity sequence

$$\Upsilon_V = \left(\underbrace{\bar{0}, \ldots, \bar{0}}_{n}, \underbrace{\bar{1}, \ldots, \bar{1}}_{m}, \bar{0}, \underbrace{\bar{1}, \ldots, \bar{1}}_{m}, \underbrace{\bar{0}, \ldots, \bar{0}}_{n} \right) \tag{5.59}$$

and the following choice of θ_i's:

$$\theta = \theta_V = \left(\underbrace{1, \ldots, 1}_{n+m+1}, \underbrace{-1, \ldots, -1}_{m}, \underbrace{1, \ldots, 1}_{n} \right). \tag{5.60}$$

In this case, we upgrade (5.50, 5.51) by adding extra bosonic superoscillators:

$$\mathbf{u} = \left(\mathbf{a}_2 \ \ldots \ \mathbf{a}_n \ \mathbf{c}_{n+1} \ \cdots \ \mathbf{c}_{n+m} \ \mathbf{a}_{n+m+1} \ \mathbf{c}_{n+m+2} \ \ldots \ \mathbf{c}_{n+2m+1} \ \mathbf{a}_{n+2m+2} \ \cdots \ \mathbf{a}_{2n+2m} \right)^t, \tag{5.61}$$

$$\bar{\mathbf{u}} = \left(\bar{\mathbf{a}}_2 \ \cdots \ \bar{\mathbf{a}}_n \ \bar{\mathbf{c}}_{n+1} \ \cdots \ \bar{\mathbf{c}}_{n+m} \ \bar{\mathbf{a}}_{n+m+1} \ \bar{\mathbf{c}}_{n+m+2} \ \cdots \ \bar{\mathbf{c}}_{n+2m+1} \ \bar{\mathbf{a}}_{n+2m+2} \ \cdots \ \bar{\mathbf{a}}_{2n+2m} \right). \tag{5.62}$$

We also modify (5.24) by introducing the following $(N + 2m - 2) \times (N + 2m - 2)$ matrix:

$$M = \left(\begin{array}{c|c|c} 0 & 0 & -\mathrm{J} \\ \hline 0 & -1 & 0 \\ \hline \mathrm{G}^{-1} & 0 & 0 \end{array} \right). \tag{5.63}$$

Conjecture 5.64 *The matrix* $\mathbf{L}(x)$ *given by* (5.53, 5.54) *with* $\mathbf{u}$ *of* (5.61)*,* $\bar{\mathbf{u}}$ *of* (5.62)*, M of* (5.63) *is a solution to the RTT-relation* (2.8) *with the R-matrix* (3.12)*, hence, is a Lax matrix.*

Remark 5.65 (a) This has been confirmed for $n, m \leq 2$, but a general proof is currently missing.

(b) For $m = 0$, we recover precisely the B_n-type Lax matrix of [12, (9.1)].

5.4 Non-degenerate quadratic Lax matrix through the factorisation

In this subsection, we present a factorisation formula for quadratic Lax matrices that yields a non-degenerate quadratic orthosymplectic Lax matrix of superoscillator type. We uniformly treat both cases of even and odd N, assuming the validity of Conjecture 5.64.

The factorisation is analogous to that for the linear case from Sect. 4.2. As before, we first introduce another solution to the same RTT-relation via a proper conjugation of the degenerate Lax matrix $\mathbf{L}(x)$ from Proposition 5.52 or Conjecture 5.64. To this end, recall the matrix $\tilde{\mathrm{J}}$ of (3.27):

$$\tilde{\mathrm{J}} = \left(\begin{array}{c|c|c} 0 & 0 & 1 \\ \hline 0 & \mathrm{Id}_{N+2m-2} & 0 \\ \hline 1 & 0 & 0 \end{array} \right). \tag{5.66}$$

The R-matrix (3.12) commutes with $\tilde{J} \otimes \tilde{J}$: for even N this is proved in Lemma 3.26, while for odd N the argument is the same. Thus, we get another solution to the same RTT-relation (2.8) via

$$\tilde{L}(x) = \tilde{J}L(x)\tilde{J}^{-1} = \tilde{J}L(x)\tilde{J}. \tag{5.67}$$

We further apply the particle-hole transformation

$$\mathbf{a}_i \mapsto \bar{\mathbf{a}}_{i'}, \qquad \bar{\mathbf{a}}_i \mapsto -\mathbf{a}_{i'}, \tag{5.68}$$

$$\mathbf{c}_i \mapsto \theta_i \bar{\mathbf{c}}_{i'}, \qquad \bar{\mathbf{c}}_i \mapsto \theta_i \mathbf{c}_{i'}, \tag{5.69}$$

to obtain the following Lax matrix:

$$\hat{\tilde{L}}(y) = \tilde{L}(y)|_{p.h.}$$

$$= \left(\begin{array}{c|c|c} 1 & \bar{\mathbf{u}} & \frac{1}{2}\bar{\mathbf{u}}M\bar{\mathbf{u}}^t \\ \hline \mathbf{u} & y\mathrm{Id}_{N+2m-2} + \mathbf{u}\bar{\mathbf{u}} & yM\bar{\mathbf{u}}^t + \frac{1}{2}\mathbf{u}\bar{\mathbf{u}}M\bar{\mathbf{u}}^t \\ \hline \frac{1}{2}\mathbf{u}^t M\mathbf{u} & y\mathbf{u}^t M + \frac{1}{2}\mathbf{u}^t M\mathbf{u}\bar{\mathbf{u}} & y(y-\kappa+1) + y\mathbf{u}^t MM\bar{\mathbf{u}}^t + \frac{1}{4}\mathbf{u}^t M\mathbf{u}\bar{\mathbf{u}}M\bar{\mathbf{u}}^t \end{array}\right). \tag{5.70}$$

Similarly to (2.27, 4.51, 5.21), let us factorise the product of the Lax matrices $\mathbf{L}^{[1]}(x)$ and $\hat{\tilde{\mathbf{L}}}^{[2]}(y)$ from (5.54) and (5.70), with two families of superoscillators encoded by $(\mathbf{u}_1, \bar{\mathbf{u}}_1)$ and $(\mathbf{u}_2, \bar{\mathbf{u}}_2)$, respectively. Explicitly, one obtains the following factorisation:

$$\mathbf{L}^{[1]}(x)\hat{\tilde{\mathbf{L}}}^{[2]}(y) = \mathcal{L}'(x, y)H' \tag{5.71}$$

with

$$\mathcal{L}'(x, y) = \left(\begin{array}{c|c|c} 1 & \bar{\mathbf{u}}_1' & \frac{1}{2}\bar{\mathbf{u}}_1'M(\bar{\mathbf{u}}_1')^t \\ \hline 0 & \mathrm{Id}_{N+2m-2} & M(\bar{\mathbf{u}}_1')^t \\ \hline 0 & 0 & 1 \end{array}\right) \cdot D'(x, y) \cdot \left(\begin{array}{c|c|c} 1 & -\bar{\mathbf{u}}_1' & \frac{1}{2}\bar{\mathbf{u}}_1'M(\bar{\mathbf{u}}_1')^t \\ \hline 0 & \mathrm{Id}_{N+2m-2} & -M(\bar{\mathbf{u}}_1')^t \\ \hline 0 & 0 & 1 \end{array}\right) \tag{5.72}$$

where

$$D'(x, y) = \left(\begin{array}{c|c|c} x(x-\kappa+1) & 0 & 0 \\ \hline -x\mathbf{u}_1' & xy\mathrm{Id}_{N+2m-2} & 0 \\ \hline \frac{1}{2}(\mathbf{u}_1')^t M\mathbf{u}_1' & -y(\mathbf{u}_1')^t M & y(y-\kappa+1) \end{array}\right),$$

M is given by (5.24) for $N = 2n$ or by (5.63) for $N = 2n + 1$, and

$$H' = \left(\begin{array}{c|c|c} 1 & \bar{\mathbf{u}}_2' & \frac{1}{2}\bar{\mathbf{u}}_2' M (\bar{\mathbf{u}}_2')^t \\ \hline 0 & \mathrm{Id}_{N+2m-2} & M(\bar{\mathbf{u}}_2')^t \\ \hline 0 & 0 & 1 \end{array}\right) . \tag{5.73}$$

Here, we introduced the following notation:

$$\mathbf{u}_1' = \mathbf{u}_1 - \mathbf{u}_2 , \qquad \bar{\mathbf{u}}_1' = \bar{\mathbf{u}}_1 , \tag{5.74}$$

and

$$\bar{\mathbf{u}}_2' = \bar{\mathbf{u}}_2 + \bar{\mathbf{u}}_1 , \qquad \mathbf{u}_2' = \mathbf{u}_2 , \tag{5.75}$$

cf. (2.28). Furthermore, we used the following simple properties:

$$\bar{\mathbf{u}}_2 M \bar{\mathbf{u}}_1^t = \bar{\mathbf{u}}_1 M \bar{\mathbf{u}}_2^t , \qquad \mathbf{u}_1^t M \mathbf{u}_2 = \mathbf{u}_2^t M \mathbf{u}_1 . \tag{5.76}$$

Finally, we note that there is a similarity transformation in the superoscillator space such that

$$\bar{\mathbf{u}}_i' = \mathbf{S} \bar{\mathbf{u}}_i \mathbf{S}^{-1} , \qquad \mathbf{u}_i' = \mathbf{S} \mathbf{u}_i \mathbf{S}^{-1} \quad (i = 1, 2) . \tag{5.77}$$

cf. (2.29, 2.30). For even $N = 2n$, it reads

$$\mathbf{S} = \exp\left[\sum_{i=2}^{n} \bar{\mathbf{a}}_i^{[1]} \mathbf{a}_i^{[2]} + \sum_{i=n+1}^{n+m} \bar{\mathbf{c}}_i^{[1]} \mathbf{c}_i^{[2]} + \sum_{i=n+m+1}^{n+2m} \bar{\mathbf{c}}_i^{[1]} \mathbf{c}_i^{[2]} + \sum_{i=n+2m+1}^{2n+2m-1} \bar{\mathbf{a}}_i^{[1]} \mathbf{a}_i^{[2]} \right] , \tag{5.78}$$

while for odd $N = 2n + 1$, we have

$$\mathbf{S} = \exp\left[\sum_{i=2}^{n} \bar{\mathbf{a}}_i^{[1]} \mathbf{a}_i^{[2]} + \sum_{i=n+1}^{n+m} \bar{\mathbf{c}}_i^{[1]} \mathbf{c}_i^{[2]} + \bar{\mathbf{a}}_{n+m+1}^{[1]} \mathbf{a}_{n+m+1}^{[2]} + \sum_{i=n+m+2}^{n+2m+1} \bar{\mathbf{c}}_i^{[1]} \mathbf{c}_i^{[2]} + \sum_{i=n+2m+2}^{2n+2m} \bar{\mathbf{a}}_i^{[1]} \mathbf{a}_i^{[2]} \right] . \tag{5.79}$$

We thus obtain the key result of this subsection:

Proposition 5.80 *For any $x_1, x_2 \in \mathbb{C}$, the matrix*

$$\mathcal{L}_{x_1,x_2}(x) = \left(\begin{array}{c|c|c} 1 & \bar{\mathbf{u}}_1 & \frac{1}{2}\bar{\mathbf{u}}_1 M \bar{\mathbf{u}}_1^t \\ \hline 0 & \mathrm{Id}_{N+2m-2} & M\bar{\mathbf{u}}_1^t \\ \hline 0 & 0 & 1 \end{array}\right) \cdot D_{x_1,x_2}(x) \cdot \left(\begin{array}{c|c|c} 1 & -\bar{\mathbf{u}}_1 & \frac{1}{2}\bar{\mathbf{u}}_1 M \bar{\mathbf{u}}_1^t \\ \hline 0 & \mathrm{Id}_{N+2m-2} & -M\bar{\mathbf{u}}_1^t \\ \hline 0 & 0 & 1 \end{array}\right) \tag{5.81}$$

with

$$D_{x_1,x_2}(x) = \left(\begin{array}{c|c|c} (x+x_1)(x+x_1-\kappa+1) & 0 & 0 \\ \hline -(x+x_1)\mathbf{u}_1 & (x+x_1)(x+x_2)\mathrm{Id}_{N+2m-2} & 0 \\ \hline \frac{1}{2}\mathbf{u}_1^t M \mathbf{u}_1 & -(x+x_2)\mathbf{u}_1^t M & (x+x_2)(x+x_2-\kappa+1) \end{array} \right)$$

is a solution to the RTT-relation (2.8) *with the R-matrix* (3.12)*, hence, is a Lax matrix. Furthermore, it obeys the factorisation formula*

$$\mathbf{L}^{[1]}(x+x_1)\hat{\tilde{\mathbf{L}}}^{[2]}(x+x_2) = \mathbf{S}\mathcal{L}_{x_1,x_2}(x)H\mathbf{S}^{-1} \tag{5.82}$$

with $\mathbf{L}(y)$ *of* (5.54)*,* $\hat{\tilde{\mathbf{L}}}(y)$ *of* (5.70)*,* $\mathbf{S}$ *of* (5.78, 5.79)*, and* H *given by*

$$H = \left(\begin{array}{c|c|c} 1 & \bar{\mathbf{u}}_2 & \frac{1}{2}\bar{\mathbf{u}}_2 M \bar{\mathbf{u}}_2^t \\ \hline 0 & \mathrm{Id}_{N+2m-2} & M\bar{\mathbf{u}}_2^t \\ \hline 0 & 0 & 1 \end{array} \right). \tag{5.83}$$

Remark 5.84 For $m = 0$, we recover the $\mathfrak{so}_N$-type Lax matrix $\mathcal{L}_{-x_1,-x_2}(x)$ of [12, (7.3)].

5.5 Full non-degenerate quadratic Lax matrix through the factorisation

For even $N = 2n$, the constructions from the previous subsection admit straightforward generalizations when replacing $\mathbf{L}(x)$ with $\mathbf{L}_{x_1,x_2}(x)$ of (5.45):

Theorem 5.85 *For* $N = 2n$ *and any* $x_1, x_2, y_1, y_2 \in \mathbb{C}$*, the matrix*

$$\mathcal{L}_{x_1,x_2,y_1,y_2}(x) =$$
$$\left(\begin{array}{c|c|c} 1 & \bar{\mathbf{u}}_1 & \frac{1}{2}\bar{\mathbf{u}}_1 M \bar{\mathbf{u}}_1^t \\ \hline 0 & \mathrm{Id}_{2n+2m-2} & M\bar{\mathbf{u}}_1^t \\ \hline 0 & 0 & 1 \end{array} \right) \cdot D_{x_1,x_2,y_1,y_2}(x) \cdot \left(\begin{array}{c|c|c} 1 & -\bar{\mathbf{u}}_1 & \frac{1}{2}\bar{\mathbf{u}}_1 M \bar{\mathbf{u}}_1^t \\ \hline 0 & \mathrm{Id}_{2n+2m-2} & -M\bar{\mathbf{u}}_1^t \\ \hline 0 & 0 & 1 \end{array} \right)$$
$$\tag{5.86}$$

with

$D_{x_1,x_2,y_1,y_2}(x)$

$$= \left(\begin{array}{c|c|c} (x+y_1)(x+y_1-\kappa+1) & 0 & 0 \\ \hline -\mathcal{L}_{x_1,x_2}(x+y_1)\mathbf{u}_1 & (x+y_2)\mathcal{L}_{x_1,x_2}(x+y_1) & 0 \\ \hline \frac{1}{2}\mathbf{u}_1^t M \mathbf{u}_1 & -(x+y_2)\mathbf{u}_1^t M & (x+y_2)(x+y_2-\kappa+1) \end{array} \right)$$

and $\mathcal{L}_{x_1,x_2}(x)$ being the linear $\mathfrak{osp}(2n-2|2m)$ type Lax matrix of (5.47), *is a solution to the RTT-relation* (2.8) *with the R-matrix* (3.12). *Furthermore, it obeys the factorisation formula*

$$\mathbf{L}^{[1]}_{x_1,x_2}(x+y_1)\hat{\tilde{\mathbf{L}}}^{[2]}(x+y_2) = \mathbf{S}\mathcal{L}_{x_1,x_2,y_1,y_2}(x)H\mathbf{S}^{-1} \tag{5.87}$$

with $\hat{\tilde{\mathbf{L}}}(x)$ of (5.70), *H of* (5.83), *and $\mathbf{L}_{x_1,x_2}(x)$ of* (5.45).

Remark 5.88 For $m = 0$, we recover the D_n-type Lax matrix $\mathfrak{L}_{n,s}(x)$ of [9, (5.36, 5.37)], depending on the extra parameters x_1, x_2, when setting $x_1 = s$, $x_2 = -s-n+2$, $y_1 = -x_1$, $y_2 = -x_2$.

Remark 5.89 According to Remark 3.16, the Lax matrix $\mathbf{L}_{x_1,x_2}(x)$ from Theorem 5.44(a), the Lax matrix $\mathbf{L}(x)$ from Proposition 5.52, and its generalization for odd N from Conjecture 5.64 give rise to the corresponding degenerate quadratic Lax matrices for any other $\mathbb{Z}_2$-grading satisfying $|v_1| = \bar{0}$. Likewise, the Lax matrices $\mathcal{L}_{x_1,x_2}(x)$ from Proposition 5.80 and $\mathcal{L}_{x_1,x_2,y_1,y_2}(x)$ from Theorem 5.85 give rise to the corresponding non-degenerate quadratic Lax matrices for all $\mathbb{Z}_2$-gradings of V.

Remark 5.90 Similarly to Remarks 2.34, 4.60, we can obtain the degenerate Lax matrices (5.53) and (5.70) from the non-degenerate Lax matrix (5.81) through the *renormalized limit* procedure:

$$\mathbf{L}(x+x_1) = \lim_{x_2\to\infty}\left\{\mathcal{L}_{x_1,x_2}(x)\cdot\mathrm{diag}\Big(1,\underbrace{x_2^{-1},\dots,x_2^{-1}}_{N+2m-2},x_2^{-2}\Big)\right\}\Big|_{\mathbf{u}_1\mapsto\mathbf{u},\,\bar{\mathbf{u}}_1\mapsto\bar{\mathbf{u}}},$$

$$\hat{\tilde{\mathbf{L}}}(x+x_2) = \lim_{x_1\to\infty}\left\{\mathrm{diag}\Big(x_1^{-2},\underbrace{x_1^{-1},\dots,x_1^{-1}}_{N+2m-2},1\Big)\cdot\mathcal{L}_{x_1,x_2}(x)\right\}\Big|_{\mathbf{u}_1\mapsto-\mathbf{u},\,\bar{\mathbf{u}}_1\mapsto-\bar{\mathbf{u}}}.$$

$$\tag{5.91}$$

Analogously, for even $N = 2n$, the degenerate Lax matrices (5.45) and (5.70) can be obtained as the *renormalized limits* of the full non-degenerate quadratic Lax matrix $\mathcal{L}_{x_1,x_2,y_1,y_2}(x)$ from (5.86).

Appendix A Twists for transfer matrices and Q-operators

In analogy to BCD-type spin chain transfer matrices, see [12, (8.69, 9.19)], the diagonal twists D for the finite-dimensional transfer matrices should be of the following form:

- For $N = 2n$

$$D = \mathrm{diag}\left(\tau_1, \ldots, \tau_n, \tau_{n+1}, \ldots, \tau_{n+m}, \tau_{n+m}^{-1}, \ldots, \tau_{n+1}^{-1}, \tau_n^{-1}, \ldots, \tau_1^{-1}\right) \quad \text{(A.1)}$$

- For $N = 2n + 1$

$$D = \mathrm{diag}\left(\tau_1, \ldots, \tau_n, \tau_{n+1}, \ldots, \tau_{n+m}, 1, \tau_{n+m}^{-1}, \ldots, \tau_{n+1}^{-1}, \tau_n^{-1}, \ldots, \tau_1^{-1}\right)$$
$$\text{(A.2)}$$

On the other hand, the twist D_{osc} used to construct Q-operators from the monodromy matrices

$$M(x) = \underbrace{L(x) \otimes \cdots \otimes L(x)}_{N\otimes}$$

of degenerate Lax matrices $L(x)$ is determined through the following *invariance condition*:

$$DL(x)D^{-1} = D_{\mathrm{osc}}^{-1} L(x) D_{\mathrm{osc}}, \quad \text{(A.3)}$$

cf. [12, (8.68, 9.18)]. Here, the twist D of (A.1, A.2) acts only on the matrix space, while the twist D_{osc} acts only on the oscillator space. We thus derive the following explicit formulas:

- For $N = 2n$ and the linear Lax matrix of (4.12), D_{osc} is given by

$$D_{\mathrm{osc}} = \left(\prod_{1\le i<j\le n} (\tau_i \tau_j)^{-\bar{\mathbf{a}}_{ij'} \mathbf{a}_{j'i}}\right) \left(\prod_{n+1\le i\le j\le n+m} (\tau_i \tau_j)^{-\bar{\mathbf{b}}_{ij'} \mathbf{b}_{j'i}}\right) \left(\prod_{i=1}^{n} \prod_{j=n+1}^{n+m} (\tau_i \tau_j)^{-\bar{\mathbf{c}}_{ij'} \mathbf{c}_{j'i}}\right)$$
$$\text{(A.4)}$$

which is a mixture of the corresponding D-type and C-type formulas [12, (8.66, 8.90)]

- For $N = 2n$ and the quadratic Lax matrix of (5.53, 5.54), D_{osc} is given by

$$D_{\mathrm{osc}} = \tau_1^{-\sum_{j=2}^{n}(\bar{\mathbf{a}}_j \mathbf{a}_j + \bar{\mathbf{a}}_{j'} \mathbf{a}_{j'}) - \sum_{j=n+1}^{n+m}(\bar{\mathbf{c}}_j \mathbf{c}_j + \bar{\mathbf{c}}_{j'} \mathbf{c}_{j'})} \left(\prod_{j=2}^{n} \tau_j^{\bar{\mathbf{a}}_j \mathbf{a}_j - \bar{\mathbf{a}}_{j'} \mathbf{a}_{j'}}\right) \left(\prod_{j=n+1}^{n+m} \tau_j^{\bar{\mathbf{c}}_j \mathbf{c}_j - \bar{\mathbf{c}}_{j'} \mathbf{c}_{j'}}\right)$$
$$\text{(A.5)}$$

which is an analogue of [12, (9.16)]

- For $N = 2n + 1$ and the quadratic Lax matrix of Conjecture 5.64, D_{osc} is alike given by

$$D_{\mathrm{osc}} =$$
$$\tau_1^{-\bar{\mathbf{a}}_{n+m+1} \mathbf{a}_{n+m+1} - \sum_{j=2}^{n}(\bar{\mathbf{a}}_j \mathbf{a}_j + \bar{\mathbf{a}}_{j'} \mathbf{a}_{j'}) - \sum_{j=n+1}^{n+m}(\bar{\mathbf{c}}_j \mathbf{c}_j + \bar{\mathbf{c}}_{j'} \mathbf{c}_{j'})} \left(\prod_{j=2}^{n} \tau_j^{\bar{\mathbf{a}}_j \mathbf{a}_j - \bar{\mathbf{a}}_{j'} \mathbf{a}_{j'}}\right) \left(\prod_{j=n+1}^{n+m} \tau_j^{\bar{\mathbf{c}}_j \mathbf{c}_j - \bar{\mathbf{c}}_{j'} \mathbf{c}_{j'}}\right)$$
$$\text{(A.6)}$$

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Declarations

Conflicts of interests The authors state that there are no conflicts of interest.

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