TWO-PARAMETER QUANTUM GROUPS AND R-MATRICES: CLASSICAL TYPES

IAN MARTIN AND ALEXANDER TSYMBALIUK

To our parents

Abstract. We construct finite R-matrices for the first fundamental representation V of two-parameter quantum groups $U_{r,s}(\mathfrak{g})$ for classical $\mathfrak{g}$, both through the decomposition of $V \otimes V$ into irreducibles $U_{r,s}(\mathfrak{g})$-submodules as well as by evaluating the universal R-matrix. The latter is crucially based on the construction of dual PBW-type bases of $U^{\pm}_{r,s}(\mathfrak{g})$ consisting of the ordered products of quantum root vectors defined via (r, s)-bracketings and combinatorics of standard Lyndon words. We further derive explicit formulas for affine R-matrices, both through the Yang-Baxterization technique of [GWX] and as the unique intertwiner between the tensor product of $V(u)$ and $V(v)$, viewed as modules over two-parameter quantum affine algebras $U_{r,s}(\widehat{\mathfrak{g}})$ for classical $\mathfrak{g}$. The latter generalizes the formulas of [Jim] for one-parametric quantum affine algebras.

Contents

1. Introduction

1.1. Summary.

Let $\mathfrak{g}$ be a Kac-Moody Lie algebra of finite type. Then, it admits a root space decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ with $\mathfrak{n}^\pm = \oplus_{\alpha \in \Phi^+} \mathbb{C} \cdot e_{\pm\alpha}$ corresponding to a polarization of the root system $\Phi = \Phi^+ \sqcup (-\Phi^+)$. The elements $e_{\pm\alpha}$ are called **root vectors**. Thus, $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}^+)$ and the ordered products in $\{e_{\pm\alpha}\}_{\alpha \in \Phi^+}$ form a basis of $U(\mathfrak{n}^\pm)$ for any total order on $\Phi^\pm$. The root vectors can actually be normalized so that[1]

$$[e_\alpha, e_\beta] = e_\alpha e_\beta - e_\beta e_\alpha \in \mathbb{Z}^\times \cdot e_{\alpha+\beta} \qquad \text{for all} \quad \alpha, \beta \in \Phi^+ \quad \text{such that} \quad \alpha + \beta \in \Phi^+. \tag{1.1}$$

This inductively recovers all root vectors from the generators $\{e_i\}_{i \in I}$, corresponding to simple roots $\{\alpha_i\}_{i \in I}$. When $\mathfrak{g}$ is a Kac-Moody algebra of affine type, the root subspaces corresponding to imaginary roots are no longer one-dimensional. However, the theory of such algebras and their representations are well-understood due to their alternative realization as central extensions of the loop algebras $L\bar{\mathfrak{g}}$ for $\bar{\mathfrak{g}}$ of finite type:

$$0 \to \mathbb{C} \to \mathfrak{g} \to L\bar{\mathfrak{g}} \to 0. \tag{1.2}$$

For any Kac-Moody algebra $\mathfrak{g}$, Drinfeld and Jimbo simultaneously introduced the quantum groups $U_q(\mathfrak{g})$ which quantize the universal enveloping algebras of $\mathfrak{g}$. Similarly to $U(\mathfrak{g})$, the quantum groups admit a triangular decomposition $U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^+)$. Furthermore, $U_q(\mathfrak{n}^\pm)$ admit PBW-type bases

$$U_q(\mathfrak{n}^\pm) \ = \bigoplus_{\gamma_1 \geq \dots \geq \gamma_k \in \Phi^+} \mathbb{C}(q) \cdot e_{\pm\gamma_1} \dots e_{\pm\gamma_k} \tag{1.3}$$

formed by the ordered products of **q-deformed root vectors** $e_{\pm\alpha} \in U_q(\mathfrak{n}^\pm)$, defined via Lusztig's braid group action, which requires one to choose a reduced decomposition of the longest element w_0 in the Weyl group W of $\mathfrak{g}$. It is well-known ([P]) that this choice precisely ensures that the order $\geq$ on Φ^+ is convex, in the sense of Definition 5.7. Moreover, the q-deformed root vectors satisfy a q-analogue of relation (1.1), where α, β and $\alpha + \beta$ are any positive roots satisfying $\alpha < \alpha + \beta < \beta$ and the minimality property (5.9):

$$[e_\alpha, e_\beta]_q = e_\alpha e_\beta - q^{(\alpha,\beta)} e_\beta e_\alpha \in \mathbb{Z}[q, q^{-1}]^\times \cdot e_{\alpha+\beta} \tag{1.4}$$

where $(\cdot, \cdot)$ denotes the scalar product corresponding to the root system of type $\mathfrak{g}$. Therefore, the q-deformed root vectors can be defined (up to scalar multiple) as iterated q-commutators of the Drinfeld-Jimbo generators e_i, using the combinatorics of the root system and the chosen convex order on Φ^+.

There is however a purely combinatorial approach to the construction of PBW-type bases of $U_q(\mathfrak{n}^\pm)$, cf. (1.3), that goes back to the works of [K1, K2, Le, Ro]. To this end, recall Lalonde-Ram's bijection [LR]:

$$\ell \colon \Phi^+ \xrightarrow{\sim} \Big\{ \text{standard Lyndon words in } I \Big\}. \tag{1.5}$$

We note that in the context of (1.5), the notion of standard Lyndon words intrinsically depends on a fixed total order of the indexing set I of simple roots. Furthermore, (1.5) gives rise to a total order on Φ^+ via:

$$\alpha < \beta \quad \iff \quad \ell(\alpha) < \ell(\beta) \text{ lexicographically.}$$

It was shown in [Ro] (see [Le, Proposition 26]) that this total order is convex, and hence can be applied to obtain root vectors $e_{\pm\alpha} \in U_q(\mathfrak{n}^\pm)$ for all $\alpha \in \Phi^+$, cf. (1.4), thus eliminating Lusztig's braid group action.

When $\mathfrak{g}$ is of finite type and q is not a root of unity, the representation theory of $U_q(\mathfrak{g})$ is completely parallel to that of $\mathfrak{g}$. On the other hand, to develop the representation theory of $U_q(\mathfrak{g})$ for affine $\mathfrak{g}$ one needs an alternative "new Drinfeld" realization $U_q^{\mathrm{Dr}}(\mathfrak{g})$ from [D], a q-analogue of (1.2). The isomorphism

$$\Psi \colon U_q^{\mathrm{Dr}}(\mathfrak{g}) \xrightarrow{\sim} U_q(\mathfrak{g}) \qquad \text{for affine} \quad \mathfrak{g} \tag{1.6}$$

was constructed in [B] using an affine braid group action, while Ψ^{-1} was stated in [D] using **q-bracketings**.

One of the key features of quantum groups is that they are actually quasitriangular Hopf algebras. The corresponding universal R-matrices $\mathcal{R} \in U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})$ (one needs to consider a completion here) satisfy

$$\textbf{quantum Yang-Baxter equation} \quad \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{1.7}$$

In particular, for any two finite-dimensional $U_q(\mathfrak{g})$-modules V, W one obtains a $U_q(\mathfrak{g})$-module intertwiner

$$\hat{R}_{VW} = (\rho_W \otimes \rho_V)(\mathcal{R}) \circ \tau \colon V \otimes W \xrightarrow{\sim} W \otimes V, \tag{1.8}$$

[1] Throughout the paper, we use $R^\times$ to denote the set of nonzero elements of any ring R.

where $\tau\colon V\otimes W\to W\otimes V$ is the flip map $v\otimes w\mapsto w\otimes v$, and $\rho_V\colon U_q(\mathfrak{g})\to \mathrm{End}(V), \rho_W\colon U_q(\mathfrak{g})\to \mathrm{End}(W)$.

In fact, quantum groups first appeared in the *quantum inverse scattering method*, the study of exactly solvable statistical models and quantum integrable systems arising through the quantum Yang-Baxter equation (1.7). In this context, one starts with a solution of (1.7) or its version with a spectral parameter

$$\mathcal{R}_{12}(x)\mathcal{R}_{13}(xy)\mathcal{R}_{23}(y)=\mathcal{R}_{23}(y)\mathcal{R}_{13}(xy)\mathcal{R}_{12}(x), \tag{1.9}$$

and defines the algebra $U_q^{\mathrm{RTT}}(\mathfrak{g})$ via the so-called *RTT-relations*, see [FRT]. The isomorphisms

$$\Upsilon\colon U_q(\mathfrak{g})\xrightarrow{\sim} U_q^{\mathrm{RTT}}(\mathfrak{g}) \qquad \text{for finite type } \mathfrak{g} \tag{1.10}$$

and

$$\Upsilon\colon U_q^{\mathrm{Dr}}(\mathfrak{g})\xrightarrow{\sim} U_q^{\mathrm{RTT}}(\mathfrak{g}) \qquad \text{for affine type } \mathfrak{g} \tag{1.11}$$

were first constructed in [DF] for types $A_n, A_n^{(1)}$ through the Gauss decomposition of the generating matrices. For other classical Lie algebras and their affinizations, such isomorphisms were first discovered in [HM] and were revised much more recently in [JLM1, JLM2].

The theory of multiparameter quantum groups goes back to the early 90s, see e.g. [AST, Re, Ta]. However, the current interest in the subject stems from the papers [BW1]–[BW3], which study the two-parameter quantum group $U_{r,s}(\mathfrak{gl}_n)$ and provide a further application to pointed finite-dimensional Hopf algebras. In [BW2], they developed the theory of finite-dimensional representations in a complete analogy with the one-parameter case, computed the two-parameter R-matrix for the first fundamental $U_{r,s}(\mathfrak{gl}_n)$-representation, and used it to establish the Schur-Weyl duality between $U_{r,s}(\mathfrak{gl}_n)$ and a two-parameter Hecke algebra.

The above works of Benkart and Witherspoon stimulated an increased interest in the theory of two-parameter quantum groups. In particular, the definition and the basic structural results on $U_{r,s}(\mathfrak{g})$ for other classical simple Lie algebras $\mathfrak{g}$ were provided in [BGH1, BGH2]. Since then, multiple papers have treated such algebras case-by-case; we refer the reader to [HP] for a more uniform treatment and complete references.

The generalization of this theory from simple finite-dimensional Lie algebras to affine Lie algebras started with the work [HRZ] (which however had some gaps in the exposition, see [Ts] for an alternative treatment of the PBW results stated in [HRZ] without any proof). Subsequently, some attempts were made to provide a uniform Drinfeld-Jimbo presentation of such algebras, establishing the triangular decomposition and the Drinfeld double construction for them. More importantly, a new Drinfeld realization of these algebras $U_{r,s}(\widehat{\mathfrak{g}})$ was established on a case-by-case basis for $\mathfrak{g}$ being of type A_n (see [HRZ]), types D_n and E_6 (see [HZ2]), type G_2 (see [GHZ]), and type C_n (see [HZ1]). However, we note a caveat in this treatment: while a surjective homomorphism from the Drinfeld-Jimbo to the new Drinfeld realization is constructed similarly to Ψ^{-1} of (1.6), there is no proper proof of its injectivity. The aforementioned new Drinfeld realization of $U_{r,s}(\widehat{\mathfrak{g}})$ was used to construct the vertex representations of $U_{r,s}(\widehat{\mathfrak{g}})$ in an analogy with the one-parameter case (cf. [FJ]). Finally, the FRT-formalism for two-parameter quantum groups was carried out for $U_{r,s}(\mathfrak{gl}_n)$ and $U_{r,s}(\widehat{\mathfrak{gl}}_n)$ in [JL1, JL2], establishing the two-parameter analogues of (1.10, 1.11) for types A_n and $A_n^{(1)}$.

In this work (followed up by [MT2]), we develop the FRT-formalism for both $U_{r,s}(\mathfrak{g}), U_{r,s}(\widehat{\mathfrak{g}})$ when $\mathfrak{g}$ is an orthogonal or symplectic Lie algebra. The present note is mostly concerned with the derivation of finite and affine R-matrices (denoted by R and $R(u)$), while in [MT2] we use $R, R(u)$ to construct (1.10, 1.11), naturally generalizing [JLM1, JLM2] to the two-parameter setup. The latter, in particular, immediately provides the new Drinfeld realization of $U_{r,s}(\widehat{\mathfrak{g}})$ (which seemed to be missing for B_n). Let us outline the key ingredients.

To derive the finite R-matrix $\hat{R}_{VV}$, we factorize the universal R-matrix into the "local" ones (one factor for each positive root of $\mathfrak{g}$) and then evaluate the result on the first fundamental representation V of $U_{r,s}(\mathfrak{g})$. Due to the absence of Lusztig's braid group action on $U_{r,s}(\mathfrak{g})$ (noted first in [BGH1]), we use the aforementioned construction of orthogonal dual bases of positive and negative subalgebras of $U_{r,s}(\mathfrak{g})$ through standard Lyndon words and the technique of quantum shuffle algebras, which goes back to [K1, K2, Le, Ro] in the one-parameter setup, to [CHW] in the super case, and finally to [BH, BKL, HW1, HW2] and our accompanying note [MT1] for the two-parameter case. We note, however, that once the explicit formula is obtained, one can directly check that it coincides with the R-matrix $\hat{R}_{VV}$ by verifying that it intertwines the $U_{r,s}(\mathfrak{g})$-action on $V\otimes V$ and acts by the same scalars on the highest weight vectors of $V\otimes V$ as $\hat{R}_{VV}$.

To derive the affine R-matrix $\hat{R}_{VV}(u/v)$, we use the *Yang-Baxterization* technique of [GWX]. Since $\hat{R}_{VV}$ has three distinct eigenvalues (in contrast to the case $\mathfrak{g}=\mathfrak{sl}_n$ when it has only two distinct eigenvalues), there are 12 possible resulting operators, and in each case one needs to check some extra conditions to guarantee that they satisfy the Yang-Baxter equation. Instead, once the explicit formula for the correctly chosen one is derived, it is straightforward to check that it intertwines the $U_{r,s}(\widehat{\mathfrak{g}})$-actions on the tensor products of evaluation modules $V(u)\otimes V(v)\to V(v)\otimes V(u)$. The results of [Jim] then guarantee that the space of all such intertwiners is at most one-dimensional, and therefore the operator $\hat{R}(u/v)$ constructed through the Yang-Baxterization coincides with $\hat{R}_{VV}(u/v)$, thus producing a solution of (1.9) by further composing with the flip map τ.

While we were finishing the present note, closely related preprints [HJZ, HXZ1, HXZ2] appeared on arXiv. Though we communicated our results to one of those authors back in February 2024, it is a pity they decided not to consolidate our papers to be posted simultaneously. Partially due to this flaw, the present note is separated from a more straightforward part [MT2] that will be posted later.

1.2. **Outline.**

The structure of the present paper is the following:

- In Section 2.1, we recall two-parametric quantum groups $U_{r,s}(\mathfrak{g})$ for simple finite-dimensional Lie algebras $\mathfrak{g}$, see Definition 2.2, and summarize their basic properties (including the pairing of Proposition 2.5).
- In Section 3, we explicitly construct the first fundamental representations of $U_{r,s}(\mathfrak{g})$ for classical $\mathfrak{g}$, see Propositions 3.1–3.4. We further decompose the tensor product $V\otimes V$ into irreducible $U_{r,s}(\mathfrak{g})$-submodules, see Proposition 3.9. The proof of the latter is derived, through a reduction to the Lie algebra limit, by providing explicitly the corresponding highest weight vectors.
- In Section 4, we evaluate the universal intertwiner $\hat{R}_{VV}$ from Theorem 4.1 on the first fundamental $U_{r,s}(\mathfrak{g})$-representations from Section 3 for $\mathfrak{g}=\mathfrak{so}_{2n+1},\mathfrak{sp}_{2n},\mathfrak{so}_{2n}$, see Theorems 4.4–4.6. This generalizes the corresponding formula of Theorem 4.3 discovered first in [BW2]. Our proof is slightly different though, as we only match the eigenvalues of the three highest weight vectors in $V\otimes V$ featured in Proposition 3.9 (see Lemmas 4.8–4.10), and then verify the intertwining property with the action of f_i's (see Lemma 4.11).
- In Section 5, we provide an alternative proof of the formulas for $\hat{R}_{VV}$ from Theorems 4.3–4.6 by showing that they arise as the product of "local" operators parametrized by the positive roots of Φ. Since Lusztig's braid group action has no analogue for $U_{r,s}(\mathfrak{g})$, we instead use a combinatorial approach to the construction of orthogonal PBW bases, see Theorem 5.12 (which constitutes the main result of [MT1]). To this end, we construct the (quantum) root vectors iteratively by using the combinatorics of standard Lyndon words, recalled in Subsections 5.1 and 5.3. The factorization formula that results from these considerations is stated in Theorem 5.13. We conclude the section with a case-by-case treatment of each classical series, providing proofs of the formulas (4.6, 4.7, 4.9, 4.11) that are more conceptual than those presented in Section 4.
- In Section 6, we introduce the two-parameter quantum affine algebras $U_{r,s}(\widehat{\mathfrak{g}})$, see Definition 6.2, which is in agreement with [HRZ, HZ1, HZ2] for $\mathfrak{g}$ of types A_n,C_n,D_n. We also introduce their counterparts $U'_{r,s}(\widehat{\mathfrak{g}})$ without *degree generators* and extend the first fundamental $U_{r,s}(\mathfrak{g})$-representations ρ from Propositions 3.1–3.4 to evaluation $U'_{r,s}(\widehat{\mathfrak{g}})$-representations in Propositions 6.4–6.7. The latter ones are upgraded to $U_{r,s}(\widehat{\mathfrak{g}})$-representations in Proposition 6.8. The main results of this section are Theorems 6.10–6.12, which evaluate the universal intertwiner of $U_{r,s}(\widehat{\mathfrak{g}})$ on the tensor product of two such representations for $\mathfrak{g}=\mathfrak{so}_{2n+1},\mathfrak{sp}_{2n},\mathfrak{so}_{2n}$. This generalizes the corresponding formula (6.22) of Theorem 6.9, first discovered in [JL2]. According to [Jim], composing $\hat{R}(z)$ of (6.22)–(6.25) with a flip map τ produces solutions of the Yang-Baxter relation with a spectral parameter (1.9). While the proofs are straightforward, the origin of these formulas (whose one-parameter counterparts were discovered in [Jim]) is postponed till Section 7.
- In Section 7, we derive the formulas (6.22)–(6.25) through the *Yang-Baxterization* technique of [GWX].
- In Appendix A, we realize $U_q(\mathfrak{g})$ with $q=r^{1/2}s^{-1/2}$ as a subalgebra (of an extension) of $U_{r,s}(\mathfrak{g})$.
- In Appendix B, we discuss the relation between the two-parameter R-matrices of $U_{r,s}(\mathfrak{g})$ and $U_{r,s}(\widehat{\mathfrak{g}})$ and their one-parameter counterparts of [Jim] via a classical twist construction of [Re]. While in A-type they can be matched as shown in Proposition B.1, we emphasize a surprising discrepancy in BCD-types.

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2. Notations and definitions

Throughout the paper, we will work over an algebraically closed field $\mathbb{K} \supset \mathbb{C}(r,s)$, where r and s are indeterminates.

2.1. **Two-parameter finite quantum groups.**

Let E be a Euclidean space with a symmetric bilinear form $(\cdot,\cdot)$, and $\Phi \subset E$ be an indecomposable reduced root system with an ordered set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$. Let $\mathfrak{g}$ be the complex simple Lie algebra corresponding to this root system. Let $C = (c_{ij})_{i,j=1}^n$ be the Cartan matrix of $\mathfrak{g}$, explicitly given by $c_{ij} = \frac{2(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$, and let $d_i = \frac{1}{2}(\alpha_i,\alpha_i)$ where $(\cdot,\cdot)$ is normalized so that the short roots have square length 2. We denote the root and weight lattices of $\mathfrak{g}$ by Q and P, respectively:

$$\bigoplus_{i=1}^n \mathbb{Z}\alpha_i = Q \subset P = \bigoplus_{i=1}^n \mathbb{Z}\varpi_i \qquad \text{with} \quad (\alpha_i, \varpi_j) = d_i\delta_{ij}.$$

Having fixed above the order on the set of simple roots Π, we consider the (modified) Ringel bilinear form $\langle\cdot,\cdot\rangle$ on Q, such that (unless $\{i,j\} = \{n-1,n\}$ in type D_n) we have:

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} d_i c_{ij} & \text{if } \ i < j \\ d_i & \text{if } \ i = j \\ 0 & \text{if } \ i > j \end{cases}, \tag{2.1}$$

while in the remaining case of D_n-type system, we set:

$$\langle \alpha_{n-1}, \alpha_n \rangle = \langle \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n \rangle = -1, \qquad \langle \alpha_n, \alpha_{n-1} \rangle = \langle \varepsilon_{n-1} + \varepsilon_n, \varepsilon_{n-1} - \varepsilon_n \rangle = 1. \tag{2.2}$$

We note that $(\mu,\nu) = \langle \mu,\nu \rangle + \langle \nu,\mu \rangle$ for any $\mu,\nu \in Q$.

Remark 2.1. *The modification* (2.2) *is made to ensure that the algebra* $U_{r,s}(\mathfrak{so}_{2n})$ *defined below matches its original definition in* [BGH1].

We shall also need two-parameter analogues of q-integers and q-factorials (cf. [BW2, (2.2)]):

$$[m]_{r,s} = \frac{r^m - s^m}{r-s} = r^{m-1} + r^{m-2}s + \ldots + rs^{m-2} + s^{m-1} \quad \text{for all} \quad m \in \mathbb{N},$$

$$[m]_{r,s}! = [m]_{r,s}[m-1]_{r,s} \cdots [1]_{r,s} \qquad \text{for} \quad m > 0, \qquad [0]_{r,s}! = 1,$$

and most importantly two-parameter analogues of Gaussian binomial coefficients:

$$\begin{bmatrix} m \\ k \end{bmatrix}_{r,s} = \frac{[m]_{r,s}!}{[m-k]_{r,s}![k]_{r,s}!} \qquad \text{for all} \quad 0 \leq k \leq m. \tag{2.3}$$

Finally, in analogy with the one-parameter case (see [Jan, §4.2]), we define

$$\begin{aligned} r_\gamma &= r^{(\gamma,\gamma)/2}, \qquad s_\gamma = s^{(\gamma,\gamma)/2} \qquad \text{for all} \quad \gamma \in \Phi, \\ r_i &= r_{\alpha_i} = r^{d_i}, \qquad s_i = s_{\alpha_i} = s^{d_i} \qquad \text{for all} \quad 1 \leq i \leq n. \end{aligned} \tag{2.4}$$

We are ready now to introduce the main actor of this paper, the two-parameter quantum group of $\mathfrak{g}$, following [HP]. While this definition is uniform for all types, we will make it more explicit for classical Lie algebras $\mathfrak{g}$ in Subsection 2.3.

Definition 2.2. *The **two-parameter quantum group** $U_{r,s}(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$ is the associative $\mathbb{K}$-algebra generated by $\{e_i, f_i, \omega_i^{\pm 1}, (\omega_i')^{\pm 1}\}_{i=1}^n$ with the following defining relations (for all $1 \leq i, j \leq n$):*

$$[\omega_i, \omega_j] = [\omega_i, \omega_j'] = [\omega_i', \omega_j'] = 0, \qquad \omega_i^{\pm 1}\omega_i^{\mp 1} = 1 = (\omega_i')^{\pm 1}(\omega_i')^{\mp 1}, \tag{2.5}$$

$$\omega_i e_j = r^{\langle \alpha_j, \alpha_i\rangle} s^{-\langle \alpha_i, \alpha_j\rangle} e_j \omega_i, \qquad \omega_i f_j = r^{-\langle \alpha_j, \alpha_i\rangle} s^{\langle \alpha_i, \alpha_j\rangle} f_j \omega_i, \tag{2.6}$$

$$\omega_i' e_j = r^{-\langle \alpha_i, \alpha_j\rangle} s^{\langle \alpha_j, \alpha_i\rangle} e_j \omega_i', \qquad \omega_i' f_j = r^{\langle \alpha_i, \alpha_j\rangle} s^{-\langle \alpha_j, \alpha_i\rangle} f_j \omega_i', \tag{2.7}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{\omega_i - \omega_i'}{r_i - s_i}, \tag{2.8}$$

and quantum (r, s)-Serre relations

$$\begin{aligned} &\sum_{k=0}^{1-c_{ij}} (-1)^k \begin{bmatrix} 1-c_{ij} \\ k \end{bmatrix}_{r_i, s_i} (r_i s_i)^{\frac{1}{2}k(k-1)} (rs)^{k\langle \alpha_j, \alpha_i\rangle} e_i^{1-c_{ij}-k} e_j e_i^k = 0 \qquad \forall\, i \neq j, \\ &\sum_{k=0}^{1-c_{ij}} (-1)^k \begin{bmatrix} 1-c_{ij} \\ k \end{bmatrix}_{r_i, s_i} (r_i s_i)^{\frac{1}{2}k(k-1)} (rs)^{k\langle \alpha_j, \alpha_i\rangle} f_i^k f_j f_i^{1-c_{ij}-k} = 0 \qquad \forall\, i \neq j. \end{aligned} \tag{2.9}$$

Remark 2.3. *We note that this definition does depend on the choice of an order of Π. We shall make a standard choice for classical $\mathfrak{g}$ in the end of this section, thus matching the rest of the literature* [BW1, BGH1].

We note that the algebra $U_{r,s}(\mathfrak{g})$ is Q-graded via

$$\deg(e_i) = \alpha_i, \quad \deg(f_i) = -\alpha_i, \quad \deg(\omega_i) = 0, \quad \deg(\omega_i') = 0 \qquad \text{for all} \quad 1 \leq i \leq n.$$

For $\mu \in Q$, let $U_{r,s}(\mathfrak{g})_\mu$ (or simply $(U_{r,s})_\mu$) denote the degree μ component of $U_{r,s}(\mathfrak{g})$ under this Q-grading.

Analogously to the one-parameter case (cf. [Jan, §4.11]), there is a Hopf algebra structure on $U_{r,s}(\mathfrak{g})$, where the coproduct Δ, counit ϵ, and antipode S are defined on generators by the following formulas:

$$\begin{aligned} &\Delta(\omega_i^{\pm 1}) = \omega_i^{\pm 1} \otimes \omega_i^{\pm 1} && \epsilon(\omega_i^{\pm 1}) = 1 && S(\omega_i^{\pm 1}) = \omega_i^{\mp 1} \\ &\Delta((\omega_i')^{\pm 1}) = (\omega_i')^{\pm 1} \otimes (\omega_i')^{\pm 1} && \epsilon((\omega_i')^{\pm 1}) = 1 && S((\omega_i')^{\pm 1}) = (\omega_i')^{\mp 1} \\ &\Delta(e_i) = e_i \otimes 1 + \omega_i \otimes e_i && \epsilon(e_i) = 0 && S(e_i) = -\omega_i^{-1} e_i \\ &\Delta(f_i) = 1 \otimes f_i + f_i \otimes \omega_i' && \epsilon(f_i) = 0 && S(f_i) = -f_i (\omega_i')^{-1} \end{aligned}$$

Remark 2.4. *The simplest way to see how the definition above generalizes the usual Drinfeld-Jimbo one-parametric quantum groups $U_q(\mathfrak{g})$ (cf.* [Jan, §4]*) is to work in the numeric setup. To this end, let $r, s \in \mathbb{C}\backslash\{0\}$ with $r^2 \neq s^2$ and define $U_{r,s}(\mathfrak{g})$ as in Definition 2.2, but now viewed as an algebra over $\mathbb{K} = \mathbb{C}$. Then, for any $q \in \mathbb{C}$ with $q^4 \neq 1$, there is a natural Hopf algebra epimorphism*

$$\pi \colon U_{q,q^{-1}}(\mathfrak{g}) \twoheadrightarrow U_q(\mathfrak{g}) \quad \text{given by} \quad e_i \mapsto E_i, \ f_i \mapsto F_i, \ \omega_i \mapsto K_i, \ \omega_i' \mapsto K_i^{-1} \quad \text{for all} \quad 1 \leq i \leq n.$$

Moreover, the kernel of π is the two-sided ideal $\mathfrak{I}$ generated by $\{\omega_i' - \omega_i^{-1}\}_{i=1}^n$. Thus, we get:

$$\text{Hopf algebra isomorphism} \quad U_{q,q^{-1}}(\mathfrak{g})/\mathfrak{I} \xrightarrow{\sim} U_q(\mathfrak{g}). \tag{2.10}$$

We shall also realize $U_q(\mathfrak{g})$ (with $q = r^{1/2}s^{-1/2}$) as a subalgebra of an extension of $U_{r,s}(\mathfrak{g})$ in Appendix A.

Let us also define several subalgebras of $U_{r,s}(\mathfrak{g})$:

- the "positive" subalgebra $U_{r,s}^+(\mathfrak{g})$, generated by $\{e_i\}_{i=1}^n$,
- the "negative" subalgebra $U_{r,s}^-(\mathfrak{g})$, generated by $\{f_i\}_{i=1}^n$,
- the "Cartan" subalgebra $U_{r,s}^0(\mathfrak{g})$, generated by $\{\omega_i^{\pm 1}, (\omega_i')^{\pm 1}\}_{i=1}^n$,
- the "non-negative subalgebra" $U_{r,s}^{\geq}(\mathfrak{g})$, generated by $\{e_i, \omega_i^{\pm 1}\}_{i=1}^n$,
- the "non-positive subalgebra" $U_{r,s}^{\leq}(\mathfrak{g})$, generated by $\{f_i, (\omega_i')^{\pm 1}\}_{i=1}^n$.

When $\mathfrak{g}$ is clear from the context, we will use $U_{r,s}$ instead of $U_{r,s}(\mathfrak{g})$, and similarly for the above subalgebras.

For any $\mu = \sum_{i=1}^n k_i \alpha_i \in Q$, we define $\omega_\mu, \omega_\mu' \in U_{r,s}^0(\mathfrak{g})$ via:

$$\omega_\mu = \omega_1^{k_1}\omega_2^{k_2}\cdots\omega_n^{k_n}, \qquad \omega_\mu' = (\omega_1')^{k_1}(\omega_2')^{k_2}\cdots(\omega_n')^{k_n}.$$

2.2. **Hopf pairing.**

One of the basic structural properties of $U_{r,s}(\mathfrak{g})$ is that it may be realized as a Drinfel'd double of its subalgebras $U^{\leq}_{r,s}(\mathfrak{g})$ and $U^{\geq}_{r,s}(\mathfrak{g})$ with respect to the Hopf algebra pairing. This was established case-by-case in the literature, and we shall rather just refer to [HP]:

Proposition 2.5. *There exists a unique bilinear pairing*

$$(\cdot,\cdot)\colon U^{\leq}_{r,s}(\mathfrak{g}) \times U^{\geq}_{r,s}(\mathfrak{g}) \to \mathbb{K} \tag{2.11}$$

satisfying the following structural properties

$$(yy', x) = (y \otimes y', \Delta(x)), \qquad (y, xx') = (\Delta(y), x' \otimes x) \qquad \forall\, x, x' \in U^{\geq}_{r,s}(\mathfrak{g}),\ y, y' \in U^{\leq}_{r,s}(\mathfrak{g}), \tag{2.12}$$

as well as being given on the generators by:

$$\begin{aligned} &(f_i, \omega_j) = 0, \qquad (\omega'_i, e_i) = 0, \qquad (f_i, e_j) = \delta_{ij} \frac{1}{s_i - r_i} \qquad \text{for all} \quad 1 \leq i, j \leq n, \\ &(\omega'_\lambda, \omega_\mu) = r^{\langle \lambda, \mu \rangle} s^{-\langle \mu, \lambda \rangle} \qquad \text{for all} \quad \lambda, \mu \in Q. \end{aligned} \tag{2.13}$$

We can also formally extend this pairing to the weight lattice P as follows:

$$(\omega'_\lambda, \omega_\mu) = \prod_{i,j=1}^{n} (\omega'_i, \omega_j)^{\lambda_i \mu_j} \qquad \text{for any weights} \quad \lambda = \sum_{i=1}^{n} \lambda_i \alpha_i \in P,\ \mu = \sum_{i=1}^{n} \mu_i \alpha_i \in P. \tag{2.14}$$

Although λ_i, μ_i may not be integers, the expression above still makes sense because $\mathbb{K}$ is algebraically closed.

Remark 2.6. *As mentioned in the beginning of this Subsection, the above pairing allows for the realization of any two-parameter quantum group $U_{r,s}(\mathfrak{g})$ as a Drinfel'd double of its Hopf subalgebras $U^{\leq}_{r,s}(\mathfrak{g})$, $U^{\geq}_{r,s}(\mathfrak{g})$ with respect to the pairing $(\cdot,\cdot)$ of (2.11).*

Let us list several basic properties of this pairing that will be needed later.

- First, if $x \in U_{r,s}(\mathfrak{g})_\mu$ and $\nu \in Q$, then we have:

$$\omega_\nu x \omega_\nu^{-1} = (\omega'_\mu, \omega_\nu) x, \qquad \omega'_\nu x (\omega'_\nu)^{-1} = (\omega'_\nu, \omega_\mu)^{-1} x. \tag{2.15}$$

- Second, the pairing $(\cdot,\cdot)$ is of homogeneous degree zero, i.e.

$$(y, x) = 0 \quad \text{for} \quad x \in U^{\geq}_{r,s}(\mathfrak{g})_\mu,\ y \in U^{\leq}_{r,s}(\mathfrak{g})_{-\nu} \quad \text{with} \quad \mu \neq \nu. \tag{2.16}$$

- Third, similarly to the one-parameter case (cf. [Jan, §6.14–6.15]), we have

$$\Delta(x) \in x \otimes 1 + \bigoplus_{0<\nu<\mu} U^{+}_{r,s}(\mathfrak{g})_{\mu-\nu} \omega_\nu \otimes U^{+}_{r,s}(\mathfrak{g})_\nu + \omega_\mu \otimes x, \tag{2.17}$$

$$\Delta(y) \in y \otimes \omega'_\mu + \bigoplus_{0<\nu<\mu} U^{-}_{r,s}(\mathfrak{g})_{-\nu} \otimes U^{-}_{r,s}(\mathfrak{g})_{-(\mu-\nu)} \omega'_\nu + 1 \otimes y, \tag{2.18}$$

for any $x \in U^{+}_{r,s}(\mathfrak{g})_\mu$ and $y \in U^{-}_{r,s}(\mathfrak{g})_{-\mu}$. Here, we use the standard order $\leq$ on Q:

$$\nu \leq \mu \iff \mu - \nu = \sum_i k_i \alpha_i \quad \text{with} \quad k_i \in \mathbb{Z}_{\geq 0}. \tag{2.19}$$

Then, combining the properties (2.16)–(2.18) with the defining properties (2.12, 2.13), we obtain:

$$\begin{aligned} &(\omega'_\mu y, x) = (\omega'_\mu, \omega_\nu)(y, x), \qquad (y\omega'_\mu, x) = (y, x), \\ &(y, \omega_\nu x) = (\omega'_\mu, \omega_\nu)(y, x), \qquad (y, x\omega_\nu) = (y, x), \end{aligned} \tag{2.20}$$

for any $x \in U^{+}_{r,s}(\mathfrak{g})_\nu$ and $y \in U^{-}_{r,s}(\mathfrak{g})_{-\mu}$.

2.3. **Classical types.**

Since in this paper we are only interested in the classical Lie algebras $\mathfrak{g}$, it will be helpful to have more explicit formulas for the bilinear form, avoiding the use of the form $\langle\cdot,\cdot\rangle$ on Q defined in (2.1, 2.2). To this end, let us first recall the explicit realization of the classical root systems as well as specify the choice of simple roots for them:

- *A_n-type* (corresponding to $\mathfrak{g}\simeq\mathfrak{sl}_{n+1}$).
  Let $\{\varepsilon_i\}_{i=1}^{n+1}$ be an orthonormal basis of $\mathbb{R}^{n+1}$. Then, we have

$$\begin{aligned}\Phi_{A_n}&=\{\varepsilon_i-\varepsilon_j\,|\,1\leq i\neq j\leq n+1\}\subset\mathbb{R}^{n+1},\\ \Pi_{A_n}&=\{\alpha_i=\varepsilon_i-\varepsilon_{i+1}\}_{i=1}^{n}.\end{aligned}\tag{2.21}$$

- *B_n-type* (corresponding to $\mathfrak{g}\simeq\mathfrak{so}_{2n+1}$).
  Let $\{\varepsilon_i\}_{i=1}^{n}$ be an orthogonal basis of $\mathbb{R}^n$ with $(\varepsilon_i,\varepsilon_i)=2$ for all i. Then, we have

$$\begin{aligned}\Phi_{B_n}&=\{\pm\varepsilon_i\pm\varepsilon_j\,|\,1\leq i<j\leq n\}\cup\{\pm\varepsilon_i\,|\,1\leq i\leq n\}\subset\mathbb{R}^{n},\\ \Pi_{B_n}&=\{\alpha_i=\varepsilon_i-\varepsilon_{i+1}\}_{i=1}^{n-1}\cup\{\alpha_n=\varepsilon_n\}.\end{aligned}\tag{2.22}$$

- *C_n-type* (corresponding to $\mathfrak{g}\simeq\mathfrak{sp}_{2n}$).
  Let $\{\varepsilon_i\}_{i=1}^{n}$ be an orthonormal basis of $\mathbb{R}^n$. Then, we have

$$\begin{aligned}\Phi_{C_n}&=\{\pm\varepsilon_i\pm\varepsilon_j\,|\,1\leq i<j\leq n\}\cup\{\pm2\varepsilon_i\,|\,1\leq i\leq n\}\subset\mathbb{R}^{n},\\ \Pi&=\{\alpha_i=\varepsilon_i-\varepsilon_{i+1}\}_{i=1}^{n-1}\cup\{\alpha_n=2\varepsilon_n\}.\end{aligned}\tag{2.23}$$

- *D_n-type* (corresponding to $\mathfrak{g}\simeq\mathfrak{so}_{2n}$).
  Let $\{\varepsilon_i\}_{i=1}^{n}$ be an orthonormal basis of $\mathbb{R}^n$. Then, we have

$$\begin{aligned}\Phi_{D_n}&=\{\pm\varepsilon_i\pm\varepsilon_j\,|\,1\leq i<j\leq n\}\subset\mathbb{R}^{n},\\ \Pi_{D_n}&=\{\alpha_i=\varepsilon_i-\varepsilon_{i+1}\}_{i=1}^{n-1}\cup\{\alpha_n=\varepsilon_{n-1}+\varepsilon_n\}.\end{aligned}\tag{2.24}$$

Remark 2.7. *We note that $(\varepsilon_i,\varepsilon_i)=2$ in type B_n in agreement with our scaling of the form $(\cdot,\cdot)$.*

Then we have the following respective formulas for the pairing of Cartan elements (we note that while the second formula follows from the first one in each case, it will be convenient to use both later on):

- *A_n-type*

$$\begin{aligned}(\omega'_\lambda,\omega_i)&=r^{(\varepsilon_i,\lambda)}s^{(\varepsilon_{i+1},\lambda)},\\ (\omega'_i,\omega_\lambda)&=r^{-(\varepsilon_{i+1},\lambda)}s^{-(\varepsilon_i,\lambda)}.\end{aligned}\tag{2.25}$$

- *B_n-type*

$$\begin{aligned}(\omega'_\lambda,\omega_i)&=\begin{cases}r^{(\varepsilon_i,\lambda)}s^{(\varepsilon_{i+1},\lambda)} & \text{if } 1\leq i<n\\ r^{(\varepsilon_n,\lambda)}(rs)^{-\lambda_n} & \text{if } i=n\end{cases},\\ (\omega'_i,\omega_\lambda)&=\begin{cases}r^{-(\varepsilon_{i+1},\lambda)}s^{-(\varepsilon_i,\lambda)} & \text{if } 1\leq i<n\\ s^{-(\varepsilon_n,\lambda)}(rs)^{\lambda_n} & \text{if } i=n\end{cases}.\end{aligned}\tag{2.26}$$

- *C_n-type*

$$\begin{aligned}(\omega'_\lambda,\omega_i)&=\begin{cases}r^{(\varepsilon_i,\lambda)}s^{(\varepsilon_{i+1},\lambda)} & \text{if } 1\leq i<n\\ r^{2(\varepsilon_n,\lambda)}(rs)^{-2\lambda_n} & \text{if } i=n\end{cases},\\ (\omega'_i,\omega_\lambda)&=\begin{cases}r^{-(\varepsilon_{i+1},\lambda)}s^{-(\varepsilon_i,\lambda)} & \text{if } 1\leq i<n\\ s^{-2(\varepsilon_n,\lambda)}(rs)^{2\lambda_n} & \text{if } i=n\end{cases}.\end{aligned}\tag{2.27}$$

- *D_n-type*

$$\begin{aligned}(\omega'_\lambda,\omega_i)&=\begin{cases}r^{(\varepsilon_i,\lambda)}s^{(\varepsilon_{i+1},\lambda)} & \text{if } 1\leq i<n\\ r^{(\varepsilon_{n-1},\lambda)}s^{-(\varepsilon_n,\lambda)}(rs)^{-2\lambda_{n-1}} & \text{if } i=n\end{cases},\\ (\omega'_i,\omega_\lambda)&=\begin{cases}r^{-(\varepsilon_{i+1},\lambda)}s^{-(\varepsilon_i,\lambda)} & \text{if } 1\leq i<n\\ r^{(\varepsilon_n,\lambda)}s^{-(\varepsilon_{n-1},\lambda)}(rs)^{2\lambda_{n-1}} & \text{if } i=n\end{cases}.\end{aligned}\tag{2.28}$$

3. Column-vector representations

Let $N = n+1$ for A_n-type, $N = 2n+1$ for B_n-type, and $N = 2n$ for C_n- and D_n-types. Let $V = \mathbb{K}^N$ with standard basis vectors $\{v_i\}_{i=1}^N$. In this Section, we construct an action of $U_{r,s}(\mathfrak{g})$ on V for any classical $\mathfrak{g}$ by specifying explicitly $\rho\colon U_{r,s}(\mathfrak{g}) \to \mathrm{End}(V)$, and further decompose $V \otimes V$ into irreducible submodules.

3.1. First fundamental representations.

While the A-type representation goes back to [BW2], we decided to include it since it serves as a prototype for the other classical types, new in the literature.

Proposition 3.1. [BW2, §1] *The following defines a representation $\rho\colon U_{r,s}(\mathfrak{sl}_{n+1}) \to \mathrm{End}(V)$:*

$$\rho(e_i) = E_{i,i+1}\,, \qquad \rho(f_i) = E_{i+1,i}\,,$$
$$\rho(\omega_i) = rE_{ii} + sE_{i+1,i+1} + \sum_{1\le j\le n+1}^{j\neq i,i+1} E_{jj}\,, \qquad \rho(\omega_i') = sE_{ii} + rE_{i+1,i+1} + \sum_{1\le j\le n+1}^{j\neq i,i+1} E_{jj}\,.$$

In what follows, we shall use the involution $'$ on the indexing set $\{1,\dots,N\}$ defined via:

$$i' := N+1-i \qquad \text{for all} \quad 1 \le i \le N.$$

Proposition 3.2. *(Type B_n) The following defines a representation $\rho\colon U_{r,s}(\mathfrak{so}_{2n+1}) \to \mathrm{End}(V)$:*

$$\rho(e_i) = E_{i,i+1} - E_{(i+1)',i'}, \quad 1 \le i \le n\,,$$
$$\rho(f_i) = \begin{cases} E_{i+1,i} - (rs)^{-2}E_{i',(i+1)'} & \text{if} \quad 1 \le i < n \\ (r^{-1}+s^{-1})E_{n+1,n} - (r^{-1}+s^{-1})E_{n',n+1} & \text{if} \quad i = n \end{cases},$$
$$\rho(\omega_i) = \begin{cases} r^2E_{ii} + s^2E_{i+1,i+1} + r^{-2}E_{i'i'} + s^{-2}E_{(i+1)',(i+1)'} + \sum_{1\le j\le n}^{j\neq i,i+1}(E_{jj}+E_{j'j'}) + E_{n+1,n+1} & \text{if} \quad 1 \le i < n \\ rs^{-1}E_{nn} + E_{n+1,n+1} + r^{-1}sE_{n'n'} + \sum_{j=1}^{n-1}\left(r^{-1}s^{-1}E_{jj} + rsE_{j'j'}\right) & \text{if} \quad i = n \end{cases},$$
$$\rho(\omega_i') = \begin{cases} s^2E_{ii} + r^2E_{i+1,i+1} + s^{-2}E_{i'i'} + r^{-2}E_{(i+1)',(i+1)'} + \sum_{1\le j\le n}^{j\neq i,i+1}(E_{jj}+E_{j'j'}) + E_{n+1,n+1} & \text{if} \quad 1 \le i < n \\ r^{-1}sE_{nn} + E_{n+1,n+1} + rs^{-1}E_{n'n'} + \sum_{j=1}^{n-1}\left(r^{-1}s^{-1}E_{jj} + rsE_{j'j'}\right) & \text{if} \quad i = n \end{cases}.$$

Proof. The proof is straightforward, as we just need to verify that the above linear operators satisfy the defining relations (2.5)–(2.9). The relation (2.5) is obvious since all the operators $\rho(\omega_i), \rho(\omega_j')$ act diagonally in the basis $\{v_k\}_{k=1}^N$. To check the first equality of (2.6), we note that both sides act trivially on v_k unless $k \in \{j+1, j'\}$. In the latter case, one needs to compare the ratios of eigenvalues of $\rho(\omega_i)$ on v_j and v_{j+1}, or $v_{(j+1)'}$ and $v_{j'}$, to the pairing (ω_j', ω_i). The other three relations of (2.6)–(2.7) are verified analogously. The relation $[\rho(e_i), \rho(f_j)] = 0$ for $i \neq j$ is obvious as both $\rho(e_i)\rho(f_j)$ and $\rho(f_j)\rho(e_i)$ then act trivially on V. On the other hand, for $1 \le i < n$, the commutator $[\rho(e_i), \rho(f_i)]$ acts diagonally in the basis $\{v_k\}_{k=1}^N$, with nonzero eigenvalues $1, -1, r^{-2}s^{-2}, -r^{-2}s^{-2}$ only for $k = i, i+1, (i+1)', i'$, respectively, which exactly coincide with the eigenvalues of $\frac{\rho(\omega_i)-\rho(\omega_i')}{r^2-s^2}$. Likewise, the commutator $[\rho(e_n), \rho(f_n)]$ acts diagonally in the basis v_k with nonzero eigenvalues $r^{-1}+s^{-1}, -r^{-1}-s^{-1}$ only for $k = n, n'$, respectively, which exactly coincide with the eigenvalues of $\frac{\rho(\omega_n)-\rho(\omega_n')}{r-s}$. Finally, the Serre relations (2.9) hold as each summand acts trivially on all v_k. ∎

Proposition 3.3. *(Type C_n) The following defines a representation $\rho\colon U_{r,s}(\mathfrak{sp}_{2n}) \to \mathrm{End}(V)$:*

$$\rho(e_i) = \begin{cases} E_{i,i+1} - E_{(i+1)',i'} & \text{if} \quad 1 \leq i < n \\ E_{nn'} & \text{if} \quad i = n \end{cases},$$

$$\rho(f_i) = \begin{cases} E_{i+1,i} - (rs)^{-1} E_{i',(i+1)'} & \text{if} \quad 1 \leq i < n \\ (rs)^{-1} E_{n'n} & \text{if} \quad i = n \end{cases},$$

$$\rho(\omega_i) = \begin{cases} rE_{ii} + sE_{i+1,i+1} + r^{-1}E_{i'i'} + s^{-1}E_{(i+1)',(i+1)'} + \sum\limits_{1\leq j\leq n}^{j\neq i,i+1} (E_{jj} + E_{j'j'}) & \text{if} \quad 1 \leq i < n \\ rs^{-1}E_{nn} + r^{-1}sE_{n'n'} + \sum\limits_{j=1}^{n-1} \left(r^{-1}s^{-1}E_{jj} + rsE_{j'j'}\right) & \text{if} \quad i = n \end{cases},$$

$$\rho(\omega_i') = \begin{cases} sE_{ii} + rE_{i+1,i+1} + s^{-1}E_{i'i'} + r^{-1}E_{(i+1)',(i+1)'} + \sum\limits_{1\leq j\leq n}^{j\neq i,i+1} (E_{jj} + E_{j'j'}) & \text{if} \quad 1 \leq i < n \\ r^{-1}sE_{nn} + rs^{-1}E_{n'n'} + \sum\limits_{j=1}^{n-1} \left(r^{-1}s^{-1}E_{jj} + rsE_{j'j'}\right) & \text{if} \quad i = n \end{cases}.$$

Proof. The proof is analogous to that of Proposition 3.2; we leave details to the interested reader. ∎

Proposition 3.4. *(Type D_n) For $n \geq 2$, the following defines a representation $\rho\colon U_{r,s}(\mathfrak{so}_{2n}) \to \mathrm{End}(V)$:*

$$\rho(e_i) = \begin{cases} E_{i,i+1} - E_{(i+1)',i'} & \text{if} \quad 1 \leq i < n \\ (rs)^{-1}E_{n-1,n'} - E_{n,(n-1)'} & \text{if} \quad i = n \end{cases},$$

$$\rho(f_i) = \begin{cases} E_{i+1,i} - (rs)^{-1} E_{i',(i+1)'} & \text{if} \quad 1 \leq i < n \\ E_{n',n-1} - E_{(n-1)',n} & \text{if} \quad i = n \end{cases},$$

$$\rho(\omega_i) = \begin{cases} rE_{ii} + sE_{i+1,i+1} + r^{-1}E_{i'i'} + s^{-1}E_{(i+1)',(i+1)'} + \sum\limits_{1\leq j\leq n}^{j\neq i,i+1} (E_{jj} + E_{j'j'}) & \text{if} \quad 1 \leq i < n \\ s^{-1}E_{n-1,n-1} + rE_{nn} + sE_{(n-1)',(n-1)'} + r^{-1}E_{n'n'} + \sum\limits_{j=1}^{n-2} \left(r^{-1}s^{-1}E_{jj} + rsE_{j'j'}\right) & \text{if} \quad i = n \end{cases},$$

$$\rho(\omega_i') = \begin{cases} sE_{ii} + rE_{i+1,i+1} + s^{-1}E_{i'i'} + r^{-1}E_{(i+1)',(i+1)'} + \sum\limits_{1\leq j\leq n}^{j\neq i,i+1} (E_{jj} + E_{j'j'}) & \text{if} \quad 1 \leq i < n \\ r^{-1}E_{n-1,n-1} + sE_{nn} + rE_{(n-1)',(n-1)'} + s^{-1}E_{n'n'} + \sum\limits_{j=1}^{n-2} \left(r^{-1}s^{-1}E_{jj} + rsE_{j'j'}\right) & \text{if} \quad i = n \end{cases}.$$

Proof. The proof is analogous to that of Proposition 3.2; we leave details to the interested reader. ∎

The classification of finite-dimensional $U_{r,s}(\mathfrak{g})$-modules for classical $\mathfrak{g}$ is completely parallel to that of one-parameter quantum groups (cf. [Jan, §5]). For A-type, this has been carried out in [BW2], while for BCD-types this constitutes the major result of [BGH2]. Let us recall only the notions relevant to the rest of this section. A vector v in a $U_{r,s}(\mathfrak{g})$-module V is said to have **weight** $\lambda \in P$ if

$$\omega_i v = (\omega_\lambda', \omega_i) v \qquad \text{and} \qquad \omega_i' v = (\omega_i', \omega_\lambda)^{-1} v \qquad \text{for all} \quad 1 \leq i \leq n. \tag{3.1}$$

Let $V[\lambda]$ denote the subspace of all weight λ vectors in V. The following result is straightforward:

Lemma 3.5. *(a) For the vector representation V from Propositions 3.2 in type B_n, we have:*

$$V = V[0] \oplus \bigoplus_{i=1}^{n} \left(V[\varepsilon_i] \oplus V[-\varepsilon_i]\right) \quad \text{with} \quad V[0] = \mathbb{K}v_{n+1},\ V[\varepsilon_i] = \mathbb{K}v_i,\ V[-\varepsilon_i] = \mathbb{K}v_{i'} \quad \text{for} \quad 1 \leq i \leq n.$$

(b) For the vector representation V from Proposition 3.3 in type C_n, we have:

$$V = \bigoplus_{i=1}^{n} \big(V[\varepsilon_i] \oplus V[-\varepsilon_i]\big) \quad \text{with} \quad V[\varepsilon_i] = \mathbb{K}v_i,\ V[-\varepsilon_i] = \mathbb{K}v_{i'} \quad \text{for} \quad 1 \le i \le n.$$

(c) For the vector representation V from Proposition 3.4 in type D_n, we have:

$$V = \bigoplus_{i=1}^{n} \big(V[\varepsilon_i] \oplus V[-\varepsilon_i]\big) \quad \text{with} \quad V[\varepsilon_i] = \mathbb{K}v_i,\ V[-\varepsilon_i] = \mathbb{K}v_{i'} \quad \text{for} \quad 1 \le i \le n.$$

We note that $\varepsilon_1 = \varpi_1$ is the first fundamental weight of $\mathfrak{g}$. Since the vector v_1 is clearly annihilated by all $\rho(e_i)$ and $\rho(U^0_{r,s}(\mathfrak{g}))$ acts diagonally on the basis $\{v_k\}_{k=1}^N$ with distinct joint eigenvalues, we conclude:

Corollary 3.6. *For $\mathfrak{g}$ being one of the Lie algebras $\mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}$, the $U_{r,s}(\mathfrak{g})$-module V constructed respectively in Propositions 3.2, 3.3, 3.4 is isomorphic to $L(\varpi_1)$, the first fundamental $U_{r,s}(\mathfrak{g})$-representation.*

3.2. **Decomposition of the tensor square.**

In the rest of this section, we shall study the decomposition of the tensor square $V \otimes V$ into irreducible $U_{r,s}(\mathfrak{g})$-submodules. In type A_n, this has been carried out in [BW2, Proposition 5.3]:

Proposition 3.7. [BW2] *The $U_{r,s}(\mathfrak{sl}_{n+1})$-module $V \otimes V$ decomposes into the direct sum of two irreducible modules $V \otimes V \simeq L(2\varepsilon_1) \oplus L(\varepsilon_1 + \varepsilon_2)$ with the highest weight vectors $v_1 \otimes v_1$ and $v_1 \otimes v_2 - rv_2 \otimes v_1$.*

In contrast, we shall show in Proposition 3.9 that $V \otimes V$ actually decomposes into the direct sum of three irreducible $U_{r,s}(\mathfrak{g})$-submodules in the remaining classical types B_n, C_n, D_n (with $n \ge 2$ for types C_n, D_n). To this end, we start by establishing the following preliminary result (which is of independent interest):

Proposition 3.8. *If $\lambda \in P^+$ is a dominant integral weight, then the irreducible $U_{r,s}(\mathfrak{g})$-module $L(\lambda)$ of the highest weight λ has the same dimension as the corresponding irreducible $\mathfrak{g}$-module.*

Proof. The idea of the proof is to specialize r to q and s to q^{-1}, and then appeal to the analogous result in the one-parameter case. The argument presented below closely follows that of [Jan, §5.12–5.15].

Let us first set up some notation. Let $M(\lambda) = U_{r,s}(\mathfrak{g}) \otimes_{U^{\geq}_{r,s}(\mathfrak{g})} \mathbb{K}$ be the Verma $U_{r,s}(\mathfrak{g})$-module of highest weight $\lambda \in P^+$, and let $V = L(\lambda)$ be its finite-dimensional irreducible quotient, with highest weight vector denoted by v_λ (cf. [BW2, BGH2]). Let $\mathbb{F} = \mathbb{C}(r)$ and $A = \mathbb{F}[s, s^{-1}]$. For any sequence $\mathcal{I} = (i_1, \dots, i_k) \in \{1, 2, \dots, n\}^k$, set $f_{\mathcal{I}} = f_{i_1} \cdots f_{i_k}$, so that V is spanned by all the vectors $f_{\mathcal{I}} v_\lambda$. Since V is finite-dimensional and Q-graded, we note that $f_{\mathcal{I}} v_\lambda = 0$ for all but finitely many sequences $\mathcal{I}$. Therefore the A-submodule

$$V_A = \sum_{\mathcal{I}} A f_{\mathcal{I}} v_\lambda$$

is finitely generated, and is clearly also torsion-free. Thus, V_A is a free A-module since A is a PID. Furthermore, one can easily check that the natural map $\mathbb{K} \otimes_A V_A \to V$ is an isomorphism (since $\mathbb{K}$ contains the field of fractions of A), so that $\mathrm{rk}_A V_A = \dim_{\mathbb{K}} V$. On the other hand, we have an isomorphism $\mathbb{F} \xrightarrow{\sim} \mathbb{C}(q)$ given by $r \mapsto q$, which may be further extended to a homomorphism $A \to \mathbb{C}(q)$ via $s \mapsto q^{-1}$, making $\mathbb{C}(q)$ into an A-module. Consider the module $\overline{V} = \mathbb{C}(q) \otimes_A V_A$, which has an obvious $\mathbb{C}(q)$-vector space structure. Note that any basis of V_A over A yields a basis of $\overline{V}$ over $\mathbb{C}(q)$. Thus, it suffices to make $\overline{V}$ into a $U_q(\mathfrak{g})$-module in such a way that $\overline{V} \simeq L(\lambda)$ as $U_q(\mathfrak{g})$-modules, where $U_q(\mathfrak{g})$ is the Drinfeld-Jimbo quantum group over $\mathbb{C}(q)$ (as the irreducible highest weight modules over $\mathfrak{g}$ and $U_q(\mathfrak{g})$ have the same dimension, cf. [Jan, Theorem 5.15]).

Let us first show that V_A is actually stable under the action of all the generators $f_i, e_i, \omega_i^{\pm 1}, (\omega_i')^{\pm 1}$. For f_i, this is obvious. Moreover, since all the pairings $(\omega_\mu', \omega_i)^{\pm 1}$, $(\omega_i, \omega_\mu')^{\pm 1}$ belong to A, the fact that V_A is stable under $\omega_i^{\pm 1}, (\omega_i')^{\pm 1}$ follows from the fact that V_A has a weight space decomposition. Finally, to prove stability of V_A under e_i, we first note that if $v \in V_\mu$, then

$$\frac{\omega_i - \omega_i'}{r_i - s_i} v = \frac{r^{\langle \mu, \alpha_i \rangle} s^{-\langle \alpha_i, \mu \rangle} - r^{-\langle \alpha_i, \mu \rangle} s^{\langle \mu, \alpha_i \rangle}}{r_i - s_i} v = (rs)^{-\langle \alpha_i, \mu \rangle} \frac{r^{(\alpha_i, \mu)} - s^{(\alpha_i, \mu)}}{r_i - s_i} v,$$

which belongs to V_A since (α_i, μ) is divisible by d_i. Evoking $e_j v_\lambda = 0$ for all j, we thus get:

$$e_j f_{i_1} \cdots f_{i_k} v_\lambda = \sum_{1 \le s \le k}^{i_s = j} f_{i_1} \cdots f_{i_{s-1}} \frac{\omega_{i_s} - \omega_{i_s}'}{r_i - s_i} f_{i_{s+1}} \cdots f_{i_k} v_\lambda,$$

which is in V_A by what we have already proved.

It remains to note that the factors $r^{\langle\alpha_j,\alpha_i\rangle}s^{-\langle\alpha_i,\alpha_j\rangle}$ and $r^{-\langle\alpha_i,\alpha_j\rangle}s^{\langle\alpha_j,\alpha_i\rangle}$ specialize to $q^{(\alpha_i,\alpha_j)}$ and $q^{-(\alpha_i,\alpha_j)}$, respectively, under the specialization $r\mapsto q, s\mapsto q^{-1}$. Therefore, the operators e_i, f_i, ω_i, ω_i' on $\overline{V}$ satisfy the same relations as the generators E_i, F_i, K_i, K_i^{-1} of $U_q(\mathfrak{g})$, so that $\overline{V}$ is a $U_q(\mathfrak{g})$-module. Since $\overline{V}$ is also a finite-dimensional highest weight module with the highest weight λ, we have $\overline{V}\simeq L(\lambda)$ as $U_q(\mathfrak{g})$-modules. ∎

Proposition 3.9. *For $\mathfrak{g}$ being one of the Lie algebras $\mathfrak{so}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n}$ (where $n\geq 2$ if $\mathfrak{g}=\mathfrak{sp}_{2n}$ or $\mathfrak{g}=\mathfrak{so}_{2n}$) and V being the corresponding $U_{r,s}(\mathfrak{g})$-representation from Propositions 3.2, 3.3, 3.4, we have the following decomposition into irreducibles:*

$$V\otimes V\simeq L(2\varepsilon_1)\oplus L(\varepsilon_1+\varepsilon_2)\oplus L(0). \tag{3.2}$$

Proof. We shall only present complete details for $\mathfrak{g}=\mathfrak{so}_{2n+1}$, the other cases being analogous.

Type B_n.

Let us first show that the following are highest weight vectors for the $U_{r,s}(\mathfrak{so}_{2n+1})$-action on $V\otimes V$:

$$\begin{aligned} &w_1=v_1\otimes v_1,\quad w_2=v_1\otimes v_2-(\omega'_{\varepsilon_1},\omega_1)v_2\otimes v_1=\begin{cases} v_1\otimes v_2-rs^{-1}v_2\otimes v_1 & \text{if } n=1\\ v_1\otimes v_2-r^2v_2\otimes v_1 & \text{if } n>1\end{cases},\\ &w_3=\sum_{i=1}^{n}r^{2(i-1)}v_i\otimes v_{i'}+r^{2n-1}s^{-1}v_{n+1}\otimes v_{n+1}+\sum_{i=1}^{n}r^{2n-1}s^{2(i-n)-1}v_{i'}\otimes v_i. \end{aligned} \tag{3.3}$$

The equality $e_i(w_1)=0$ for all i follows immediately from:

$$e_i\cdot v_j=\begin{cases} v_i & \text{if } j=i+1\\ -v_{(i+1)'} & \text{if } j=i'\\ 0 & \text{otherwise}\end{cases}.$$

Likewise, since $e_i(v_2)=0$ unless $i=1$ and $e_1(v_2)=v_1, \omega_1(v_1)=(\omega'_{\varepsilon_1},\omega_1)v_1$, we also obtain:

$$e_i(w_2)=\omega_i(v_1)\otimes e_i(v_2)-(\omega'_{\varepsilon_1},\omega_1)e_i(v_2)\otimes v_1=0\qquad\text{for all}\quad 1\leq i\leq n.$$

It remains to check that w_3 is a highest weight vector. Indeed, for $k<n$ we get:

$$\begin{aligned} &e_k(w_3)=r^{2k}v_k\otimes v_{(k+1)'}-r^{2n-1}s^{2(k-n)-1}v_{(k+1)'}\otimes v_k-r^{2(k-1)}\omega_k(v_k)\otimes v_{(k+1)'}+r^{2n-1}s^{2(k-n)+1}\omega_k(v_{(k+1)'})\otimes v_k\\ &=r^{2k}v_k\otimes v_{(k+1)'}-r^{2(k-1)}r^2v_k\otimes v_{(k+1)'}-r^{2n-1}s^{2(k-n)-1}v_{(k+1)'}\otimes v_k+r^{2n-1}s^{2(k-n)+1}s^{-2}v_{(k+1)'}\otimes v_k=0, \end{aligned}$$

and similarly for $k=n$ we have:

$$\begin{aligned} e_n(w_3)&=r^{2n-1}s^{-1}v_n\otimes v_{n+1}-r^{2n-1}s^{-1}v_{n+1}\otimes v_n-r^{2(n-1)}\omega_n(v_n)\otimes v_{n+1}+r^{2n-1}s^{-1}\omega_n(v_{n+1})\otimes v_n\\ &=r^{2n-1}s^{-1}v_n\otimes v_{n+1}-r^{2n-1}s^{-1}v_{n+1}\otimes v_n-rs^{-1}\cdot r^{2(n-1)}v_n\otimes v_{n+1}+r^{2n-1}s^{-1}v_{n+1}\otimes v_n=0. \end{aligned}$$

It is clear that w_1, w_2, w_3 are linearly independent, and since w_1 has weight $2\varepsilon_1$, w_2 has weight $\varepsilon_1+\varepsilon_2$, and w_3 has weight 0, the result of the Proposition will follow if we can match the corresponding dimensions:

$$\dim L(\varepsilon_1)+\dim L(\varepsilon_1+\varepsilon_2)+\dim L(0)=(2n+1)^2. \tag{3.4}$$

In view of Proposition 3.8, we can prove this by using the Weyl dimension formula. In type B_n, we have

$$\rho=\tfrac{1}{2}\sum_{\alpha\in\Phi^+}\alpha=\sum_{i=1}^{n}\left(n+\tfrac{1}{2}-i\right)\varepsilon_i,$$

and therefore

$$(\rho,\varepsilon_i-\varepsilon_j)=j-i,\qquad(\rho,\varepsilon_i+\varepsilon_j)=2n+1-i-j,\qquad(\rho,\varepsilon_i)=n+\tfrac{1}{2}-i.$$

Consequently, we obtain:

$$\prod_{\alpha\in\Phi^+}(\rho,\alpha)=\left(\prod_{i=1}^{n}\left(n+\tfrac{1}{2}-i\right)\right)\left(\prod_{1\leq i<j\leq n}(j-i)(2n+1-j-i)\right).$$

A similar computation yields

$$\prod_{\alpha\in\Phi^+}(2\varepsilon_1+\rho,\alpha)=\left(\prod_{i=2}^{n}(1+i)(2n+2-i)(n+\tfrac{1}{2}-i)\right)(n+\tfrac{3}{2})\left(\prod_{2\le i<j\le n}(j-i)(2n+1-i-j)\right),$$

and thus we get

$$\dim L(2\varepsilon_1)=\frac{\prod_{\alpha\in\Phi^+}(2\varepsilon_1+\rho,\alpha)}{\prod_{\alpha\in\Phi^+}(\rho,\alpha)}=n(2n+3).$$

Similarly, we obtain:

$$\dim L(\varepsilon_1+\varepsilon_2)=n(2n+1).$$

This completes the proof of the equality (3.4) and hence of the Proposition, since $\dim L(0)=1$.

Type C_n.

The highest weight vectors for $U_{r,s}(\mathfrak{sp}_{2n})$-action on $V\otimes V$ are:

$$w_1=v_1\otimes v_1,\quad w_2=v_1\otimes v_2-(\omega'_{\varepsilon_1},\omega_1)v_2\otimes v_1=v_1\otimes v_2-rv_2\otimes v_1,$$
$$w_3=\sum_{i=1}^{n}\left(r^{i-1}v_i\otimes v_{i'}-r^n s^{i-n-1}v_{i'}\otimes v_i\right), \tag{3.5}$$

and we have

$$\dim L(2\varepsilon_1)=n(2n+1),\qquad \dim L(\varepsilon_1+\varepsilon_2)=(2n+1)(n-1),\qquad \dim L(0)=1.$$

Type D_n.

The highest weight vectors $U_{r,s}(\mathfrak{so}_{2n})$-action on $V\otimes V$ are:

$$w_1=v_1\otimes v_1,\quad w_2=v_1\otimes v_2-(\omega'_{\varepsilon_1},\omega_1)v_2\otimes v_1=v_1\otimes v_2-rv_2\otimes v_1,$$
$$w_3=\sum_{i=1}^{n}\left(r^{i-1}v_i\otimes v_{i'}+r^{n-1}s^{i-n}v_{i'}\otimes v_i\right), \tag{3.6}$$

and we have

$$\dim L(2\varepsilon_1)=(2n-1)(n+1),\qquad \dim L(\varepsilon_1+\varepsilon_2)=n(2n-1),\qquad \dim L(0)=1.$$

■

4. R-Matrices

In this Section, we evaluate the $U_{r,s}(\mathfrak{g})$-module isomorphism $V\otimes V\xrightarrow{\sim}V\otimes V$ for $\mathfrak{g}$ one of $\mathfrak{so}_{2n+1},\mathfrak{sp}_{2n},\mathfrak{so}_{2n}$ and their first fundamental representation V from Section 3, arising through the universal R-matrix. This produces two-parametric solutions of the quantum Yang-Baxter equation for classical Lie algebras, cf. (1.7):

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}. \tag{4.1}$$

4.1. Universal construction.

Let us first recall the general construction (1.8) of a $U_{r,s}(\mathfrak{g})$-module isomorphism $V\otimes W\to W\otimes V$ arising through the universal R-matrix (see [BW1, §4] for $U_{r,s}(\mathfrak{sl}_n)$ and [BGH2, §3] for other classical $U_{r,s}(\mathfrak{g})$, both modeled after the treatment of one-parameter case in [Jan, §7]). To this end, we pick dual bases $\{x_i^\mu\}$ and $\{y_i^\mu\}$ of $U^+_{r,s}(\mathfrak{g})_\mu$ and $U^-_{r,s}(\mathfrak{g})_{-\mu}$ with respect to the Hopf pairing (2.11), and set

$$\Theta=1+\sum_{\mu>0}\Theta_\mu\qquad\text{with}\qquad\Theta_\mu=\sum_i y_i^\mu\otimes x_i^\mu. \tag{4.2}$$

Let $\tau\colon V\otimes W\to W\otimes V$ be the flip map $v\otimes w\mapsto w\otimes v$. Finally, consider $f\colon P\times P\to\mathbb{K}^\times$ satisfying

$$f(\lambda+\nu,\mu)=f(\lambda,\mu)f(\nu,\mu),\qquad f(\lambda,\mu+\nu)=f(\lambda,\mu)f(\lambda,\nu),$$
$$f(\lambda,\alpha_i)=(\omega'_i,\omega_\lambda)^{-1},\qquad f(\alpha_i,\mu)=(\omega'_\mu,\omega_i)^{-1}, \tag{4.3}$$

for all $\lambda,\mu\in P$, $\nu\in Q$, and $\alpha_i\in\Pi$. Then, for any two $U_{r,s}(\mathfrak{g})$-modules V and W with weight space decomposition, we define a linear map $\widetilde{f}\colon V\otimes W\to V\otimes W$ via $\widetilde{f}(v\otimes w)=f(\lambda,\mu)\cdot v\otimes w$ if $v\in V[\lambda], w\in W[\mu]$.

Theorem 4.1. *For any finite-dimensional $U_{r,s}(\mathfrak{g})$-modules V and W, the map*

$$\hat{R}_{VW} = \Theta \circ \widetilde{f} \circ \tau \colon V \otimes W \to W \otimes V \tag{4.4}$$

is an isomorphism of $U_{r,s}(\mathfrak{g})$-modules.

Let $R_{VW} = \hat{R}_{WV} \circ \tau = \Theta \circ \widetilde{f} \colon V \otimes W \to V \otimes W$. Given finite-dimensional $U_{r,s}(\mathfrak{g})$-modules V_1, V_2, V_3, define three endomorphisms of $V_1 \otimes V_2 \otimes V_3$: $R_{12} = R_{V_1,V_2} \otimes \mathrm{Id}_{V_3}, R_{23} = \mathrm{Id}_{V_1} \otimes R_{V_2,V_3}, R_{13} = (\mathrm{Id} \otimes \tau) R_{12} (\mathrm{Id} \otimes \tau)$. We likewise define linear operators $\hat{R}_{12}, \hat{R}_{23}, \hat{R}_{13}$. According to [BW1, BGH2], modelled after [Jan], we have:

$$\begin{aligned} R_{12}R_{13}R_{23} &= R_{23}R_{13}R_{12} \colon V_1 \otimes V_2 \otimes V_3 \to V_1 \otimes V_2 \otimes V_3, \\ \hat{R}_{12}\hat{R}_{23}\hat{R}_{12} &= \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \colon V_1 \otimes V_2 \otimes V_3 \to V_3 \otimes V_2 \otimes V_1. \end{aligned} \tag{4.5}$$

In particular, we obtain a whole family of solutions of the quantum Yang-Baxter equation:

Corollary 4.2. *For any finite-dimensional $U_{r,s}(\mathfrak{g})$-module V, the operator $R_{VV} = \hat{R}_{VV} \circ \tau$ satisfies (4.1).*

4.2. **Explicit R-matrices.**

For the representation V of Proposition 3.1, the explicit formula for $\hat{R}_{VV}$ was obtained in [BW2, §5]:

Theorem 4.3. *(Type A_n) The $U_{r,s}(\mathfrak{sl}_{n+1})$-module isomorphism $\hat{R}_{VV} \colon V \otimes V \xrightarrow{\sim} V \otimes V$ from Theorem 4.1 for the $U_{r,s}(\mathfrak{sl}_{n+1})$-module V from Proposition 3.1 coincides with the following operator:*

$$\hat{R} = \sum_{i=1}^{n+1} E_{ii} \otimes E_{ii} + r \sum_{1 \le i < j \le n+1} E_{ji} \otimes E_{ij} + s^{-1} \sum_{1 \le i < j \le n+1} E_{ij} \otimes E_{ji} + (1 - rs^{-1}) \sum_{1 \le i < j \le n+1} E_{jj} \otimes E_{ii}. \tag{4.6}$$

The main results of this Section generalize the above formula to the other classical series.

Theorem 4.4. *(Type B_n) The $U_{r,s}(\mathfrak{so}_{2n+1})$-module isomorphism $\hat{R}_{VV} \colon V \otimes V \xrightarrow{\sim} V \otimes V$ from Theorem 4.1 for the $U_{r,s}(\mathfrak{so}_{2n+1})$-module V from Proposition 3.2 coincides with the following operator:*

$$\begin{aligned} \hat{R} = {} & r^{-1}s \sum_{1 \le i \le 2n+1}^{i \ne n+1} E_{ii} \otimes E_{ii} + E_{n+1,n+1} \otimes E_{n+1,n+1} + rs^{-1} \sum_{1 \le i \le 2n+1}^{i \ne n+1} E_{ii'} \otimes E_{i'i} \\ & + \sum_{1 \le i,j \le 2n+1}^{j \ne i,i'} a_{ij} E_{ij} \otimes E_{ji} + (r^2 - s^2)(rs)^{-1} \sum_{i=1}^{n} (r^{2(n-i)+1} s^{2(i-n)-1} - 1) E_{i'i'} \otimes E_{ii} \\ & + (s^2 - r^2)(rs)^{-1} \sum_{i>j}^{j \ne i'} E_{ii} \otimes E_{jj} + (r^2 - s^2)(rs)^{-1} \sum_{i<j}^{j \ne i'} t_i t_j^{-1} E_{i'j} \otimes E_{ij'}, \end{aligned} \tag{4.7}$$

with the constants t_i and a_{ij} given explicitly by:

$$t_i = \begin{cases} s^{2(i-n)-1} & \text{if } i < n+1 \\ s^{-1} & \text{if } i = n+1 \\ r^{2(n+1-i)+1} & \text{if } i > n+1 \end{cases}, \qquad a_{ij} = \begin{cases} (rs)^{-\sigma_i \sigma_j} & \text{if } i < j, j' \text{ or } i > j, j' \\ (rs)^{\sigma_i \sigma_j} & \text{if } j < i < j' \text{ or } j' < i < j \end{cases}, \tag{4.8}$$

where we set

$$\sigma_i = \begin{cases} -1 & \text{if } i < n+1 \\ 0 & \text{if } i = n+1 \\ 1 & \text{if } i > n+1 \end{cases}.$$

Theorem 4.5. *(Type C_n) The $U_{r,s}(\mathfrak{sp}_{2n})$-module isomorphism $\hat{R}_{VV}: V\otimes V \xrightarrow{\sim} V\otimes V$ from Theorem 4.1 for the $U_{r,s}(\mathfrak{sp}_{2n})$-module V from Proposition 3.3 coincides with the following operator:*

$$\begin{aligned}\hat{R} =& r^{-1/2}s^{1/2}\sum_{i=1}^{2n} E_{ii}\otimes E_{ii} + r^{1/2}s^{-1/2}\sum_{i=1}^{2n} E_{ii'}\otimes E_{i'i} \\ &+ \sum_{1\leq i,j\leq 2n}^{j\neq i,i'} a_{ij}E_{ij}\otimes E_{ji} + (s-r)(rs)^{-1/2}\sum_{i=1}^{n}(r^{n+1-i}s^{i-n-1}+1)E_{i'i'}\otimes E_{ii} \\ &+ (s-r)(rs)^{-1/2}\sum_{i>j}^{j\neq i'} E_{ii}\otimes E_{jj} + (r-s)(rs)^{-1/2}\sum_{i<j}^{j\neq i'} t_i t_j^{-1}E_{i'j}\otimes E_{ij'}\,,\end{aligned} \tag{4.9}$$

with the constants t_i and a_{ij} given explicitly by:

$$t_i = \begin{cases} s^{i-n-1} & \text{if } i\leq n \\ -r^{n-i} & \text{if } i>n \end{cases}, \qquad a_{ij} = \begin{cases} (rs)^{-\frac{1}{2}\sigma_i\sigma_j} & \text{if } i<j,j' \text{ or } i>j,j' \\ (rs)^{\frac{1}{2}\sigma_i\sigma_j} & \text{if } j<i<j' \text{ or } j'<i<j \end{cases}, \tag{4.10}$$

where we set

$$\sigma_i = \begin{cases} 1 & \text{if } i\leq n \\ -1 & \text{if } i>n \end{cases}.$$

Theorem 4.6. *(Type D_n) The $U_{r,s}(\mathfrak{so}_{2n})$-module isomorphism $\hat{R}_{VV}: V\otimes V \xrightarrow{\sim} V\otimes V$ from Theorem 4.1 for the $U_{r,s}(\mathfrak{so}_{2n})$-module V from Proposition 3.4 coincides with the following operator:*

$$\begin{aligned}\hat{R} =& r^{-1/2}s^{1/2}\sum_{i=1}^{2n} E_{ii}\otimes E_{ii} + r^{1/2}s^{-1/2}\sum_{i=1}^{2n} E_{ii'}\otimes E_{i'i} \\ &+ \sum_{1\leq i,j\leq 2n}^{j\neq i,i'} a_{ij}E_{ij}\otimes E_{ji} + (s-r)(rs)^{-1/2}\sum_{i=1}^{n}(1-r^{n-i}s^{i-n})E_{i'i'}\otimes E_{ii} \\ &+ (s-r)(rs)^{-1/2}\sum_{i>j}^{j\neq i'} E_{ii}\otimes E_{jj} + (r-s)(rs)^{-1/2}\sum_{i<j}^{j\neq i'} t_i t_j^{-1}E_{i'j}\otimes E_{ij'}\,,\end{aligned} \tag{4.11}$$

with the constants t_i and a_{ij} given explicitly by:

$$t_i = \begin{cases} s^{i-n} & \text{if } i\leq n \\ r^{n+1-i} & \text{if } i>n \end{cases}, \qquad a_{ij} = \begin{cases} (rs)^{-\frac{1}{2}\sigma_i\sigma_j} & \text{if } i<j,j' \text{ or } i>j,j' \\ (rs)^{\frac{1}{2}\sigma_i\sigma_j} & \text{if } j<i<j' \text{ or } j'<i<j \end{cases}, \tag{4.12}$$

where we set

$$\sigma_i = \begin{cases} 1 & \text{if } i\leq n \\ -1 & \text{if } i>n \end{cases}.$$

Remark 4.7. *Although the proofs of Theorems 4.4–4.6 share many similarities with that of Theorem 4.3, there are slight differences. Indeed, instead of explicitly determining bases of each irreducible component from the decomposition (3.2) and computing the action of $\hat{R}$ on these basis vectors, we rather verify that operators $\hat{R}$ from (4.7, 4.9, 4.11) do commute with the action of all f_i acting on $V\otimes V$, and act on the three highest weight vectors by the desired scalars. This approach is essential to our construction of $\hat{R}(z)$ in Section 6.*

4.3. **Proofs of explicit formulas.**

To prove theorems in the previous subsection, we shall first study the operators $\hat{R}$ featured in (4.7, 4.9, 4.11). Our first technical result evaluates $\hat{R}$-action on the three highest weight vectors of $V\otimes V$, see Proposition 3.9.

Lemma 4.8. *(Type B_n) The highest weight vectors $w_1, w_2, w_3 \in V\otimes V$ from (3.3) are eigenvectors of the operator $\hat{R}$ from (4.7), with respective eigenvalues $\lambda_1 = r^{-1}s, \lambda_2 = -rs^{-1}, \lambda_3 = r^{2n}s^{-2n}$.*

Lemma 4.9. *(Type C_n) The highest weight vectors $w_1, w_2, w_3 \in V\otimes V$ from (3.5) are eigenvectors for the operator $\hat{R}$ from (4.9), with respective eigenvalues $\lambda_1 = r^{-1/2}s^{1/2}, \lambda_2 = -r^{1/2}s^{-1/2}, \lambda_3 = -r^{n+1/2}s^{-n-1/2}$.*

Lemma 4.10. *(Type D_n) The highest weight vectors $w_1, w_2, w_3 \in V \otimes V$ from (3.6) are eigenvectors of the operator $\hat{R}$ from (4.11), with respective eigenvalues $\lambda_1 = r^{-1/2}s^{1/2}, \lambda_2 = -r^{1/2}s^{-1/2}, \lambda_3 = r^{n-1/2}s^{-n+1/2}$.*

Since the proofs of all three results are completely analogous, we shall prove only the first one.

Proof of Lemma 4.8. For w_1, we clearly get:

$$\hat{R}(v_1 \otimes v_1) = r^{-1}s\, v_1 \otimes v_1.$$

Likewise, if $n > 1$, then for w_2 we obtain:

$$\hat{R}(v_1 \otimes v_2 - r^2 v_2 \otimes v_1) = rsv_2 \otimes v_1 - rs^{-1}v_1 \otimes v_2 - (s^2 - r^2)rs^{-1}v_2 \otimes v_1 = -rs^{-1}(v_1 \otimes v_2 - r^2 v_2 \otimes v_1).$$

If $n = 1$, then we have $a_{12} = a_{21} = 1$, and so we get:

$$\hat{R}(v_1 \otimes v_2 - rs^{-1}v_2 \otimes v_1) = v_2 \otimes v_1 - rs^{-1}v_1 \otimes v_2 - rs^{-1}(r^{-1}s - rs^{-1})v_2 \otimes v_1 = -rs^{-1}(v_1 \otimes v_2 - rs^{-1}v_2 \otimes v_1).$$

Finally, for w_3 we obtain:

$$\begin{aligned}
&\hat{R}\left(\sum_{i=1}^{n} r^{2(i-1)} v_i \otimes v_{i'} + r^{2n-1}s^{-1}v_{n+1} \otimes v_{n+1} + \sum_{i=1}^{n} r^{2n-1}s^{-2(n-i)-1} v_{i'} \otimes v_i\right) \\
&= \sum_{i=1}^{n} r^{2i-1}s^{-1} v_{i'} \otimes v_i + r^{2n-1}s^{-1}v_{n+1} \otimes v_{n+1} + \sum_{i=1}^{n} r^{2n}s^{2(i-n)-2} v_i \otimes v_{i'} \\
&\quad + (rs^{-1} - r^{-1}s)\sum_{i=1}^{n}(r^{2(n-i)+1}s^{2(i-n)-1} - 1)r^{2n-1}s^{2(i-n)-1} v_{i'} \otimes v_i \\
&\quad + (rs^{-1} - r^{-1}s)\sum_{i=1}^{n} r^{2n-1}s^{2(i-n)-1} v_{i'} \otimes v_i + (rs^{-1} - r^{-1}s)\sum_{i=1}^{n}\sum_{j=1}^{i-1} r^{2(i-1)}s^{2(j-i)} v_{j'} \otimes v_j \\
&\quad + (rs^{-1} - r^{-1}s)\sum_{i=1}^{n}\sum_{1 \le j \le (i+1)'}^{j \ne i} t_j t_{i'}^{-1} r^{2n-1}s^{2(i-n)-1} v_{j'} \otimes v_j\,,
\end{aligned}$$

where the last three summands arise from the action of the last sum in (4.7).

The last two summands simplify as follows:

$$\sum_{i=1}^{n}\sum_{j=1}^{i-1} r^{2(i-1)}s^{2(j-i)} v_{j'} \otimes v_j = \sum_{j=1}^{n-1}\left(\sum_{i=j+1}^{n} r^{2(i-1)}s^{2(j-i)}\right) v_{j'} \otimes v_j = \sum_{j=1}^{n} r^{2j}s^{2(j-n)}[n-j]_{r^2,s^2} v_{j'} \otimes v_j$$

and

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{1 \le j \le (i+1)'}^{j \ne i} t_j t_{i'}^{-1} r^{2n-1}s^{2(i-n)-1} v_{j'} \otimes v_j &= \sum_{i=1}^{n}\sum_{j=1}^{n} r^{4n-2i}s^{2(i+j-2n-1)} v_{j'} \otimes v_j - \sum_{i=1}^{n} r^{4n-2i}s^{4(i-n)-2} v_{i'} \otimes v_i \\
&\quad + \sum_{i=1}^{n}\sum_{j=i+1}^{n} r^{2(j-i+n)-1}s^{2(i-n)-1} v_j \otimes v_{j'} + \sum_{i=1}^{n} r^{4n-2i}s^{2(i-n-1)} v_{n+1} \otimes v_{n+1} \\
= \sum_{j=1}^{n}(r^{2n}s^{2j-4n}[n]_{r^2,s^2} - r^{4n-2j}s^{4(j-n)-2}) v_{j'} \otimes v_j + \sum_{j=1}^{n} r^{2n+1}s^{1-2n}[j-1]_{r^2,s^2} v_j \otimes v_{j'} \\
+ r^{2n}s^{-2n}[n]_{r^2,s^2} v_{n+1} \otimes v_{n+1}.
\end{aligned}$$

Thus, the coefficient of $v_{i'} \otimes v_i$ in $\hat{R}(w_3)$ for $1 \le i \le n$ equals

$$\begin{aligned}
&r^{2i-1}s^{-1} + (rs^{-1} - r^{-1}s)(r^{2(n-i)+1}s^{2(i-n)-1} - 1)r^{2n-1}s^{2(i-n)-1} + (rs^{-1} - r^{-1}s)r^{2n-1}s^{2(i-n)-1} \\
&+ r^{2i-1}s^{2(i-n)-1}(r^{2(n-i)} - s^{2(n-i)}) + r^{2n-1}s^{2i-4n-1}(r^{2n} - s^{2n}) - r^{4n-2i-1}s^{4(i-n)-3}(r^2 - s^2) \\
&= r^{4n-1}s^{2i-4n-1} = r^{2n}s^{-2n} \cdot r^{2n-1}s^{2(i-n)-1},
\end{aligned}$$

as desired. Likewise, the coefficient of $v_i \otimes v_{i'}$ in $\hat{R}(w_3)$ for $1 \le i \le n$ equals $r^{2(n+i-1)}s^{-2n} = r^{2n}s^{-2n} \cdot r^{2(i-1)}$, while the coefficient of $v_{n+1} \otimes v_{n+1}$ equals $r^{4n-1}s^{-2n-1} = r^{2n}s^{-2n} \cdot r^{2n-1}s^{-1}$. This completes the proof. ∎

Next, we verify that the operators $\hat{R}$ are indeed $U_{r,s}(\mathfrak{g})$-module homomorphisms.

Lemma 4.11. *The operators $\hat{R}\colon V\otimes V\to V\otimes V$ from (4.7, 4.9, 4.11) are isomorphisms of $U_{r,s}(\mathfrak{g})$-modules.*

Proof. In each case, it suffices to verify $\Delta(f_k)\hat{R}=\hat{R}\Delta(f_k)\in\mathrm{End}(V\otimes V)$ for all k, since Lemmas 4.8–4.10 then imply that $\hat{R}$ acts as a nonzero scalar on each irreducible component of $V\otimes V$ (see Proposition 3.9). We will present this verification only in type B_n, since the arguments in the remaining cases are similar.

To make the computations more manageable, it will be helpful to break the operator $\hat{R}$ from (4.7) into the following six pieces:

$$R_1 = r^{-1}s\sum_{1\le i\le 2n+1}^{i\neq n+1} E_{ii}\otimes E_{ii} + E_{n+1,n+1}\otimes E_{n+1,n+1},$$

$$R_2 = rs^{-1}\sum_{1\le i\le 2n+1}^{i\neq n+1} E_{ii'}\otimes E_{i'i},$$

$$R_3 = \sum_{1\le i,j\le 2n+1}^{j\neq i,i'} a_{ij}E_{ij}\otimes E_{ji},$$

$$R_4 = (r^2-s^2)(rs)^{-1}\sum_{i=1}^{n}(r^{2(n-i)+1}s^{2(i-n)-1}-1)E_{i'i'}\otimes E_{ii},$$

$$R_5 = (s^2-r^2)(rs)^{-1}\sum_{i>j}^{j\neq i'} E_{ii}\otimes E_{jj},$$

$$R_6 = (r^2-s^2)(rs)^{-1}\sum_{i<j}^{j\neq i'} t_it_j^{-1}E_{i'j}\otimes E_{ij'}.$$

Now, for $k<n$, the matrix of $\Delta(f_k)$ is

$$\begin{aligned}\Delta(f_k) = {} & 1\otimes E_{k+1,k} - (rs)^{-2}1\otimes E_{k',(k+1)'} + s^2E_{k+1,k}\otimes E_{kk} + r^2E_{k+1,k}\otimes E_{k+1,k+1} + s^{-2}E_{k+1,k}\otimes E_{k'k'}\\
& + r^{-2}E_{k+1,k}\otimes E_{(k+1)',(k+1)'} + \sum_{1\le j\le n}^{j\neq k,k+1}(E_{k+1,k}\otimes E_{jj} + E_{k+1,k}\otimes E_{j'j'}) + E_{k+1,k}\otimes E_{n+1,n+1}\\
& - r^{-2}E_{k',(k+1)'}\otimes E_{kk} - s^{-2}E_{k',(k+1)'}\otimes E_{k+1,k+1} - r^{-2}s^{-4}E_{k',(k+1)'}\otimes E_{k'k'}\\
& - r^{-4}s^{-2}E_{k',(k+1)'}\otimes E_{(k+1)',(k+1)'} - (rs)^{-2}\sum_{1\le j\le n}^{j\neq k,k+1}(E_{k',(k+1)'}\otimes E_{jj} + E_{k',(k+1)'}\otimes E_{j'j'})\\
& - (rs)^{-2}E_{k',(k+1)'}\otimes E_{n+1,n+1}.\end{aligned}$$

Thus, by direct computation, we get

$$\begin{aligned}R_1\Delta(f_k) = {} & r^{-1}sE_{k+1,k+1}\otimes E_{k+1,k} - r^{-3}s^{-1}E_{k'k'}\otimes E_{k',(k+1)'}\\
& + rsE_{k+1,k}\otimes E_{k+1,k+1} - r^{-3}s^{-3}E_{k',(k+1)'}\otimes E_{k'k'},\end{aligned}$$

$$\begin{aligned}\Delta(f_k)R_1 = {} & r^{-1}sE_{kk}\otimes E_{k+1,k} - r^{-3}s^{-1}E_{(k+1)',(k+1)'}\otimes E_{k',(k+1)'}\\
& + r^{-1}s^3E_{k+1,k}\otimes E_{kk} - r^{-5}s^{-1}E_{k',(k+1)'}\otimes E_{(k+1)',(k+1)'},\\
R_2\Delta(f_k) = {} & rs^{-1}E_{k+1,(k+1)'}\otimes E_{(k+1)',k} - r^{-1}s^{-3}E_{k',k}\otimes E_{k,(k+1)'}\\
& + (rs)^{-1}E_{(k+1)',k}\otimes E_{k+1,(k+1)'} - (rs)^{-1}E_{k,(k+1)'}\otimes E_{k'k},\end{aligned}$$

$$\begin{aligned}\Delta(f_k)R_2 = {} & rs^{-1}E_{k'k}\otimes E_{k+1,k'} - r^{-1}s^{-3}E_{k+1,(k+1)'}\otimes E_{k',k+1}\\
& + rs^{-3}E_{k+1,k'}\otimes E_{k'k} - rs^{-3}E_{k',k+1}\otimes E_{k+1,(k+1)'},\end{aligned}$$

$$\begin{aligned}R_3\Delta(f_k)-\Delta(f_k)R_3 &= rsE_{k+1,k}\otimes E_{kk}+(rs)^{-1}E_{k+1,k'}\otimes E_{k'k}-(rs)^{-1}E_{k',k+1}\otimes E_{k+1,(k+1)'}\\ &-(rs)^{-3}E_{k',(k+1)'}\otimes E_{(k+1)',(k+1)'}+r^{-1}sE_{kk}\otimes E_{k+1,k}+rs^{-1}E_{k'k}\otimes E_{k+1,k'}\\ &-rsE_{k+1,k}\otimes E_{k+1,k+1}-(rs)^{-1}E_{(k+1)',k}\otimes E_{k+1,(k+1)'}\\ &+(rs)^{-1}E_{k,(k+1)'}\otimes E_{k'k}+(rs)^{-3}E_{k',(k+1)'}\otimes E_{k'k'}-r^{-1}s^{-3}E_{k+1,(k+1)'}\otimes E_{k',k+1}\\ &-r^{-3}s^{-1}E_{(k+1)',(k+1)'}\otimes E_{k',(k+1)'}-rs^{-1}E_{k+1,k+1}\otimes E_{k+1,k}\\ &-r^{-1}sE_{k+1,(k+1)'}\otimes E_{(k+1)',k}+r^{-3}s^{-1}E_{k'k}\otimes E_{k,(k+1)'}\\ &+r^{-1}s^{-3}E_{k'k'}\otimes E_{k',(k+1)'},\end{aligned}$$

$$\begin{aligned}R_4\Delta(f_k) &= (r^2-s^2)(rs)^{-1}(r^{2(n-k-1)+1}s^{2(k+1-n)-1}-1)E_{(k+1)',(k+1)'}\otimes E_{k+1,k}\\ &-(r^2-s^2)r^{-3}s^{-1}(r^{2(n-k)+1}s^{2(k-n)-1}-1)E_{k',(k+1)'}\otimes E_{kk},\end{aligned}$$

$$\begin{aligned}\Delta(f_k)R_4 &= (r^2-s^2)(rs)^{-1}(r^{2(n-k)+1}s^{2(k-n)-1}-1)E_{k'k'}\otimes E_{k+1,k}\\ &-(r^2-s^2)r^{-1}s^{-3}(r^{2(n-k)-1}s^{2(k-n)+1}-1)E_{k',(k+1)'}\otimes E_{k+1,k+1},\end{aligned}$$

$$\begin{aligned}\Delta(f_k)R_5-R_5\Delta(f_k) &= (rs^{-1}-r^{-1}s)E_{k'k'}\otimes E_{k+1,k}+(r^{-1}s-rs^{-1})E_{k+1,k+1}\otimes E_{k+1,k}\\ &+(r^{-1}s-rs^{-1})E_{(k+1)',(k+1)'}\otimes E_{k+1,k}-(r^{-3}s^{-1}-r^{-1}s^{-3})E_{k'k'}\otimes E_{k',(k+1)'}\\ &-(r^{-1}s^3-rs)E_{k+1,k}\otimes E_{kk}+(r^{-1}s^{-1}-rs^{-3})E_{k',(k+1)'}\otimes E_{k+1,k+1}\\ &-(r^{-3}s-r^{-1}s^{-1})E_{k',(k+1)'}\otimes E_{kk}+(r^{-5}s^{-1}-r^{-3}s^{-3})E_{k',(k+1)'}\otimes E_{(k+1)',(k+1)'},\end{aligned}$$

$$\begin{aligned}R_6\Delta(f_k) &= (r^2-s^2)(rs)^{-1}r^{2(n-k)-1}s^{2(k-n)-1}E_{k',(k+1)'}\otimes E_{kk}\\ &-(r^2-s^2)(rs)^{-1}s^{2(k-n)+1}r^{2(n-k)-1}E_{(k+1)',(k+1)'}\otimes E_{k+1,k}\\ &-(rs^{-1}-r^{-1}s)E_{k+1,(k+1)'}\otimes E_{(k+1)',k}+(r^{-1}s^{-3}-r^{-3}s^{-1})E_{k'k}\otimes E_{k,(k+1)'},\end{aligned}$$

$$\begin{aligned}\Delta(f_k)R_6 &= (r^2-s^2)(rs)^{-1}r^{2(n-k)-1}s^{2(k-n)-1}E_{k',(k+1)'}\otimes E_{k+1,k+1}\\ &-(r^2-s^2)(rs)^{-1}r^{2(n-k)+1}s^{2(k-n)-1}E_{k'k'}\otimes E_{k+1,k}\\ &+(rs^{-3}-r^{-1}s^{-1})E_{k',k+1}\otimes E_{k+1,(k+1)'}-(rs^{-3}-r^{-1}s^{-1})E_{k+1,k'}\otimes E_{k'k}.\end{aligned}$$

In particular, we obtain

$$\begin{aligned}R_4\Delta(f_k)+R_6\Delta(f_k) &= -(rs^{-1}-r^{-1}s)E_{(k+1)',(k+1)'}\otimes E_{k+1,k}+(r^{-1}s^{-1}-r^{-3}s)E_{k',(k+1)'}\otimes E_{kk}\\ &-(rs^{-1}-r^{-1}s)E_{k+1,(k+1)'}\otimes E_{(k+1)',k}+(r^{-1}s^{-3}-r^{-3}s^{-1})E_{k'k}\otimes E_{k,(k+1)'}\end{aligned}$$

and

$$\begin{aligned}\Delta(f_k)R_4+\Delta(f_k)R_6 &= -(rs^{-1}-r^{-1}s)E_{k'k'}\otimes E_{k+1,k}+(rs^{-3}-r^{-1}s^{-1})E_{k',(k+1)'}\otimes E_{k+1,k+1}\\ &+(rs^{-3}-r^{-1}s^{-1})E_{k',k+1}\otimes E_{k+1,(k+1)'}-(rs^{-3}-r^{-1}s^{-1})E_{k+1,k'}\otimes E_{k'k}.\end{aligned}$$

From the computations above, we finally get

$$\begin{aligned}&R_3\Delta(f_k)-\Delta(f_k)R_3+R_1\Delta(f_k)+R_2\Delta(f_k)-\Delta(f_k)R_1-\Delta(f_k)R_2\\ &=(r^{-1}s^{-3}-r^{-3}s^{-1})E_{k'k'}\otimes E_{k',(k+1)'}+(rs^{-1}-r^{-1}s)E_{k+1,(k+1)'}\otimes E_{(k+1)',k}\\ &\quad+(r^{-3}s^{-1}-r^{-1}s^{-3})E_{k'k}\otimes E_{k,(k+1)'}+(rs-r^{-1}s^3)E_{k+1,k}\otimes E_{kk}\\ &\quad+(r^{-1}s^{-1}-rs^{-3})E_{k+1,k'}\otimes E_{k'k}+(rs^{-3}-r^{-1}s^{-1})E_{k',k+1}\otimes E_{k+1,(k+1)'}\\ &\quad+(r^{-5}s^{-1}-r^{-3}s^{-3})E_{k',(k+1)'}\otimes E_{(k+1)',(k+1)'}+(r^{-1}s-rs^{-1})E_{k+1,k+1}\otimes E_{k+1,k}\end{aligned}\tag{4.13}$$

and

$$\begin{aligned}
&\Delta(f_k)R_5 - R_5\Delta(f_k) + \Delta(f_k)R_4 + \Delta(f_k)R_6 - R_4\Delta(f_k) - R_6\Delta(f_k) \\
&= (r^{-1}s - rs^{-1})E_{k+1,k+1} \otimes E_{k+1,k} + (r^{-1}s^{-3} - r^{-3}s^{-1})E_{k'k'} \otimes E_{k',(k+1)'} \\
&\quad + (rs - r^{-1}s^{3})E_{k+1,k} \otimes E_{kk} + (r^{-5}s^{-1} - r^{-3}s^{-3})E_{k',(k+1)'} \otimes E_{(k+1)',(k+1)'} \\
&\quad + (rs^{-3} - r^{-1}s^{-1})E_{k',k+1} \otimes E_{k+1,(k+1)'} + (r^{-1}s^{-1} - rs^{-3})E_{k+1,k'} \otimes E_{k'k} \\
&\quad + (rs^{-1} - r^{-1}s)E_{k+1,(k+1)'} \otimes E_{(k+1)',k} + (r^{-3}s^{-1} - r^{-1}s^{-3})E_{k'k} \otimes E_{k,(k+1)'}.
\end{aligned} \tag{4.14}$$

The right-hand sides of (4.13) and (4.14) are obviously equal, which implies $\hat{R}\Delta(f_k) = \Delta(f_k)\hat{R}$ for $k < n$.

The computation for $k = n$ is similar. This completes the proof of the lemma. ∎

Now we are ready to prove the main results of this section, Theorems 4.4, 4.5, 4.6. We present full details only for the first one, since the other two are completely analogous (details are left to the interested reader).

Proof of Theorem 4.4. According to Theorem 4.1 and Proposition 3.9, the action of $\hat{R}_{VV}$ on the tensor product $V \otimes V$ is uniquely determined by the eigenvalues of the highest weight vectors w_1, w_2, w_3 from (3.3) under its action. We shall now verify that these eigenvalues are precisely equal to λ_1, λ_2, λ_3 from Lemma 4.8.

The eigenvalue $\widetilde{\lambda}_1$ of the $\hat{R}_{VV}$-action on w_1 is equal to $f(\varepsilon_1, \varepsilon_1)$. Since $\varepsilon_1 = \alpha_1 + \ldots + \alpha_n$, $f(\varepsilon_1, \alpha_i) = (\omega_i', \omega_{\varepsilon_1})^{-1}$, and a computation using (2.26) yields $(\omega_1', \omega_{\varepsilon_1}) = s^{-2}$, $(\omega_n', \omega_{\varepsilon_1}) = rs$, $(\omega_i', \omega_{\varepsilon_1}) = 1$ if $1 < i < n$, we thus obtain $\widetilde{\lambda}_1 = r^{-1}s = \lambda_1$.

The eigenvalue $\widetilde{\lambda}_2$ of the $\hat{R}_{VV}$-action on w_2 equals the coefficient of $v_1 \otimes v_2$ in $\hat{R}_{VV}(w_2)$, and the latter appears only from applying $\widetilde{f} \circ \tau$ to the multiple of $v_2 \otimes v_1$. Thus, we have $\widetilde{\lambda}_2 = -rs^{-1}f(\varepsilon_1, 0) = -rs^{-1} = \lambda_2$ if $n = 1$. On the other hand, if $n > 1$, then $\widetilde{\lambda}_2 = -r^2 f(\varepsilon_1, \varepsilon_2)$, and since $\varepsilon_2 = \alpha_2 + \ldots + \alpha_n$, a similar calculation to the one above yields $\widetilde{\lambda}_2 = -r^2 \cdot (rs)^{-1} = -rs^{-1} = \lambda_2$.

The eigenvalue $\widetilde{\lambda}_3$ of the $\hat{R}_{VV}$-action on w_3 equals the coefficient of $v_1 \otimes v_{1'}$ in $\hat{R}_{VV}(w_3)$. The latter appears only from applying $\widetilde{f} \circ \tau$ to the multiple of $v_{1'} \otimes v_1$, so that $\widetilde{\lambda}_3 = r^{2n-1}s^{1-2n}f(\varepsilon_1, -\varepsilon_1)$. As $f(\varepsilon_1, -\varepsilon_1) = f(\varepsilon_1, \varepsilon_1)^{-1} = rs^{-1}$, we thus get $\widetilde{\lambda}_3 = r^{2n}s^{-2n} = \lambda_3$. ∎

Remark 4.12. *The above proofs of Theorems 4.4–4.6 are quite elementary, but they require knowing the correct formulas for $\hat{R}$ in the first place. In the next Section, we provide the conceptual origin of these formulas by factorizing them into an ordered product of "local" operators, one for each positive root of $\mathfrak{g}$.*

5. PBW bases, orthogonality, and factorization

In this Section, we present the factorization formulas for $\hat{R}$ from (4.7, 4.9, 4.11). In the absence of Lusztig's braid group action on $U_{r,s}(\mathfrak{g})$, one rather needs to use the combinatorial construction of orthogonal dual bases of $U^+_{r,s}(\mathfrak{g})$ and $U^-_{r,s}(\mathfrak{g})$, based on the combinatorics of standard Lyndon words, cf. [K1, K2, Le, Ro].

5.1. Standard Lyndon words.

Let $I = \{1, 2, \ldots, n\}$ be a finite ordered alphabet parametrizing the simple roots of $\mathfrak{g}$, and let I^* be the set of all finite length words in the alphabet I. For $u = [i_1 \ldots i_k] \in I^*$, we define its *length* by $|u| = k$. We introduce the *lexicographical order* on I^* in a standard way:

$$[i_1 \ldots i_k] < [j_1 \ldots j_l] \quad \text{if} \quad \begin{cases} i_1 = j_1, \ldots, i_a = j_a, i_{a+1} < j_{a+1} \text{ for some } a \geq 0 \\ \text{or} \\ i_1 = j_1, \ldots, i_k = j_k \text{ and } k < l \end{cases}.$$

For a word $w = [i_1 \ldots i_k] \in I^*$, the subwords:

$$w_{a|} = [i_1 \ldots i_a] \qquad \text{and} \qquad w_{|a} = [i_{k-a+1} \ldots i_k] \tag{5.1}$$

with $0 \leq a \leq k$ will be called a *prefix* and a *suffix* of w, respectively. We call such a prefix or a suffix *proper* if $0 < a < k$. We start with the following important definition:

Definition 5.1. *A word w is **Lyndon** if it is smaller than all of its proper suffixes:*

$$w < w_{|a} \qquad \text{for all} \quad 0 < a < |w|. \tag{5.2}$$

We recall the following two basic facts from the theory of Lyndon words:

Proposition 5.2. *([Lo, Proposition 5.1.3]) Any Lyndon word ℓ has a factorization:*

$$\ell = \ell_1 \ell_2 \tag{5.3}$$

defined by the property that ℓ_1 is the longest proper prefix of ℓ which is also a Lyndon word. Then, ℓ_2 is also a Lyndon word.

The factorization (5.3) is called a **standard factorization** of a Lyndon word.

Proposition 5.3. *([Lo, Proposition 5.1.5]) Any word w has a unique factorization:*

$$w = \ell_1 \dots \ell_k \tag{5.4}$$

where $\ell_1 \geq \dots \geq \ell_k$ are all Lyndon words.

The factorization (5.4) is called a **canonical factorization**.

Let $\mathfrak{n}^+$ be a Lie subalgebra of $\mathfrak{g}$ generated by all $\{e_i\}_{i=1}^n$. The **standard bracketing** of a Lyndon word ℓ (with respect to the Lie algebra $\mathfrak{n}^+$) is defined inductively by the following procedure:

- $e_{[i]} = e_i \in \mathfrak{n}^+$ for $i \in I$,
- $e_{[\ell]} = [e_{[\ell_1]}, e_{[\ell_2]}] \in \mathfrak{n}^+$, where $\ell = \ell_1 \ell_2$ is the standard factorization (5.3).

The following definition is due to [LR]:

Definition 5.4. *A Lyndon word ℓ is called **standard Lyndon** if e_ℓ cannot be expressed as a linear combination of e_m for various Lyndon words $m > \ell$.*

The major importance of this definition is due to the following result:

Theorem 5.5. *([LR]) The set $\big\{e_{[\ell]} | \ell-\text{standard Lyndon word}\big\}$ provides a basis of $\mathfrak{n}^+$.*

Due to a root space decomposition $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ with all $\mathfrak{g}_\alpha$ being one-dimensional, we get

$$\ell \colon \Phi^+ \xrightarrow{\sim} \big\{\text{standard Lyndon words}\big\}, \tag{5.5}$$

the so-called *Lalonde-Ram*'s bijection, mentioned in (1.5). This bijection was described explicitly in [Le]:

Proposition 5.6. *([Le, Proposition 25]) The bijection ℓ is inductively given as follows:*

- *for simple roots, we have $\ell(\alpha_i) = [i]$,*
- *for other positive roots, the value of $\ell(\alpha)$ is determined using Leclerc's algorithm:*

$$\ell(\alpha) = \max \Big\{ \ell(\gamma_1)\ell(\gamma_2) \,\Big|\, \alpha = \gamma_1 + \gamma_2 \,,\, \gamma_1, \gamma_2 \in \Phi^+ \,,\, \ell(\gamma_1) < \ell(\gamma_2) \Big\}. \tag{5.6}$$

We shall also need one more important property of ℓ. To the end, let us recall:

Definition 5.7. *A total order on the set of positive roots Φ^+ is **convex** if:*

$$\alpha < \alpha + \beta < \beta \tag{5.7}$$

for all $\alpha < \beta \in \Phi^+$ such that $\alpha + \beta$ is also a root.

The following result is [Le, Proposition 28], where it is attributed to [Ro] (see also [NT, Proposition 2.34]):

Proposition 5.8. *Consider the order on Φ^+ induced from the lexicographical order on standard Lyndon words:*

$$\alpha < \beta \quad \Longleftrightarrow \quad \ell(\alpha) < \ell(\beta) \ \text{ lexicographically.} \tag{5.8}$$

This order is convex.

Finally, recall that given any convex order on Φ^+, a pair (α, β) of positive roots is called a **minimal pair** for $\gamma = \alpha + \beta \in \Phi^+$ if

$$\alpha < \beta \quad \text{and} \quad \nexists\, \alpha < \alpha' < \gamma < \beta' < \beta \quad \text{such that} \quad \alpha' + \beta' = \gamma. \tag{5.9}$$

The following result goes back to [Ro, Le] (cf. [NT, Proposition 2.38]):

Proposition 5.9. *For any $\gamma \in \Phi^+$, consider the standard factorization (5.3), so that $\ell_1 = \ell(\alpha)$ and $\ell_2 = \ell(\beta)$. Then (α, β) is a minimal pair for $\gamma = \alpha + \beta$.*

5.2. **Convex orders for classical types.**

Let Φ be a root system of classical type, and choose the order of I exactly as in Subsection 2.3. Combining Propositions 5.6 and 5.8, we obtain the following explicit convex orders on the sets Φ^+ of positive roots:

- *Type* A_n

$$\alpha_1 < \alpha_1+\alpha_2 < \cdots < \alpha_1+\ldots+\alpha_n < \alpha_2 < \cdots < \alpha_{n-1} < \alpha_{n-1}+\alpha_n < \alpha_n. \tag{5.10}$$

- *Type* B_n

$$\begin{aligned} &\alpha_1 < \alpha_1+\alpha_2 < \cdots < \alpha_1+\ldots+\alpha_n \\ &\quad < \alpha_1+\ldots+\alpha_{n-1}+2\alpha_n < \cdots < \alpha_1+2\alpha_2+\ldots+2\alpha_n < \\ &\alpha_2 < \cdots < \alpha_{n-1} < \alpha_{n-1}+\alpha_n < \alpha_{n-1}+2\alpha_n < \alpha_n. \end{aligned} \tag{5.11}$$

- *Type* C_n

$$\begin{aligned} &\alpha_1 < \alpha_1+\alpha_2 < \cdots < \alpha_1+\ldots+\alpha_{n-1} < 2\alpha_1+\ldots+2\alpha_{n-1}+\alpha_n < \alpha_1+\ldots+\alpha_n \\ &\quad < \alpha_1+\ldots+\alpha_{n-2}+2\alpha_{n-1}+\alpha_n < \cdots < \alpha_1+2\alpha_2+\ldots+2\alpha_{n-1}+\alpha_n < \\ &\alpha_2 < \cdots < \alpha_{n-1} < 2\alpha_{n-1}+\alpha_n < \alpha_{n-1}+\alpha_n < \alpha_n. \end{aligned} \tag{5.12}$$

- *Type* D_n

$$\begin{aligned} &\alpha_1 < \alpha_1+\alpha_2 < \cdots < \alpha_1+\ldots+\alpha_{n-2}+\alpha_{n-1} < \alpha_1+\ldots+\alpha_{n-2}+\alpha_n < \alpha_1+\ldots+\alpha_n \\ &\quad < \alpha_1+\ldots+\alpha_{n-3}+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n < \cdots < \alpha_1+2\alpha_2+\ldots+2\alpha_{n-2}+\alpha_{n-1}+\alpha_n < \\ &\alpha_2 < \cdots < \alpha_{n-2} < \alpha_{n-2}+\alpha_{n-1} < \alpha_{n-2}+\alpha_n < \alpha_{n-2}+\alpha_{n-1}+\alpha_n < \alpha_{n-1} < \alpha_n. \end{aligned} \tag{5.13}$$

Remark 5.10. *An important feature of these convex orders on root systems of type X_n (with $X = A, B, C, D$) is their telescopic structure, that is, erasing all roots containing α_1 provides the order alike on the rank 1 smaller root system of type X_{n-1}. This will significantly simplify our calculations in Subsection 5.4.*

Remark 5.11. *It is a classical result, due to* [P]*, that the convex orders on Φ^+ are in bijection with the reduced decompositions of the longest element w_0 of the Weyl group W of the root system Φ. In particular, the convex orders* (5.10)–(5.13) *correspond respectively to the following reduced decompositions of w_0 ($s_i = s_{\alpha_i}$):*

- $w_0 = (s_1 s_2 \ldots s_n)(s_1 s_2 \ldots s_{n-1}) \ldots (s_1 s_2)(s_1)$,
- $w_0 = (s_1 \ldots s_{n-1} s_n s_{n-1} \ldots s_1)(s_2 \ldots s_{n-1} s_n s_{n-1} \ldots s_2) \ldots (s_{n-1} s_n s_{n-1})(s_n)$,
- $w_0 = (s_1 \ldots s_{n-1} s_n s_{n-1} \ldots s_1)(s_2 \ldots s_{n-1} s_n s_{n-1} \ldots s_2) \ldots (s_{n-1} s_n s_{n-1})(s_n)$,
- $w_0 = \begin{cases} (s_1 \ldots s_{n-1} s_n s_{n-2} \ldots s_1)(s_2 \ldots s_{n-2} s_n s_{n-1} \ldots s_2) \ldots (s_{n-2} s_{n-1} s_n s_{n-2})(s_n s_{n-1}) & n\text{–odd} \\ (s_1 \ldots s_{n-1} s_n s_{n-2} \ldots s_1)(s_2 \ldots s_{n-2} s_n s_{n-1} \ldots s_1) \ldots (s_{n-2} s_n s_{n-1} s_{n-2})(s_{n-1} s_n) & n\text{–even} \end{cases}$.

Let us set up convenient notation for the positive roots in each type, and identify the minimal pairs arising through the standard factorization of the corresponding standard Lyndon words, see Proposition 5.9.

Type A_n. Let $\gamma_{ij} = \alpha_i + \ldots + \alpha_j$ for $1 \le i \le j \le n$, so that $\Phi^+ = \{\gamma_{ij}\}_{i \le j}$. For roots $\gamma = \gamma_{ij}$ with $i < j$, the minimal pair arising through the standard factorization of $\ell(\gamma)$ is $(\alpha, \beta) = (\gamma_{i,j-1}, \alpha_j)$.

Type B_n. Let $\gamma_{ij} = \alpha_i + \ldots + \alpha_j$ for $1 \le i \le j \le n$, and let $\beta_{ij} = \alpha_i + \ldots + \alpha_{j-1} + 2\alpha_j + \ldots + 2\alpha_n$ for $1 \le i < j \le n$. Let us now indicate the minimal pairs arising through the standard factorization of $\ell(\gamma)$:

- for the roots $\gamma = \gamma_{ij}$ with $i < j$, the minimal pair is $(\alpha, \beta) = (\gamma_{i,j-1}, \alpha_j)$;
- for the roots $\gamma = \beta_{in}$ with $1 \le i < n$, the minimal pair is $(\alpha, \beta) = (\gamma_{in}, \alpha_n)$;
- for the roots $\gamma = \beta_{ij}$ with $i < j < n$, the minimal pair is $(\alpha, \beta) = (\beta_{i,j+1}, \alpha_j)$.

Type C_n. Let $\gamma_{ij} = \alpha_i + \ldots + \alpha_j$ for $1 \le i \le j \le n$, and let $\beta_{ij} = \alpha_i + \ldots + \alpha_{j-1} + 2\alpha_j + \ldots + 2\alpha_{n-1} + \alpha_n$ for $1 \le i \le j < n$. Let us now indicate the minimal pairs arising through the standard factorization of $\ell(\gamma)$:

- for the roots $\gamma = \gamma_{ij}$ with $i < j$, the minimal pair is $(\alpha, \beta) = (\gamma_{i,j-1}, \alpha_j)$;
- for the roots $\gamma = \beta_{ij}$ with $i < j < n-1$, the minimal pair is $(\alpha, \beta) = (\beta_{i,j+1}, \alpha_j)$;
- for the roots $\gamma = \beta_{ii}$ with $1 \le i < n$, the minimal pair is $(\alpha, \beta) = (\gamma_{i,n-1}, \gamma_{in})$;
- for the roots $\gamma = \beta_{i,n-1}$ with $1 \le i < n-1$, the minimal pair is $(\alpha, \beta) = (\gamma_{in}, \alpha_{n-1})$.

Type D_n. Following (2.24), let $\gamma_{ij} = \varepsilon_i - \varepsilon_{j+1}$ for $1 \leq i \leq j < n$, and let $\beta_{ij} = \varepsilon_i + \varepsilon_j$ for $1 \leq i < j \leq n$. Then $\gamma_{ij} = \alpha_i + \ldots + \alpha_j$ for $1 \leq i \leq j < n$, $\beta_{in} = \alpha_i + \ldots + \alpha_{n-2} + \alpha_n$, $\beta_{i,n-1} = \alpha_i + \ldots + \alpha_n$, and finally $\beta_{ij} = \alpha_i + \ldots + \alpha_{j-1} + 2\alpha_j + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ for $j < n-1$. Let us now indicate the minimal pairs arising through the standard factorization of $\ell(\gamma)$:

- for the roots $\gamma = \gamma_{ij}$ with $i < j$, the minimal pair is $(\alpha, \beta) = (\gamma_{i,j-1}, \alpha_j)$;
- for the roots $\gamma = \beta_{in}$ with $i \leq n-2$, the minimal pair is $(\alpha, \beta) = (\gamma_{i,n-2}, \alpha_n)$;
- for the roots $\gamma = \beta_{ij}$ with $i < j < n$, the minimal pair is $(\alpha, \beta) = (\beta_{i,j+1}, \alpha_j)$.

5.3. **Root vectors and the PBW theorem.**

The following construction of (quantum) **root vectors** e_γ, f_γ goes back to [K1, K2, Le, Ro] in the one-parameter setup, to [CHW] in the super setup, and to [BH, BKL, HW1, HW2] in the two-parameter setup. For $\gamma = \alpha_i \in \Pi$, we set

$$e_{\alpha_i} = e_i, \qquad f_{\alpha_i} = f_i.$$

By induction on the height of a root, for any $\gamma \in \Phi^+ \backslash \Pi$, define (cf. (1.4)):

$$e_\gamma = e_\alpha e_\beta - (\omega'_\beta, \omega_\alpha) e_\beta e_\alpha \,, \qquad f_\gamma = f_\beta f_\alpha - (\omega'_\alpha, \omega_\beta)^{-1} f_\alpha f_\beta, \tag{5.14}$$

where the minimal pair (α, β) of γ corresponds to the standard factorization of $\ell(\gamma)$, see Proposition 5.9.

The following is the two-parameter version (a detailed proof of which is presented in Theorem 7.1 of the companion paper [MT1] crucially utilizing the technique of appropriate quantum shuffle algebras) of the classical result of [K1, K2, Le, Ro] (see also [CHW] where it was adapted to the super case):

Theorem 5.12. *(a) The ordered products*

$$\left\{ \prod_{\gamma \in \Phi^+}^{\leftarrow} e_\gamma^{m_\gamma} \,\Big|\, m_\gamma \geq 0 \right\} \quad \text{and} \quad \left\{ \prod_{\gamma \in \Phi^+}^{\leftarrow} f_\gamma^{m_\gamma} \,\Big|\, m_\gamma \geq 0 \right\} \tag{5.15}$$

are bases for $U^+_{r,s}(\mathfrak{g})$ and $U^-_{r,s}(\mathfrak{g})$, respectively. Here and below, the arrow $\leftarrow$ over the product signs refers to the total order (5.8) on Φ^+.

(b) The Hopf pairing (2.11) is orthogonal with respect to these bases. More explicitly, we have:

$$\left(\prod_{\gamma \in \Phi^+}^{\leftarrow} f_\gamma^{n_\gamma}, \prod_{\gamma \in \Phi^+}^{\leftarrow} e_\gamma^{m_\gamma} \right) = \prod_{\gamma \in \Phi^+} \Big(\delta_{n_\gamma, m_\gamma} (f_\gamma^{m_\gamma}, e_\gamma^{m_\gamma}) \Big). \tag{5.16}$$

(c) For each $\gamma \in \Phi^+$ and $m \geq 0$, we have (cf. notation (2.4)):

$$(f_\gamma^m, e_\gamma^m) = s_\gamma^{-m(m-1)/2} c_\gamma^m [m]_{r_\gamma, s_\gamma}! \tag{5.17}$$

The constants c_γ featured in this formula are defined recursively by

$$c_\gamma = \left((1 - \delta_{p_{\alpha,\beta},0}) [p_{\alpha,\beta} + 1]^2_{r_\alpha, s_\alpha} \frac{(s_\alpha - r_\alpha)(s_\beta - r_\beta)}{(s_\gamma - r_\gamma)} + (\omega'_\beta, \omega_\alpha) - (\omega'_\alpha, \omega_\beta)^{-1} \right) c_\alpha c_\beta, \tag{5.18}$$

where (α, β) is the minimal pair of Proposition 5.9 corresponding to the standard factorization of $\ell(\gamma)$, and

$$p_{\alpha,\beta} = \max \big\{ k \geq 0 \,|\, \alpha - k\beta \in \Phi \big\}.$$

As an immediate corollary, we obtain the following **factorization formula**:

Theorem 5.13. *The operator Θ of (4.2) can be factorized as follows:*

$$\Theta = \prod_{\gamma \in \Phi^+}^{\leftarrow} \left(\sum_{m \geq 0} \frac{1}{(f_\gamma^m, e_\gamma^m)} f_\gamma^m \otimes e_\gamma^m \right), \tag{5.19}$$

with e_γ, f_γ defined in (5.14) and (f_γ^m, e_γ^m) evaluated in (5.17).

Remark 5.14. *In fact, Θ can be expressed in an even more compact form by using (5.19) in conjunction with (r,s)-**exponential***

$$\exp_{r,s}(z) = \sum_{m\geq 0} s^{m(m-1)/2} \frac{z^m}{[m]_{r,s}!}.$$

Indeed, for each $\gamma \in \Phi^+$, we have

$$\sum_{m\geq 0} \frac{1}{(f_\gamma^m, e_\gamma^m)} f_\gamma^m \otimes e_\gamma^m = \sum_{m\geq 0} \frac{s_\gamma^{m(m-1)/2}}{c_\gamma^m [m]_{r_\gamma,s_\gamma}!} f_\gamma^m \otimes e_\gamma^m = \exp_{r_\gamma,s_\gamma}(c_\gamma^{-1} f_\gamma \otimes e_\gamma). \tag{5.20}$$

Thus, the factorization formula (5.19) simplifies as follows:

$$\Theta = \prod_{\gamma\in\Phi^+}^{\longleftarrow} \exp_{r_\gamma,s_\gamma}(c_\gamma^{-1} f_\gamma \otimes e_\gamma).$$

We conclude this subsection by presenting explicit formulas for (f_γ^m, e_γ^m) in all classical types:

- *Type A_n*

$$(f_{\gamma_{ij}}^m, e_{\gamma_{ij}}^m) = (-1)^m s^{-m(m-1)/2} \frac{[m]_{r,s}!}{(r-s)^m} \qquad \text{for} \quad 1\leq i\leq j\leq n. \tag{5.21}$$

- *Type B_n*

$$\begin{aligned}
(f_{\gamma_{ij}}^m, e_{\gamma_{ij}}^m) &= (-1)^m s^{-m(m-1)} \frac{[m]_{r^2,s^2}!}{(r^2-s^2)^m} \qquad \text{for} \quad 1\leq i\leq j<n,\\
(f_{\gamma_{in}}^m, e_{\gamma_{in}}^m) &= (-1)^m s^{-m(m-1)/2} \frac{[m]_{r,s}!}{(r-s)^m} \qquad \text{for} \quad 1\leq i\leq n,\\
(f_{\beta_{ij}}^m, e_{\beta_{ij}}^m) &= (-1)^m [2]_{r,s}^{2m} (rs)^{-2m(n-j)} s^{-m(m-1)} \frac{[m]_{r^2,s^2}!}{(r^2-s^2)^m} \qquad \text{for} \quad 1\leq i<j\leq n.
\end{aligned} \tag{5.22}$$

- *Type C_n*

$$\begin{aligned}
(f_{\gamma_{ij}}^m, e_{\gamma_{ij}}^m) &= (-1)^m s^{-m(m-1)/2} \frac{[m]_{r,s}!}{(r-s)^m} \qquad \text{for} \quad 1\leq i\leq j\leq n \quad \text{with} \quad (i,j)\neq(n,n),\\
(f_{\gamma_{nn}}^m, e_{\gamma_{nn}}^m) &= (-1)^m s^{-m(m-1)} \frac{[m]_{r^2,s^2}!}{(r^2-s^2)^m},\\
(f_{\beta_{ii}}^m, e_{\beta_{ii}}^m) &= (-1)^m [2]_{r,s}^{2m} s^{-m(m-1)} \frac{[m]_{r^2,s^2}!}{(r^2-s^2)^m} \qquad \text{for} \quad 1\leq i<n,\\
(f_{\beta_{ij}}^m, e_{\beta_{ij}}^m) &= (-1)^m (rs)^{-m(n-j)} s^{-m(m-1)/2} \frac{[m]_{r,s}!}{(r-s)^m} \qquad \text{for} \quad 1\leq i<j<n.
\end{aligned} \tag{5.23}$$

- *Type D_n*

$$\begin{aligned}
(f_{\gamma_{ij}}^m, e_{\gamma_{ij}}^m) &= (-1)^m s^{-m(m-1)/2} \frac{[m]_{r,s}!}{(r-s)^m} \qquad \text{for} \quad 1\leq i\leq j<n,\\
(f_{\beta_{ij}}^m, e_{\beta_{ij}}^m) &= (-1)^m (rs)^{-m(n-j)} s^{-m(m-1)/2} \frac{[m]_{r,s}!}{(r-s)^m} \qquad \text{for} \quad 1\leq i<j\leq n.
\end{aligned} \tag{5.24}$$

5.4. R-Matrix computation.

We shall now use the factorization formula (5.19) to compute Θ and $\hat{R}_{VV}$ for all classical types. Throughout this subsection, we will use the more convenient notation (cf. (5.20))

$$\Theta_\gamma = \sum_{m\geq 0} \frac{1}{(f_\gamma^m, e_\gamma^m)} f_\gamma^m \otimes e_\gamma^m \qquad \text{for any} \quad \gamma\in\Phi^+, \tag{5.25}$$

so that equation (5.19) becomes

$$\Theta = \prod_{\gamma\in\Phi^+}^{\longleftarrow} \Theta_\gamma.$$

5.4.1. *Factorized formula in type A_n.*

We start by evaluating the action of $\{e_\gamma, f_\gamma\}_{\gamma\in\Phi^+}$ on the $U_{r,s}(\mathfrak{sl}_{n+1})$-representation V from Proposition 3.1:

Lemma 5.15. *$\rho(e_{\gamma_{ij}}) = E_{i,j+1}$ and $\rho(f_{\gamma_{ij}}) = E_{j+1,i}$ for any $1 \le i \le j \le n$.*

Proof. The proof is straightforward and proceeds by induction on the height $|\gamma_{ij}| = j - i + 1$, where we use

$$e_{\gamma_{ij}} = e_{\gamma_{i,j-1}} e_j - (\omega'_j, \omega_i \dots \omega_{j-1}) e_j e_{\gamma_{i,j-1}}, \qquad f_{\gamma_{ij}} = f_j f_{\gamma_{i,j-1}} - (\omega'_i \dots \omega'_{j-1}, \omega_j)^{-1} f_{\gamma_{i,j-1}} f_j$$

and the explicit $U_{r,s}(\mathfrak{sl}_{n+1})$-action on V, cf. (5.14) and Proposition 3.1. ∎

Clearly, $\rho(e_{\gamma_{ij}})^2 = \rho(f_{\gamma_{ij}})^2 = 0$ for all $i \le j$, so that

$$\Theta_{\gamma_{ij}} = 1 \otimes 1 + (s - r) E_{j+1,i} \otimes E_{i,j+1}$$

on the $U_{r,s}(\mathfrak{sl}_{n+1})$-representation $V \otimes V$. To evaluate Θ, let us first set

$$\Theta_i = \Theta_{\gamma_{in}} \Theta_{\gamma_{i,n-1}} \dots \Theta_{\gamma_{i,i+1}} \Theta_{\alpha_i},$$

so that

$$\Theta = \Theta_n \Theta_{n-1} \dots \Theta_1.$$

Since $E_{j+1,i} \otimes E_{i,j+1} \cdot E_{k+1,i} \otimes E_{i,k+1} = 0$ for any $i \le k < j$, we thus obtain

$$\Theta_i = 1 \otimes 1 + (s - r) \sum_{j=i+1}^{n} E_{j+1,i} \otimes E_{i,j+1}.$$

Moreover, since $E_{k+1,j} \otimes E_{j,k+1} \cdot E_{\ell+1,i} \otimes E_{i,\ell+1} = 0$ for any $k \ge j > i$ and $\ell \ge i$, we get

$$\Theta = \Theta_n \dots \Theta_1 = 1 \otimes 1 + (s - r) \sum_{1 \le i < j \le n+1} E_{ji} \otimes E_{ij}. \tag{5.26}$$

For type A_n, the function f of (4.3) takes the following values on the weights of V, see [BW1, Lemma 4.4]:

$$f(\varepsilon_i, \varepsilon_j) = \begin{cases} s^{-1} & \text{if } \ i < j \\ 1 & \text{if } \ i = j \\ r & \text{if } \ i > j \end{cases}.$$

Combining this formula with the formula (5.26) and the flip map τ, we recover the explicit formula (4.6) for $\hat{R}_{VV} = \Theta \circ \widetilde{f} \circ \tau \colon V \otimes V \to V \otimes V$, thus providing an alternative proof of Theorem 4.3, and giving a factorization of the R-matrix of [BW2] into "local" operators, parametrized by the set of positive roots Φ^+.

5.4.2. *Factorized formula in type B_n.*

We start by evaluating the action of $\{e_\gamma, f_\gamma\}_{\gamma\in\Phi^+}$ on the $U_{r,s}(\mathfrak{so}_{2n+1})$-module V from Proposition 3.2:

Lemma 5.16. *(a) For $i \le j \le n$, we have*

$$\rho(e_{\gamma_{ij}}) = E_{i,j+1} - s^{2(j-i)} E_{(j+1)',i'},$$
$$\rho(f_{\gamma_{ij}}) = \begin{cases} E_{j+1,i} - s^{2(i-j)} (rs)^{-2} E_{i',(j+1)'} & \text{if } \ j < n \\ (r^{-1} + s^{-1})(E_{n+1,i} - s^{2(i-n)} E_{i',n+1}) & \text{if } \ j = n \end{cases}.$$

(b) For $i < j \le n$, we have

$$\rho(e_{\beta_{ij}}) = (-1)^{n+1-j} (E_{ij'} - s^{2(n-i)} r^{-2(n-j)} (rs) E_{ji'}),$$
$$\rho(f_{\beta_{ij}}) = (-1)^{n+1-j} (r^{-1} + s^{-1})^2 \cdot s^{2(j-n)} (r^{2(j-n)} E_{j'i} - s^{2(i-n)} (rs) E_{i'j}).$$

Proof. The proof is straightforward and proceeds by an increasing induction on j (from $j = i$ up to $j = n$) for the roots $\gamma = \gamma_{ij}$, and then by a descending induction on j (from $j = n$ till $j = i + 1$) for the roots $\gamma = \beta_{ij}$. Here, we use the explicit $U_{r,s}(\mathfrak{so}_{2n+1})$-action on V from Proposition 3.2, the explicit list of minimal pairs (α, β) as specified in Subsection 5.2, and finally the inductive construction (5.14). ∎

According to Lemma 5.16(a), we have $\rho(e_{\gamma_{ij}})^2 = \rho(f_{\gamma_{ij}})^2 = 0$ for $1 \le i \le j < n$, so that

$$\Theta_{\gamma_{ij}} = 1 \otimes 1 + (s^2 - r^2)\left(E_{j+1,i} - s^{2(i-j)}(rs)^{-2}E_{i',(j+1)'}\right) \otimes \left(E_{i,j+1} - s^{2(j-i)}E_{(j+1)',i'}\right) \tag{5.27}$$

for any $1 \le i \le j < n$. In contrast, for $j = n$, we have:

$$\rho(e_{\gamma_{in}})^2 = -s^{2(n-i)}E_{ii'}, \quad \rho(f_{\gamma_{in}})^2 = -s^{2(i-n)}(r+s)^2(rs)^{-2}E_{i'i}, \quad \rho(e_{\gamma_{in}})^3 = \rho(f_{\gamma_{in}})^3 = 0.$$

Therefore, we obtain:

$$\begin{aligned}\Theta_{\gamma_{in}} = 1 \otimes 1 + (s^2 - r^2)(rs)^{-1}\left(E_{n+1,i} - s^{2(i-n)}E_{i',n+1}\right) \otimes \left(E_{i,n+1} - s^{2(n-i)}E_{n+1,i'}\right) \\ + r^{-2}s^{-1}(r^2 - s^2)(r - s)E_{i'i} \otimes E_{ii'}.\end{aligned} \tag{5.28}$$

According to Lemma 5.16(b), we also have $\rho(e_{\beta_{ij}})^2 = \rho(f_{\beta_{ij}})^2 = 0$ for all $i < j \le n$, so that

$$\Theta_{\beta_{ij}} = 1 \otimes 1 + (s^2 - r^2)(rs)^{-2}\left(r^{2(j-n)}E_{j'i} - s^{2(i-n)}(rs)E_{i'j}\right) \otimes \left(r^{2(n-j)}E_{ij'} - s^{2(n-i)}(rs)E_{ji'}\right). \tag{5.29}$$

To evaluate Θ, let us first set

$$\Theta''_i = \Theta_{\beta_{i,i+1}}\Theta_{\beta_{i,i+2}}\dots\Theta_{\beta_{in}}, \quad \Theta'_i = \Theta_{\gamma_{in}}\Theta_{\gamma_{i,n-1}}\dots\Theta_{\gamma_{i,i+1}}\Theta_{\alpha_i}, \quad \Theta_i = \Theta''_i\Theta'_i,$$

so that

$$\Theta = \Theta_n\Theta_{n-1}\dots\Theta_1.$$

We will now show by induction on $n - k$ that

(5.30)

$$\begin{aligned}\Theta^{(k)} := \Theta_n\Theta_{n-1}\dots\Theta_k &= 1 \otimes 1 + c\sum_{i=k}^{n}\left(r^{-1}s - r^{2(n-i)}s^{2(i-n)}\right)E_{i'i} \otimes E_{ii'} \\ &+ c\sum_{k\le i<j\le n}\left(rsE_{ji} \otimes E_{ij} - r^{2(j-i)-1}sE_{ji} \otimes E_{j'i'} - r^{-1}s^{2(i-j)+1}E_{i'j'} \otimes E_{ij} + (rs)^{-1}E_{i'j'} \otimes E_{j'i'}\right) \\ &+ c\sum_{i=k}^{n}\left(E_{n+1,i} \otimes E_{i,n+1} - r^{2(n-i)}E_{n+1,i} \otimes E_{n+1,i'} - s^{-2(n-i)}E_{i',n+1} \otimes E_{i,n+1} + E_{i',n+1} \otimes E_{n+1,i'}\right) \\ &+ c\sum_{k\le i<j\le n}\left((rs)^{-1}E_{j'i} \otimes E_{ij'} - r^{2(n-i)}s^{2(j-n)}E_{j'i} \otimes E_{ji'} - r^{2(n-j)}s^{2(i-n)}E_{i'j} \otimes E_{ij'} + rsE_{i'j} \otimes E_{ji'}\right),\end{aligned}$$

where $c = (s^2 - r^2)(rs)^{-1}$. For $k = 1$, this provides the desired formula for $\Theta = \Theta^{(1)}$.

Let us start by computing a single Θ_i. First, it is easy to see from (5.27) and (5.28) that the only non-zero products in $\Theta_{\gamma_{in}}\Theta_{\gamma_{i,n-1}}\dots\Theta_{\alpha_i}$ involve $1 \otimes 1$, and thus

$$\begin{aligned}\Theta'_i &= 1 \otimes 1 + c(r^{-1}s - 1)E_{i'i} \otimes E_{ii'} \\ &+ c\sum_{j=i+1}^{n}\left(rsE_{ji} \otimes E_{ij} - rs^{2(j-i)-1}E_{ji} \otimes E_{j'i'} - r^{-1}s^{2(i-j)+1}E_{i'j'} \otimes E_{ij} + (rs)^{-1}E_{i'j'} \otimes E_{j'i'}\right) \\ &+ c\left(E_{n+1,i} \otimes E_{i,n+1} - s^{2(n-i)}E_{n+1,i} \otimes E_{n+1,i'} - s^{2(i-n)}E_{i',n+1} \otimes E_{i,n+1} + E_{i',n+1} \otimes E_{n+1,i'}\right).\end{aligned}$$

For similar reasons, we have

$$\begin{aligned}\Theta''_i &= 1 \otimes 1 \\ &+ c\sum_{j=i+1}^{n}\left((rs)^{-1}E_{j'i} \otimes E_{ij'} - r^{2(j-n)}s^{2(n-i)}E_{j'i} \otimes E_{ji'} - r^{2(n-j)}s^{2(i-n)}E_{i'j} \otimes E_{ij'} + rsE_{i'j} \otimes E_{ji'}\right).\end{aligned}$$

On the other hand, when computing the product $\Theta''_i\Theta'_i$, there are some non-zero products not involving $1 \otimes 1$, namely $cr^{2(n-j)}s^{2(i-n)}E_{i'j} \otimes E_{ij'} \cdot crs^{2(j-i)-1}E_{ji} \otimes E_{j'i'}$ for $i < j \le n$. The overall contribution of

those equals

$$c^2 r^{2n+1} s^{-2n-1} \left(\sum_{j=i+1}^{n} r^{-2j} s^{2j} \right) E_{i'i} \otimes E_{ii'} = c^2 r s^{2(i-n)+1} [n-i]_{r^2,s^2} E_{i'i} \otimes E_{ii'} \\ = c \left(1 - r^{2(n-i)} s^{2(i-n)} \right) E_{i'i} \otimes E_{ii'},$$

and therefore we have

(5.31)

$$\Theta_i = \Theta_i'' \Theta_i' = 1 \otimes 1 + c \left(r^{-1} s - r^{2(n-i)} s^{2(i-n)} \right) E_{i'i} \otimes E_{ii'} \\ + c \sum_{j=i+1}^{n} \left(rs E_{ji} \otimes E_{ij} - r s^{2(j-i)-1} E_{ji} \otimes E_{j'i'} - r^{-1} s^{2(i-j)+1} E_{i'j'} \otimes E_{ij} + (rs)^{-1} E_{i'j'} \otimes E_{j'i'} \right) \\ + c \left(E_{n+1,i} \otimes E_{i,n+1} - s^{2(n-i)} E_{n+1,i} \otimes E_{n+1,i'} - s^{2(i-n)} E_{i',n+1} \otimes E_{i,n+1} + E_{i',n+1} \otimes E_{n+1,i'} \right) \\ + c \sum_{j=i+1}^{n} \left((rs)^{-1} E_{j'i} \otimes E_{ij'} - r^{2(j-n)} s^{2(n-i)} E_{j'i} \otimes E_{ji'} - r^{2(n-j)} s^{2(i-n)} E_{i'j} \otimes E_{ij'} + rs E_{i'j} \otimes E_{ji'} \right).$$

In particular, for $i = n$ we get

$$\Theta_n = 1 \otimes 1 + c(r^{-1} s - 1) E_{i'i} \otimes E_{ii'} \\ + c \left(E_{n+1,n} \otimes E_{n,n+1} - E_{n+1,n} \otimes E_{n+1,n'} - E_{n',n+1} \otimes E_{n,n+1} + E_{n',n+1} \otimes E_{n+1,n'} \right),$$

which agrees with the claimed formula (5.30) when $k = n$, thus establishing the base of our induction.

Let us now prove the step of induction in (5.30). It suffices to treat the $k = 1$ case, due to the telescopic structure of the action on V and order on Φ^+, see Remark 5.10. It thus remains to evaluate $\Theta^{(1)} = \Theta^{(2)} \cdot \Theta_1$, where $\Theta^{(2)}$ is given by (5.30) and Θ_1 was just evaluated above. In addition to the terms of this product that involve $1 \otimes 1$, we get the following extra summands:

$$\left(-c \sum_{i=2}^{n} s^{2(i-n)} E_{i',n+1} \otimes E_{i,n+1} \right) \left(-c s^{2(n-1)} E_{n+1,1} \otimes E_{n+1,1'} \right) = (r^2 - s^2)^2 \sum_{i=2}^{n} r^{-2} s^{2i-4} E_{i'1} \otimes E_{i1'}, \tag{5.32a}$$

$$\left(-c \sum_{2 \le i < j \le n} r^{2(n-i)} s^{2(j-n)} E_{j'i} \otimes E_{ji'} \right) \left(-c \sum_{i=2}^{n} r s^{2i-3} E_{i1} \otimes E_{i'1'} \right) \\ = c^2 \sum_{i=2}^{n-1} \sum_{j=i+1}^{n} r^{2(n-i)+1} s^{2(j+i-n-1)-1} E_{j'1} \otimes E_{j1'} \\ = c^2 \sum_{j=3}^{n} \left(\sum_{i=2}^{j-1} r^{2(n-i)+1} s^{2(j+i-n-1)-1} \right) E_{j'1} \otimes E_{j1'} \\ = (r^2 - s^2)^2 \sum_{j=2}^{n} r^{2(n-j)+1} s^{2(j-n)-1} [j-2]_{r^2,s^2} E_{j'1} \otimes E_{j1'}, \tag{5.32b}$$

$$\left(-c \sum_{2 \le i < j \le n} r^{2(n-j)} s^{2(i-n)} E_{i'j} \otimes E_{ij'} \right) \left(-c \sum_{j=2}^{n} r s^{2j-3} E_{j1} \otimes E_{j'1'} \right) \\ = (r^2 - s^2)^2 \sum_{i=2}^{n-1} \left(\sum_{j=i+1}^{n} r^{2(n-j)-1} s^{2(i+j-n-2)-1} \right) E_{i'1} \otimes E_{i1'} \\ = (r^2 - s^2)^2 \sum_{i=2}^{n} r^{-1} s^{4i-2n-3} [n-i]_{r^2,s^2} E_{i'1} \otimes E_{i1'}, \tag{5.32c}$$

$$\left(c\sum_{i=2}^{n}(r^{-1}s-r^{2(n-i)}s^{2(i-n)})E_{i'i}\otimes E_{ii'}\right)\left(-c\sum_{i=2}^{n}rs^{2(i-1)-1}E_{i1}\otimes E_{i'1'}\right)$$
$$=(r^2-s^2)^2\sum_{i=2}^{n}(r^{2(n-i)-1}s^{4i-2n-5}-r^{-2}s^{2(i-2)})E_{i'1}\otimes E_{i1'}, \tag{5.32d}$$

$$\left(-c\sum_{2\le i<j\le n}r^{-1}s^{2(i-j)+1}E_{i'j'}\otimes E_{ij}\right)\left(-c\sum_{j=2}^{n}r^{2(j-n)}s^{2(n-1)}E_{j'1}\otimes E_{j1'}\right)$$
$$=(r^2-s^2)^2\sum_{i=2}^{n-1}\left(\sum_{j=i+1}^{n}r^{2(j-n-1)-1}s^{2(n+i-j-1)-1}\right)E_{i'1}\otimes E_{i1'}$$
$$=(r^2-s^2)^2\sum_{i=2}^{n}r^{2(i-n)-1}s^{2i-3}[n-i]_{r^2,s^2}E_{i'1}\otimes E_{i1'}, \tag{5.32e}$$

$$\left(-c\sum_{2\le i<j\le n}r^{2(j-i)-1}sE_{ji}\otimes E_{j'i'}\right)\left(-c\sum_{i=2}^{n}rs^{2i-3}E_{i1}\otimes E_{i'1'}\right)$$
$$=(r^2-s^2)^2\sum_{i=2}^{n}\sum_{j=i+1}^{n}r^{2(j-i-1)}s^{2(i-2)}E_{j1}\otimes E_{j'1'}$$
$$=(r^2-s^2)^2\sum_{j=3}^{n}\left(\sum_{i=2}^{j-1}r^{2(j-i-1)}s^{2(i-2)}\right)E_{j1}\otimes E_{j'1'}$$
$$=(r^2-s^2)^2\sum_{j=2}^{n}[j-2]_{r^2,s^2}E_{j1}\otimes E_{j'1'}, \tag{5.32f}$$

$$\left(-c\sum_{i=2}^{n}r^{2(n-i)}E_{n+1,i}\otimes E_{n+1,i'}\right)\left(-c\sum_{i=2}^{n}rs^{2i-3}E_{i1}\otimes E_{i'1'}\right)$$
$$=(r^2-s^2)^2\left(\sum_{i=2}^{n}r^{2(n-i)-1}s^{2(i-2)-1}\right)E_{n+1,1}\otimes E_{n+1,1'}$$
$$=(r^2-s^2)^2(rs)^{-1}[n-1]_{r^2,s^2}E_{n+1,1}\otimes E_{n+1,1'}. \tag{5.32g}$$

Thus, the overall contribution of the terms $\{E_{j1}\otimes E_{j'1'}\}_{1<j<n+1}$ into Θ equals

$$\sum_{j=2}^{n}(r^2-s^2)s^{2(j-2)}E_{j1}\otimes E_{j'1'}+\sum_{j=3}^{n}(r^2-s^2)^2[j-2]_{r^2,s^2}E_{j1}\otimes E_{j'1'}=(r^2-s^2)\sum_{j=2}^{n}r^{2(j-2)}E_{j1}\otimes E_{j'1'},$$

where the first summand arises from $1\otimes 1\cdot\Theta_1$ and the second from (5.32f). Likewise, the overall coefficient of $E_{n+1,1}\otimes E_{n+1,1'}$ in Θ is

$$(r^2-s^2)(rs)^{-1}s^{2(n-1)}+(r^2-s^2)^2(rs)^{-1}[n-1]_{r^2,s^2}=(r^2-s^2)(rs)^{-1}r^{2(n-1)},$$

where the first summand arises from $1\otimes 1\cdot\Theta_1$ and the second from (5.32g). Finally, the remaining terms (5.32a)–(5.32e) contribute to the coefficients of $\{E_{j'1}\otimes E_{j1'}\}_{2\le j\le n}$. Combining these with the corresponding terms from $1\otimes 1\cdot\Theta_1$, one eventually arrives at

$$(r^2-s^2)(rs)^{-1}\sum_{i=2}^{n}r^{2(n-1)}s^{2(i-n)}E_{i'1}\otimes E_{i1'}.$$

Thus, (5.32a)–(5.32g) and their counterparts from $1\otimes 1\cdot\Theta_1$ match the corresponding part in the right-hand side of (5.30) for $k=1$. On the other hand, it is easy to see that the remaining terms arising from $\Theta^{(2)}\cdot 1\otimes 1$ and $1\otimes 1\cdot\Theta_1$ exactly match with the remaining terms in the right-hand side of (5.30) for $k=1$. This completes our proof of the induction step, thus establishing (5.30).

To derive the formula for $\hat{R}_{VV}$ it only remains to evaluate the values of f from (4.3) on the weights of V. In accordance with (4.3), we have

$$f(\lambda,\mu)=(\omega'_\mu,\omega_\lambda)^{-1},$$

where we extend the Hopf pairing to the weight lattice as in (2.14). From the formulas (2.26), the equality $\varepsilon_k=\alpha_k+\ldots+\alpha_n$, and the basic properties of the Hopf pairing, we derive:

$$f(\varepsilon_i,\varepsilon_j)=f(-\varepsilon_i,-\varepsilon_j)=\begin{cases}(rs)^{-1} & \text{if } i<j\\ r^{-1}s & \text{if } i=j\\ rs & \text{if } i>j\end{cases}, \tag{5.33}$$

while the remaining values are then determined by:

$$f(\varepsilon_i,-\varepsilon_j)=f(-\varepsilon_i,\varepsilon_j)=f(\varepsilon_i,\varepsilon_j)^{-1},\qquad f(0,0)=f(0,\pm\varepsilon_i)=f(\pm\varepsilon_i,0)=1. \tag{5.34}$$

We also note that, if we set $\varepsilon_{n+1}=0$ and $\varepsilon_{i'}=-\varepsilon_i$ for all $1\le i\le n$, then we have

$$f(\varepsilon_i,\varepsilon_j)=a_{ij}\qquad \text{for all}\quad i\neq j,j', \tag{5.35}$$

where a_{ij} are given by (4.8). Combining the formulas (5.33, 5.34) with the formula (5.30) for $k=1$ and the flip map τ, we recover the explicit formula (4.7) for $\hat{R}_{VV}=\Theta\circ\widetilde{f}\circ\tau\colon V\otimes V\to V\otimes V$, thus providing an alternative proof of Theorem 4.4.

5.4.3. *Factorized formula in type C_n.*

The calculations in this case are very similar to those for type B_n, so we highlight only the main points. As before, we start by deriving explicit formulas for the action of $\{e_\gamma,f_\gamma\}_{\gamma\in\Phi^+}$ on the $U_{r,s}(\mathfrak{sp}_{2n})$-representation V from Proposition 3.3:

Lemma 5.17. *(a) For $i\le j\le n$, we have*

$$\rho(e_{\gamma_{ij}})=\begin{cases}E_{i,j+1}-s^{j-i}E_{(j+1)',i'} & \text{if } j<n\\ E_{in'}+s^{n+1-i}E_{ni'} & \text{if } j=n\end{cases},$$

$$\rho(f_{\gamma_{ij}})=\begin{cases}E_{j+1,i}-s^{i-j}(rs)^{-1}E_{i',(j+1)'} & \text{if } j<n\\ (rs)^{-1}E_{n'i}+s^{i-n-1}E_{i'n} & \text{if } j=n\end{cases}.$$

(b) For $1\le i<n$, we have

$$\rho(e_{\beta_{ii}})=s^{n-i}(r+s)E_{ii'},$$
$$\rho(f_{\beta_{ii}})=s^{i-n}(r^{-1}+s^{-1})E_{i'i}.$$

(c) For $1\le i<j<n$, we have

$$\rho(e_{\beta_{ij}})=(-1)^{n-j}(E_{ij'}+r^{j-n}s^{n+1-i}E_{ji'}),$$
$$\rho(f_{\beta_{ij}})=(-s)^{j-n}(r^{j-n}(rs)^{-1}E_{j'i}+s^{i-n-1}E_{i'j}).$$

Similarly to the above treatment of B_n-type, we define

$$\Theta_i=\Theta_{\beta_{i,i+1}}\ldots\Theta_{\beta_{i,n-1}}\Theta_{\gamma_{in}}\Theta_{\beta_{ii}}\Theta_{\gamma_{i,n-1}}\ldots\Theta_{\gamma_{i,i+1}}\Theta_{\alpha_i}\qquad\text{for}\quad 1\le i\le n,$$

where each factor is evaluated through (5.25) by using Lemma 5.17 and formulas (5.23). The following formula is derived completely analogously to (5.31):

$$\begin{aligned}\Theta_i&=1\otimes 1+c(r^{n-i+1}s^{i-n}+s)E_{i'i}\otimes E_{ii'}\\ &+c\sum_{j=i+1}^{n}\left(rsE_{ji}\otimes E_{ij}-rs^{j-i}E_{ji}\otimes E_{j'i'}-s^{i-j+1}E_{i'j'}\otimes E_{ij}+E_{i'j'}\otimes E_{j'i'}\right)\\ &+c\sum_{j=i+1}^{n}\left(E_{j'i}\otimes E_{ij'}+r^{j-n}s^{n+1-i}E_{j'i}\otimes E_{ji'}+r^{n-j+1}s^{i-n}E_{i'j}\otimes E_{ij'}+rsE_{i'j}\otimes E_{ji'}\right),\end{aligned} \tag{5.36}$$

where $c = (s-r)(rs)^{-1}$. Consider the following family of operators $\Theta^{(k)} = \Theta_n \Theta_{n-1} \dots \Theta_k$, so that $\Theta = \Theta^{(1)}$. We claim that $\Theta^{(k)}$ is explicitly given by the following formula:

$$\begin{aligned}\Theta^{(k)} &= 1 \otimes 1 + c \sum_{i=k}^{n} (r^{n-i+1} s^{i-n} + s) E_{i'i} \otimes E_{ii'} \\ &+ c \sum_{k \le i < j \le n} \left(rs E_{ji} \otimes E_{ij} - r^{j-i} s E_{ji} \otimes E_{j'i'} - s^{i-j+1} E_{i'j'} \otimes E_{ij} + E_{i'j'} \otimes E_{j'i'} \right) \\ &+ c \sum_{k \le i < j \le n} \left(E_{j'i} \otimes E_{ij'} + r^{n+1-i} s^{j-n} E_{j'i} \otimes E_{ji'} + r^{n-j+1} s^{i-n} E_{i'j} \otimes E_{ij'} + rs E_{i'j} \otimes E_{ji'} \right).\end{aligned} \tag{5.37}$$

The proof proceeds by an induction on $n-k$, with the base case $k=n$ following from (5.36). As per the step of induction, we note that when opening brackets in $\Theta^{(1)} = \Theta^{(2)} \Theta_1$, besides for the summands where one of the terms is $1 \otimes 1$, we get the following additional terms:

$$(s-r) \sum_{j=2}^{n} r^{n-1} s^{j-n-1} (1 - (r^{-1}s)^{j-2}) E_{j'1} \otimes E_{j1'},$$

$$(s-r) \sum_{j=2}^{n} r^{n-j} s^{2j-2-n} (1 - r^{j-n} s^{n-j}) E_{j'1} \otimes E_{j1'},$$

$$-(r-s)^2 \sum_{j=2}^{n} s^{j-3} (r^{n-j} s^{j-n} + r^{-1} s) E_{j'1} \otimes E_{j1'},$$

$$(s-r) \sum_{j=2}^{n} r^{j-n-1} s^{n-1} (r^{n-j} s^{j-n} - 1) E_{j'1} \otimes E_{j1'},$$

$$(s-r) \sum_{j=2}^{n} r^{j-2} (r^{2-j} s^{j-2} - 1) E_{j1} \otimes E_{j'1'}.$$

Combining these summands with the appropriate terms from $1 \otimes 1 \cdot \Theta_1$ and $\Theta^{(2)} \cdot 1 \otimes 1$ matches precisely the right-hand side of (5.37) for $k=1$, thus providing the formula for Θ.

From (2.27), the equality $\varepsilon_k = \alpha_k + \ldots + \alpha_{n-1} + \frac{1}{2}\alpha_n$, and basic properties of the Hopf pairing, we derive:

$$f(\varepsilon_i, \varepsilon_j) = f(-\varepsilon_i, -\varepsilon_j) = \begin{cases} (rs)^{-1/2} & \text{if } \ i < j \\ (r^{-1}s)^{1/2} & \text{if } \ i = j \\ (rs)^{1/2} & \text{if } \ i > j \end{cases}, \tag{5.39}$$

while the remaining values are then determined by:

$$f(\varepsilon_i, -\varepsilon_j) = f(-\varepsilon_i, \varepsilon_j) = f(\varepsilon_i, \varepsilon_j)^{-1}. \tag{5.40}$$

Furthermore, setting $\varepsilon_{i'} = -\varepsilon_i$ for all $1 \le i \le n$, we have

$$f(\varepsilon_i, \varepsilon_j) = a_{ij} \qquad \text{for all} \quad i \neq j, j', \tag{5.41}$$

where a_{ij} are given by (4.10). Combining the formulas (5.39, 5.40) with the formula (5.37) for $k=1$ and the flip map τ, we recover the explicit formula (4.9) for $\hat{R}_{VV} = \Theta \circ \widetilde{f} \circ \tau \colon V \otimes V \to V \otimes V$, thus providing an alternative proof of Theorem 4.5.

5.4.4. *Factorized formula in type D_n.*

As in the previous types, we start by deriving explicit formulas for the action of $\{e_\gamma, f_\gamma\}_{\gamma \in \Phi^+}$ on the $U_{r,s}(\mathfrak{so}_{2n})$-representation V from Proposition 3.4:

Lemma 5.18. *(a) For $1 \le i \le j < n$, we have*

$$\rho(e_{\gamma_{ij}}) = E_{i,j+1} - s^{j-i} E_{(j+1)',i'},$$
$$\rho(f_{\gamma_{ij}}) = E_{j+1,i} - s^{i-j} (rs)^{-1} E_{i',(j+1)'}.$$

(b) For $1 \le i < j \le n$, we have

$$\rho(e_{\beta_{ij}}) = (-1)^{n-j}((rs)^{-1}E_{ij'} - r^{j-n}s^{n-i-1}E_{ji'}),$$
$$\rho(f_{\beta_{ij}}) = (-1)^{n-j}((rs)^{j-n}E_{j'i} - s^{i+j+1-2n}E_{i'j}).$$

Similarly to the previous types, we define

$$\Theta_i = \Theta_{\beta_{i,i+1}} \dots \Theta_{\beta_{i,n-1}} \Theta_{\beta_{in}} \Theta_{\gamma_{i,n-1}} \Theta_{\gamma_{i,n-2}} \dots \Theta_{\gamma_{i,i+1}} \Theta_{\alpha_i} \qquad \text{for} \quad 1 \le i < n,$$

and derive the following counterpart of the formulas (5.31, 5.36):

$$\begin{aligned}\Theta_i &= 1 \otimes 1 + c(s - r^{n-i}s^{i+1-n})E_{i'i} \otimes E_{ii'} \\ &+ c\sum_{j=i+1}^{n} (rsE_{ji} \otimes E_{ij} - rs^{j-i}E_{ji} \otimes E_{j'i'} - s^{i-j+1}E_{i'j'} \otimes E_{ij} + E_{i'j'} \otimes E_{j'i'}) \\ &+ c\sum_{j=i+1}^{n} (E_{j'i} \otimes E_{ij'} - r^{j+1-n}s^{n-i}E_{j'i} \otimes E_{ji'} - r^{n-j}s^{i+1-n}E_{i'j} \otimes E_{ij'} + rsE_{i'j} \otimes E_{ji'}),\end{aligned} \tag{5.42}$$

where $c = (s-r)(rs)^{-1}$. Consider the following family of operators $\Theta^{(k)} = \Theta_{n-1} \dots \Theta_k$, so that $\Theta = \Theta^{(1)}$. We claim that $\Theta^{(k)}$ is explicitly given by the following formula:

$$\begin{aligned}\Theta^{(k)} &= 1 \otimes 1 + c\sum_{i=k}^{n}(s - r^{n-i}s^{i+1-n})E_{i'i} \otimes E_{ii'} \\ &+ c\sum_{k \le i < j \le n} \left(rsE_{ji} \otimes E_{ij} - r^{j-i}sE_{ji} \otimes E_{j'i'} - s^{i-j+1}E_{i'j'} \otimes E_{ij} + E_{i'j'} \otimes E_{j'i'}\right) \\ &+ c\sum_{k \le i < j \le n} \left(E_{j'i} \otimes E_{ij'} - r^{n-i}s^{j+1-n}E_{j'i} \otimes E_{ji'} - r^{n-j}s^{i+1-n}E_{i'j} \otimes E_{ij'} + rsE_{i'j} \otimes E_{ji'}\right).\end{aligned} \tag{5.43}$$

The proof proceeds by an induction on $n-k-1$, with the base case $k = n-1$ following from (5.42). As per the step of induction, we note that when opening brackets in $\Theta^{(1)} = \Theta^{(2)}\Theta_1$, besides for the summands where one of the terms is $1 \otimes 1$, we get the following additional terms:

$$\sum_{j=2}^{n}(r-s)r^{n-2}s^{j-n}(1 - r^{2-j}s^{j-2})E_{j'1} \otimes E_{j1'},$$

$$\sum_{j=2}^{n}(r-s)r^{n-j-1}s^{2j-1-n}(1 - r^{j-n}s^{n-j})E_{j'1} \otimes E_{j1'},$$

$$\sum_{j=2}^{n}(r-s)^2r^{-1}s^{j-2}(r^{n-j}s^{j-n} - 1)E_{j'1} \otimes E_{j1'},$$

$$\sum_{j=2}^{n}(r-s)r^{j-n}s^{n-2}(r^{n-j}s^{j-n} - 1)E_{j'1} \otimes E_{j1'},$$

$$\sum_{j=2}^{n}(r-s)r^{j-2}(1 - r^{2-j}s^{j-2})E_{j1} \otimes E_{j'1'}.$$

Combining these summands with the appropriate terms from $1 \otimes 1 \cdot \Theta_1$ and $\Theta^{(2)} \cdot 1 \otimes 1$ matches precisely the right-hand side of (5.43) for $k = 1$, thus providing the formula for Θ.

Since f is again given by (5.39), we recover the explicit formula (4.11) for $\hat{R}_{VV} = \Theta \circ \widetilde{f} \circ \tau \colon V \otimes V \to V \otimes V$, thus providing an alternative proof of Theorem 4.6.

6. R-Matrices with a spectral parameter

In this Section, we generalize the previous constructions to the affine setup of two-parameter quantum affine algebras, which were introduced and studied in the literature case-by-case, see [HRZ, HZ1, HZ2, GHZ].

6.1. **Two-parameter quantum affine algebras and evaluation modules.**

We start with a uniform definition of $U_{r,s}(\widehat{\mathfrak{g}})$. To this end, let $\theta \in \Phi^+$ be the highest root of $\mathfrak{g}$, so that the new simple root α_0 of $\widehat{\mathfrak{g}}$ is given by $\alpha_0 = \delta - \theta$. Define the matrix of structural constants $\Omega = \big(\Omega_{ij}\big)_{i,j=0}^{n}$ by:

$$\Omega_{ij} = (\omega_i', \omega_j), \quad \Omega_{0j} = (\omega_{-\theta}', \omega_j), \quad \Omega_{i0} = (\omega_i', \omega_{-\theta}), \quad \Omega_{00} = (\omega_{-\theta}', \omega_{-\theta}) \quad \text{for all} \quad 1 \le i, j \le n. \tag{6.1}$$

We also recall that the Cartan matrix $\hat{C} = (c_{ij})_{i,j=0}^{n}$ for $\widehat{\mathfrak{g}}$ is given by

$$c_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \qquad c_{0j} = \frac{2(-\theta, \alpha_j)}{(\theta, \theta)}, \qquad c_{i0} = \frac{2(\alpha_i, -\theta)}{(\alpha_i, \alpha_i)}, \qquad c_{00} = 2 \quad \text{for all} \quad 1 \le i, j \le n. \tag{6.2}$$

Thus, $\hat{C}$ is the extended Cartan matrix of the Cartan matrix C of $\mathfrak{g}$ from Section 2.1.

Remark 6.1. *For the reader's convenience, we specify the new values of $\hat{C}$ and Ω in classical types:*

- *Type A_n $(n \ge 2)$*
  $\Omega_{00} = rs^{-1}, \Omega_{01} = r^{-1}, \Omega_{0n} = s, \Omega_{10} = s, \Omega_{n0} = r^{-1}$, *and* $\Omega_{0i} = \Omega_{i0} = 1$ *for* $1 < i < n$;
  $c_{01} = c_{0n} = c_{10} = c_{n0} = -1$ *and* $c_{0i} = c_{i0} = 0$ *for* $1 < i < n$.
- *Type B_n $(n \ge 3)$*
  $\Omega_{00} = r^2s^{-2}, \Omega_{01} = r^{-2}s^{-2}, \Omega_{02} = r^{-2}, \Omega_{0n} = r^2s^2, \Omega_{10} = r^2s^2, \Omega_{20} = s^2, \Omega_{n0} = r^{-2}s^{-2}$, *and* $\Omega_{0i} = \Omega_{i0} = 1$ *for* $2 < i < n$;
  $c_{02} = c_{20} = -1$, $c_{01} = c_{10} = 0$, *and* $c_{0i} = c_{i0} = 0$ *for* $2 < i \le n$.
- *Type C_n $(n \ge 2)$*
  $\Omega_{00} = r^2s^{-2}, \Omega_{01} = r^{-2}, \Omega_{0n} = r^2s^2, \Omega_{10} = s^2, \Omega_{n0} = r^{-2}s^{-2}$, *and* $\Omega_{0i} = \Omega_{i0} = 1$ *for* $1 < i < n$;
  $c_{01} = -1, c_{10} = -2$, *and* $c_{0i} = c_{i0} = 0$ *for* $1 < i \le n$.
- *Type D_n $(n \ge 4)$*
  $\Omega_{00} = rs^{-1}, \Omega_{01} = r^{-1}s^{-1}, \Omega_{02} = r^{-1}, \Omega_{0n} = r^2s^2, \Omega_{10} = rs, \Omega_{20} = s, \Omega_{n0} = r^{-2}s^{-2}$, *and* $\Omega_{0i} = \Omega_{i0} = 1$ *for* $2 < i < n$;
  $c_{02} = c_{20} = -1$, $c_{01} = c_{10} = 0$, *and* $c_{0i} = c_{i0} = 0$ *for* $2 < i \le n$.

Definition 6.2. *The **two-parameter quantum affine algebra** $U_{r,s}(\widehat{\mathfrak{g}})$ is the associative $\mathbb{K}$-algebra generated by $\{e_i, f_i, \omega_i^{\pm 1}, (\omega_i')^{\pm 1}\}_{i=0}^{n} \cup \{\gamma^{\pm}, (\gamma')^{\pm 1}\} \cup \{D^{\pm 1}, (D')^{\pm 1}\}$ with the following defining relations:*

$$\begin{aligned} &D^{\pm 1} \cdot D^{\mp 1} = 1, \quad (D')^{\pm 1} \cdot (D')^{\mp 1} = 1, \quad DD' = D'D, \\ &[D, \omega_i] = 0, \quad [D, \omega_i'] = 0, \quad [D', \omega_i] = 0, \quad [D', \omega_i'] = 0, \\ &De_iD^{-1} = r_i^{\delta_{0i}} e_i, \quad Df_iD^{-1} = r_i^{-\delta_{0i}} f_i, \quad D'e_i(D')^{-1} = s_i^{\delta_{0i}} e_i, \quad D'f_i(D')^{-1} = s_i^{-\delta_{0i}} f_i, \end{aligned} \tag{6.3}$$

$$\gamma = \omega_\delta = \omega_0\omega_\theta, \ \gamma' = \omega_\delta' = \omega_0'\omega_\theta' - \text{central elements}, \tag{6.4}$$

$$[\omega_i, \omega_j] = [\omega_i, \omega_j'] = [\omega_i', \omega_j'] = 0, \qquad \omega_i^{\pm 1}\omega_i^{\mp 1} = 1 = (\omega_i')^{\pm 1}(\omega_i')^{\mp 1}, \tag{6.5}$$

$$\omega_i e_j = \Omega_{ji} e_j \omega_i, \qquad \omega_i f_j = \Omega_{ji}^{-1} f_j \omega_i, \tag{6.6}$$

$$\omega_i' e_j = \Omega_{ij}^{-1} e_j \omega_i', \qquad \omega_i' f_j = \Omega_{ij} f_j \omega_i', \tag{6.7}$$

$$e_if_j - f_je_i = \delta_{ij} \frac{\omega_i - \omega_i'}{r_i - s_i}, \tag{6.8}$$

$$\begin{aligned} &\sum_{k=0}^{1-c_{ij}} (-1)^k \begin{bmatrix} 1-c_{ij} \\ k \end{bmatrix}_{r_i, s_i} (r_is_i)^{\frac{1}{2}k(k-1)} \Omega_{ji}^k s_i^{kc_{ij}} e_i^{1-c_{ij}-k} e_j e_i^k = 0 \qquad \forall\, i \ne j, \\ &\sum_{k=0}^{1-c_{ij}} (-1)^k \begin{bmatrix} 1-c_{ij} \\ k \end{bmatrix}_{r_i, s_i} (r_is_i)^{\frac{1}{2}k(k-1)} \Omega_{ji}^k s_i^{kc_{ij}} f_i^k f_j f_i^{1-c_{ij}-k} = 0 \qquad \forall\, i \ne j. \end{aligned} \tag{6.9}$$

It is often more convenient to work with the version of $U_{r,s}(\widehat{\mathfrak{g}})$ without the degree generators D, D':

Definition 6.3. *Let $U_{r,s}'(\widehat{\mathfrak{g}})$ be the associative $\mathbb{K}$-algebra generated by $\{e_i, f_i, \omega_i^{\pm 1}, (\omega_i')^{\pm 1}\}_{i=0}^{n} \cup \{\gamma^{\pm}, (\gamma')^{\pm 1}\}$ with the defining relations (6.4)–(6.9).*

We now extend the $U_{r,s}(\mathfrak{g})$-modules V from Section 3 to $U'_{r,s}(\widehat{\mathfrak{g}})$-modules $V(u)$ with $u \in \mathbb{C}^\times$, depending on two additional parameters $a, b \in \mathbb{C}^\times$. We shall further extend them to $U_{r,s}(\widehat{\mathfrak{g}})$-modules $V(u)$ with u viewed as an indeterminate. We call the resulting $U'_{r,s}(\widehat{\mathfrak{g}})$- and $U_{r,s}(\widehat{\mathfrak{g}})$-modules as *evaluation representations* $\rho_u^{a,b}$.

Proposition 6.4. *($A_n, n \geq 1$) For any nonzero $a, b \in \mathbb{C}$ set $c = rs \cdot ab$. Then the $U_{r,s}(\mathfrak{sl}_{n+1})$-action ρ on V from Proposition 3.1 can be extended to a $U'_{r,s}(\widehat{\mathfrak{sl}}_{n+1})$-action $\rho_u^{a,b}$ on the vector space $V(u) = V$ by setting*

$$\rho_u^{a,b}(x) = \rho(x) \qquad \text{for all} \quad x \in \{e_i, f_i, \omega_i, \omega_i'\}_{i=1}^n \tag{6.10}$$

and defining the action of $e_0, f_0, \omega_0, \omega_0', \gamma, \gamma'$ via

$$\begin{aligned} &\rho_u^{a,b}(e_0) = au \cdot E_{n+1,1}, \quad \rho_u^{a,b}(f_0) = bu^{-1} \cdot E_{1,n+1}, \quad \rho_u^{a,b}(\gamma) = c\,\mathrm{Id} = \rho_u^{a,b}(\gamma'), \\ &\rho_u^{a,b}(\omega_0) = c\left(r^{-1}E_{11} + r^{-1}s^{-1}\sum_{i=2}^n E_{ii} + s^{-1}E_{n+1,n+1}\right), \\ &\rho_u^{a,b}(\omega_0') = c\left(s^{-1}E_{11} + r^{-1}s^{-1}\sum_{i=2}^n E_{ii} + r^{-1}E_{n+1,n+1}\right). \end{aligned} \tag{6.11}$$

Proposition 6.5. *($B_n, n \geq 2$) For any nonzero $a, b \in \mathbb{C}$ set $c = (rs)^2 ab$. Then the $U_{r,s}(\mathfrak{so}_{2n+1})$-action ρ on V from Proposition 3.2 can be extended to a $U'_{r,s}(\widehat{\mathfrak{so}}_{2n+1})$-action $\rho_u^{a,b}$ on the vector space $V(u) = V$ by setting*

$$\rho_u^{a,b}(x) = \rho(x) \qquad \text{for all} \quad x \in \{e_i, f_i, \omega_i, \omega_i'\}_{i=1}^n \tag{6.12}$$

and defining the action of $e_0, f_0, \omega_0, \omega_0', \gamma, \gamma'$ via

$$\begin{aligned} &\rho_u^{a,b}(e_0) = au \cdot \left(E_{1'2} - r^2s^2E_{2'1}\right), \quad \rho_u^{a,b}(f_0) = bu^{-1} \cdot \left(E_{21'} - E_{12'}\right), \quad \rho_u^{a,b}(\gamma) = c\,\mathrm{Id} = \rho_u^{a,b}(\gamma'), \\ &\rho_u^{a,b}(\omega_0) = c\left(s^2E_{11} + r^{-2}E_{22} + \sum_{i=3}^n \left(r^{-2}s^{-2}E_{ii} + r^2s^2E_{i'i'}\right) + r^2E_{2'2'} + s^{-2}E_{1'1'} + E_{n+1,n+1}\right), \\ &\rho_u^{a,b}(\omega_0') = c\left(r^2E_{11} + s^{-2}E_{22} + \sum_{i=3}^n \left(r^{-2}s^{-2}E_{ii} + r^2s^2E_{i'i'}\right) + s^2E_{2'2'} + r^{-2}E_{1'1'} + E_{n+1,n+1}\right). \end{aligned} \tag{6.13}$$

Proposition 6.6. *($C_n, n \geq 2$) For any nonzero $a, b \in \mathbb{C}$ set $c = rs \cdot ab$. Then the $U_{r,s}(\mathfrak{sp}_{2n})$-action ρ on V from Proposition 3.3 can be extended to a $U'_{r,s}(\widehat{\mathfrak{sp}}_{2n})$-action $\rho_u^{a,b}$ on the vector space $V(u) = V$ by setting*

$$\rho_u^{a,b}(x) = \rho(x) \qquad \text{for all} \quad x \in \{e_i, f_i, \omega_i, \omega_i'\}_{i=1}^n \tag{6.14}$$

and defining the action of $e_0, f_0, \omega_0, \omega_0', \gamma, \gamma'$ via

$$\begin{aligned} &\rho_u^{a,b}(e_0) = au \cdot E_{1'1}, \quad \rho_u^{a,b}(f_0) = bu^{-1} \cdot E_{11'}, \quad \rho_u^{a,b}(\gamma) = c\,\mathrm{Id} = \rho_u^{a,b}(\gamma'), \\ &\rho_u^{a,b}(\omega_0) = c\left(r^{-1}sE_{11} + \sum_{i=2}^n \left(r^{-1}s^{-1}E_{ii} + rsE_{i'i'}\right) + rs^{-1}E_{1'1'}\right), \\ &\rho_u^{a,b}(\omega_0') = c\left(rs^{-1}E_{11} + \sum_{i=2}^n \left(r^{-1}s^{-1}E_{ii} + rsE_{i'i'}\right) + r^{-1}sE_{1'1'}\right). \end{aligned} \tag{6.15}$$

Proposition 6.7. *($D_n, n \geq 3$) For any nonzero $a, b \in \mathbb{C}$ set $c = rs \cdot ab$. Then the $U_{r,s}(\mathfrak{so}_{2n})$-action ρ on V from Proposition 3.4 can be extended to a $U'_{r,s}(\widehat{\mathfrak{so}}_{2n})$-action $\rho_u^{a,b}$ on the vector space $V(u) = V$ by setting*

$$\rho_u^{a,b}(x) = \rho(x) \qquad \text{for all} \quad x \in \{e_i, f_i, \omega_i, \omega_i'\}_{i=1}^n \tag{6.16}$$

and defining the action of $e_0, f_0, \omega_0, \omega_0', \gamma, \gamma'$ via

$$\begin{aligned} &\rho_u^{a,b}(e_0) = au \cdot \left(E_{1'2} - rsE_{2'1}\right), \quad \rho_u^{a,b}(f_0) = bu^{-1} \cdot \left(E_{21'} - E_{12'}\right), \quad \rho_u^{a,b}(\gamma) = c\,\mathrm{Id} = \rho_u^{a,b}(\gamma'), \\ &\rho_u^{a,b}(\omega_0) = c\left(sE_{11} + r^{-1}E_{22} + \sum_{i=3}^n \left(r^{-1}s^{-1}E_{ii} + rsE_{i'i'}\right) + rE_{2'2'} + s^{-1}E_{1'1'}\right), \\ &\rho_u^{a,b}(\omega_0') = c\left(rE_{11} + s^{-1}E_{22} + \sum_{i=3}^n \left(r^{-1}s^{-1}E_{ii} + rsE_{i'i'}\right) + sE_{2'2'} + r^{-1}E_{1'1'}\right). \end{aligned} \tag{6.17}$$

These evaluation $U'_{r,s}(\widehat{\mathfrak{g}})$-modules $\rho_u^{a,b}$ can be naturally upgraded to $U_{r,s}(\widehat{\mathfrak{g}})$-modules in the standard way:

Proposition 6.8. *Let u be an indeterminate and redefine the vector space via $V(u) = V \otimes_{\mathbb{C}} \mathbb{C}[u,u^{-1}]$ accordingly. Then, the formulas defining $\rho_u^{a,b}$ on the generators from Propositions 6.4–6.7 together with*

$$\rho_u^{a,b}(D)(v\otimes u^k) = r_0^k \cdot v\otimes u^k, \qquad \rho_u^{a,b}(D')(v\otimes u^k) = s_0^k\cdot v\otimes u^k \qquad \text{for all} \quad v\in V,\ k\in\mathbb{Z} \tag{6.18}$$

give rise to the same-named action $\rho_u^{a,b}$ of $U_{r,s}(\widehat{\mathfrak{g}})$ on $V(u)$.

The proofs of all these results are straightforward, cf. our proof of Proposition 3.2.

6.2. **Affine R-matrices.**

Let $U_{r,s}^{\geq}(\widehat{\mathfrak{g}})$ be the subalgebra of $U_{r,s}(\widehat{\mathfrak{g}})$ generated by $\{e_i,\omega_i^{\pm 1},\gamma^{\pm 1},D^{\pm 1}\}_{i=0}^n$ and similarly let $U_{r,s}^{\leq}(\widehat{\mathfrak{g}})$ be the subalgebra generated by $\{f_i,(\omega_i')^{\pm 1},(\gamma')^{\pm 1},(D')^{\pm 1}\}_{i=0}^n$. Likewise, we define the subalgebras $U_{r,s}^{'\geq}(\widehat{\mathfrak{g}})$ and $U_{r,s}^{'\leq}(\widehat{\mathfrak{g}})$ of $U'_{r,s}(\widehat{\mathfrak{g}})$. We note that the same formulas as in Section 2.1 for $U_{r,s}(\mathfrak{g})$ can be used to define the Hopf algebra structures on both $U_{r,s}(\widehat{\mathfrak{g}})$ and $U'_{r,s}(\widehat{\mathfrak{g}})$, so that $U_{r,s}^{\geq}(\widehat{\mathfrak{g}})$, $U_{r,s}^{\leq}(\widehat{\mathfrak{g}})$ and $U_{r,s}^{'\geq}(\widehat{\mathfrak{g}})$, $U_{r,s}^{'\leq}(\widehat{\mathfrak{g}})$ are also Hopf subalgebras of $U_{r,s}(\widehat{\mathfrak{g}})$ and $U'_{r,s}(\widehat{\mathfrak{g}})$. Finally, parallel to Proposition 2.5, one has bilinear Hopf pairings

$$\begin{aligned} &(\cdot,\cdot)\colon U_{r,s}^{\leq}(\widehat{\mathfrak{g}})\times U_{r,s}^{\geq}(\widehat{\mathfrak{g}})\to\mathbb{K},\\ &(\cdot,\cdot)\colon U_{r,s}^{'\leq}(\widehat{\mathfrak{g}})\times U_{r,s}^{'\geq}(\widehat{\mathfrak{g}})\to\mathbb{K}. \end{aligned} \tag{6.19}$$

We note that the second of these pairings is actually degenerate as $(\gamma'-1,U_{r,s}^{'\geq}(\widehat{\mathfrak{g}}))=0=(U_{r,s}^{'\leq}(\widehat{\mathfrak{g}}),\gamma-1)$. On the other hand (which is one of the key reasons to add the generators D,D'), the first pairing in (6.19) is non-degenerate, and hence allows to realize the two-parameter quantum affine algebra $U_{r,s}(\widehat{\mathfrak{g}})$ as a Drinfel'd double of its Hopf subalgebras $U_{r,s}^{\leq}(\widehat{\mathfrak{g}})$, $U_{r,s}^{\geq}(\widehat{\mathfrak{g}})$ with respect to the pairing above.

The above discussion yields the universal R-matrix for $U_{r,s}(\widehat{\mathfrak{g}})$, which induces intertwiners $V\otimes W\xrightarrow{\sim} W\otimes V$ for suitable $U_{r,s}(\widehat{\mathfrak{g}})$-modules V,W, akin to Section 4.1. In order to not overburden the paper, we choose to skip the detailed presentation on this standard but rather technical discussion. Instead, we shall now proceed directly to the main goal of this paper–the evaluation of such intertwiners when $V=\rho_u^{a,b}$, $W=\rho_v^{a,b}$ are the modules from Section 6.1. In this context, we are looking for $U_{r,s}(\widehat{\mathfrak{g}})$-module intertwiners $\hat{R}(u/v)$ satisfying

$$\hat{R}(u/v)\circ(\rho_u^{a,b}\otimes\rho_v^{a,b})(x) = (\rho_v^{a,b}\otimes\rho_u^{a,b})(x)\circ\hat{R}(u/v) \tag{6.20}$$

for all $x\in U_{r,s}(\widehat{\mathfrak{g}})$ (equivalently, one can rather request $x\in U'_{r,s}(\widehat{\mathfrak{g}})$ in the context of $U'_{r,s}(\widehat{\mathfrak{g}})$-modules). According to [Jim, Proposition 2], it suffices to check the validity of (6.20) only for $x=f_i\,(0\leq i\leq n)$. In fact, the space of such solutions is one-dimensional, see [Jim, Proposition 1], which is essentially due to the irreducibility of the tensor product $\rho_u^{a,b}\otimes\rho_v^{a,b}$ (which still holds when viewing them as $U'_{r,s}(\widehat{\mathfrak{g}})$-modules as long as u,v are *generic*), in contrast to Proposition 3.9. As an immediate corollary, see [Jim, Proposition 3], the operator $R(u/v)=\hat{R}(u/v)\circ\tau$ satisfies the aforementioned Yang-Baxter relation with a spectral parameter:

$$\begin{aligned} &R_{12}(v/w)R_{13}(u/w)R_{23}(u/v) = R_{23}(u/v)R_{13}(u/w)R_{12}(v/w),\\ &\hat{R}_{12}(v/w)\hat{R}_{23}(u/w)\hat{R}_{12}(u/v) = \hat{R}_{23}(u/v)\hat{R}_{12}(u/w)\hat{R}_{23}(v/w), \end{aligned} \tag{6.21}$$

with notation as in Subsection 4.1. We shall now present explicit formulas for such $\hat{R}(z)$ in all classical types, generalizing [Jim] for the one-parameter setup[2]. We note that the origin of these formulas will be explained in the next section, where they will be derived through the so-called *Yang-Baxterization* technique of [GWX]. However, once the formulas are guessed, the above discussions imply that it suffices to check that they satisfy (6.20) for $x=f_i\ (0\leq i\leq n)$.

We start with the simplest case of A-type (part (b) of which goes back to [JL2, §2]):

Theorem 6.9. *(Type A_n) (a) Let $z=u/v$. For $U_{r,s}(\widehat{\mathfrak{sl}}_{n+1})$-modules $\rho_u^{a,b}$ and $\rho_v^{a,b}$ from Proposition 6.4, the following operator $\hat{R}(z)$ satisfies (6.20) whenever $ab=(rs)^{-1}$:*

$$\begin{aligned} \hat{R}(z) = (1-zrs^{-1})\sum_{i=1}^{n+1}E_{ii}\otimes E_{ii} + (1-z)r\sum_{i>j}E_{ij}\otimes E_{ji} + (1-z)s^{-1}\sum_{i<j}E_{ij}\otimes E_{ji}\\ + (1-rs^{-1})\sum_{i>j}E_{ii}\otimes E_{jj} + (1-rs^{-1})z\sum_{i<j}E_{ii}\otimes E_{jj}. \end{aligned} \tag{6.22}$$

[2]We note that a twist of [Re] is needed to recover the formulas of [Jim, §3] due to a different coproduct [Jim, (2.10)] on $U_q(\widehat{\mathfrak{g}})$.

(b) The operator $R(z) = \hat{R}(z) \circ \tau$ given explicitly by

$$R(z) = (1 - zrs^{-1}) \sum_{i=1}^{n+1} E_{ii} \otimes E_{ii} + (1-z)r \sum_{i>j} E_{ii} \otimes E_{jj} + (1-z)s^{-1} \sum_{i<j} E_{ii} \otimes E_{jj}$$
$$+ (1 - rs^{-1}) \sum_{i>j} E_{ij} \otimes E_{ji} + (1 - rs^{-1}) z \sum_{i<j} E_{ij} \otimes E_{ji}$$

satisfies the Yang-Baxter equation with a spectral parameter (6.21).

The main results of this Section generalize the above theorem to the other classical series.

Theorem 6.10. *(Type B_n) (a) Let $z = u/v$. For $U_{r,s}(\widehat{\mathfrak{so}}_{2n+1})$-modules $\rho_u^{a,b}$ and $\rho_v^{a,b}$ from Proposition 6.5, the following operator $\hat{R}(z)$ satisfies* (6.20) *whenever $ab = (rs)^{-2}$:*

$$\hat{R}(z) = (z - r^{-2}s^2)(z-\xi) \sum_{1 \le i \le 2n+1}^{i \neq n+1} E_{ii} \otimes E_{ii} + \sum_{1 \le i,j \le 2n+1}^{j \neq i, i'} a_{ij}(z) E_{ij} \otimes E_{ji} + (1 - r^{-2}s^2)(z-\xi) \sum_{i>j}^{j \neq i'} E_{ii} \otimes E_{jj}$$
$$+ (1 - r^{-2}s^2) z (z-\xi) \sum_{i<j}^{j \neq i'} E_{ii} \otimes E_{jj} + \sum_{i,j=1}^{2n+1} b_{ij}(z)\, E_{i'j} \otimes E_{ij'}, \tag{6.23}$$

where $\xi = r^{-2n+1}s^{2n-1}$,

$$a_{ij}(z) = \begin{cases} r^{-1}s(z-1)(z-\xi)(rs)^{-\sigma_i\sigma_j} & \text{if } i < j, j' \text{ or } i > j, j' \\ r^{-1}s(z-1)(z-\xi)(rs)^{\sigma_i\sigma_j} & \text{if } j < i < j' \text{ or } j' < i < j \end{cases},$$

$$b_{ij}(z) = \begin{cases} (r^{-2}s^2 z - \xi)(z-1) & \text{if } j = i,\ i \neq n+1 \\ r^{-1}s(z-1)(z-\xi) + (r^{-2}s^2 - 1)(\xi - 1)z & \text{if } i = j = n+1 \\ (r^{-2}s^2 - 1)(\xi t_i t_j^{-1}(z-1) - \delta_{ij'}(z-\xi)) & \text{if } i < j \\ (r^{-2}s^2 - 1)z(t_i t_j^{-1}(z-1) - \delta_{ij'}(z-\xi)) & \text{if } i > j \end{cases},$$

with t_i, σ_i precisely as in (4.8).

(b) The operator $R(z) = \hat{R}(z) \circ \tau$ is a solution of the Yang-Baxter equation with a spectral parameter (6.21).

Theorem 6.11. *(Type C_n) (a) Let $z = u/v$. For $U_{r,s}(\widehat{\mathfrak{sp}}_{2n})$-modules $\rho_u^{a,b}$ and $\rho_v^{a,b}$ from Proposition 6.6, the following operator $\hat{R}(z)$ satisfies* (6.20) *whenever $ab = (rs)^{-1}$:*

$$\hat{R}(z) = (z - r^{-1}s)(z-\xi) \sum_{i=1}^{2n} E_{ii} \otimes E_{ii} + \sum_{1 \le i,j \le 2n}^{j \neq i, i'} a_{ij}(z) E_{ij} \otimes E_{ji} + (1 - r^{-1}s)(z-\xi) \sum_{i>j}^{j \neq i'} E_{ii} \otimes E_{jj}$$
$$+ (1 - r^{-1}s) z (z-\xi) \sum_{i<j}^{j \neq i'} E_{ii} \otimes E_{jj} + \sum_{i,j=1}^{2n} b_{ij}(z) E_{i'j} \otimes E_{ij'}, \tag{6.24}$$

where $\xi = r^{-n-1}s^{n+1}$,

$$a_{ij}(z) = \begin{cases} r^{-1/2}s^{1/2}(z-1)(z-\xi)(rs)^{-\frac{1}{2}\sigma_i\sigma_j} & \text{if } i < j, j' \text{ or } i > j, j' \\ r^{-1/2}s^{1/2}(z-1)(z-\xi)(rs)^{\frac{1}{2}\sigma_i\sigma_j} & \text{if } j < i < j' \text{ or } j' < i < j \end{cases},$$

$$b_{ij}(z) = \begin{cases} (r^{-1}sz - \xi)(z-1) & \text{if } j = i \\ (r^{-1}s - 1)(\xi t_i t_j^{-1}(z-1) - \delta_{ij'}(z-\xi)) & \text{if } i < j \\ (r^{-1}s - 1)z(t_i t_j^{-1}(z-1) - \delta_{ij'}(z-\xi)) & \text{if } i > j \end{cases},$$

with t_i, σ_i precisely as in (4.10).

(b) The operator $R(z) = \hat{R}(z) \circ \tau$ is a solution of the Yang-Baxter equation with a spectral parameter (6.21).

Theorem 6.12. *(Type D_n) (a) Let $z = u/v$. For $U_{r,s}(\widehat{\mathfrak{so}}_{2n})$-modules $\rho_u^{a,b}$ and $\rho_v^{a,b}$ from Proposition 6.7, the following operator $\hat{R}(z)$ satisfies (6.20) whenever $ab = (rs)^{-1}$:*

$$\begin{aligned}\hat{R}(z) = (z - r^{-1}s)(z-\xi)\sum_{i=1}^{2n} E_{ii}\otimes E_{ii} + \sum_{1\le i,j\le 2n}^{j\neq i,i'} a_{ij}(z)E_{ij}\otimes E_{ji} + (1-r^{-1}s)(z-\xi)\sum_{i>j}^{j\neq i'} E_{ii}\otimes E_{jj} \\ + (1-r^{-1}s)z(z-\xi)\sum_{i<j}^{j\neq i'} E_{ii}\otimes E_{jj} + \sum_{i,j=1}^{2n} b_{ij}(z)E_{i'j}\otimes E_{ij'},\end{aligned} \tag{6.25}$$

where $\xi = r^{-n+1}s^{n-1}$,

$$a_{ij}(z) = \begin{cases} r^{-1/2}s^{1/2}(z-1)(z-\xi)(rs)^{-\frac{1}{2}\sigma_i\sigma_j} & \text{if } i<j,j' \text{ or } i>j,j' \\ r^{-1/2}s^{1/2}(z-1)(z-\xi)(rs)^{\frac{1}{2}\sigma_i\sigma_j} & \text{if } j<i<j' \text{ or } j'<i<j \end{cases},$$

$$b_{ij}(z) = \begin{cases} (r^{-1}sz-\xi)(z-1) & \text{if } j=i \\ (r^{-1}s-1)(\xi t_i t_j^{-1}(z-1) - \delta_{ij'}(z-\xi)) & \text{if } i<j \\ (r^{-1}s-1)z(t_it_j^{-1}(z-1) - \delta_{ij'}(z-\xi)) & \text{if } i>j \end{cases},$$

with t_i, σ_i precisely as in (4.12).

(b) The operator $R(z) = \hat{R}(z)\circ\tau$ is a solution of the Yang-Baxter equation with a spectral parameter (6.21).

Remark 6.13. *The careful reader may have noticed that for $r = q$ and $s = q^{-1}$, the formula (6.23) does not precisely match with the corresponding R-matrix of [Jim]. This discrepancy occurs because [Jim] uses a coproduct Δ' on $U_q(\mathfrak{g})$ that differs from the more standard Δ of [Jan]. One may check that*

$$\Delta'(x) = \tilde{f}^{1/2}\Delta(x)\tilde{f}^{-1/2} \qquad \text{for all } x\in U_q(\mathfrak{g}),$$

where both sides are regarded as operators on $M\otimes M'$ for any finite-dimensional (type 1) $U_q(\mathfrak{g})$-modules M and M' (see [Jan, Chapter 7] for the notation). Using this relationship, one may then show that

$$\hat{R}' = \tilde{f}^{1/2}\hat{R}\tilde{f}^{-1/2}$$

is the corresponding intertwiner relative to the coproduct Δ' of [Jim]. The Yang-Baxterization results of Section 7, then imply that the one-parameter specialization of our (6.23)–(6.25) should be related to the corresponding formulas in [Jim] via conjugation by $\tilde{f}^{1/2}$. The latter can be verified directly case-by-case.

6.3. **Proofs of explicit formulas.**

We shall now present the proofs of the theorems from the previous Subsection. We start with A-type:

Proof of Theorem 6.9. We shall only prove (a), since (b) follows from it (see also our discussion in Section 7).

As follows from Section 7, the operator $\hat{R}(uv^{-1})$ of (6.22) is a linear combination of $\hat{R}$ from (4.6) and its inverse $\hat{R}^{-1}$, hence it satisfies (6.20) for $x = e_i, f_i$ $(1\le i\le n)$. For the remaining generators, it will be helpful to record the explicit action of $\hat{R}(uv^{-1})$ on the basis vectors $v_i\otimes v_j$ of $V(u)\otimes V(v)$:

$$\hat{R}(uv^{-1})\, v_i\otimes v_j = \begin{cases} (1-uv^{-1}rs^{-1})v_i\otimes v_i & \text{if } i=j \\ (1-rs^{-1})v_i\otimes v_j + s^{-1}(1-uv^{-1})v_j\otimes v_i & \text{if } i>j \\ uv^{-1}(1-rs^{-1})v_i\otimes v_j + r(1-uv^{-1})v_j\otimes v_i & \text{if } i<j \end{cases}. \tag{6.26}$$

In particular, $\hat{R}(uv^{-1})$ preserves the weight subspaces, so that (6.20) holds also for $x = \omega_i, \omega_i'$ $(0\le i\le n)$.

We shall now verify (6.20) for $x = e_0$. First, we note that

$$a^{-1}(\rho_u\otimes\rho_v)(e_0)(v_i\otimes v_j) = \begin{cases} 0 & \text{if } i\neq 1,\ j\neq 1 \\ u\, v_{n+1}\otimes v_j & \text{if } i=1,\ j\neq 1 \\ u\, v_{n+1}\otimes v_1 + vr^{-1}\, v_1\otimes v_{n+1} & \text{if } i=j=1 \\ vs^{-1}\, v_{n+1}\otimes v_{n+1} & \text{if } i=n+1,\ j=1 \\ vr^{-1}s^{-1}\, v_i\otimes v_{n+1} & \text{if } i\neq 1,n+1,\ j=1 \end{cases}, \tag{6.27}$$

and we have similar formulas for $a^{-1}(\rho_v\otimes\rho_u)(e_0)(v_i\otimes v_j)$ with u and v interchanged. As shorthand, we shall use the notation E_{uv} and E_{vu} to denote the operators $a^{-1}(\rho_u\otimes\rho_v)(e_0)$ and $a^{-1}(\rho_v\otimes\rho_u)(e_0)$, respectively.

According to (6.26, 6.27), we clearly have $\hat{R}(uv^{-1})(E_{uv}(v_i \otimes v_j)) = 0 = E_{vu}(\hat{R}(uv^{-1})(v_i \otimes v_j))$ for $i, j \neq 1$. We now consider the remaining cases:

- If $i = 1$ and $j \neq 1, n+1$, then

$$\hat{R}(uv^{-1})(E_{uv}(v_1 \otimes v_j)) = s^{-1}u(1 - uv^{-1})v_j \otimes v_{n+1} + u(1 - rs^{-1})v_{n+1} \otimes v_j,$$

  which is equal to

$$\begin{aligned} E_{vu}\hat{R}(uv^{-1})(v_1 \otimes v_j) &= E_{vu}\left(r(1 - uv^{-1})v_j \otimes v_1 + uv^{-1}(1 - rs^{-1})v_1 \otimes v_j\right) \\ &= s^{-1}u(1 - uv^{-1})v_j \otimes v_{n+1} + u(1 - rs^{-1})v_{n+1} \otimes v_j. \end{aligned}$$

- If $i = 1$ and $j = n+1$, then

$$\hat{R}(uv^{-1})(E_{uv}(v_1 \otimes v_{n+1})) = u(1 - uv^{-1}rs^{-1})v_{n+1} \otimes v_{n+1},$$

  which is equal to

$$\begin{aligned} E_{vu}\hat{R}(uv^{-1})(v_1 \otimes v_{n+1}) &= rs^{-1}u(1 - uv^{-1})v_{n+1} \otimes v_{n+1} + u(1 - rs^{-1})v_{n+1} \otimes v_{n+1} \\ &= u(1 - uv^{-1}rs^{-1})v_{n+1} \otimes v_{n+1}. \end{aligned}$$

- If $i = j = 1$, then we have

$$\begin{aligned} &\hat{R}(uv^{-1})(E_{uv}(v_1 \otimes v_1)) = \hat{R}(uv^{-1})(uv_{n+1} \otimes v_1 + vr^{-1}v_1 \otimes v_{n+1}) \\ &= us^{-1}(1 - uv^{-1})v_1 \otimes v_{n+1} + u(1 - rs^{-1})v_{n+1} \otimes v_1 \\ &\quad + ur^{-1}(1 - rs^{-1})v_1 \otimes v_{n+1} + v(1 - uv^{-1})v_{n+1} \otimes v_1 \\ &= ur^{-1}(1 - uv^{-1}rs^{-1})v_1 \otimes v_{n+1} + (v - urs^{-1})v_{n+1} \otimes v_1, \end{aligned}$$

  which is equal to

$$\begin{aligned} E_{vu}\hat{R}(uv^{-1})(v_1 \otimes v_1) &= E_{vu}(1 - uv^{-1}rs^{-1})v_1 \otimes v_1 \\ &= (v - urs^{-1})v_{n+1} \otimes v_1 + ur^{-1}(1 - uv^{-1}rs^{-1})v_1 \otimes v_{n+1}. \end{aligned}$$

- If $i = n+1$ and $j = 1$, then

$$\hat{R}(uv^{-1})(E_{uv}(v_{n+1} \otimes v_1)) = vs^{-1}(1 - uv^{-1}rs^{-1})v_{n+1} \otimes v_{n+1},$$

  which is equal to

$$\begin{aligned} E_{vu}\hat{R}(uv^{-1})(v_{n+1} \otimes v_1) &= E_{vu}\left(s^{-1}(1 - uv^{-1})v_1 \otimes v_{n+1} + (1 - rs^{-1})v_{n+1} \otimes v_1\right) \\ &= vs^{-1}(1 - uv^{-1})v_{n+1} \otimes v_{n+1} + us^{-1}(1 - rs^{-1})v_{n+1} \otimes v_{n+1} \\ &= vs^{-1}(1 - uv^{-1}rs^{-1})v_{n+1} \otimes v_{n+1}. \end{aligned}$$

- Finally, if $i \neq 1, n+1$ and $j = 1$, then

$$\hat{R}(uv^{-1})(E_{uv}(v_i \otimes v_1)) = vs^{-1}(1 - uv^{-1})v_{n+1} \otimes v_i + ur^{-1}s^{-1}(1 - rs^{-1})v_i \otimes v_{n+1},$$

  which is equal to

$$\begin{aligned} E_{vu}\hat{R}(uv^{-1})(v_i \otimes v_1) &= E_{vu}\left(s^{-1}(1 - uv^{-1})v_1 \otimes v_i + (1 - rs^{-1})v_i \otimes v_1\right) \\ &= vs^{-1}(1 - uv^{-1})v_{n+1} \otimes v_i + ur^{-1}s^{-1}(1 - rs^{-1})v_i \otimes v_{n+1}. \end{aligned}$$

This completes our verification of (6.20) for $x = e_0$.

The verification for $x = f_0$ is completely analogous. This completes the proof of Theorem 6.9(a). ∎

Let us now present the proof of Theorem 6.10 (the proofs of Theorems 6.11, 6.12 are completely analogous).

Proof of Theorem 6.10. As follows from Section 7, the operator $\hat{R}(uv^{-1})$ of (6.23) is a linear combination of $\hat{R}$ from (4.7), its inverse $\hat{R}^{-1}$, and the identity operator Id. Hence, it satisfies (6.20) for $x = e_i, f_i$ $(1 \leq i \leq n)$. Moreover, $\hat{R}(uv^{-1})$ clearly preserves the weight subspaces, so it also satisfies (6.20) for $x = \omega_i, \omega_i'$ $(0 \leq i \leq n)$.

We shall now verify (6.20) for $x = f_0$, proceeding similarly to our proof of Lemma 4.11. To this end, let $F_{uv} = b^{-1}(\rho_u^{a,b} \otimes \rho_v^{a,b})(f_0)$ and $F_{vu} = b^{-1}(\rho_v^{a,b} \otimes \rho_u^{a,b})(f_0)$, where we assume that $ab = (rs)^{-2}$, so that

$c = (rs)^2 ab = 1$. Then we need to verify that $\hat{R}(uv^{-1})F_{uv} = F_{vu}\hat{R}(uv^{-1}) \in \mathrm{End}(V \otimes V)$. First, let us record the explicit formula for $F_{uv} \in \mathrm{End}(V \otimes V)$:

$$\begin{aligned}
F_{uv} &= v^{-1} \cdot 1 \otimes E_{21'} - v^{-1} \cdot 1 \otimes E_{12'} + u^{-1}r^2 E_{21'} \otimes E_{11} + u^{-1}s^{-2} E_{21'} \otimes E_{22} + u^{-1}s^2 E_{21'} \otimes E_{2'2'} \\
&+ u^{-1}r^{-2} E_{21'} \otimes E_{1'1'} + u^{-1} E_{21'} \otimes E_{n+1,n+1} + \sum_{i=3}^{n} \left(u^{-1}(rs)^{-2} E_{21'} \otimes E_{ii} + u^{-1}(rs)^2 E_{21'} \otimes E_{i'i'} \right) \\
&- u^{-1}r^2 E_{12'} \otimes E_{11} - u^{-1}s^{-2} E_{12'} \otimes E_{22} - u^{-1}s^2 E_{12'} \otimes E_{2'2'} - u^{-1}r^{-2} E_{12'} \otimes E_{1'1'} \\
&- u^{-1} E_{12'} \otimes E_{n+1,n+1} - \sum_{i=3}^{n} \left(u^{-1}(rs)^{-2} E_{12'} \otimes E_{ii} + u^{-1}(rs)^2 E_{12'} \otimes E_{i'i'} \right),
\end{aligned}$$

while F_{vu} is given by the same formula with u and v interchanged. As in the proof of Lemma 4.11, it will be helpful to break the operator $\hat{R}(uv^{-1})$ from (6.23) into the following six terms:

$$\begin{aligned}
R_1(uv^{-1}) &= (uv^{-1} - r^{-2}s^2)(uv^{-1} - \xi) \sum_{1 \le i \le 2n+1}^{i \ne n+1} E_{ii} \otimes E_{ii} \\
&\quad + \left(r^{-1}s(uv^{-1} - 1)(uv^{-1} - \xi) + (r^{-2}s^2 - 1)(\xi - 1)uv^{-1} \right) E_{n+1,n+1} \otimes E_{n+1,n+1}, \\
R_2(uv^{-1}) &= (r^{-2}s^2 uv^{-1} - \xi)(uv^{-1} - 1) \sum_{1 \le i \le 2n+1}^{i \ne n+1} E_{i'i} \otimes E_{ii'}, \\
R_3(uv^{-1}) &= \sum_{1 \le i,j \le 2n+1}^{j \ne i,i'} a_{ij}(uv^{-1}) E_{ij} \otimes E_{ji}, \\
R_4(uv^{-1}) &= (1 - r^{-2}s^2)(uv^{-1} - \xi) \sum_{i>j}^{j \ne i'} E_{ii} \otimes E_{jj}, \\
R_5(uv^{-1}) &= (1 - r^{-2}s^2)uv^{-1}(uv^{-1} - \xi) \sum_{i<j}^{j \ne i'} E_{ii} \otimes E_{jj}, \\
R_6(uv^{-1}) &= \sum_{1 \le i,j \le 2n+1}^{i \ne j} b_{ij}(uv^{-1}) E_{i'j} \otimes E_{ij'}.
\end{aligned}$$

Then a direct computation yields

$$\begin{aligned}
&R_1(uv^{-1})F_{uv} = v^{-1}(uv^{-1} - r^{-2}s^2)(uv^{-1} - \xi)E_{22} \otimes E_{21'} - v^{-1}(uv^{-1} - r^{-2}s^2)(uv^{-1} - \xi)E_{11} \otimes E_{12'} \\
&+ u^{-1}s^{-2}(uv^{-1} - r^{-2}s^2)(uv^{-1} - \xi)E_{21'} \otimes E_{22} - u^{-1}r^2(uv^{-1} - r^{-2}s^2)(uv^{-1} - \xi)E_{12'} \otimes E_{11},
\end{aligned}$$

$$\begin{aligned}
&F_{vu}R_1(uv^{-1}) = u^{-1}(uv^{-1} - r^{-2}s^2)(uv^{-1} - \xi)E_{1'1'} \otimes E_{21'} - u^{-1}(uv^{-1} - r^{-2}s^2)(uv^{-1} - \xi)E_{2'2'} \otimes E_{12'} \\
&+ v^{-1}r^{-2}(uv^{-1} - r^{-2}s^2)(uv^{-1} - \xi)E_{21'} \otimes E_{1'1'} - v^{-1}s^2(uv^{-1} - r^{-2}s^2)(uv^{-1} - \xi)E_{12'} \otimes E_{2'2'},
\end{aligned}$$

$$\begin{aligned}
&R_2(uv^{-1})F_{uv} = v^{-1}(r^{-2}s^2uv^{-1} - \xi)(uv^{-1} - 1)E_{22'} \otimes E_{2'1'} - v^{-1}(r^{-2}s^2uv^{-1} - \xi)(uv^{-1} - 1)E_{11'} \otimes E_{1'2'} \\
&+ u^{-1}s^2(r^{-2}s^2uv^{-1} - \xi)(uv^{-1} - 1)E_{2'1'} \otimes E_{22'} - u^{-1}r^{-2}(r^{-2}s^2uv^{-1} - \xi)(uv^{-1} - 1)E_{1'2'} \otimes E_{11'},
\end{aligned}$$

$$\begin{aligned}
&F_{vu}R_2(uv^{-1}) = u^{-1}(r^{-2}s^2uv^{-1} - \xi)(uv^{-1} - 1)E_{11'} \otimes E_{21} - u^{-1}(r^{-2}s^2uv^{-1} - \xi)(uv^{-1} - 1)E_{22'} \otimes E_{12} \\
&+ v^{-1}r^2(r^{-2}s^2uv^{-1} - \xi)(uv^{-1} - 1)E_{21} \otimes E_{11'} - v^{-1}s^{-2}(r^{-2}s^2uv^{-1} - \xi)(uv^{-1} - 1)E_{12} \otimes E_{22'},
\end{aligned}$$

$$
\begin{aligned}
&R_3(uv^{-1})F_{uv} - F_{vu}R_3(uv^{-1}) = s^2v^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{21}\otimes E_{11'}\\
&+ r^{-2}v^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{21'}\otimes E_{1'1'} - r^{-2}v^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{12}\otimes E_{22'}\\
&- s^2v^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{12'}\otimes E_{2'2'} + u^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{11'}\otimes E_{21}\\
&+ r^{-2}s^2u^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{1'1'}\otimes E_{21'} - r^{-2}u^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{21'}\otimes E_{22}\\
&- s^2u^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{2'1'}\otimes E_{22'} - u^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{22'}\otimes E_{12}\\
&- r^{-2}s^2u^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{2'2'}\otimes E_{12'} + s^2u^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{12'}\otimes E_{11}\\
&+ r^{-2}u^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{1'2'}\otimes E_{11'} - v^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{22}\otimes E_{21'}\\
&- r^{-2}s^2v^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{22'}\otimes E_{2'1'} + v^{-1}(uv^{-1}-1)(uv^{-1}-\xi)E_{11}\otimes E_{12'}\\
&+ v^{-1}r^{-2}s^2(uv^{-1}-1)(uv^{-1}-\xi)E_{11'}\otimes E_{1'2'},
\end{aligned}
$$

$$
\begin{aligned}
&F_{vu}R_5(uv^{-1}) - R_4(uv^{-1})F_{uv} = (r^{-2}s^2-1)(uv^{-1}-\xi)v^{-1}E_{1'1'}\otimes E_{21'}\\
&+ (1-r^{-2}s^2)(uv^{-1}-\xi)v^{-1}E_{22}\otimes E_{21'} + (1-r^{-2}s^2)(uv^{-1}-\xi)v^{-1}E_{2'2'}\otimes E_{21'}\\
&+ (1-r^{-2}s^2)(uv^{-1}-\xi)v^{-1}E_{22}\otimes E_{12'} + (1-r^{-2}s^2)(uv^{-1}-\xi)v^{-1}E_{2'2'}\otimes E_{12'}\\
&+ (r^{-2}s^2-1)(uv^{-1}-\xi)v^{-1}E_{11}\otimes E_{12'} + (s^2-r^2)(uv^{-1}-\xi)u^{-1}E_{21'}\otimes E_{11}\\
&+ (r^{-4}s^2-r^{-2})(uv^{-1}-\xi)uv^{-2}E_{12'}\otimes E_{1'1'},
\end{aligned}
$$

$$
\begin{aligned}
&F_{vu}R_4(uv^{-1}) - R_5(uv^{-1})F_{uv} = (r^{-2}s^2-1)(uv^{-1}-\xi)uv^{-2}E_{11}\otimes E_{21'}\\
&+ (r^{-4}s^2-r^{-2})(uv^{-1}-\xi)v^{-1}E_{21'}\otimes E_{1'1'} + (s^{-2}-r^{-2})(uv^{-1}-\xi)v^{-1}E_{12'}\otimes E_{22}\\
&+ (s^2-r^{-2}s^4)(uv^{-1}-\xi)v^{-1}E_{12'}\otimes E_{2'2'} + (r^{-2}s^2-1)(uv^{-1}-\xi)u^{-1}E_{1'1'}\otimes E_{12'}\\
&+ (s^{-2}-r^{-2})(uv^{-1}-\xi)v^{-1}E_{21'}\otimes E_{22} + (s^2-r^{-2}s^4)(uv^{-1}-\xi)v^{-1}E_{21'}\otimes E_{2'2'}\\
&+ (s^2-r^2)(uv^{-1}-\xi)v^{-1}E_{12'}\otimes E_{11},
\end{aligned}
$$

$$
\begin{aligned}
&R_6(uv^{-1})F_{uv} = (r^{-4}s^2-r^{-2})(uv^{-1}-1)v^{-1}E_{1'2'}\otimes E_{11'} - (r^{-2}s^2-1)(uv^{-1}-1)v^{-1}\xi E_{22'}\otimes E_{2'1'}\\
&- (r^{-2}s^2-1)(uv^{-1}-\xi)v^{-1}E_{2'2'}\otimes E_{21'} - (r^{-2}s^4-s^2)(uv^{-1}-\xi)v^{-1}E_{21'}\otimes E_{2'2'}\\
&- (r^{-2}s^4-s^2)(uv^{-1}-1)v^{-1}E_{2'1'}\otimes E_{22'} + (r^{-2}s^2-1)(uv^{-1}-1)\xi v^{-1}E_{11'}\otimes E_{1'2'}\\
&- (r^{-2}s^2-1)(uv^{-1}-1)(uv^{-1}-\xi)u^{-1}E_{1'1'}\otimes E_{12'} + (r^{-2}s^2-1)(uv^{-1}-\xi)v^{-1}E_{1'1'}\otimes E_{12'}\\
&+ (r^{-4}s^2-r^{-2})(uv^{-1}-1)(uv^{-1}-\xi)v^{-1}E_{12'}\otimes E_{1'1'} + (r^{-4}s^2-r^{-2})(uv^{-1}-\xi)v^{-1}E_{12'}\otimes E_{1'1'},
\end{aligned}
$$

$$
\begin{aligned}
&F_{vu}R_6(uv^{-1}) = (r^2-s^2)(uv^{-1}-\xi)u^{-1}E_{21'}\otimes E_{11} + (r^{-2}s^2-1)(uv^{-1}-1)v^{-1}E_{22'}\otimes E_{12}\\
&+ (r^{-2}s^2-1)(uv^{-1}-\xi)v^{-1}E_{22}\otimes E_{12'} + (r^2-s^2)(uv^{-1}-1)\xi v^{-1}E_{21}\otimes E_{11'}\\
&+ (r^{-2}-s^{-2})(uv^{-1}-\xi)v^{-1}E_{12'}\otimes E_{22} + (r^{-2}-s^{-2})(uv^{-1}-1)\xi v^{-1}E_{12}\otimes E_{22'}\\
&- (r^{-2}s^2-1)(uv^{-1}-1)v^{-1}E_{11'}\otimes E_{21} - (r^{-2}s^2-1)(uv^{-1}-\xi)uv^{-2}E_{11}\otimes E_{21'}.
\end{aligned}
$$

Combining the expressions above, we get

$$
\begin{aligned}
&R_3(uv^{-1})F_{uv} - F_{vu}R_3(uv^{-1}) + R_1(uv^{-1})F_{uv} + R_2(uv^{-1})F_{uv} - F_{vu}R_1(uv^{-1}) - F_{vu}R_2(uv^{-1})\\
&= (1-r^{-2}s^2)(uv^{-1}-\xi)v^{-1}E_{22}\otimes E_{21'} + (r^{-2}s^2-1)(uv^{-1}-\xi)v^{-1}E_{11}\otimes E_{12'}\\
&\quad + (s^{-2}-r^{-2})(uv^{-1}-\xi)v^{-1}E_{21'}\otimes E_{22} + (s^2-r^2)(uv^{-1}-\xi)v^{-1}E_{12'}\otimes E_{11}\\
&\quad + (r^{-2}s^2-1)(uv^{-1}-1)\xi v^{-1}E_{22'}\otimes E_{2'1'} + (1-r^{-2}s^2)(uv^{-1}-1)\xi v^{-1}E_{11'}\otimes E_{1'2'}\\
&\quad + (r^{-2}s^4-s^2)(uv^{-1}-1)v^{-1}E_{2'1'}\otimes E_{22'} + (r^{-2}-r^{-4}s^2)(uv^{-1}-1)v^{-1}E_{1'2'}\otimes E_{11'}\\
&\quad + (r^{-2}s^2-1)(uv^{-1}-\xi)v^{-1}E_{1'1'}\otimes E_{21'} + (1-r^{-2}s^2)(uv^{-1}-\xi)v^{-1}E_{2'2'}\otimes E_{12'}\\
&\quad + (r^{-4}s^2-r^{-2})(uv^{-1}-\xi)v^{-1}E_{21'}\otimes E_{1'1'} + (s^2-r^{-2}s^4)(uv^{-1}-\xi)v^{-1}E_{12'}\otimes E_{2'2'}\\
&\quad + (1-r^{-2}s^2)(uv^{-1}-1)v^{-1}E_{11'}\otimes E_{21} + (r^{-2}s^2-1)(uv^{-1}-1)v^{-1}E_{22'}\otimes E_{12}\\
&\quad + (r^2-s^2)(uv^{-1}-1)\xi v^{-1}E_{21}\otimes E_{11'} + (r^{-2}-s^{-2})(uv^{-1}-1)\xi v^{-1}E_{12}\otimes E_{22'}
\end{aligned}
\tag{6.28}
$$

and

$$
\begin{aligned}
&(F_{vu}R_5(uv^{-1}) - R_4(uv^{-1})F_{uv}) + (F_{vu}R_4(uv^{-1}) - R_5(uv^{-1})F_{uv}) + F_{vu}R_6(uv^{-1}) - R_6(uv^{-1})F_{uv}\\
&= (r^{-2}s^2-1)(uv^{-1}-\xi)v^{-1}E_{1'1'}\otimes E_{21'} + (1-r^{-2}s^2)(uv^{-1}-\xi)v^{-1}E_{22}\otimes E_{21'}\\
&\quad + (1-r^{-2}s^2)(uv^{-1}-\xi)v^{-1}E_{2'2'}\otimes E_{12'} + (r^{-2}s^2-1)(uv^{-1}-\xi)v^{-1}E_{11}\otimes E_{12'}\\
&\quad + (r^{-4}s^2-r^{-2})(uv^{-1}-\xi)v^{-1}E_{21'}\otimes E_{1'1'} + (s^2-r^{-2}s^4)(uv^{-1}-\xi)v^{-1}E_{12'}\otimes E_{2'2'}\\
&\quad + (s^{-2}-r^{-2})(uv^{-1}-\xi)v^{-1}E_{21'}\otimes E_{22} + (s^2-r^2)(uv^{-1}-\xi)v^{-1}E_{12'}\otimes E_{11}\\
&\quad + (r^{-2}s^2-1)(uv^{-1}-1)v^{-1}E_{22'}\otimes E_{12} + (r^2-s^2)(uv^{-1}-1)\xi v^{-1}E_{21}\otimes E_{11'}\\
&\quad + (r^{-2}-s^{-2})(uv^{-1}-1)\xi v^{-1}E_{12}\otimes E_{22'} + (1-r^{-2}s^2)(uv^{-1}-1)v^{-1}E_{11'}\otimes E_{21}\\
&\quad + (r^{-2}-r^{-4}s^2)(uv^{-1}-1)v^{-1}E_{1'2'}\otimes E_{11'} + (r^{-2}s^2-1)(uv^{-1}-1)\xi v^{-1}E_{22'}\otimes E_{2'1'}\\
&\quad + (r^{-2}s^4-s^2)(uv^{-1}-1)v^{-1}E_{2'1'}\otimes E_{22'} + (1-r^{-2}s^2)(uv^{-1}-1)\xi v^{-1}E_{11'}\otimes E_{1'2'}.
\end{aligned}
\tag{6.29}
$$

The right-hand sides of (6.28) and (6.29) are clearly equal, which completes the proof of (6.20) for $x = f_0$.

In the same way, one can verify (6.20) for $x = e_0$ (although according to [Jim], it is actually sufficient to check (6.20) only for $x = f_i$). This completes the proof of part (a) of Theorem 6.10. The proof of part (b) actually follows from (a), as in [Jim, Proposition 3]. ∎

Remark 6.14. *Similarly to Remark 4.12, we note that the above proofs of Theorems 6.10–6.12 are quite elementary, but they require knowing the correct formulas for $\hat{R}(z)$ at the first place. In the next Section, we present the origin of these formulas, by using the Yang-Baxterization technique of* [GWX].

7. Yang-Baxterization

In this last Section, we present a natural derivation of the rather complicated formulas (6.22)–(6.25) from their finite counterparts (4.6, 4.7, 4.9, 4.11). This is based on a so-called *Yang-Baxterization* technique of [GWX], which produces $\hat{R}(z)$ satisfying (6.21) from $\hat{R}$ satisfying (4.5) when the latter has 2 or 3 eigenvalues.

Yang-Baxterization in A-type.

For a uniform presentation, we start by recalling the derivation of (6.22) through that technique. As noted in [BW2], the R-matrix $\hat{R}$ of (4.6) is diagonalizable with two eigenvalues $\lambda_1 = -rs^{-1}, \lambda_2 = 1$, so that

$$\hat{R}^{-1} = -\lambda_1^{-1}\lambda_2^{-1}\hat{R} + (\lambda_1^{-1}+\lambda_2^{-1})\mathrm{Id}.$$

In that setup, the Yang-Baxterization of [GWX, (3.15)] produces the following solution of (6.21):

$$\hat{R}(z) = \lambda_2^{-1}\hat{R} + z\lambda_1\hat{R}^{-1}. \tag{7.1}$$

Combining (7.1) with (4.6) and the above formula for $\hat{R}^{-1}$, one immediately derives the operator (6.22).

Yang-Baxterization in BCD-types.

Let us treat the other three classical series. To this end, we recall that the R-matrices $\hat{R}$ of (4.7, 4.9, 4.11) have three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$, in accordance with Lemmas 4.8, 4.9, 4.10. In that setup, the Yang-Baxterization of [GWX, (3.29), (3.31)] produces the following two solutions to (6.21):

$$\hat{R}(z) = \lambda_1 z(z-1)\hat{R}^{-1} + \left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_2}{\lambda_3}\right) z\mathrm{Id} - \frac{1}{\lambda_3}(z-1)\hat{R}, \tag{7.2}$$

$$\hat{R}(z) = \lambda_1 z(z-1)\hat{R}^{-1} + \left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_1^2}{\lambda_2\lambda_3}\right) z\mathrm{Id} - \frac{\lambda_1}{\lambda_2\lambda_3}(z-1)\hat{R}, \tag{7.3}$$

provided that $\hat{R}$ satisfies the additional relations of [GWX, (3.27)] (cf. correction [GWX, (A.9)]), which, in particular, hold whenever $\hat{R}$ is a representation of a *Birman-Wenzl algebra*.

Remark 7.1. *For the purpose of the present section, we shall not really need to verify these additional relations, since according to Theorems 6.10–6.12 the constructed $\hat{R}(z)$ do manifestly satisfy the relation* (6.21).

To apply formulas above, it remains to evaluate $\hat{R}^{-1}$. To this end, we consider the $\mathbb{C}$-algebra automorphism

$$\sigma\colon U_{r,s}(\mathfrak{g}) \to U_{r,s}(\mathfrak{g}) \quad \text{given by} \quad e_i \mapsto e_i, \quad f_i \mapsto f_i, \quad \omega_i \mapsto \omega_i', \quad \omega_i' \mapsto \omega_i, \quad r \mapsto s, \quad s \mapsto r. \tag{7.4}$$

Evoking the notation (4.2), we define

$$\overline{\Theta} = \sum_{\mu \geq 0} \overline{\Theta}_\mu \qquad \text{with} \qquad \overline{\Theta}_\mu = (\sigma \otimes \sigma)(\Theta_\mu) \quad \text{for all} \quad \mu. \tag{7.5}$$

We also introduce another coproduct homomorphism $\overline{\Delta}\colon U_{r,s}(\mathfrak{g}) \to U_{r,s}(\mathfrak{g}) \otimes U_{r,s}(\mathfrak{g})$ via

$$\overline{\Delta} = (\sigma \otimes \sigma) \circ \Delta \circ \sigma^{-1}. \tag{7.6}$$

Then, for any finite-dimensional $U_{r,s}(\mathfrak{g})$-modules V and W, we have

$$\Delta(u) \circ \Theta = \Theta \circ \overline{\Delta}(u)\colon V \otimes W \to V \otimes W \qquad \text{for all} \quad u \in U_{r,s}(\mathfrak{g}),$$

cf. [BW1, Lemma 4.10] and [BGH2, Lemma 3.3]. Applying $\sigma \otimes \sigma$ to the equality above, we then get $\overline{\Delta}(\sigma(u))\overline{\Theta} = \overline{\Theta}\Delta(\sigma(u))$. Since σ is an automorphism, the last equality can be written as $\overline{\Delta}(u)\overline{\Theta} = \overline{\Theta}\Delta(u)$ for any $u \in U_{r,s}(\mathfrak{g})$. Let us now also show that $\overline{\Delta}(u)\widetilde{f} = \widetilde{f}\Delta^{\mathrm{op}}(u)$ on $V \otimes W$. It suffices to verify this formula when u is one of the generators. For $u = \omega_i$ or ω_i', this is obvious. For $u = e_i$ and any $v \in V[\lambda]$, $w \in V[\mu]$:

$$\overline{\Delta}(e_i)\widetilde{f}(v \otimes w) = f(\lambda, \mu)(e_i v \otimes w + (\omega_i', \omega_\lambda)^{-1} v \otimes e_i w)$$

and

$$\widetilde{f}(\Delta^{\mathrm{op}}(e_i)(v \otimes w)) = f(\lambda, \mu + \alpha_i) v \otimes e_i w + f(\lambda + \alpha_i, \mu)(\omega_\mu', \omega_i) e_i v \otimes w,$$

which are equal due to the properties (4.3) satisfied by f. The proof for $u = f_i$ is completely analogous. Putting all of this together, we find:

$$\tau \circ \widetilde{f}^{-1} \circ \overline{\Theta} \circ \Delta(u) = \tau \circ \widetilde{f}^{-1} \circ \overline{\Delta}(u) \circ \overline{\Theta} = \tau \circ \Delta^{\mathrm{op}}(u) \circ \widetilde{f}^{-1} \circ \overline{\Theta} = \Delta(u) \circ \tau \circ \widetilde{f}^{-1} \circ \overline{\Theta}$$

as linear maps $V \otimes W \to W \otimes V$. Thus $\overline{R} = \tau \circ \widetilde{f}^{-1} \circ \overline{\Theta}\colon V \otimes W \to W \otimes V$ is a $U_{r,s}(\mathfrak{g})$-module isomorphism.

Specializing now to the case where $V = W$ is one of the representations from Propositions 3.2–3.4, one can easily see from the defining formulas that $\rho(\sigma(u)) = \widetilde{\sigma}(\rho(u))$ for any $u \in U_{r,s}(\mathfrak{g})$. Here, we regard $\rho(u)$ as an element of $\mathrm{Mat}_N(\mathbb{K})$, and $\widetilde{\sigma}\colon \mathrm{Mat}_N(\mathbb{K}) \to \mathrm{Mat}_N(\mathbb{K})$ is the $\mathbb{C}$-algebra automorphism defined by

$$\widetilde{\sigma}(r^k s^\ell E_{ij}) = r^\ell s^k E_{ij} \qquad \forall\, k, \ell \in \mathbb{Z},\ 1 \leq i, j \leq N.$$

By abuse of notation, we shall use $\widetilde{\sigma}$ to denote similar $\mathbb{C}$-algebra automorphisms $\mathrm{Mat}_N(\mathbb{K})^{\otimes 2} \to \mathrm{Mat}_N(\mathbb{K})^{\otimes 2}$ and $\mathbb{K} \to \mathbb{K}$. To obtain explicit formulas for $\overline{R}$, which we present in equations (7.7)–(7.9) below, we just need to evaluate $\tau \circ \widetilde{f}^{-1} \circ \widetilde{\sigma}(R \circ \tau \circ \widetilde{f}^{-1})$ from the respective formulas for R given in (4.7, 4.9, 4.11), see (4.4). Most of the terms transform easily into the corresponding terms in (7.7)–(7.9), due to the equalities

$$\widetilde{\sigma}(f(\varepsilon_j, \varepsilon_i)) = f(\varepsilon_i, \varepsilon_j)^{-1} \qquad \text{for all} \quad i, j,$$

but some additional explanation is necessary for two of them. First, the term $\sum_{i \neq j, j'} a_{ij} E_{ij} \otimes E_{ji}$ transforms into the corresponding one in (7.7)–(7.9) due to the following identities satisfied by a_{ij} of (4.8, 4.10, 4.12):

$$a_{ij} = f(\varepsilon_i, \varepsilon_j) \qquad \text{and} \qquad a_{ji}^{-1} = a_{ij} \qquad \text{for all} \quad i \neq j, j'.$$

For the former equality, see (5.35) and (5.41). Second, the term $\sum_{i < j \neq i'} t_i t_j^{-1} E_{i'j} \otimes E_{ij'}$ transforms into the corresponding one in (7.7)–(7.9) due to the additional observations that, unless $i = n+1$ or $j = n+1$ in type B_n, we have (with $\varepsilon_{i'} = -\varepsilon_i$ as defined prior to (5.35) and (5.41))

$$f(-\varepsilon_i, \varepsilon_i) = f(\varepsilon_j, -\varepsilon_j) \qquad \text{and} \qquad \widetilde{\sigma}(t_{i'} t_{j'}^{-1}) = t_i t_j^{-1} \qquad \text{for all} \quad 1 \leq i, j \leq N.$$

The former equality follows from our formulas (5.33, 5.34) for type B_n and (5.39, 5.40) for types C_n, D_n. When $i = n+1$ or $j = n+1$ in type B_n, we rather use the following equalities:

$$f(0,0) = 1, \qquad \widetilde{\sigma}(t_{n+1}) r s^{-1} = t_{n+1}, \qquad \widetilde{\sigma}(t_{i'}) = t_i \quad \text{for} \quad i \neq n+1.$$

These results allow us to prove the following lemmas:

Lemma 7.2. *(Type B_n) The inverse of the operator $\hat{R}\colon V\otimes V\to V\otimes V$ from Theorem 4.4 is equal to*

$$\begin{aligned}\overline{R}=\tau\circ\widetilde{f}^{-1}\circ\overline{\Theta}=&rs^{-1}\sum_{1\le i\le 2n+1}^{i\neq n+1}E_{ii}\otimes E_{ii}+E_{n+1,n+1}\otimes E_{n+1,n+1}+r^{-1}s\sum_{1\le i\le 2n+1}^{i\neq n+1}E_{ii'}\otimes E_{i'i}\\&+\sum_{1\le i,j\le 2n+1}^{j\neq i,i'}a_{ij}E_{ij}\otimes E_{ji}+(s^2-r^2)(rs)^{-1}\sum_{i=1}^{n}(r^{2(i-n)-1}s^{2(n-i)+1}-1)E_{ii}\otimes E_{i'i'}\\&+(r^2-s^2)(rs)^{-1}\sum_{i<j}^{j\neq i'}E_{ii}\otimes E_{jj}+(s^2-r^2)(rs)^{-1}\sum_{i>j}^{j\neq i'}t_it_j^{-1}E_{i'j}\otimes E_{ij'},\end{aligned}\tag{7.7}$$

with the constants t_i and a_{ij} given explicitly by (4.8).

Lemma 7.3. *(Type C_n) The inverse of the operator $\hat{R}\colon V\otimes V\to V\otimes V$ from Theorem 4.5 is equal to*

$$\begin{aligned}\overline{R}=\tau\circ\widetilde{f}^{-1}\circ\overline{\Theta}=&r^{1/2}s^{-1/2}\sum_{i=1}^{2n}E_{ii}\otimes E_{ii}+r^{-1/2}s^{1/2}\sum_{i=1}^{2n}E_{ii'}\otimes E_{i'i}\\&+\sum_{1\le i,j\le 2n}^{j\neq i,i'}a_{ij}E_{ij}\otimes E_{ji}+(r-s)(rs)^{-1/2}\sum_{i=1}^{n}(r^{i-n-1}s^{n+1-i}+1)E_{ii}\otimes E_{i'i'}\\&+(r-s)(rs)^{-1/2}\sum_{i<j}^{j\neq i'}E_{ii}\otimes E_{jj}+(s-r)(rs)^{-1/2}\sum_{i>j}^{j\neq i'}t_it_j^{-1}E_{i'j}\otimes E_{ij'},\end{aligned}\tag{7.8}$$

with the constants t_i and a_{ij} given explicitly by (4.10).

Lemma 7.4. *(Type D_n) The inverse of the operator $\hat{R}\colon V\otimes V\to V\otimes V$ from Theorem 4.6 is equal to*

$$\begin{aligned}\overline{R}=\tau\circ\tilde{f}^{-1}\circ\overline{\Theta}=&r^{1/2}s^{-1/2}\sum_{i=1}^{2n}E_{ii}\otimes E_{ii}+r^{-1/2}s^{1/2}\sum_{i=1}^{2n}E_{ii'}\otimes E_{i'i}\\&+\sum_{1\le i,j\le 2n}^{j\neq i,i'}a_{ij}E_{ij}\otimes E_{ji}+(r-s)(rs)^{-1/2}\sum_{i=1}^{n}(1-r^{i-n}s^{n-i})E_{ii}\otimes E_{i'i'}\\&+(r-s)(rs)^{-1/2}\sum_{i<j}^{j\neq i'}E_{ii}\otimes E_{jj}+(s-r)(rs)^{-1/2}\sum_{i>j}^{j\neq i'}t_it_j^{-1}E_{i'j}\otimes E_{ij'},\end{aligned}\tag{7.9}$$

with the constants t_i and a_{ij} given explicitly by (4.12).

Since $\overline{R}$ is a $U_{r,s}(\mathfrak{g})$-module intertwiner by the above discussions, it suffices to verify that the eigenvalues of $\overline{R}$ on the highest weight vectors w_1, w_2, w_3 from our proof of Proposition 3.9 are inverse to those of $\hat{R}$ as specified in Lemmas 4.8–4.10. As the arguments are very similar, we shall only present the proof in type B_n.

Proof of Lemma 7.2. For $w_1=v_1\otimes v_1$, the eigenvalue of $\hat{R}$ is $\lambda_1=r^{-1}s$, while we clearly have:

$$\overline{R}(w_1)=\overline{R}(v_1\otimes v_1)=rs^{-1}\,v_1\otimes v_1=\lambda_1^{-1}w_1.$$

For w_2, we have:

$$\overline{R}(w_2)=\overline{R}(v_1\otimes v_2-rs^{-1}v_2\otimes v_1)=(rs^{-1}-r^{-1}s)v_1\otimes v_2+v_2\otimes v_1-rs^{-1}v_1\otimes v_2=-r^{-1}s(v_1\otimes v_2-rs^{-1}v_2\otimes v_1)\tag{7.10}$$

for $n=1$, and

$$\overline{R}(w_2)=\overline{R}(v_1\otimes v_2-r^2v_2\otimes v_1)=(rs^{-1}-r^{-1}s)v_1\otimes v_2+rsv_2\otimes v_1-rs^{-1}v_1\otimes v_2=-r^{-1}s(v_1\otimes v_2-r^2v_2\otimes v_1)$$

for $n>1$. The corresponding eigenvalue for $\hat{R}$ is $\lambda_2=-rs^{-1}$, so the claim is true in this case as well.

Finally, the eigenvalue of the $\overline{R}$-action on w_3 equals the ratio of coefficients of $v_{1'}\otimes v_1$ in $\overline{R}(w_3)$ and w_3. To compute the former, we note that only the third summand of (7.7) makes a nontrivial contribution of $r^{-1}s\cdot v_{1'}\otimes v_1$. As the coefficient of $v_{1'}\otimes v_1$ in w_3 equals $r^{2n-1}s^{-2n+1}$, we conclude that $\overline{R}(w_3)=r^{-2n}s^{2n}w_3$. Thus, the eigenvalue of w_3 for $\overline{R}$ is indeed equal to $\lambda_3^{-1}=r^{-2n}s^{2n}$. ∎

With these preliminaries out of the way, we can now present our formulas for $\hat{R}(z)$, which we obtain from (7.2) in types B_n and D_n, and from (7.3) in type C_n (the key reason to use a different formula in type C_n is because (7.2) does not produce a solution that also satisfies the intertwining property (6.20)):

- *Type B_n*

$$\hat{R}(z) = r^{-1}sz(z-1)\overline{R} + (1-r^{-2n+1}s^{2n-1})(1-r^{-2}s^2)z\mathrm{Id} - r^{-2n}s^{2n}(z-1)\hat{R},$$

  which after a direct computation simplifies to (6.23).
- *Type C_n*

$$\hat{R}(z) = r^{-1/2}s^{1/2}z(z-1)\overline{R} + (1-r^{-n-1}s^{n+1})(1-r^{-1}s)z\mathrm{Id} - r^{-n-3/2}s^{n+3/2}(z-1)\hat{R},$$

  which after a direct computation simplifies to (6.24).
- *Type D_n*

$$\hat{R}(z) = r^{-1/2}s^{1/2}z(z-1)\overline{R} + (1-r^{-n+1}s^{n-1})(1-r^{-1}s)z\mathrm{Id} - r^{-n+1/2}s^{n-1/2}(z-1)\hat{R},$$

  which after a direct computation simplifies to (6.25).

Appendix A. Two-parameter vs one-parameter quantum groups

In this Appendix, we realize $U_q(\mathfrak{g})$ with $q = r^{1/2}s^{-1/2}$ as a subalgebra of $U_{r,s}(\mathfrak{g})$, extended by adjoining $\{\omega_i^{\pm 1/2}, (\omega_i')^{\pm 1/2}\}_{i=1}^n$, in contrast to the specialization realization (2.10). To do so, we consider

(A.1) **modified generators** $\widetilde{e}_i = e_i \cdot \omega_i^{-1/2}, \quad \widetilde{f}_i = s_i \cdot f_i (\omega_i')^{-1/2}, \quad \widetilde{\omega}_i = \omega_i^{1/2}(\omega_i')^{-1/2} \quad \forall\, 1 \leq i \leq n.$

The following is the key result of this Appendix:

Proposition A.1. *The elements (A.1) satisfy the defining relations of Drinfeld-Jimbo generators for $U_q(\mathfrak{g})$.*

Proof. The proof consists of a straightforward verification of all of the defining relations.

- The commutativity $[\widetilde{\omega}_i, \widetilde{\omega}_j] = 0$ is a direct consequence of (2.5).
- Let us now check $\widetilde{\omega}_i \widetilde{e}_j = q^{(\alpha_i,\alpha_j)} \widetilde{e}_j \widetilde{\omega}_i$. Using (2.6) and (2.7), we obtain

$$\widetilde{\omega}_i \widetilde{e}_j = \omega_i^{1/2}(\omega_i')^{-1/2}\widetilde{e}_j = (r^{\langle \alpha_i,\alpha_j\rangle + \langle \alpha_j,\alpha_i\rangle} s^{-\langle \alpha_j,\alpha_i\rangle - \langle \alpha_i,\alpha_j\rangle})^{1/2} \widetilde{e}_j \widetilde{\omega}_i = q^{(\alpha_i,\alpha_j)} \widetilde{e}_j \widetilde{\omega}_i,$$

  with the last equality due to $(\alpha_i, \alpha_j) = \langle \alpha_i, \alpha_j \rangle + \langle \alpha_j, \alpha_i \rangle$.
- The verification of $\widetilde{\omega}_i \widetilde{f}_j = q^{-(\alpha_i,\alpha_j)} \widetilde{f}_j \widetilde{\omega}_i$ is analogous to the above calculation.
- For $i \neq j$, we have $\omega_i^{-1/2} \widetilde{f}_j = (\omega_j', \omega_i)^{1/2} \widetilde{f}_j \omega_i^{-1/2}$ and $(\omega_j')^{-1/2} \widetilde{e}_i = (\omega_j', \omega_i)^{1/2} \widetilde{e}_i (\omega_j')^{-1/2}$, so that

$$[\widetilde{e}_i, \widetilde{f}_j] = s_j (\omega_j', \omega_i)^{1/2} \cdot [e_i, f_j] \cdot (\omega_i \omega_j')^{-1/2} = 0,$$

  with the last equality due to (2.8).
- For $i = j$, we likewise obtain

$$[\widetilde{e}_i, \widetilde{f}_i] = q_i s_i \cdot [e_i, f_i] \cdot (\omega_i \omega_i')^{-1/2} = \frac{q_i s_i}{r_i - s_i} \cdot (\widetilde{\omega}_i - \widetilde{\omega}_i^{-1}) = \frac{\widetilde{\omega}_i - \widetilde{\omega}_i^{-1}}{q_i - q_i^{-1}},$$

  where we used (2.8) as well as the equalities $(\omega_i', \omega_i) = q^{2d_i} = q_i^2$ and $q_i s_i = (rs)^{d_i/2}$.
- Finally, let us show that $\widetilde{e}_i$ satisfy the q-Serre relation. Fix $i \neq j$, and set $m = 1 - c_{ij}$. First, we note

$$\left(\widetilde{e}_i \omega_i^{1/2}\right)^m = q_i^{\frac{1}{2}m(m-1)} \widetilde{e}_i^m \omega_i^{m/2},$$

  so that

$$\left(\widetilde{e}_i \omega_i^{1/2}\right)^{m-k} \left(\widetilde{e}_j \omega_j^{1/2}\right) \left(\widetilde{e}_i \omega_i^{1/2}\right)^k = q_i^{\frac{1}{2}m(m-1)} r^{\frac{1}{2}(m-k)\langle \alpha_j,\alpha_i\rangle + \frac{1}{2}k\langle \alpha_i,\alpha_j\rangle} s^{-\frac{1}{2}(m-k)\langle \alpha_i,\alpha_j\rangle - \frac{1}{2}k\langle \alpha_j,\alpha_i\rangle} \cdot \widetilde{e}_i^{m-k} \widetilde{e}_j \widetilde{e}_i^k \omega_i^{m/2} \omega_j^{1/2}.$$

Thus, plugging $e_i = \widetilde{e}_i \omega_i^{1/2}$ and $e_j = \widetilde{e}_j \omega_j^{1/2}$ into the first (r,s)-Serre relation of (2.9) and using the fact that $1 - m = c_{ij} = \frac{1}{d_i}(\alpha_i, \alpha_j) = \frac{1}{d_i}\left(\langle \alpha_i, \alpha_j \rangle + \langle \alpha_j, \alpha_i \rangle\right)$, we get

$$\begin{aligned}
0 &= \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{r_i, s_i} (r_i s_i)^{\frac{1}{2}k(k-1)} (rs)^{k\langle \alpha_j, \alpha_i \rangle} \, (\widetilde{e}_i \omega_i^{1/2})^{m-k} (\widetilde{e}_j \omega_j^{1/2}) (\widetilde{e}_i \omega_i^{1/2})^k \\
&= \sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{r_i, s_i} (r_i s_i)^{\frac{1}{2}k(k-1)} q_i^{\frac{1}{2}m(m-1)} r^{\frac{1}{2}m\langle \alpha_j, \alpha_i \rangle} s^{-\frac{1}{2}m\langle \alpha_i, \alpha_j \rangle} (r_i s_i)^{\frac{1}{2}k(1-m)} \, \widetilde{e}_i^{m-k} \widetilde{e}_j \widetilde{e}_i^k \omega_i^{m/2} \omega_j^{1/2} \\
&= q_i^{\frac{1}{2}m(m-1)} r^{\frac{1}{2}m\langle \alpha_j, \alpha_i \rangle} s^{-\frac{1}{2}m\langle \alpha_i, \alpha_j \rangle} \cdot \left(\sum_{k=0}^{m} (-1)^k \begin{bmatrix} m \\ k \end{bmatrix}_{r_i, s_i} (r_i s_i)^{\frac{1}{2}k(k-m)} \widetilde{e}_i^{m-k} \widetilde{e}_j \widetilde{e}_i^k \right) \cdot \omega_i^{m/2} \omega_j^{1/2}.
\end{aligned}$$

Combining the above with the identity

$$\begin{bmatrix} m \\ k \end{bmatrix}_{q_i} = (r_i s_i)^{-\frac{1}{2}k(m-k)} \begin{bmatrix} m \\ k \end{bmatrix}_{r_i, s_i},$$

we thus obtain the desired q-Serre relation:

$$\sum_{k=0}^{1-c_{ij}} (-1)^k \begin{bmatrix} 1-c_{ij} \\ k \end{bmatrix}_{q_i} \widetilde{e}_i^{1-c_{ij}-k} \widetilde{e}_j \widetilde{e}_i^k = 0.$$

- The verification of the fact that $\widetilde{f}_i$ satisfy the q-Serre relations is completely analogous.

This completes our proof of the proposition. ∎

Remark A.2. *The above result admits a natural generalization to the affine setup of $U'_{r,s}(\widehat{\mathfrak{g}})$ and $U_{r,s}(\widehat{\mathfrak{g}})$. For the former case, we use exactly the same formulas (A.1) but now with $0 \leq i \leq n$ (the proof applies without changes, provided that we set $\langle \alpha_0, \alpha_i \rangle = \langle -\theta, \alpha_i \rangle$, $\langle \alpha_i, \alpha_0 \rangle = \langle \alpha_i, -\theta \rangle$, and $\langle \alpha_0, \alpha_0 \rangle = \langle -\theta, -\theta \rangle$). To generalize this to $U_{r,s}(\widehat{\mathfrak{g}})$ we set $\widetilde{D} = D^{1/2}(D')^{-1/2}$, so that $\widetilde{D}\widetilde{e}_i = q_0^{\delta_{0i}} \widetilde{e}_i \widetilde{D}$, $\widetilde{D}\widetilde{f}_i = q_0^{-\delta_{0i}} \widetilde{f}_i \widetilde{D}$, and $\widetilde{D}\widetilde{\omega}_i = \widetilde{\omega}_i \widetilde{D}$.*

As an important application of the Proposition above, let us relate the root vectors $\{e_\gamma, f_\gamma\}_{\gamma \in \Phi^+}$ of $U_{r,s}(\mathfrak{g})$ from (5.14) to their counterparts $\{\widetilde{e}_\gamma, \widetilde{f}_\gamma\}_{\gamma \in \Phi^+}$ of $U_q(\mathfrak{g})$ defined via

$$\widetilde{e}_\gamma = \widetilde{e}_\alpha \widetilde{e}_\beta - q^{(\alpha,\beta)} \widetilde{e}_\beta \widetilde{e}_\alpha \,, \qquad \widetilde{f}_\gamma = \widetilde{f}_\beta \widetilde{f}_\alpha - q^{-(\alpha,\beta)} \widetilde{f}_\alpha \widetilde{f}_\beta, \tag{A.2}$$

where the minimal pair (α, β) of γ corresponds to the standard factorization of $\ell(\gamma)$, see Proposition 5.9. Here, the quantum parameter $q = r^{1/2} s^{-1/2}$ and we view $U_q(\mathfrak{g})$ as a subalgebra of $U_{r,s}(\mathfrak{g})$ via Proposition A.1:

$$U_{r^{1/2} s^{-1/2}}(\mathfrak{g}) \hookrightarrow U_{r,s}(\mathfrak{g}).$$

To state the result, we introduce the constants $\{\kappa_\gamma\}_{\gamma \in \Phi^+}$ as follows:

- *Type A_n*

$$\kappa_{\gamma_{ij}} = s^{\frac{1}{2}(j-i)} \qquad \text{for} \quad 1 \leq i \leq j \leq n.$$

- *Type B_n*

$$\begin{aligned}
\kappa_{\gamma_{ij}} &= s^{j-i} \qquad \text{for} \quad 1 \leq i \leq j \leq n, \\
\kappa_{\beta_{ij}} &= r^{\frac{1}{2}+j-n} s^{n+\frac{1}{2}-i} \qquad \text{for} \quad 1 \leq i < j \leq n.
\end{aligned}$$

- *Type C_n*

$$\begin{aligned}
\kappa_{\gamma_{ij}} &= s^{\frac{1}{2}(j-i)} \qquad \text{for} \quad 1 \leq i \leq j < n, \\
\kappa_{\gamma_{in}} &= s^{\frac{1}{2}(n+1-i-\delta_{in})} \qquad \text{for} \quad 1 \leq i \leq n, \\
\kappa_{\beta_{ij}} &= r^{\frac{1}{2}(j-n)} s^{\frac{1}{2}(n+1-i)} \qquad \text{for} \quad 1 \leq i < j < n, \\
\kappa_{\beta_{ii}} &= r^{\frac{1}{2}} s^{n-i+\frac{1}{2}} \qquad \text{for} \quad 1 \leq i \leq n.
\end{aligned}$$

- *Type D_n*

$$\begin{aligned}
\kappa_{\gamma_{ij}} &= s^{\frac{1}{2}(j-i)} \qquad \text{for} \quad 1 \leq i \leq j < n, \\
\kappa_{\beta_{ij}} &= r^{\frac{1}{2}(j-n)} s^{\frac{1}{2}(n-1-i)} \qquad \text{for} \quad 1 \leq i < j \leq n.
\end{aligned}$$

For any $\gamma = \sum_{i=1}^n k_i\alpha_i \in \Phi^+$, we also define $d_\gamma = \prod_{i=1}^n s_i^{k_i}$. Then, we have the following result:

Proposition A.3. *For any positive root $\gamma \in \Phi^+$ of $\mathfrak{g}$, we have:*

$$\widetilde{e}_\gamma = \kappa_\gamma^{-1} \cdot e_\gamma \omega_\gamma^{-1/2}, \qquad \widetilde{f}_\gamma = d_\gamma \kappa_\gamma^{-1} \cdot f_\gamma (\omega'_\gamma)^{-1/2}. \tag{A.3}$$

Proof. Let us prove the first formula by an induction on the height of γ. If $\gamma = \alpha_i$, then $\kappa_\gamma = 1$, hence the above equality follows from Proposition A.1. For the step of induction, pick a non-simple root $\gamma \in \Phi^+$ and let (α, β) be the minimal pair used in (5.14) and (A.2). By the induction hypothesis, we have:

$$e_\alpha = \kappa_\alpha \cdot \widetilde{e}_\alpha \omega_\alpha^{1/2} \qquad \text{and} \qquad e_\beta = \kappa_\beta \cdot \widetilde{e}_\beta \omega_\beta^{1/2}. \tag{A.4}$$

Furthermore, the constants $\{\kappa_\gamma\}$ above are chosen to satisfy the following equality:

$$\kappa_{\alpha+\beta} = \kappa_\alpha \kappa_\beta (\omega'_\beta, \omega_\alpha)^{1/2}. \tag{A.5}$$

Combining (A.4, A.5) with (5.14) and (A.2), we obtain:

$$\begin{aligned} e_\gamma = e_\alpha e_\beta - (\omega'_\beta, \omega_\alpha) e_\beta e_\alpha &= \kappa_\alpha \kappa_\beta \Big((\omega'_\beta, \omega_\alpha)^{1/2} \widetilde{e}_\alpha \widetilde{e}_\beta - (\omega'_\beta, \omega_\alpha)(\omega'_\alpha, \omega_\beta)^{1/2} \widetilde{e}_\beta \widetilde{e}_\alpha \Big) \cdot \omega_{\alpha+\beta}^{1/2} \\ &= \kappa_\alpha \kappa_\beta (\omega'_\beta, \omega_\alpha)^{1/2} \Big(\widetilde{e}_\alpha \widetilde{e}_\beta - (\omega'_\beta, \omega_\alpha)^{1/2} (\omega'_\alpha, \omega_\beta)^{1/2} \widetilde{e}_\beta \widetilde{e}_\alpha \Big) \cdot \omega_{\alpha+\beta}^{1/2} = \kappa_{\alpha+\beta} e_\gamma \omega_\gamma^{1/2}, \end{aligned}$$

with the last equality due to the identity

$$\big((\omega'_\beta, \omega_\alpha) \cdot (\omega'_\alpha, \omega_\beta) \big)^{1/2} = (r^{1/2} s^{-1/2})^{(\alpha,\beta)} = q^{(\alpha,\beta)}.$$

This completes our proof of the induction step and hence of the first formula in (A.3). The proof of the second formula in (A.3) is completely analogous. ∎

Appendix B. Two-parameter vs one-parameter R-matrices

In this Appendix, we discuss the relation between the two-parameter finite and affine R-matrices evaluated in this paper and their one-parameter counterparts through the classical twist construction of [Re].

A-type.

Let $R(z) = \hat{R}(z) \circ \tau$ be the affine R-matrix from Theorem 6.9(b), cf. [JL2, Remark 2.3]:

$$\begin{aligned} R(z) = (1 - zrs^{-1}) \sum_i E_{ii} \otimes E_{ii} + (1-z) r \sum_{i>j} E_{ii} \otimes E_{jj} + (1-z) s^{-1} \sum_{i<j} E_{ii} \otimes E_{jj} \\ + (1 - rs^{-1}) \sum_{i>j} E_{ij} \otimes E_{ji} + (1 - rs^{-1}) z \sum_{i<j} E_{ij} \otimes E_{ji}. \end{aligned} \tag{B.1}$$

Note that setting $z \mapsto 0$ recovers the finite two-parameter R-matrix $R = \hat{R} \circ \tau$ with $\hat{R}$ from Theorem 4.3:

$$R = \sum_i E_{ii} \otimes E_{ii} + r \sum_{i>j} E_{ii} \otimes E_{jj} + s^{-1} \sum_{i<j} E_{ii} \otimes E_{jj} + (1 - rs^{-1}) \sum_{i>j} E_{ij} \otimes E_{ji}. \tag{B.2}$$

Specializing $r \mapsto q, s \mapsto q^{-1}$ in the formulas above, we obtain standard one-parameter R-matrices, cf. [DF]:

$$\bar{R}(z) = (1 - zq^2) \sum_i E_{ii} \otimes E_{ii} + (1-z) q \sum_{i \neq j} E_{ii} \otimes E_{jj} + (1-q^2) \sum_{i>j} E_{ij} \otimes E_{ji} + (1-q^2) z \sum_{i<j} E_{ij} \otimes E_{ji} \tag{B.3}$$

and

$$\bar{R} = \sum_i E_{ii} \otimes E_{ii} + q \sum_{i \neq j} E_{ii} \otimes E_{jj} + (1-q^2) \sum_{i>j} E_{ij} \otimes E_{ji}. \tag{B.4}$$

Recall the standard twist construction of [Re]. Given a quasitriangular Hopf algebra (H, Δ, ϵ) with an R-matrix R, and an element $\mathcal{F} \in H \otimes H$ (or some appropriate completion) satisfying the following conditions:

$$(\Delta \otimes \mathrm{Id})\mathcal{F} = \mathcal{F}_{13}\mathcal{F}_{23}, \quad (\mathrm{Id} \otimes \Delta)\mathcal{F} = \mathcal{F}_{13}\mathcal{F}_{12}, \quad \mathcal{F}_{12}\mathcal{F}_{13}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{13}\mathcal{F}_{12}, \quad \mathcal{F}_{12}\mathcal{F}_{21} = 1, \tag{B.5}$$

one constructs a twisted quasitriangular Hopf algebra $(H, \Delta^{(\mathcal{F})}, \epsilon^{(\mathcal{F})})$ with an R-matrix $R^{(\mathcal{F})}$ as follows:

$$\Delta^{(\mathcal{F})} = \mathcal{F} \Delta \mathcal{F}^{-1}, \qquad \epsilon^{(\mathcal{F})} = \epsilon, \qquad R^{(\mathcal{F})} = \mathcal{F}^{-1} R \mathcal{F}^{-1}. \tag{B.6}$$

The main result of this Appendix realizes (B.1) and (B.2) as such twists of (B.3) and (B.4), respectively. To this end, generalizing [Re, §3][3], we look for $\mathcal{F}$ in the following form:

$$\mathcal{F} = \exp\left(\sum_{1\leq i,j\leq n+1} \phi_{ij} E_{ii}\otimes E_{jj}\right) \quad \text{with} \quad \phi_{ij} = -\phi_{ji} \quad \text{for all} \quad 1\leq i,j\leq n+1. \tag{B.7}$$

Proposition B.1. *For $q = r^{1/2}s^{-1/2}$ and a skew-symmetric matrix (ϕ_{ij}) such that $\exp(2\phi_{ij}) = r^{1/2}s^{1/2}$ for $i>j$, we have $R = \mathcal{F}^{-1}\bar{R}\mathcal{F}^{-1}$ and $R(z) = \mathcal{F}^{-1}\bar{R}(z)\mathcal{F}^{-1}$ with $\mathcal{F}$ given by (B.7).*

Proof. This is a direct calculation, based on the formula $\mathcal{F}(v_i\otimes v_j) = \exp(\phi_{ij})\cdot v_i\otimes v_j$ for any i,j. ∎

BCD-**types.**

The generalization of the above result to other classical types **fails** on the nose. Let us illustrate this for type B_n. The two-parameter finite R-matrix $R = \hat{R}\circ\tau$, with $\hat{R}$ evaluated in Theorem 4.4, is given by:

$$R = r^{-1}s\sum_{i\neq n+1} E_{ii}\otimes E_{ii} + E_{n+1,n+1}\otimes E_{n+1,n+1} + rs^{-1}\sum_{i\neq n+1} E_{ii}\otimes E_{i'i'} - \sum_{i>j\neq i'} (rs^{-1}-r^{-1}s)E_{ij}\otimes E_{ji}$$
$$+ \sum_{j\neq i,i'} a_{ij}E_{ii}\otimes E_{jj} + \sum_{i\leq n}(rs^{-1}-r^{-1}s)((rs^{-1})^{2n-2i+1}-1)E_{i'i}\otimes E_{ii'} + \sum_{i<j\neq i'}(rs^{-1}-r^{-1}s)\frac{t_i}{t_j}E_{i'j'}\otimes E_{ij}.$$

Specializing $r\mapsto q$ and $s\mapsto q^{-1}$, we obtain the standard one-parameter R-matrix:

$$\bar{R} = q^{-2}\sum_{i\neq n+1} E_{ii}\otimes E_{ii} + E_{n+1,n+1}\otimes E_{n+1,n+1} + q^2\sum_{i\neq n+1} E_{ii}\otimes E_{i'i'} - \sum_{i>j\neq i'}(q^2-q^{-2})E_{ij}\otimes E_{ji}$$
$$+ \sum_{j\neq i,i'} E_{ii}\otimes E_{jj} + \sum_{i\leq n}(q^2-q^{-2})(q^{2(2n-2i+1)}-1)E_{i'i}\otimes E_{ii'} + \sum_{i<j\neq i'}(q^2-q^{-2})\frac{\bar{t}_i}{\bar{t}_j}E_{i'j'}\otimes E_{ij},$$

where $\bar{t}_i$ denotes the above specialization of t_i. However, there is no skew-symmetric $(\phi_{ij})_{i,j=1}^{2n+1}$ such that

$$\text{the diagonal twist} \quad \mathcal{F} = \exp\left(\sum_{i,j}\phi_{ij}E_{ii}\otimes E_{jj}\right) \quad \text{satisfies} \quad R = \mathcal{F}^{-1}\cdot\bar{R}\cdot\mathcal{F}^{-1},$$

alike to Proposition B.1. To see this, let us compare the respective summands in R and $\mathcal{F}^{-1}\cdot\bar{R}\cdot\mathcal{F}^{-1}$. The identification of the first and second summands amounts to $\phi_{ii}=0$ for all i. Identification of the third summands amounts to $\phi_{ii'}=0$. Identification of the fourth summands amounts to $\phi_{ij}+\phi_{ji}=0$ for $i>j\neq i'$. Identification of the sixth summands amounts to $\phi_{ii'}+\phi_{i'i}=0$ for $i\leq n$. Identification of the fifth summands amounts to the equalities $\exp(2\phi_{ij}) = a_{ij}^{-1}$ for $j\neq i,i'$, which are compatible as $a_{ij}a_{ji}=1$. These already determine the matrix $(\phi_{i,j})$ uniquely. Yet, with this choice, the last summands **do not match**.

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I.M.: Purdue University, Department of Mathematics, West Lafayette, IN, USA
Email address: mart2151@purdue.edu

A.T.: Purdue University, Department of Mathematics, West Lafayette, IN, USA
Email address: sashikts@gmail.com