ORTHOSYMPLECTIC R-MATRICES

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Abstract. We present a formula for trigonometric orthosymplectic R-matrices associated with any parity sequence, and establish their factorization into the ordered product of q-exponents parametrized by positive roots in the corresponding reduced root systems. The latter is crucially based on the construction of orthogonal bases of the positive subalgebra through q-bracketings and combinatorics of dominant Lyndon words, as developed in [7]. We further evaluate the affine orthosymplectic R-matrices, establishing their intertwining property as well as matching them with those obtained through the Yang-Baxterization technique of [17]. This reproduces the celebrated formulas of [21] for classical BCD types and the formula of [31] for the standard parity sequence.

Contents

1. Introduction

1.1. **Summary.**

For classical Lie algebras $\mathfrak{g}$, the quantum groups $U_q^{\mathrm{rtt}}(\mathfrak{g})$ first implicitly appeared in the work of Faddeev's school on the *quantum inverse scattering method*, see e.g. [11]. In this *RLL realization*, the algebra generators are encoded by two square matrices $L^{\pm}$ subject to the famous *RLL-relations*

$$RL_1^{\pm}L_2^{\pm} = L_2^{\pm}L_1^{\pm}R\,, \qquad RL_1^{+}L_2^{-} = L_2^{-}L_1^{+}R$$

(and some additional relations to kill the center), where R is a solution of the *Yang-Baxter equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\,. \tag{1.1}$$

This is a natural analogue of the matrix realization of classical Lie algebras, and it manifestly exhibits the Hopf algebra structure, with the coproduct Δ, antipode S, and counit ϵ given by

$$\Delta(L^{\pm}) = L^{\pm}\otimes L^{\pm}\,, \qquad S(L^{\pm}) = (L^{\pm})^{-1}\,, \qquad \epsilon(L^{\pm}) = \mathrm{I}\,.$$

The uniform definition of quantum groups $U_q^{\mathrm{DJ}}(\mathfrak{g})$ for any Kac-Moody Lie algebra $\mathfrak{g}$ was provided independently by Drinfeld [9] and Jimbo [20], and is usually referred to as the *Drinfeld-Jimbo realization*. In this presentation, the generators $e_i, f_i, k_i^{\pm 1} = q^{\pm h_i}$ are labeled by simple roots α_i of $\mathfrak{g}$, while the Hopf algebra structure is given formally by the assignment on the generators. In A-type, the corresponding isomorphism $U_q^{\mathrm{rtt}}(\mathfrak{gl}_n) \simeq U_q^{\mathrm{DJ}}(\mathfrak{gl}_n)$, and subsequently its $\mathfrak{sl}_n$-counterpart, were constructed in [8, §2] by considering the Gauss decomposition of the generator matrices $L^{\pm}$.

The next important class of Kac-Moody Lie algebras is the so-called affine Lie algebras $\widehat{\mathfrak{g}}$, which admit a similar Chevalley-Serre type presentation associated with extended Dynkin diagrams. It is well-known that they are central extensions of the corresponding loop algebra $L\mathfrak{g} = \mathfrak{g}\otimes\mathbb{C}[t,t^{-1}]$:

$$0 \to \mathbb{C}\cdot c \to \widehat{\mathfrak{g}} \to L\mathfrak{g} \to 0\,.$$

The aforementioned construction of [11] was extended to the loop setup of $L\mathfrak{g}$ in [12] by crucially replacing the R-matrices satisfying (1.1) with parameter-dependent R-matrices $R(z)$ satisfying

$$R_{12}(z)R_{13}(zw)R_{23}(w) = R_{23}(w)R_{13}(zw)R_{12}(z)\,, \tag{1.2}$$

the so-called *Yang-Baxter equation (with a spectral parameter)*. The generators of these algebras $U_q^{\mathrm{rtt}}(L\mathfrak{g})$ are now encoded by two square matrices $L^{\pm}(z)$ subject to analogous *RLL-relations*

$$R(z/w)L_1^{\pm}(z)L_2^{\pm}(w) = L_2^{\pm}(w)L_1^{\pm}(z)R(z/w)\,, \qquad R(z/w)L_1^{+}(z)L_2^{-}(w) = L_2^{-}(w)L_1^{+}(z)R(z/w)\,.$$

Finally, this was generalized to $\widehat{\mathfrak{g}}$ in [38], thus producing $U_q^{\mathrm{rtt}}(\widehat{\mathfrak{g}})$ by incorporating the central charge. For classical $\mathfrak{g}$, this construction is an exact affine analogue of the construction from [11].

There is yet another realization [10] of quantum affine groups $U_q(\widehat{\mathfrak{g}})$, which is usually referred to as the *new Drinfeld realization* (a.k.a. *current realization*). The isomorphism $U_q(\widehat{\mathfrak{g}}) \simeq U_q^{\mathrm{DJ}}(\widehat{\mathfrak{g}})$ was stated in [10] without a proof, while the complete details appeared a decade later in the work of Beck and Damiani. In A-type, the corresponding isomorphism $U_q^{\mathrm{rtt}}(\widehat{\mathfrak{gl}}_n) \simeq U_q(\widehat{\mathfrak{gl}}_n)$, and subsequently its $\widehat{\mathfrak{sl}}_n$-counterpart, were first constructed in [8] by considering the Gauss decomposition of the generator matrices $L^{\pm}(z)$, similarly to the finite type. For affinizations of other classical Lie algebras such isomorphisms were first discovered in [18] and were revised more recently in [23, 24].

The above results also admit *rational* counterparts, with quantum loop/affine groups replaced by the Yangians $Y^J_\hbar(\mathfrak{g})$ (in the *J-realization*), first introduced in [9]. The representation theory of these algebras has been developed using their alternative *(new) Drinfeld realization* $Y_\hbar(\mathfrak{g})$ proposed in [10], though it should be noted that the Hopf algebra structure is much more involved in this presentation. One can also adapt [11, 12] to define $Y^{\mathrm{rtt}}_\hbar(\mathfrak{g})$ in the *RTT-realization*. In this presentation, the algebra generators are encoded by a square matrix $T(u)$ subject to a single *RTT-relation* (together with an extra central reduction)

$$R(u-v)T_1(u)T_2(v) = T_2(v)T_1(u)R(u-v)$$

where the R-matrix $R(u)$ is again a solution of the *Yang-Baxter equation*

$$R_{12}(u)R_{13}(u+v)R_{23}(v) = R_{23}(v)R_{13}(u+v)R_{12}(u)\,.$$

For classical series, the relevant R-matrices $R(u)$ are the Yang's matrix in type A, and the Zamolodchikov-Zamolodchikov's matrix in types BCD. The Hopf algebra structure on such $Y^{\mathrm{rtt}}_\hbar(\mathfrak{g})$ is especially simple, with the coproduct Δ, antipode S, and counit ϵ given explicitly by

$$\Delta(T(u)) = T(u)\otimes T(u)\,, \qquad S(T(u)) = T^{-1}(u)\,, \qquad \epsilon(T(u)) = \mathrm{I}\,.$$

These features make the RTT-realization to be well-suited both for the representation theory as well as the study of corresponding integrable systems. An explicit isomorphism $Y^{\mathrm{rtt}}_\hbar(\mathfrak{g}) \simeq Y_\hbar(\mathfrak{g})$ is again constructed using the Gauss decomposition of the generator matrix $T(u)$. For A-type this was carried out in [4], for BCD-types it was carried a decade later in [22], while a less explicit isomorphism in general types was established in [40]. Finally, we note that the RTT realization of the (antidominantly) shifted Yangians $Y_\mu(\mathfrak{g})$ from [3] was recently obtained in [13, 14] for classical $\mathfrak{g}$, thus significantly simplifying some of their basic structures as well as producing integrable systems on the corresponding quantized Coulomb branches of $3d$ $\mathcal{N}=4$ quiver gauge theories.

The theory of quantum groups and Yangians associated with Lie superalgebras is still far from a full development. While the Drinfeld-Jimbo realization of quantum finite and affine supergroups was proposed two decades ago in [42, 43], there is no uniform (new) Drinfeld realization of such algebras in affine types, besides for A-type. A novel feature of Lie superalgebras is that they admit several non-isomorphic Dynkin diagrams. The isomorphism of the Lie superalgebras corresponding to different Dynkin diagrams of the same finite/affine type was established in [27, Appendix]. Upgrading the latter to the isomorphism of quantum finite/affine superalgebras associated with different Dynkin diagrams is a highly non-trivial technical task that constitutes one of the major results of [43]. The renewed interest in quantum supergroups over the last decade is often motivated by intriguing predictions in string theory. In particular, the recent work [41] established a certain duality between $U_q(\mathfrak{osp}(2m+1|2n))$ and $U_{-q}(\mathfrak{osp}(2n+1|2m))$ generalizing a conjecture of [32].

Likewise, there is no J- or new Drinfeld realizations of superYangians. The cases studied mostly up to date involve rather the RTT realization. The general linear RTT Yangians $Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}(n|m))$ and the orthosymplectic RTT Yangians $Y^{\mathrm{rtt}}_\hbar(\mathfrak{osp}(N|2m))$ first appeared in [34] and [1], respectively, using the super-analogues of the Yang's and Zamolodchikov-Zamolodchikov's rational R-matrices. In the above classical types, the underlying R-matrices posses natural symmetries, which yield isomorphisms of $Y^{\mathrm{rtt}}_\hbar(\mathfrak{g})$ associated with different Dynkin diagrams. In the recent work [15], the new Drinfeld realization of orthosymplectic superYangian $Y_\hbar(\mathfrak{osp}(V))$ was finally obtained for any Dynkin diagram, generalizing the one of [33] for the case of the distinguished Dynkin diagram (we note that the orthosymplectic type simultaneously resembles all three classical types B, C, D). The key idea of [15] was to derive all the defining relations (including the higher order Serre relations) from the RTT-relations by performing the Gauss decomposition of the generating matrix, thus generalizing the derivation of the new Drinfeld realization of $Y^{\mathrm{rtt}}_\hbar(\mathfrak{gl}(n|m))$ from [36, 39].

In this note, we evaluate finite and affine R-matrices associated with the orthosymplectic Lie algebras and any Dynkin diagram. In analogy with the aforementioned orthosymplectic superYangian case, in the sequel note [25] we shall derive the new Drinfeld realization of the orthosymplectic quantum affine algebras, thus generalizing the BCD-types of [23, 24]. While for the distinguished Dynkin diagram, the corresponding R-matrices were presented almost 20 years ago in [31] (also cf. [16]), we hope that the present work also adds more in understanding the origin of these formulas. Our presentation follows closely [29], the recent joint work of I. Martin and the second author. To this end, we first evaluate the finite orthosymplectic R-matrices, also presenting their factorization into "local q-exponents". While the latter resembles the classical factorization of finite R-matrices for (non-super) quantum groups $U_q^{\mathrm{DJ}}(\mathfrak{g})$ (see [26]), our proof is quite different as it relies on the combinatorial shuffle approach. We then apply the Yang-Baxterization technique of [17] to get several potential candidates for the affine $R(z)$, and finally choose the correct one that intertwines the tensor product of two evaluation modules. We note that a combination of the present work and [29, 30] allows to evaluate finite and affine R-matrices for two-parameter orthosymplectic quantum groups and subsequently derive their new Drinfeld presentation.

1.2. **Outline.**

The structure of the present paper is the following:

• In Section 2, we recall the basic conventions on superalgebras as well as definitions of orthosymplectic Lie superalgebras $\mathfrak{osp}(V)$ and their Drinfeld-Jimbo type quantizations $U_q(\mathfrak{osp}(V))$.

• In Section 3, we explicitly construct the first fundamental representation V of $U_q(\mathfrak{osp}(V))$, see Proposition 3.1. We further present three highest weight vectors w_1, w_2, w_3 in $V \otimes V$, see Proposition 3.2 (and Subsection B.2), such that w_1, w_2 and yet another vector $\widetilde{w}_3$ or $\hat{w}_3$ (or actually w_3 unless $n = m$) generate the entire tensor square $V \otimes V$ under the $U_q(\mathfrak{osp}(V))$-action.

• In Section 4, we evaluate the universal intertwiner $\hat{R}_{VV}$ from Proposition 4.3 on the first fundamental $U_q(\mathfrak{osp}(V))$-representation from Proposition 3.1, see Theorems 4.6. This generalizes the formula of [31] for the standard parity sequence. Our proof is quite different though, as we directly verify the intertwining property, see Proposition 4.8 (and Subsection 4.3 for its proof), and match the action on the generating three vectors from Proposition 3.2, see Propositions 4.9, 4.11.

• In Section 5, we present an alternative proof of the formula for R_{VV} from Theorem 4.6 by factorizing it into the "local" operators parametrized by the positive roots of the reduced root system $\bar{\Phi}$. To do so, we use a combinatorial construction of orthogonal PBW bases of the positive subalgebra of $U_q(\mathfrak{osp}(V))$ developed in [7], see Proposition 5.6. While the setup of [7] slightly differs from ours (in that they use a different pairing, coproduct, and a twisted product), we relate the two explicitly, thus obtaining dual bases of the positive and negative subalgebras of $U_q(\mathfrak{osp}(V))$ with respect to the bialgebra pairing (4.3), see Theorem 5.16. This implies the factorization formula of Theorem 5.18, cf. Remark 5.19, thus providing a conceptual origin of (4.13) from Theorem 4.6.

• In Section 6, we extend the first fundamental $U_q(\mathfrak{osp}(V))$-module from Proposition 3.1 to evaluation modules over the orthosymplectic quantum affine supergroup $U_q(\widehat{\mathfrak{osp}}(V))$ and its reduced version $U'_q(\widehat{\mathfrak{osp}}(V))$ in Propositions 6.1–6.2. The main result of this Section is Theorem 6.3 which evaluates the universal intertwiner of $U_q(\widehat{\mathfrak{osp}}(V))$ on the tensor product of two such representations, generalizing the orthogonal and symplectic types due to [21]. According to [21], this produces a solution of the Yang-Baxter relation with a spectral parameter, cf. (6.11). While the proof of Theorem 6.3 is straightforward (see Subsection 6.4), we derived the formula from its finite counterpart (Theorem 4.6) through the *Yang-Baxterization* technique of [17], cf. Proposition 6.8.

• In Appendix A, we present a similar treatment for A-type quantum finite and affine supergroups, and derive the corresponding finite and affine R-matrices in Theorems A.3 and A.18, respectively. In Subsection A.6, we also present a factorization formula for the corresponding finite R-matrix, which seems to be missing in the literature. We also emphasize that the affine R-matrix can be obtained from the finite one through the Yang-Baxterization of [17], up to a prefactor.

• In Appendix B, we present a direct tedious verification of the "generating" property of $V \otimes V$ by the two highest weight vectors in type A and three vectors in orthosymplectic type (which can be chosen to be the highest weight vectors unless $n = m$), cf. Propositions A.2(b) and 3.2(b,c). Our analysis emphasizes an importance of the special case $n = m$, when $V \otimes V$ is not semisimple.

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2. Orthosymplectic Lie superalgebras and quantum groups

In this Section, we recall the definition of orthosymplectic Lie superalgebras and associated quantum groups, following [6].

2.1. Setup and notations.

Fix non-negative integers m,n so that n is even, and set $N=m+n$ as well as $s=\lfloor \frac{N}{2} \rfloor$. We shall assume that $N>2$. We equip the index set $\mathbb{I}=\{1,2,\ldots,N\}$ with an involution $'$ via:

$$i \mapsto i' := N+1-i\,.$$

Consider a superspace $V=\mathbb{C}^{m|n}$ with a homogeneous $\mathbb{C}$-basis $\{v_1,v_2,\ldots,v_N\}$ such that each v_i is either *even* (that is $v_i \in V_{\bar 0}$) or *odd* (that is $v_i \in V_{\bar 1}$), the dimensions are $\dim(V_{\bar 0})=m, \dim(V_{\bar 1})=n$, and the vectors $v_i, v_{i'}$ have the same parity for any $i\in\mathbb{I}$ (in particular, v_{s+1} is even for odd N). The latter condition means that

$$\bar i = \bar{i'}\,, \tag{2.1}$$

where for $i\in\mathbb{I}$, we define its $\mathbb{Z}_2$-parity $\bar i\in\mathbb{Z}_2$ via:

$$\bar i = |v_i| = \begin{cases} \bar 0 & \text{if } v_i \text{ is even} \\ \bar 1 & \text{if } v_i \text{ is odd} \end{cases}.$$

The choice of $\mathbb{Z}_2$-parity of the basis vectors can be encoded by the *parity sequence*

$$\gamma_V := (|v_1|,\ldots,|v_s|) = (\bar 1,\ldots,\bar s) \in \{\bar 0, \bar 1\}^s.$$

We also choose a sequence $\vartheta_V := (\vartheta_1,\vartheta_2,\ldots,\vartheta_N)$ of ±1's satisfying

$$\vartheta_i = \begin{cases} 1 & \text{if } \bar i = \bar 0 \\ -\vartheta_{i'} & \text{if } \bar i = \bar 1 \end{cases} \tag{2.2}$$

(we do not assume that $\vartheta_i=1$ for $i\le s$). Under the conventions $(-1)^{\bar 0}=1, (-1)^{\bar 1}=-1$, we get

$$\vartheta_i^2 = 1 \qquad \text{and} \qquad \vartheta_i\vartheta_{i'} = (-1)^{\bar i} \qquad \text{for any} \qquad i\in\mathbb{I}\,.$$

Any superalgebra $A=A_{\bar 0}\oplus A_{\bar 1}$ can be equipped with a natural *Lie superalgebra* structure via:

$$[x,x'] = \mathrm{ad}_x(x') := xx' - (-1)^{|x|\cdot|x'|}x'x \tag{2.3}$$

defined for homogeneous $x,x'\in A$ with $|x|,|x'|$ denoting their $\mathbb{Z}_2$-grading, and extended linearly. Given two superspaces $A=A_{\bar 0}\oplus A_{\bar 1}$ and $B=B_{\bar 0}\oplus B_{\bar 1}$, their tensor product $A\otimes B$ is also a superspace with $(A\otimes B)_{\bar 0} = (A_{\bar 0}\otimes B_{\bar 0})\oplus(A_{\bar 1}\otimes B_{\bar 1})$ and $(A\otimes B)_{\bar 1} = (A_{\bar 0}\otimes B_{\bar 1})\oplus(A_{\bar 1}\otimes B_{\bar 0})$. Furthermore, if A and B are superalgebras, then $A\otimes B$ is made into a superalgebra, called the *graded tensor product* of the superalgebras A and B, via the following multiplication:

$$(x\otimes y)(x'\otimes y') = (-1)^{|y|\cdot|x'|}(xx')\otimes(yy') \quad \text{for any homogeneous} \quad x,x'\in A,\ y,y'\in B\,. \tag{2.4}$$

We will use only the graded tensor products of superalgebras, unless stated otherwise.

2.2. Orthosymplectic Lie superalgebras.

Consider the set $\mathfrak{gl}(V)$ of all linear endomorphisms of V. Using the basis $\{v_1,v_2,\ldots,v_N\}$ of V, we can identify $\mathfrak{gl}(V)$ with the set of all $N\times N$ matrices. It can be made into a Lie superalgebra, called the *general linear Lie superalgebra*, by defining the $\mathbb{Z}_2$-grading

$$|E_{ij}| := \bar i + \bar j$$

and consequently with the Lie superbracket given by (cf. (2.3))

$$[E_{ij},E_{ab}] = \delta_{ja}E_{ib} - \delta_{bi}(-1)^{(\bar i+\bar j)(\bar a+\bar b)}E_{aj}\,,$$

where we use the basis $\{E_{ij}\}_{i,j=1}^N$ of elementary matrices with 1 at the (i,j)-entry and 0 elsewhere.

Consider a bilinear form $B_G\colon V\times V\to\mathbb{C}$ defined by the anti-diagonal matrix (cf. (2.2))

$$G = \sum_{i=1}^N \vartheta_i E_{i'i}\,.$$

The *orthosymplectic Lie superalgebra* $\mathfrak{osp}(V)$ associated with the bilinear form B_G is the Lie subalgebra of $\mathfrak{gl}(V)$ consisting of all matrices $X = \sum_{i,j} x_{ij} E_{ij} \in \mathfrak{gl}(V)$ preserving B_G, i.e. satisfying

$$X^{\mathrm{st}} G + GX = 0$$

where the *supertransposition* of X is defined via

$$X^{\mathrm{st}} := \sum_{i,j=1}^{N} (-1)^{\bar{j}(\bar{i}+\bar{j})} x_{ij} E_{ji} \,. \tag{2.5}$$

Thus, $\mathfrak{osp}(V)$ is spanned by the elements

$$X_{ij} = E_{ij} - (-1)^{\bar{i}(\bar{i}+\bar{j})} \vartheta_i \vartheta_j E_{j'i'} \qquad \forall\, 1 \leq i,j \leq N \,. \tag{2.6}$$

We note that $X_{j'i'} = -(-1)^{\bar{i}(\bar{i}+\bar{j})} \vartheta_i \vartheta_j \cdot X_{ij}$. The following elements form a basis of $\mathfrak{osp}(V)$:

$$\{X_{ij} \,|\, i+j < N+1\} \cup \{X_{ii'} \,|\, \bar{i} = \bar{1},\, 1 \leq i \leq s\} \,.$$

We choose the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{osp}(V)$ to consist of all diagonal matrices. Thus, $\mathfrak{h}$ has a basis $\{X_{ii}\}_{i=1}^{s}$. Consider the linear functionals $\{\varepsilon_r\}_{r=1}^{N}$ on $\mathfrak{gl}(V)$ defined by $\varepsilon_r(\sum_{i,j} x_{ij} E_{ij}) = x_{rr}$. Their restrictions to $\mathfrak{h}$ satisfy

$$\varepsilon_i|_{\mathfrak{h}} = -\varepsilon_{i'}|_{\mathfrak{h}} \quad \text{for any } i \,, \qquad \varepsilon_{s+1}|_{\mathfrak{h}} = 0 \quad \text{for odd } N \,. \tag{2.7}$$

Therefore, $\{\varepsilon_i\}_{i=1}^{s}$ give rise to a basis of $\mathfrak{h}^*$ that is dual to the basis $\{X_{ii}\}_{i=1}^{s}$ of $\mathfrak{h}$. The computation $[X_{ii}, X_{ab}] = (\varepsilon_a - \varepsilon_b)(X_{ii}) X_{ab}$ shows that X_{ab} is a root vector corresponding to the root $\varepsilon_a - \varepsilon_b$. Hence, we get the *root space decomposition* $\mathfrak{osp}(V) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{osp}(V)_\alpha$ with the root system

$$\Phi = \{\varepsilon_a - \varepsilon_b \,|\, a+b < N+1,\, a \neq b\} \cup \{2\varepsilon_a \,|\, \bar{a} = \bar{1}\} \,. \tag{2.8}$$

We further have a decomposition $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ into *even* and *odd* roots. We also choose the following polarization of Φ:

$$\begin{aligned} \Phi^+ &= \{\varepsilon_a - \varepsilon_b \,|\, a < b < a'\} \cup \{2\varepsilon_a \,|\, \bar{a} = \bar{1},\, a \leq s\} \,, \\ \Phi^- &= \{\varepsilon_a - \varepsilon_b \,|\, b < a < b'\} \cup \{2\varepsilon_a \,|\, \bar{a} = \bar{1},\, a' \leq s\} \,. \end{aligned} \tag{2.9}$$

Let $\bar{\Phi} = \bar{\Phi}_{\bar{0}} \cup \bar{\Phi}_{\bar{1}}$ be the *reduced* root system of $\mathfrak{osp}(V)$, that is:

$$\bar{\Phi} = \left\{ \gamma \in \Phi \,|\, \tfrac{1}{2}\gamma \notin \Phi \right\} \,, \quad \bar{\Phi}_{\bar{0}} = \bar{\Phi} \cap \Phi_{\bar{0}} \,, \quad \bar{\Phi}_{\bar{1}} = \bar{\Phi} \cap \Phi_{\bar{1}} \,. \tag{2.10}$$

2.3. **Chevalley-Serre type presentation.**

Consider the non-degenerate *supertrace* bilinear form $(\cdot,\cdot)\colon \mathfrak{osp}(V) \times \mathfrak{osp}(V) \to \mathbb{C}$ defined by

$$(X,Y) = \tfrac{1}{2} \mathrm{sTr}(XY) \,.$$

Its restriction to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{osp}(V)$ is non-degenerate, thus giving rise to an identification $\mathfrak{h} \simeq \mathfrak{h}^*$ via $\varepsilon_i \leftrightarrow (-1)^{\bar{i}} X_{ii}$ and inducing a bilinear form $(\cdot,\cdot)\colon \mathfrak{h}^* \times \mathfrak{h}^* \to \mathbb{C}$ such that

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij} (-1)^{\bar{i}} \qquad \text{for any} \qquad 1 \leq i,j \leq s \,. \tag{2.11}$$

We also recall that an odd root $\alpha \in \Phi_{\bar{1}}$ is called *isotropic* if $(\alpha,\alpha) = 0$.

Following the choice of the polarization (2.9) of the root system (2.8), the simple roots and the corresponding root vectors can be written as follows:

- Case 1: m is odd.

$$\begin{aligned} &\alpha_i = \varepsilon_i - \varepsilon_{i+1} \qquad \text{for } \ 1 \leq i \leq s \,, \\ &\mathsf{e}_i = X_{i,i+1} \qquad \text{for } \ 1 \leq i \leq s \,, \\ &\mathsf{f}_i = (-1)^{\bar{i}} X_{i+1,i} \qquad \text{for } \ 1 \leq i \leq s \,, \\ &\mathsf{h}_i = [\mathsf{e}_i, \mathsf{f}_i] = (-1)^{\bar{i}} X_{ii} - (-1)^{\overline{i+1}} X_{i+1,i+1} \qquad \text{for } \ 1 \leq i \leq s \,. \end{aligned} \tag{2.12}$$

- Case 2: m is even and $\bar{s} = \bar{0}$.

$$\begin{aligned}
\alpha_i &= \begin{cases} \varepsilon_i - \varepsilon_{i+1} & \text{if } 1 \leq i < s \\ \varepsilon_{s-1} - \varepsilon_{s+1} = \varepsilon_{s-1} + \varepsilon_s & \text{if } i = s \end{cases}, \\
\mathsf{e}_i &= \begin{cases} X_{i,i+1} & \text{if } 1 \leq i < s \\ X_{s-1,s+1} & \text{if } i = s \end{cases}, \\
\mathsf{f}_i &= \begin{cases} (-1)^{\bar{i}} X_{i+1,i} & \text{if } 1 \leq i < s \\ (-1)^{\overline{s-1}} X_{s+1,s-1} & \text{if } i = s \end{cases}, \\
\mathsf{h}_i = [\mathsf{e}_i, \mathsf{f}_i] &= \begin{cases} (-1)^{\bar{i}} X_{ii} - (-1)^{\overline{i+1}} X_{i+1,i+1} & \text{if } 1 \leq i < s \\ (-1)^{\overline{s-1}} X_{s-1,s-1} - (-1)^{\overline{s+1}} X_{s+1,s+1} & \text{if } i = s \end{cases}.
\end{aligned} \tag{2.13}$$

- Case 3: m is even and $\bar{s} = \bar{1}$.

$$\begin{aligned}
\alpha_i &= \begin{cases} \varepsilon_i - \varepsilon_{i+1} & \text{if } 1 \leq i < s \\ 2\varepsilon_s & \text{if } i = s \end{cases}, \\
\mathsf{e}_i &= \begin{cases} X_{i,i+1} & \text{if } 1 \leq i < s \\ E_{s,s+1} & \text{if } i = s \end{cases}, \\
\mathsf{f}_i &= \begin{cases} (-1)^{\bar{i}} X_{i+1,i} & \text{if } 1 \leq i < s \\ -2E_{s+1,s} & \text{if } i = s \end{cases}, \\
\mathsf{h}_i = [\mathsf{e}_i, \mathsf{f}_i] &= \begin{cases} (-1)^{\bar{i}} X_{ii} - (-1)^{\overline{i+1}} X_{i+1,i+1} & \text{if } 1 \leq i < s \\ -2X_{ss} & \text{if } i = s \end{cases}.
\end{aligned} \tag{2.14}$$

Define the *symmetrized Cartan matrix* $(a_{ij})_{i,j=1}^{s}$ of $\mathfrak{osp}(V)$ via

$$a_{ij} = (\alpha_i, \alpha_j)\,. \tag{2.15}$$

Then, the above elements $\{\mathsf{e}_i, \mathsf{f}_i, \mathsf{h}_i\}_{i=1}^{s}$ are easily seen to satisfy the Chevalley-type relations:

$$[\mathsf{h}_i, \mathsf{h}_j] = 0\,, \quad [\mathsf{h}_i, \mathsf{e}_j] = a_{ij}\mathsf{e}_j\,, \quad [\mathsf{h}_i, \mathsf{f}_j] = -a_{ij}\mathsf{f}_j\,, \quad [\mathsf{e}_i, \mathsf{f}_j] = \delta_{ij}\mathsf{h}_i\,. \tag{2.16}$$

In fact, the Lie superalgebra $\mathfrak{osp}(V)$ admits a generators-and-relations presentation, which is a special case of [44, Main Theorem]. Explicitly, it is generated by $\{\mathsf{e}_i, \mathsf{f}_i, \mathsf{h}_i\}_{i=1}^{s}$, with the $\mathbb{Z}_2$-grading

$$|\mathsf{e}_i| = |\mathsf{f}_i| = \begin{cases} \bar{0} & \text{if } \alpha_i \in \Phi_{\bar{0}} \\ \bar{1} & \text{if } \alpha_i \in \Phi_{\bar{1}} \end{cases}, \qquad |\mathsf{h}_i| = \bar{0}\,, \tag{2.17}$$

while the defining relations are (2.16) together with the *standard Serre relations* and the *higher order Serre relations*. As we shall not need the explicit form of the Serre relations, we rather refer the interested reader to [44, §3.2.1] for the exact formulas.

Remark 2.1. We note that our choice of the generators is a rescaled version of those used in [43,44], as we use the symmetrized Cartan matrix instead of the non-symmetrized one in (2.16).

2.4. **Orthosymplectic quantum groups.**

The *orthosymplectic quantum supergroup* $U_q(\mathfrak{osp}(V))$ is a natural quantization of the universal enveloping superalgebra $U(\mathfrak{osp}(V))$. Explicitly, $U_q(\mathfrak{osp}(V))$ is a $\mathbb{C}(q^{\pm 1/2})$-superalgebra generated by $\{e_i, f_i, q^{\pm h_i/2}\}_{i=1}^{s}$, with the $\mathbb{Z}_2$-grading as in (2.17), subject to the following analogues of (2.16):

$$[q^{h_i/2}, q^{h_j/2}] = 0\,, \qquad q^{\pm h_i/2} q^{\mp h_i/2} = 1\,, \tag{2.18}$$

$$q^{h_i/2} e_j q^{-h_i/2} = q^{a_{ij}/2} e_j\,, \quad q^{h_i/2} f_j q^{-h_i/2} = q^{-a_{ij}/2} f_j\,, \tag{2.19}$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}\,, \tag{2.20}$$

as well as the *standard* and the *higher order q-Serre relations*, which the interested reader may find in [42, Proposition 10.4.1], cf. [7, Proposition 2.7].

Remark 2.2. We note that our choice of the denominator $q - q^{-1}$ in (2.20) follows conventions of [43], and thus may differ from some other literature by a mere rescaling of f_j's.

Consider a lattice $P = \bigoplus_{i=1}^{s} \mathbb{Z}\varepsilon_i$, a root lattice $Q = \bigoplus_{i=1}^{s} \mathbb{Z}\alpha_i$ contained in P via (2.12)–(2.14), and set $Q^+ = \bigoplus_{i=1}^{s} \mathbb{Z}_{\geq 0}\alpha_i$. The algebra $U_q(\mathfrak{osp}(V))$ is naturally graded by Q (thus also by P) via:

$$\deg(e_i) = \alpha_i\,, \qquad \deg(f_i) = -\alpha_i\,, \qquad \deg(q^{\pm h_i/2}) = 0 \qquad \forall\, 1 \leq i \leq s\,. \tag{2.21}$$

For elements $a, b \in U_q(\mathfrak{osp}(V))$ homogeneous with respect to the $\mathbb{Z}_2$-grading and (2.21), we set:

$$[\![a, b]\!] = ab - (-1)^{|a||b|} q^{(\deg(a), \deg(b))} \cdot ba\,. \tag{2.22}$$

Moreover, the following formulas endow $U_q(\mathfrak{osp}(V))$ with a Hopf superalgebra structure:

$$\begin{aligned}\Delta(e_i) &= q^{h_i/2} \otimes e_i + e_i \otimes q^{-h_i/2}\,,\\ \Delta(f_i) &= q^{h_i/2} \otimes f_i + f_i \otimes q^{-h_i/2}\,,\\ \Delta(q^{\pm h_i/2}) &= q^{\pm h_i/2} \otimes q^{\pm h_i/2}\,,\end{aligned} \tag{2.23}$$

the counit

$$\epsilon(e_i) = 0\,, \qquad \epsilon(f_i) = 0\,, \qquad \epsilon(q^{\pm h_i/2}) = 1\,,$$

and the antipode

$$S(e_i) = -q^{-a_{ii}/2} e_i\,, \qquad S(f_i) = -q^{a_{ii}/2} f_i\,, \qquad S(q^{\pm h_i/2}) = q^{\mp h_i/2}\,.$$

Following [19], we also define a superalgebra homomorphism $\Delta^J \colon U_q(\mathfrak{osp}(V)) \to U_q(\mathfrak{osp}(V))^{\otimes 2}$ via

$$\Delta^J(e_i) = q^{h_i} \otimes e_i + e_i \otimes 1\,, \quad \Delta^J(f_i) = 1 \otimes f_i + f_i \otimes q^{-h_i}\,, \quad \Delta^J(q^{\pm h_i/2}) = q^{\pm h_i/2} \otimes q^{\pm h_i/2}\,. \tag{2.24}$$

Remark 2.3. We prefer to follow the notations from physics literature as we use $q^{\pm h_i}$ instead of the more common generators $K_i^{\pm 1}$ and use the coproduct (2.23) more often than (2.24).

3. Column-vector representations

3.1. First fundamental representations.

Henceforth, we shall use the following convention q^D for a diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_N)$:

$$q^D = q^{d_1} E_{11} + \cdots + q^{d_N} E_{NN}\,.$$

We shall also identify $\mathrm{End}(V)$ with the set of $N \times N$ matrices using the basis $\{v_1, \ldots, v_N\}$ of V.

Proposition 3.1. *The following defines a representation* $\varrho \colon U_q(\mathfrak{osp}(V)) \to \mathrm{End}(V)$*:*

$$\varrho(e_i) = \mathsf{e}_i\,, \qquad \varrho(f_i) = \kappa_i \mathsf{f}_i\,, \qquad \varrho(q^{\pm h_i/2}) = q^{\pm \mathsf{h}_i/2} \qquad \text{for} \quad 1 \leq i \leq s\,, \tag{3.1}$$

where $\{\mathsf{e}_i, \mathsf{f}_i, \mathsf{h}_i\}_{i=1}^{s}$ *denote the action of Chevalley-type generators of* $\mathfrak{osp}(V)$ *from* (2.12)–(2.14)*, and*

$$\kappa_i = \begin{cases} \frac{q+q^{-1}}{2} & \text{if } m \text{ is even, } \bar{s} = \bar{1}, \text{ and } i = s \\ 1 & \text{otherwise} \end{cases}.$$

Proof. We need to show that the operators (3.1) satisfy the defining relations (2.18)–(2.20) together with the standard and the higher order q-Serre relations. The relations (2.18) are obvious as all h_i are diagonal in the basis $\{v_1, \ldots, v_N\}$. For the first relation of (2.19), we note that its left-hand side is the conjugation of e_j by the diagonal matrix $q^{\mathsf{h}_i/2}$. Hence, it suffices to compare $q^{a_{ij}/2}$ to the ratios of the eigenvalues of $q^{\mathsf{h}_i/2}$ on the vectors $\mathsf{e}_j v_a$ and v_a (assuming the former is nonzero), which follows from the second equality of (2.16). The second relation of (2.19) is checked analogously. Finally, the relation (2.20) follows from (2.16), since h_i is diagonal with $\{0, \pm 1, \pm 2\}$ on diagonal (with ± 2 appearing only when m is even, $\bar{s} = \bar{1}$, and $i = s$) and

$$(q^r - q^{-r})/(q - q^{-1}) = r \qquad \text{for} \quad r \in \{0, \pm 1\}\,. \tag{3.2}$$

To verify the q-Serre relations, let us equip V with a natural P-grading (with P as above) via $\deg(v_j) = \varepsilon_j$ for all $1 \leq j \leq N$, where we follow (2.7) for $s < j \leq N$. Evoking the P-grading (2.21) of $U_q(\mathfrak{osp}(V))$, we see that the assignment (3.1) preserves this P-grading:

$$\deg(\varrho(x) v_j) = \deg(x) + \deg(v_j) \qquad \text{for any} \quad x \in \{e_i, f_i, q^{\pm h_i/2}\}_{i=1}^{s}\,,\; 1 \leq j \leq N\,. \tag{3.3}$$

Referring to the explicit form of all q-Serre relations, left-hand sides of which are presented in [42, Definition 4.2.1], one can easily see that all of them, besides (v), are homogeneous whose degrees are not in the set $\{\varepsilon_i - \varepsilon_j \,|\, 1 \le i, j \le N\}$. Hence, they act trivially on the superspace V.

It remains to verify only the cubic q-Serre relation (cf. notation (2.22))

$$[\![e_s, [\![e_{s-1}, e_{s-2}]\!]]\!] - [\![e_{s-1}, [\![e_s, e_{s-2}]\!]]\!] = 0 \tag{3.4}$$

arising from [42, Definition 4.2.1(v)], which occurs only when $\gamma_V = (*, \ldots, *, \bar{1}, \bar{0})$ and $N \ge 6$ is even. Evoking the above P-grading, we note that the left-hand side of (3.4) has degree $\alpha_{s-2} + \alpha_{s-1} + \alpha_s = \varepsilon_{s-2} + \varepsilon_{s-1}$, and hence acts trivially on v_j for $j \notin \{(s-1)', (s-2)'\}$. It is straightforward to check that it also acts trivially on the basis vectors $v_{(s-1)'}$ and $v_{(s-2)'}$. □

3.2. Tensor square of the first fundamental representation.

Recall that a vector w in a $U_q(\mathfrak{osp}(V))$-module W is called *highest weight vector of weight* μ if

$$e_i(w) = 0 \qquad \text{and} \qquad q^{h_i/2}(w) = q^{\mu(\mathsf{h}_i/2)} w \qquad \text{for all} \quad 1 \le i \le s\,.$$

Proposition 3.2. *(a) The following are highest weight vectors in $U_q(\mathfrak{osp}(V))$-module $V \otimes V$:*

$$\begin{aligned} w_1 &= v_1 \otimes v_1\,, \\ w_2 &= v_1 \otimes v_2 - (-1)^{\bar{1}(\bar{1}+\bar{2})} q^{(-1)^{\bar{1}}} \cdot v_2 \otimes v_1\,, \\ w_3 &= \sum_{i=1}^{N} c_i \cdot v_i \otimes v_{i'}\,, \end{aligned} \tag{3.5}$$

where the coefficients c_i's in w_3 are determined by $c_1 = 1$ and the following relations:

$$\begin{aligned} c_{a+1} &= q^{(-1)^{\bar{a}}/2 + (-1)^{\overline{a+1}}/2} \cdot \vartheta_a \vartheta_{a+1} \cdot c_a \qquad \text{for} \quad 1 \le a < s\,, \\ c_{a'} &= (-1)^{\bar{a} + \overline{a+1}} \cdot q^{(-1)^{\bar{a}}/2 + (-1)^{\overline{a+1}}/2} \cdot \vartheta_a \vartheta_{a+1} \cdot c_{(a+1)'} \qquad \text{for} \quad 1 \le a < s\,, \end{aligned} \tag{3.6}$$

as well as one of the following

$$c_{s+1} = q^{(-1)^{\bar{s}}/2} \cdot \vartheta_s \vartheta_{s+1} \cdot c_s\,, \quad c_{s+2} = (-1)^{\bar{s} + \overline{s+1}} \cdot q^{(-1)^{\bar{s}}/2} \cdot \vartheta_s \vartheta_{s+1} \cdot c_{s+1} \qquad \text{for odd } m\,, \tag{3.7}$$

$$c_{s+1} = q^{(-1)^{\bar{s}}/2 + (-1)^{\overline{s-1}}/2} \cdot \vartheta_{s-1} \vartheta_{s+1} \cdot c_{s-1} \qquad \text{for even } m \text{ and } \bar{s} = \bar{0}\,, \tag{3.8}$$

$$c_{s+1} = -q^{(-1)^{\bar{s}} \cdot 2} \cdot c_s = -q^{-2} \cdot c_s \qquad \text{for even } m \text{ and } \bar{s} = \bar{1}\,. \tag{3.9}$$

(b) For $n \neq m$, the $U_q(\mathfrak{osp}(V))$-module $V \otimes V$ is generated by these vectors w_1, w_2, w_3 of (3.5).

(c) For any n and m, the $U_q(\mathfrak{osp}(V))$-module $V \otimes V$ is generated by the vectors w_1, w_2, $\widetilde{w}_3 = v_1 \otimes v_{1'}$, as well as by the vectors w_1, w_2, $\hat{w}_3 = v_{1'} \otimes v_1$.

Proof. (a) Let us show that the vectors w_1, w_2, w_3 are indeed highest weight vectors for the action $\varrho^{\otimes 2}$ of $U_q(\mathfrak{osp}(V))$ on $V \otimes V$. First, we note that they are eigenvectors with respect to $\{q^{h_i/2}\}_{i=1}^s$:

$$\varrho^{\otimes 2}(q^{h_i/2}) w_1 = q^{2\varepsilon_1(\mathsf{h}_i/2)} w_1\,, \quad \varrho^{\otimes 2}(q^{h_i/2}) w_2 = q^{(\varepsilon_1 + \varepsilon_2)(\mathsf{h}_i/2)} w_2\,, \quad \varrho^{\otimes 2}(q^{h_i/2}) w_3 = w_3\,.$$

It remains to verify that w_1, w_2, w_3 are annihilated by all $\varrho^{\otimes 2}(e_i)$. The equality $\varrho^{\otimes 2}(e_i)(w_1) = 0$ follows from $\varrho(e_i)(v_1) = 0$. Likewise, $\varrho^{\otimes 2}(e_i)(w_2) = 0$ for $i > 1$ follows from $\varrho(e_i)v_1 = \varrho(e_i)v_2 = 0$. Meanwhile, combining $\varrho(e_1)v_2 = v_1$, $\varrho(e_1)v_1 = 0$, $\varrho(q^{h_1/2})v_1 = q^{(-1)^{\bar{1}}/2} v_1$, and (2.23), we also get:

$$\begin{aligned} \varrho^{\otimes 2}(e_1) w_2 &= (\varrho(q^{h_1/2}) \otimes \varrho(e_1))(v_1 \otimes v_2) - (-1)^{\bar{1}(\bar{1}+\bar{2})} q^{(-1)^{\bar{1}}} (\varrho(e_1) \otimes \varrho(q^{-h_1/2}))(v_2 \otimes v_1) \\ &= \Big((-1)^{(\bar{1}+\bar{2})\bar{1}} \cdot q^{(-1)^{\bar{1}}/2} - (-1)^{\bar{1}(\bar{1}+\bar{2})} q^{(-1)^{\bar{1}}} \cdot q^{-(-1)^{\bar{1}}/2} \Big) \cdot v_1 \otimes v_1 = 0\,, \end{aligned}$$

where the sign $(-1)^{\bar{1}(\bar{1}+\bar{2})}$ in the beginning of the second line is due to the conventions (2.4).

It remains to verify $\varrho^{\otimes 2}(e_a)w_3 = 0$ for all a. First, let us consider the case $1 \le a < s$. Then:

$$\begin{aligned}
\varrho^{\otimes 2}(e_a)w_3 &= (\varrho(e_a) \otimes \varrho(q^{-h_a/2}))(c_{a+1} \cdot v_{a+1} \otimes v_{(a+1)'} + c_{a'} \cdot v_{a'} \otimes v_a) \\
&\quad + (\varrho(q^{h_a/2}) \otimes \varrho(e_a))(c_{(a+1)'} \cdot v_{(a+1)'} \otimes v_{a+1} + c_a \cdot v_a \otimes v_{a'}) \\
&= c_{a+1} \cdot q^{-(-1)^{\overline{a+1}}/2} \cdot v_a \otimes v_{(a+1)'} - c_{a'} \cdot q^{-(-1)^{\overline{a}}/2} \cdot (-1)^{\overline{a}(\overline{a}+\overline{a+1})} \vartheta_a \vartheta_{a+1} \cdot v_{(a+1)'} \otimes v_a \\
&\quad + (-1)^{\overline{a+1}(\overline{a}+\overline{a+1})} \cdot c_{(a+1)'} \cdot q^{(-1)^{\overline{a+1}}/2} \cdot v_{(a+1)'} \otimes v_a \\
&\quad - (-1)^{\overline{a}(\overline{a}+\overline{a+1})} \cdot c_a \cdot q^{(-1)^{\overline{a}}/2} \cdot (-1)^{\overline{a}(\overline{a}+\overline{a+1})} \vartheta_a \vartheta_{a+1} \cdot v_a \otimes v_{(a+1)'} \,,
\end{aligned}$$

with the first signs in the last two lines due to the conventions (2.4). Evoking both defining relations (3.6), we see that the right-hand side above vanishes, and so $\varrho^{\otimes 2}(e_a)w_3 = 0$ for $1 \le a < s$.

To evaluate $\varrho^{\otimes 2}(e_s)w_3$, we have to consider three separate cases:

- Case 1: m is odd. In this case, we likewise have:

$$\begin{aligned}
\varrho^{\otimes 2}(e_s)w_3 &= (\varrho(e_s) \otimes \varrho(q^{-h_s/2}))\big(c_{s+1} \cdot v_{s+1} \otimes v_{s+1} + c_{s+2} \cdot v_{s+2} \otimes v_s\big) \\
&\quad + (\varrho(q^{h_s/2}) \otimes \varrho(e_s))(c_{s+1} \cdot v_{s+1} \otimes v_{s+1} + c_s \cdot v_s \otimes v_{s+2}) \\
&= c_{s+1} \cdot v_s \otimes v_{s+1} - c_{s+2} \cdot q^{-(-1)^{\overline{s}}/2} \cdot (-1)^{\overline{s}(\overline{s}+\overline{s+1})} \vartheta_s \vartheta_{s+1} \cdot v_{s+1} \otimes v_s \\
&\quad + (-1)^{\overline{s+1}(\overline{s}+\overline{s+1})} \cdot c_{s+1} \cdot v_{s+1} \otimes v_s \\
&\quad - (-1)^{\overline{s}(\overline{s}+\overline{s+1})} \cdot c_s \cdot q^{(-1)^{\overline{s}}/2} \cdot (-1)^{\overline{s}(\overline{s}+\overline{s+1})} \vartheta_s \vartheta_{s+1} \cdot v_s \otimes v_{s+1} \,,
\end{aligned}$$

  with the first signs in the last two lines due to the conventions (2.4). Evoking both defining relations (3.7), we see that the right-hand side above vanishes, and so $\varrho^{\otimes 2}(e_s)w_3 = 0$.
- Case 2: m is even and $\overline{s} = \overline{0}$. In this case, we obtain:

$$\begin{aligned}
\varrho^{\otimes 2}(e_s)w_3 &= (\varrho(e_s) \otimes \varrho(q^{-h_s/2}))(c_{s+1} \cdot v_{s+1} \otimes v_s + c_{s+2} \cdot v_{s+2} \otimes v_{s-1}) \\
&\quad + (\varrho(q^{h_s/2}) \otimes \varrho(e_s))(c_s \cdot v_s \otimes v_{s+1} + c_{s-1} \cdot v_{s-1} \otimes v_{s+2}) \\
&= c_{s+1} \cdot q^{-(-1)^{\overline{s}}/2} \cdot v_{s-1} \otimes v_s - c_{s+2} \cdot q^{-(-1)^{\overline{s-1}}/2} \cdot (-1)^{\overline{s-1}(\overline{s-1}+\overline{s})} \cdot \vartheta_{s-1} \vartheta_{s+1} \cdot v_s \otimes v_{s-1} \\
&\quad + (-1)^{\overline{s}(\overline{s-1}+\overline{s})} \cdot c_s \cdot q^{(-1)^{\overline{s}}/2} \cdot v_s \otimes v_{s-1} \\
&\quad - (-1)^{\overline{s-1}(\overline{s-1}+\overline{s})} \cdot c_{s-1} \cdot q^{(-1)^{\overline{s-1}}/2} \cdot (-1)^{\overline{s-1}(\overline{s-1}+\overline{s})} \cdot \vartheta_{s-1} \vartheta_{s+1} \cdot v_{s-1} \otimes v_s \,.
\end{aligned}$$

  Evoking (3.6, 3.8), we see that the right-hand side above vanishes, and so $\varrho^{\otimes 2}(e_s)w_3 = 0$.
- Case 3: m is even and $\overline{s} = \overline{1}$. In this case, we again get the desired vanishing by (3.9):

$$\begin{aligned}
\varrho^{\otimes 2}(e_s)w_3 &= (\varrho(q^{h_s/2}) \otimes \varrho(e_s))(c_s \cdot v_s \otimes v_{s+1}) + (\varrho(e_s) \otimes \varrho(q^{-h_s/2}))(c_{s+1} \cdot v_{s+1} \otimes v_s) \\
&= c_s \cdot q^{-1} \cdot v_s \otimes v_s + c_{s+1} \cdot q \cdot v_s \otimes v_s = 0 \,.
\end{aligned}$$

(b) Part (b) is established in Propositions B.2(b,d), B.3(b,e), B.4(b,e) from Appendix B.
(c) Part (c) is established in Propositions B.2(b–d), B.3(c–e), B.4(c–e) from Appendix B. □

Remark 3.3. Reversing the above calculations, we see that the only highest weight vectors in $U_q(\mathfrak{osp}(V))$-module $V \otimes V$ of weights $2\varepsilon_1, \varepsilon_1 + \varepsilon_2, 0$ are multiples of w_1, w_2, w_3, respectively.

4. R-Matrices

4.1. Universal construction.

We first recall the general construction of a $U_q(\mathfrak{osp}(V))$-module isomorphism $W_1 \otimes W_2 \to W_2 \otimes W_1$ arising through the universal R-matrix, following the treatment of [19, §7] in the non-super setup. First of all, for any superspaces A and B, we define the *graded permutation* operator $\tau = \tau_{A,B}$ via

$$\tau\colon A \otimes B \to B \otimes A \,, \qquad x \otimes y \mapsto (-1)^{|x||y|} y \otimes x \quad \text{for homogeneous } x \in A, y \in B \,. \tag{4.1}$$

We note that if both A, B are superalgebras, then τ of (4.1) is a superalgebra homomorphism. In particular, we can now define opposite coproducts Δ^{op} and $\Delta^{J,\mathrm{op}}$, which are superalgebra homomorphisms, via:

$$\Delta^{\mathrm{op}}(x) = \tau \circ \Delta(x)\,, \qquad \Delta^{J,\mathrm{op}}(x) = \tau \circ \Delta^J(x) \qquad \forall\, x \in U_q(\mathfrak{osp}(V))\,, \tag{4.2}$$

cf. (2.23, 2.24). Henceforth, we shall only work with *type* 1 P-graded $U_q(\mathfrak{osp}(V))$-modules W, i.e.

$$W = \bigoplus_{\mu\in P} W[\mu] \qquad \text{with} \qquad W[\mu] = \Big\{ w \in W \,|\, q^{h_i/2}(w) = q^{(\alpha_i/2,\mu)}w \quad \forall\, 1 \le i \le s \Big\}\,.$$

For any such $U_q(\mathfrak{osp}(V))$-modules W_1, W_2, we define a linear map $\widetilde{f}\colon W_1 \otimes W_2 \to W_1 \otimes W_2$ via

$$\widetilde{f}(w_1 \otimes w_2) = q^{-(\nu,\mu)} \cdot w_1 \otimes w_2 \qquad \text{for any} \quad w_1 \in W_1[\nu]\,,\, w_2 \in W_2[\mu]\,.$$

Let $U_q^+(\mathfrak{osp}(V))$ and $U_q^-(\mathfrak{osp}(V))$ be the subalgebras of $U_q(\mathfrak{osp}(V))$ generated by $\{e_i\}_{i=1}^s$ and $\{f_i\}_{i=1}^s$, respectively. We also define $U_q^{\geq}(\mathfrak{osp}(V))$ and $U_q^{\leq}(\mathfrak{osp}(V))$ as subalgebras of $U_q(\mathfrak{osp}(V))$ generated by $\{e_i, q^{\pm h_i/2}\}_{i=1}^s$ and $\{f_i, q^{\pm h_i/2}\}_{i=1}^s$, respectively. The basic property of all quantum (super)groups is that they are Drinfeld doubles with respect to a bialgebra pairing, which presently relies on the following result (generalizing [19, Propositions 6.12, 6.18] to the super setup):

Proposition 4.1. *There exists a unique bilinear bialgebra pairing*

$$(\cdot,\cdot)_J \colon U_q^{\leq}(\mathfrak{osp}(V)) \times U_q^{\geq}(\mathfrak{osp}(V)) \longrightarrow \mathbb{C}(q^{\pm 1/4}) \tag{4.3}$$

which satisfies the following structural properties

$$(yy', x)_J = (y \otimes y', \Delta^J(x))_J\,, \qquad (y, xx')_J = (\Delta^{J,\mathrm{op}}(y), x \otimes x')_J \tag{4.4}$$

for any $x, x' \in U_q^{\geq}(\mathfrak{osp}(V))$, $y, y' \in U_q^{\leq}(\mathfrak{osp}(V))$ with the pairing in the right-hand sides defined by

$$(x \otimes x', y \otimes y')_J = (-1)^{|x'||y|}(x,y)_J(x',y')_J \qquad \text{for any } \mathbb{Z}_2\text{−homogeneous } x, x', y, y'\,,$$

and is given on the generators by (for any $1 \le i, j \le s$):

$$\begin{gathered} (f_i, q^{\pm h_j/2})_J = 0\,, \quad (q^{\pm h_j/2}, e_i)_J = 0\,, \\ (f_i, e_j)_J = \frac{\delta_{ij}(-1)^{|f_i||e_j|}}{q^{-1}-q}\,, \quad (q^{h_i/2}, q^{h_j/2})_J = q^{-a_{ij}/4}\,. \end{gathered} \tag{4.5}$$

The pairing (4.3) is non-degenerate.

Remark 4.2. The non-degeneracy of (4.3) easily follows from the non-degeneracy of its restriction to $U_q^-(\mathfrak{osp}(V)) \times U_q^+(\mathfrak{osp}(V))$. We note that the latter is a highly non-trivial result, and is actually the main result of [42], where the q-Serre relations were precisely derived to satisfy this property.

Recall the P-grading on $U_q(\mathfrak{osp}(V))$, hence on all the subalgebras above, cf. (2.21). We note that the graded components $U_q^-(\mathfrak{osp}(V))_\nu$ and $U_q^+(\mathfrak{osp}(V))_\mu$ are all finite-dimensional. Furthermore, in accordance with (4.4, 4.5), the pairing (4.3) is graded:

$$(y,x)_J = 0 \qquad \text{for} \quad y \in U_q^-(\mathfrak{osp}(V))_\nu\,, x \in U_q^+(\mathfrak{osp}(V))_\mu \quad \text{with} \quad \mu + \nu \neq 0\,.$$

Let us pick dual bases $\{x_i^\mu\}$, $\{y_i^\mu\}$ of $U_q^+(\mathfrak{osp}(V))_\mu$, $U_q^-(\mathfrak{osp}(V))_{-\mu}$ with respect to (4.3), and set

$$\Theta = 1 + \sum_{\mu>0} \Theta_\mu \qquad \text{with} \qquad \Theta_\mu = \sum_i x_i^\mu \otimes y_i^\mu\,. \tag{4.6}$$

The following is proved completely analogously to [19, Theorem 7.3]:

Proposition 4.3. *For any $U_q(\mathfrak{osp}(V))$-modules W_1 and W_2 as above, the map*

$$\hat{R}_{W_1,W_2} = \tau \circ \widetilde{f}^{1/2} \circ \Theta \circ \widetilde{f}^{1/2} \colon W_1 \otimes W_2 \longrightarrow W_2 \otimes W_1 \tag{4.7}$$

is an isomorphism of $U_q(\mathfrak{osp}(V))$-modules.

Remark 4.4. Completely analogously to [19, Theorem 7.3], one verifies that

$$\Delta^{J,\mathrm{op}}(x)\Theta\widetilde{f} = \Theta\widetilde{f}\Delta^J(x) \qquad \forall\, x \in U_q(\mathfrak{osp}(V))\,.$$

Comparing the defining formulas of Δ^J from (2.24) to those of Δ from (2.23), it is easy to see that

$$\Delta^J(x) = \widetilde{f}^{-1/2}\Delta(x)\widetilde{f}^{1/2}\,, \qquad \Delta^{J,\mathrm{op}}(x) = \widetilde{f}^{-1/2}\Delta^{\mathrm{op}}(x)\widetilde{f}^{1/2} \qquad \forall\, x \in U_q(\mathfrak{osp}(V))\,.$$

Combining the above two equalities, we obtain:

$$\Delta^{\mathrm{op}}(x)\widetilde{f}^{1/2}\Theta\widetilde{f}^{1/2} = \widetilde{f}^{1/2}\Theta\widetilde{f}^{1/2}\Delta(x) \qquad \forall\, x \in U_q(\mathfrak{osp}(V))\,,$$

which together with $\tau(x(w_1 \otimes w_2)) = \Delta^{\mathrm{op}}(x)(\tau(w_1 \otimes w_2))$ for all $x \in U_q(\mathfrak{osp}(V)), w_1 \in W_1, w_2 \in W_2$ establishes Proposition 4.3.

Let $R_{W_1,W_2} = \tau \circ \hat{R}_{W_1,W_2} = \widetilde{f}^{1/2} \circ \Theta \circ \widetilde{f}^{1/2} \colon W_1 \otimes W_2 \to W_1 \otimes W_2$. Given $U_q(\mathfrak{osp}(V))$-modules W_1, W_2, W_3 as above, we define the following three endomorphisms of $W_1 \otimes W_2 \otimes W_3$:

$$R_{12} = R_{W_1,W_2} \otimes \mathrm{Id}_{W_3}\,, \quad R_{23} = \mathrm{Id}_{W_1} \otimes R_{W_2,W_3}\,, \quad R_{13} = (\mathrm{Id} \otimes \tau)R_{12}(\mathrm{Id} \otimes \tau)\,.$$

We likewise define linear operators $\hat{R}_{12}, \hat{R}_{23}, \hat{R}_{13}$. Then, analogously to [19], we have:

$$\begin{split} R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \colon W_1 \otimes W_2 \otimes W_3 \to W_1 \otimes W_2 \otimes W_3\,, \\ \hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \colon W_1 \otimes W_2 \otimes W_3 \to W_3 \otimes W_2 \otimes W_1\,. \end{split} \tag{4.8}$$

In particular, we obtain a whole family of solutions of the quantum Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \in \mathrm{End}(W \otimes W \otimes W)\,. \tag{4.9}$$

Corollary 4.5. *For any $U_q(\mathfrak{osp}(V))$-module W as above, R_{WW} satisfies (4.9).*

4.2. Explicit R-matrices.

Let

$$\rho = \frac{1}{2}\sum_{\alpha \in \Phi^+_{\bar 0}} \alpha - \frac{1}{2}\sum_{\alpha \in \Phi^+_{\bar 1}} \alpha \tag{4.10}$$

be the *Weyl vector* of the root system Φ, which is the graded half-sum of all positive roots. Due to [6, Proposition 1.33], we have

$$(\rho, \alpha_i) = \tfrac{1}{2}(\alpha_i, \alpha_i) \qquad \forall\, 1 \le i \le s\,. \tag{4.11}$$

In accordance with (4.1), we consider the *graded permutation operator* $\tau_{VV} \colon V \otimes V \to V \otimes V$ defined via $\tau(v_i \otimes v_j) = (-1)^{\bar{i}\bar{j}}\, v_j \otimes v_i$ for any $1 \le i, j \le N$, which is explicitly given by

$$\tau_{VV} = \sum_{i,j=1}^{N} (-1)^{\bar{j}} E_{ij} \otimes E_{ji}\,.$$

We are now ready to state our first main result:

Theorem 4.6. *The $U_q(\mathfrak{osp}(V))$-module isomorphism $\hat{R}_{VV} \colon V \otimes V \xrightarrow{\sim} V \otimes V$ from Proposition 4.3 and its inverse $\hat{R}^{-1}_{VV}$ for the $U_q(\mathfrak{osp}(V))$-module V constructed in Proposition 3.1 are given by*

$$\hat{R}_{VV} = \tau_{VV} \circ R_0 \qquad \text{and} \qquad \hat{R}^{-1}_{VV} = \tau_{VV} \circ R_\infty \tag{4.12}$$

with the following explicit operators

$$\begin{split} R_0 = \mathrm{I} + (q^{-1/2} - q^{1/2}) \sum_{i=1}^{N} (-1)^{\bar{i}} E_{ii} \otimes \Big(q^{-(\varepsilon_i,\varepsilon_i)/2} E_{ii} - q^{(\varepsilon_i,\varepsilon_i)/2} E_{i'i'}\Big) \\ + (q^{-1} - q) \sum_{i<j} (-1)^{\bar{j}} E_{ij} \otimes \Big(E_{ji} - (-1)^{\bar{j}(\bar{i}+\bar{j})} \vartheta_i \vartheta_j q^{(\rho,\varepsilon_i - \varepsilon_j)} E_{i'j'}\Big) \end{split} \tag{4.13}$$

and

$$\begin{split} R_\infty = \mathrm{I} + (q^{1/2} - q^{-1/2}) \sum_{i=1}^{N} (-1)^{\bar{i}} E_{ii} \otimes \Big(q^{(\varepsilon_i,\varepsilon_i)/2} E_{ii} - q^{-(\varepsilon_i,\varepsilon_i)/2} E_{i'i'}\Big) \\ + (q - q^{-1}) \sum_{i>j} (-1)^{\bar{j}} E_{ij} \otimes \Big(E_{ji} - (-1)^{\bar{j}(\bar{i}+\bar{j})} \vartheta_i \vartheta_j q^{(\rho,\varepsilon_i - \varepsilon_j)} E_{i'j'}\Big)\,. \end{split} \tag{4.14}$$

Remark 4.7. Evoking Remark 4.4, we can also recover the formula for the operator R defined via $R = \Theta \circ \widetilde{f}$ and its inverse R^{-1}, corresponding to the usual coproduct Δ^J, as follows:

$$R = \widetilde{f}^{-1/2} \circ R_0 \circ \widetilde{f}^{1/2} = \mathrm{I} + (q^{-1/2} - q^{1/2}) \sum_{i=1}^{N} (-1)^{\bar{i}} E_{ii} \otimes \Big(q^{-(\varepsilon_i,\varepsilon_i)/2} E_{ii} - q^{(\varepsilon_i,\varepsilon_i)/2} E_{i'i'} \Big)$$
$$+ (q^{-1} - q) \sum_{i<j} (-1)^{\bar{j}} E_{ij} \otimes \Big(E_{ji} - (-1)^{\bar{j}(\bar{i}+\bar{j})} \vartheta_i \vartheta_j q^{(\rho,\varepsilon_i-\varepsilon_j)} q^{-(\varepsilon_i,\varepsilon_i)/2} q^{(\varepsilon_j,\varepsilon_j)/2} E_{i'j'} \Big) ,$$

$$R^{-1} = \tau \circ \widetilde{f}^{-1/2} \circ R_\infty \circ \widetilde{f}^{1/2} \circ \tau = \mathrm{I} + (q^{1/2} - q^{-1/2}) \sum_{i=1}^{N} (-1)^{\bar{i}} E_{ii} \otimes \Big(q^{(\varepsilon_i,\varepsilon_i)/2} E_{ii} - q^{-(\varepsilon_i,\varepsilon_i)/2} E_{i'i'} \Big)$$
$$+ (q - q^{-1}) \sum_{i<j} (-1)^{\bar{j}} E_{ij} \otimes \Big(E_{ji} - (-1)^{\bar{j}(\bar{i}+\bar{j})} \vartheta_i \vartheta_j q^{-(\rho,\varepsilon_i-\varepsilon_j)} q^{-(\varepsilon_i,\varepsilon_i)/2} q^{(\varepsilon_j,\varepsilon_j)/2} E_{i'j'} \Big) .$$

To prove Theorem 4.6, we first establish some properties of R_0 and R_∞ defined in (4.13) and (4.14). By abuse of notation, we shall often denote $(\varrho \otimes \varrho)(\Delta(x))$ simply by $\Delta(x)$[1] or $\varrho^{\otimes 2}(x)$. We start with a straightforward result, the proof of which is postponed till the end of this Section:

Proposition 4.8. *For any element $x \in U_q(\mathfrak{osp}(V))$, the following equalities hold (cf. (4.2)):*

$$R_0 \Delta(x) = \Delta^{\mathrm{op}}(x) R_0 \qquad \text{and} \qquad R_\infty \Delta(x) = \Delta^{\mathrm{op}}(x) R_\infty . \tag{4.15}$$

Next, we evaluate the action of $\tau_{VV} R_0, \tau_{VV} R_\infty$ on the generating vectors from Proposition 3.2.

Proposition 4.9. *(a) The highest weight vectors w_1, w_2, w_3 are eigenvectors of $\tau_{VV} \circ R_0$*

$$\tau_{VV} R_0 \colon \quad w_1 \mapsto \mu_1^0 \cdot w_1 , \quad w_2 \mapsto \mu_2^0 \cdot w_2 , \quad w_3 \mapsto \mu_3^0 \cdot w_3$$

with the eigenvalues given explicitly by:

$$\mu_1^0 = (-1)^{\bar{1}} q^{-(-1)^{\bar{1}}} , \qquad \mu_2^0 = -(-1)^{\bar{1}} q^{(-1)^{\bar{1}}} , \qquad \mu_3^0 = q^{m-n-1} . \tag{4.16}$$

(b) The action of $\tau_{VV} \circ R_0$ on $\widetilde{w}_3 = v_1 \otimes v_{1'}$ is given by $\tau_{VV} R_0(\widetilde{w}_3) = (-1)^{\bar{1}} q^{(-1)^{\bar{1}}} \cdot \hat{w}_3$.

Proof. (a) We evaluate each eigenvalue separately.

- μ_1^0. For $w_1 = v_1 \otimes v_1$, the direct computation shows that
$$R_0(w_1) = v_1 \otimes v_1 + (q^{-1/2} - q^{1/2})(-1)^{\bar{1}} q^{-(-1)^{\bar{1}}/2} v_1 \otimes v_1 = q^{-(-1)^{\bar{1}}} v_1 \otimes v_1 .$$
The above equality implies the desired formula
$$\tau_{VV} R_0(w_1) = (-1)^{\bar{1}} q^{-(-1)^{\bar{1}}} v_1 \otimes v_1 .$$
- μ_2^0. For $w_2 = v_1 \otimes v_2 - (-1)^{\bar{1}(\bar{1}+\bar{2})} q^{(-1)^{\bar{1}}} \cdot v_2 \otimes v_1$, the direct computation shows that
$$R_0(w_2) = v_1 \otimes v_2 - (-1)^{\bar{1}(\bar{1}+\bar{2})} q^{(-1)^{\bar{1}}} \Big(v_2 \otimes v_1 + (q^{-1} - q)(-1)^{\bar{2}} (E_{12} \otimes E_{21})(v_2 \otimes v_1) \Big)$$
$$= v_1 \otimes v_2 - (-1)^{\bar{1}(\bar{1}+\bar{2})} q^{(-1)^{\bar{1}}} \cdot v_2 \otimes v_1 + (q - q^{-1})(-1)^{\bar{1}} q^{(-1)^{\bar{1}}} \cdot v_1 \otimes v_2$$
$$= q^{(-1)^{\bar{1}} \cdot 2} \cdot v_1 \otimes v_2 - (-1)^{\bar{1}(\bar{1}+\bar{2})} q^{(-1)^{\bar{1}}} \cdot v_2 \otimes v_1 .$$
The above equality implies the desired formula
$$\tau_{VV} R_0(w_2) = -(-1)^{\bar{1}} q^{(-1)^{\bar{1}}} \cdot v_1 \otimes v_2 + (-1)^{\bar{1}\cdot\bar{2}} q^{(-1)^{\bar{1}} \cdot 2} \cdot v_2 \otimes v_1 = -(-1)^{\bar{1}} q^{(-1)^{\bar{1}}} \cdot w_2 .$$
- μ_3^0. For $w_3 = \sum_{i=1}^N c_i \cdot v_i \otimes v_{i'}$, we note that $\tau_{VV} R_0(w_3)$ is also a linear combination of $\{v_i \otimes v_{i'}\}_{i=1}^N$. Furthermore, the intertwining property of Proposition 4.8 shows that
$$\Delta(e_a)(\tau_{VV} R_0(w_3)) = \tau_{VV} R_0(\Delta(e_a) w_3) = \tau_{VV} R_0(\varrho^{\otimes 2}(e_a) w_3) = 0 \qquad \forall\, 1 \le a \le s .$$

[1] We note that a similar convention was already used in our Remark 4.4 above.

Combining this with Remark 3.3, we see that this forces $\tau_{VV}R_0(w_3)$ to be a scalar multiple of w_3. Therefore, to find μ_3^0 it is enough to compare the coefficients of the term $v_1\otimes v_{1'}$. To this end, we note that

$$\begin{aligned} R_0(w_3) = &\sum_{1\le i\le N} c_i\cdot v_i\otimes v_{i'} - (q^{-1/2}-q^{1/2})\sum_{\substack{1\le i\le N\\ i\ne i'}} (-1)^{\bar{i}} q^{(\varepsilon_i,\varepsilon_i)/2} c_i\cdot (E_{ii}\otimes E_{i'i'})(v_i\otimes v_{i'}) \\ &+ (q^{-1}-q)\sum_{i'<i}(-1)^{\bar{i}} c_i\cdot (E_{i'i}\otimes E_{ii'})(v_i\otimes v_{i'}) \\ &- (q^{-1}-q)\sum_{j<i}(-1)^{\bar{i}\bar{j}} q^{(\rho,\varepsilon_j-\varepsilon_i)}\vartheta_j\vartheta_i c_i\cdot (E_{ji}\otimes E_{j'i'})(v_i\otimes v_{i'}) \\ = &\sum_{1\le i\le N} c_i\cdot v_i\otimes v_{i'} - (q^{-1/2}-q^{1/2})\sum_{1\le i\le N}(-1)^{\bar{i}} q^{(\varepsilon_i,\varepsilon_i)/2}\cdot c_i\cdot v_i\otimes v_{i'} \\ &+ (q^{-1}-q)\sum_{i'\le s}(-1)^{\bar{i}}\cdot c_i\cdot v_{i'}\otimes v_i - (q^{-1}-q)\sum_{j<i}(-1)^{\bar{i}} q^{(\rho,\varepsilon_j-\varepsilon_i)}\vartheta_j\vartheta_i\cdot c_i\cdot v_j\otimes v_{j'}\,. \end{aligned}$$

In particular, the coefficient of $v_{1'}\otimes v_1$ in $R_0(w_3)$ equals

$$\left(1-(q^{-1/2}-q^{1/2})(-1)^{\bar{1}} q^{(-1)^{\bar{1}}/2}\right) c_{1'} = q^{(-1)^{\bar{1}}} c_{1'}\,,$$

and therefore the coefficient of $v_1\otimes v_{1'}$ in $\tau_{VV}R_0(w_3)$ is $(-1)^{\bar{1}} q^{(-1)^{\bar{1}}} c_{1'}$. The latter implies $\mu_3^0 = (-1)^{\bar{1}} q^{(-1)^{\bar{1}}} c_{1'}/c_1$. As $c_1=1$, to deduce the last formula of (4.16) it suffices to show:

$$c_{1'} = (-1)^{\bar{1}} q^{-(-1)^{\bar{1}}} q^{m-n-1}\,. \tag{4.17}$$

The proof of (4.17) is straightforward and is based on (3.6)–(3.9). Indeed, multiplying

$$c_{a'}/c_a = (-1)^{\bar{a}+\overline{a+1}} q^{(-1)^{\bar{a}}+(-1)^{\overline{a+1}}}\cdot c_{(a+1)'}/c_{(a+1)} \qquad \text{for} \quad 1\le a<s\,,$$

due to (3.6), we find

$$c_{1'}/c_1 = (-1)^{\bar{1}+\bar{s}} q^{(-1)^{\bar{1}}+(-1)^{\bar{2}}\cdot 2+\ldots+(-1)^{\overline{s-1}}\cdot 2+(-1)^{\bar{s}}}\cdot c_{s'}/c_s\,.$$

Combining this with $\sum_{i=1}^N (-1)^{\bar{i}} = m-n$ and the explicit formula

$$c_{s'}/c_s = \begin{cases} (-1)^{\bar{s}+\overline{s+1}} q^{(-1)^{\bar{s}}} & \text{if } m \text{ is odd} \\ (-1)^{\bar{s}} & \text{if } m \text{ is even and } \bar{s}=\bar{0} \\ (-1)^{\bar{s}} q^{(-1)^{\bar{s}}\cdot 2} & \text{if } m \text{ is even and } \bar{s}=\bar{1} \end{cases},$$

due to (3.6)–(3.9), we obtain the uniform formula for $c_{1'}/c_1 = c_{1'}$ from (4.17).

(b) For $\widetilde{w}_3 = v_1\otimes v_{1'}$, the direct computation shows that

$$R_0(\widetilde{w}_3) = v_1\otimes v_{1'} - (q^{-1/2}-q^{1/2})(-1)^{\bar{1}} q^{(-1)^{\bar{1}}/2} v_1\otimes v_{1'} = q^{(-1)^{\bar{1}}} v_1\otimes v_{1'}\,.$$

The above equality implies the desired formula

$$\tau_{VV}R_0(\widetilde{w}_3) = (-1)^{\bar{1}} q^{(-1)^{\bar{1}}} v_{1'}\otimes v_1 = (-1)^{\bar{1}} q^{(-1)^{\bar{1}}}\hat{w}_3\,.$$

This completes the proof of the proposition. □

By completely analogous computations, we get the following result:

Proposition 4.10. *(a) The highest weight vectors w_1, w_2, w_3 are eigenvectors of $\tau_{VV}\circ R_\infty$*

$$\tau_{VV}R_\infty\colon\quad w_1\mapsto \mu_1^\infty\cdot w_1\,,\quad w_2\mapsto\mu_2^\infty\cdot w_2\,,\quad w_3\mapsto\mu_3^\infty\cdot w_3$$

with the eigenvalues given explicitly by:

$$\mu_1^\infty = (-1)^{\bar{1}} q^{(-1)^{\bar{1}}} = 1/\mu_1^0\,,\quad \mu_2^\infty = -(-1)^{\bar{1}} q^{-(-1)^{\bar{1}}} = 1/\mu_2^0\,,\quad \mu_3^\infty = q^{-m+n+1} = 1/\mu_3^0\,.$$

(b) The action of $\tau_{VV}\circ R_\infty$ on $\hat{w}_3 = v_{1'}\otimes v_1$ is given by $\tau_{VV}R_\infty(\hat{w}_3) = (-1)^{\bar{1}} q^{-(-1)^{\bar{1}}}\cdot\widetilde{w}_3$.

Let us now evaluate the action of $\hat{R}_{VV}$ on w_1, w_2, w_3, and $\widetilde{w}_3$:

Proposition 4.11. *(a) The highest weight vectors w_1, w_2, w_3 are eigenvectors of $\hat{R}_{VV}$ from (4.7)*

$$\hat{R}_{VV}\colon \quad w_1 \mapsto \lambda_1 w_1\,, \quad w_2 \mapsto \lambda_2 w_2\,, \quad w_3 \mapsto \lambda_3 w_3$$

with the eigenvalues given explicitly by:

$$\lambda_1 = (-1)^{\bar{1}} q^{-(-1)^{\bar{1}}} = \mu_1^0\,, \qquad \lambda_2 = -(-1)^{\bar{1}} q^{(-1)^{\bar{1}}} = \mu_2^0\,, \qquad \lambda_3 = q^{m-n-1} = \mu_3^0\,. \tag{4.18}$$

(b) The action of $\hat{R}_{VV}$ on $\widetilde{w}_3 = v_1 \otimes v_{1'}$ is given by $\hat{R}_{VV}(\widetilde{w}_3) = (-1)^{\bar{1}} q^{(-1)^{\bar{1}}} \cdot \hat{w}_3$.

Proof. (a) The intertwining property of $\hat{R}_{VV}$ from Proposition 4.3 together with Remark 3.3 implies that all three vectors w_1, w_2, w_3 are indeed eigenvectors for $\hat{R}_{VV}$. We shall now evaluate each eigenvalue separately.

- λ_1. For $w_1 = v_1 \otimes v_1$, the direct computation shows that
$$\tau_{VV} \widetilde{f}(w_1) = q^{-(\varepsilon_1,\varepsilon_1)} \tau_{VV}(v_1 \otimes v_1) = (-1)^{\bar{1}} q^{-(-1)^{\bar{1}}} w_1\,,$$
which implies the desired formula for λ_1 (as $\Theta(v_1 \otimes v_1) = v_1 \otimes v_1$).
- λ_2. The eigenvalue λ_2 of the $\hat{R}_{VV}$-action on $w_2 = v_1 \otimes v_2 - (-1)^{\bar{1}(\bar{1}+\bar{2})} \cdot q^{(-1)^{\bar{1}}} \cdot v_2 \otimes v_1$ equals the coefficient of $v_1 \otimes v_2$ in $\hat{R}_{VV}(w_2)$. The latter appears only from applying $\tau_{VV}\widetilde{f}$ to the above multiple of $v_2 \otimes v_1$ (thus implying the desired formula for λ_2):

$$\tau_{VV} \widetilde{f}\big(- (-1)^{\bar{1}(\bar{1}+\bar{2})} \cdot q^{(-1)^{\bar{1}}} \cdot v_2 \otimes v_1 \big) = \\ - (-1)^{\bar{1}(\bar{1}+\bar{2})} \cdot q^{(-1)^{\bar{1}}} q^{-(\varepsilon_2,\varepsilon_1)} \cdot \tau_{VV}(v_2 \otimes v_1) = -(-1)^{\bar{1}} q^{(-1)^{\bar{1}}} v_1 \otimes v_2\,.$$

- λ_3. The eigenvalue λ_3 of the $\hat{R}_{VV}$-action on $w_3 = v_1 \otimes v_{1'} + \sum_{i=2}^{N} c_i \cdot v_i \otimes v_{i'}$ equals the coefficient of $v_1 \otimes v_{1'}$ in $\hat{R}_{VV}(w_3)$. The latter appears only from applying $\tau_{VV}\widetilde{f}$ to the above multiple of $v_{1'} \otimes v_1$ (thus implying the desired formula for λ_3):

$$\tau_{VV} \widetilde{f}(c_{1'} \cdot (v_{1'} \otimes v_1)) = \\ c_{1'} q^{-(\varepsilon_{1'},\varepsilon_1)} \tau_{VV}(v_{1'} \otimes v_1) = (-1)^{\bar{1}} q^{(-1)^{\bar{1}}} c_{1'} \cdot v_1 \otimes v_{1'} \overset{(4.17)}{=} q^{m-n-1} \cdot v_1 \otimes v_{1'}\,.$$

(b) For $\widetilde{w}_3 = v_1 \otimes v_{1'}$, the direct computation shows that

$$\tau_{VV} \widetilde{f}(\widetilde{w}_3) = q^{-(\varepsilon_1,\varepsilon_{1'})} \tau_{VV}(v_1 \otimes v_{1'}) = (-1)^{\bar{1}} q^{(-1)^{\bar{1}}} v_{1'} \otimes v_1\,,$$

which implies the desired formula as $\Theta(v_1 \otimes v_{1'}) = v_1 \otimes v_{1'} = \widetilde{w}_3$. □

Combining the Propositions above, we can now immediately derive our main result:

Proof of Theorem 4.6. Combining the intertwining property (4.15) with the equality

$$\Delta^{\mathrm{op}}(x) = \tau_{VV}^{-1} \circ \Delta(x) \circ \tau_{VV} \in \mathrm{End}(V \otimes V)\,, \tag{4.19}$$

we obtain

$$(\tau_{VV} R_0) \circ \Delta(x) = \tau_{VV} \circ \Delta^{\mathrm{op}}(x) \circ R_0 = (\tau_{VV} \circ \Delta^{\mathrm{op}}(x) \circ \tau_{VV}^{-1}) \circ (\tau_{VV} \circ R_0) = \Delta(x) \circ (\tau_{VV} R_0)\,,$$

so that $\tau_{VV} \circ R_0 \colon V \otimes V \to V \otimes V$ is a $U_q(\mathfrak{osp}(V))$-module morphism. In fact, since R_0 specializes to the identity map I at $q = 1$, it is generically a vector space isomorphism, so that $\tau_{VV} \circ R_0$ is a $U_q(\mathfrak{osp}(V))$-module isomorphism. Likewise is the operator $\hat{R}_{VV}$, which acts on the generating vectors $w_1, w_2, \widetilde{w}_3$ (or the generating highest weight vectors w_1, w_2, w_3 unless $n = m$) of $V \otimes V$ in the same way as $\tau_{VV} \circ R_0$, due to Propositions 4.9 and 4.11. This implies $\hat{R}_{VV} = \tau_{VV} \circ R_0$.

Similar arguments also show that $\tau_{VV} \circ R_\infty$ is a $U_q(\mathfrak{osp}(V))$-module isomorphism. Since the operators $\tau_{VV} \circ R_\infty$ and $\hat{R}_{VV}^{-1}$ act in the same way on the generating vectors $w_1, w_2, \hat{w}_3$ (or the generating highest weight vectors w_1, w_2, w_3 unless $n = m$) of $V \otimes V$, due to Propositions 4.10 and 4.11, we obtain the desired equality $\hat{R}_{VV}^{-1} = \tau_{VV} \circ R_\infty$. □

Remark 4.12. The above proof of Theorems 4.6 is quite elementary, but it does require knowing the correct formulas for R_{VV} in the first place. In Section 5, we provide the conceptual origin of these formulas by factorizing them into an ordered product of "local" operators.

4.3. **Proof of the intertwining property.**

In this separate Subsection, we sketch (presenting the key formulas) the proof of Proposition 4.8.

We start with the intertwining property of R_∞. To make the computations more manageable, it is helpful to break the operator R_∞ from (4.14) into I and the following four pieces:

$$\begin{aligned}
R_1 &= (q^{1/2}-q^{-1/2}) \sum_{1\leq i\leq N} (-1)^{\overline{i}} q^{(\varepsilon_i,\varepsilon_i)/2} E_{ii}\otimes E_{ii}\,, \\
R_2 &= -(q^{1/2}-q^{-1/2}) \sum_{1\leq i\leq N} (-1)^{\overline{i}} q^{-(\varepsilon_i,\varepsilon_i)/2} E_{ii}\otimes E_{i'i'}\,, \\
R_3 &= (q-q^{-1}) \sum_{i>j} (-1)^{\overline{j}} E_{ij}\otimes E_{ji}\,, \\
R_4 &= -(q-q^{-1}) \sum_{i>j} (-1)^{\overline{i}\,\overline{j}} q^{(\rho,\varepsilon_i-\varepsilon_j)} \vartheta_i\vartheta_j E_{ij}\otimes E_{i'j'}\,,
\end{aligned}$$

so that $R_\infty = \mathrm{I}+R_1+R_2+R_3+R_4$.

- Proof of $R_\infty\Delta(e_a)=\Delta^{\mathrm{op}}(e_a)R_\infty$ for $1\leq a<s$.

  Recall the explicit formula $\Delta(e_a)=q^{h_a/2}\otimes e_a + e_a\otimes q^{-h_a/2}$ as well as

$$\varrho(q^{h_a/2}) = q^{\mathsf{h}_a/2} = \sum_{1\leq i\leq N}^{i\neq a,a',a+1,(a+1)'} E_{ii} + \\
q^{(-1)^{\overline{a}}/2}E_{aa} + q^{-(-1)^{\overline{a+1}}/2}E_{a+1,a+1} + q^{-(-1)^{\overline{a}}/2}E_{a'a'} + q^{(-1)^{\overline{a+1}}/2}E_{(a+1)',(a+1)'}\,.$$

By direct computation, we get:

$$\begin{aligned}
R_1\Delta(e_a) = (q^{1/2}-q^{-1/2})\Big\{&(-1)^{\overline{a}}q^{(-1)^{\overline{a}}}\cdot E_{aa}\otimes E_{a,a+1} \\
&- (-1)^{\overline{a+1}}(-1)^{\overline{a}(\overline{a}+\overline{a+1})}q^{(-1)^{\overline{a+1}}}\vartheta_a\vartheta_{a+1}\cdot E_{(a+1)',(a+1)'}\otimes E_{(a+1)',a'} \\
&+ (-1)^{\overline{a}}\cdot E_{a,a+1}\otimes E_{aa} - (-1)^{\overline{a+1}}(-1)^{\overline{a}(\overline{a}+\overline{a+1})}\vartheta_a\vartheta_{a+1}\cdot E_{(a+1)',a'}\otimes E_{(a+1)',(a+1)'}\Big\}\,,
\end{aligned}$$

$$\begin{aligned}
\Delta^{\mathrm{op}}(e_a)R_1 = (q^{1/2}-q^{-1/2})\Big\{&(-1)^{\overline{a+1}}q^{(-1)^{\overline{a+1}}}\cdot E_{a+1,a+1}\otimes E_{a,a+1} \\
&- (-1)^{\overline{a}\,\overline{a+1}}q^{(-1)^{\overline{a}}}\vartheta_a\vartheta_{a+1}\cdot E_{a'a'}\otimes E_{(a+1)',a'} \\
&+ (-1)^{\overline{a+1}}\cdot E_{a,a+1}\otimes E_{a+1,a+1} - (-1)^{\overline{a}\,\overline{a+1}}\vartheta_a\vartheta_{a+1}\cdot E_{(a+1)',a'}\otimes E_{a',a'}\Big\}\,,
\end{aligned}$$

$$\begin{aligned}
R_2\Delta(e_a) = -\,(q^{1/2}-q^{-1/2})\Big\{&(-1)^{\overline{a}}q^{-(-1)^{\overline{a}}}\cdot E_{a'a'}\otimes E_{a,a+1} \\
&- (-1)^{\overline{a+1}}(-1)^{\overline{a}(\overline{a}+\overline{a+1})}q^{-(-1)^{\overline{a+1}}}\vartheta_a\vartheta_{a+1}\cdot E_{a+1,a+1}\otimes E_{(a+1)',a'} \\
&+ (-1)^{\overline{a}}\cdot E_{a,a+1}\otimes E_{a'a'} - (-1)^{\overline{a+1}}(-1)^{\overline{a}(\overline{a}+\overline{a+1})}\vartheta_a\vartheta_{a+1}\cdot E_{(a+1)',a'}\otimes E_{a+1,a+1}\Big\}\,,
\end{aligned}$$

$$\begin{aligned}
\Delta^{\mathrm{op}}(e_a)R_2 = -\,(q^{1/2}-q^{-1/2})\Big\{&(-1)^{\overline{a+1}}q^{-(-1)^{\overline{a+1}}}\cdot E_{(a+1)',(a+1)'}\otimes E_{a,a+1} \\
&- (-1)^{\overline{a}\,\overline{a+1}}q^{-(-1)^{\overline{a}}}\vartheta_a\vartheta_{a+1}\cdot E_{aa}\otimes E_{(a+1)',a'} \\
&+ (-1)^{\overline{a+1}}\cdot E_{a,a+1}\otimes E_{(a+1)',(a+1)'} - (-1)^{\overline{a}\,\overline{a+1}}\vartheta_a\vartheta_{a+1}\cdot E_{(a+1)',a'}\otimes E_{aa}\Big\}\,,
\end{aligned}$$

$$
\begin{aligned}
R_3\Delta(e_a) = (q-q^{-1}) \Bigg\{ & \sum_{j<a} (-1)^{\bar{j}} \cdot (E_{aj} q^{\mathsf{h}_a/2}) \otimes E_{j,a+1} \\
& - \sum_{j<(a+1)'} (-1)^{\bar{j}} (-1)^{\bar{a}(\bar{a}+\overline{a+1})} \vartheta_a \vartheta_{a+1} \cdot (E_{(a+1)'j} q^{\mathsf{h}_a/2}) \otimes E_{ja'} \\
& + \sum_{i>a} (-1)^{\bar{a}} (-1)^{(\bar{a}+\bar{i})(\bar{a}+\overline{a+1})} \cdot E_{i,a+1} \otimes (E_{ai} q^{-\mathsf{h}_a/2}) \\
& - \sum_{i>(a+1)'} (-1)^{\bar{a}} (-1)^{\bar{i}(\bar{a}+\overline{a+1})} \vartheta_a \vartheta_{a+1} \cdot E_{ia'} \otimes (E_{(a+1)',i} q^{-\mathsf{h}_a/2}) \Bigg\} ,
\end{aligned}
$$

$$
\begin{aligned}
\Delta^{\mathrm{op}}(e_a) R_3 = (q-q^{-1}) \Bigg\{ & \sum_{i>a+1} (-1)^{\overline{a+1}} (-1)^{(\bar{a}+\overline{a+1})(\bar{i}+\overline{a+1})} \cdot (q^{-\mathsf{h}_a/2} E_{i,a+1}) \otimes E_{ai} \\
& - \sum_{i>a'} (-1)^{\bar{a}} (-1)^{\bar{i}(\bar{a}+\overline{a+1})} \vartheta_a \vartheta_{a+1} \cdot (q^{-\mathsf{h}_a/2} E_{ia'}) \otimes E_{(a+1)',i} \\
& + \sum_{j<a+1} (-1)^{\bar{j}} \cdot E_{aj} \otimes (q^{\mathsf{h}_a/2} E_{j,a+1}) \\
& - \sum_{j<a'} (-1)^{\bar{j}} (-1)^{\bar{a}(\bar{a}+\overline{a+1})} \vartheta_a \vartheta_{a+1} \cdot E_{(a+1)',j} \otimes (q^{\mathsf{h}_a/2} E_{ja'}) \Bigg\} .
\end{aligned}
$$

Also note that the difference $R_3\Delta(e_a) - \Delta^{\mathrm{op}}(e_a)R_3$ can be simplified as follows:

$$
\begin{aligned}
R_3(\Delta e_a) - \Delta^{\mathrm{op}}(e_a)R_3 = & -\Big(q^{(-1)^{\bar{a}}\cdot 3/2} - q^{-(-1)^{\bar{a}}/2}\Big) E_{aa} \otimes E_{a,a+1} \\
& + (-1)^{\bar{a}(\bar{a}+\overline{a+1})} \vartheta_a \vartheta_{a+1} \Big(q^{(-1)^{\overline{a+1}}\cdot 3/2} - q^{-(-1)^{\overline{a+1}}/2}\Big) E_{(a+1)',(a+1)'} \otimes E_{(a+1)',a'} \\
& + \Big(q^{(-1)^{\overline{a+1}}\cdot 3/2} - q^{-(-1)^{\overline{a+1}}/2}\Big) E_{a+1,a+1} \otimes E_{a,a+1} \\
& - (-1)^{\bar{a}(\bar{a}+\overline{a+1})} \vartheta_a \vartheta_{a+1} \Big(q^{(-1)^{\bar{a}}\cdot 3/2} - q^{-(-1)^{\bar{a}}/2}\Big) E_{a'a'} \otimes E_{(a+1)',a'} .
\end{aligned}
$$

To compute the last two terms involving R_4, let us first note that (4.11) implies:

$$
(\rho, \varepsilon_a - \varepsilon_{a+1}) = \frac{(\varepsilon_a - \varepsilon_{a+1}, \varepsilon_a - \varepsilon_{a+1})}{2} = \frac{(-1)^{\bar{a}} + (-1)^{\overline{a+1}}}{2} .
$$

Using this equality, one derives the following formulas:

$$
\begin{aligned}
R_4\Delta(e_a) = & -(q-q^{-1}) \Bigg\{ \sum_{i>a'} (-1)^{\bar{i}\,\bar{a}} (-1)^{\bar{a}} q^{(\rho,\varepsilon_i+\varepsilon_a)} q^{-(-1)^{\bar{a}}/2} \vartheta_i \vartheta_a \cdot E_{ia'} \otimes E_{i',a+1} \\
& - \sum_{i>a+1} (-1)^{\bar{i}\,\overline{a+1}} (-1)^{\bar{a}(\bar{a}+\overline{a+1})} q^{(\rho,\varepsilon_i-\varepsilon_{a+1})} q^{-(-1)^{\overline{a+1}}/2} \vartheta_i \vartheta_a \cdot E_{i,a+1} \otimes E_{i'a'} \\
& + \sum_{i>a} (-1)^{\bar{i}\,\overline{a+1}} (-1)^{\bar{a}(\bar{a}+\overline{a+1})} q^{(\rho,\varepsilon_i-\varepsilon_a)} q^{(-1)^{\bar{a}}/2} \vartheta_i \vartheta_a \cdot E_{i,a+1} \otimes E_{i'a'} \\
& - \sum_{i>(a+1)'} (-1)^{\bar{i}\,\bar{a}} (-1)^{\bar{a}} q^{(\rho,\varepsilon_i+\varepsilon_{a+1})} q^{(-1)^{\overline{a+1}}/2} \vartheta_i \vartheta_a \cdot E_{ia'} \otimes E_{i',a+1} \Bigg\} \\
= & -(q-q^{-1}) \Big\{ -(-1)^{\bar{a}} q^{-(-1)^{\bar{a}}/2} \cdot E_{a'a'} \otimes E_{a,a+1} \\
& + (-1)^{\overline{a+1}} (-1)^{\bar{a}(\bar{a}+\overline{a+1})} q^{-(-1)^{\overline{a+1}}/2} \vartheta_a \vartheta_{a+1} \cdot E_{a+1,a+1} \otimes E_{(a+1)',a'} \Big\}
\end{aligned}
$$

as well as

$$
\begin{aligned}
\Delta^{\mathrm{op}}(e_a)R_4 = &-(q-q^{-1})\left\{-\sum_{j<a}(-1)^{\overline{a+1}\,\overline{j}}q^{(\rho,\varepsilon_a-\varepsilon_j)}q^{-(-1)^{\overline{a}}/2}\vartheta_{a+1}\vartheta_j\cdot E_{aj}\otimes E_{(a+1)',j'}\right.\\
&\qquad + \sum_{j<(a+1)'}(-1)^{\overline{a}\,\overline{j}}(-1)^{\overline{a}\,\overline{a+1}}q^{-(\rho,\varepsilon_{a+1}+\varepsilon_j)}q^{-(-1)^{\overline{a+1}}/2}\vartheta_{a+1}\vartheta_j\cdot E_{(a+1)',j}\otimes E_{aj'}\\
&\qquad + \sum_{j<a+1}(-1)^{\overline{a+1}\,\overline{j}}q^{(\rho,\varepsilon_{a+1}-\varepsilon_j)}q^{(-1)^{\overline{a+1}}/2}\vartheta_{a+1}\vartheta_j\cdot E_{aj}\otimes E_{(a+1)',j'}\\
&\qquad \left. - \sum_{j<a'}(-1)^{\overline{a}\,\overline{j}}(-1)^{\overline{a}\,\overline{a+1}}q^{-(\rho,\varepsilon_a+\varepsilon_j)}q^{(-1)^{\overline{a}}/2}\vartheta_{a+1}\vartheta_j\cdot E_{(a+1)',j}\otimes E_{aj'}\right\}\\
= &-(q-q^{-1})\Big\{-(-1)^{\overline{a+1}}q^{-(-1)^{\overline{a+1}}/2}\cdot E_{(a+1)',(a+1)'}\otimes E_{a,a+1}\\
&\qquad + (-1)^{\overline{a+1}\,\overline{a}}q^{-(-1)^{\overline{a}}/2}\vartheta_a\vartheta_{a+1}\cdot E_{aa}\otimes E_{(a+1)',a'}\Big\}\,.
\end{aligned}
$$

Combining the above eight formulas, using the obvious equalities

$$
(q^{1/2}-q^{-1/2})(-1)^{\overline{i}} = q^{(-1)^{\overline{i}}/2} - q^{-(-1)^{\overline{i}}/2}\,, \qquad (q-q^{-1})(-1)^{\overline{i}} = q^{(-1)^{\overline{i}}} - q^{-(-1)^{\overline{i}}}\,,
$$

and collecting similar terms, we finally obtain:

$$
\sum_{k=1}^{4}\Big(R_k\Delta(e_a)-\Delta^{\mathrm{op}}(e_a)R_k\Big) = -\Delta(e_a)+\Delta^{\mathrm{op}}(e_a)\,. \tag{4.20}
$$

This establishes the claimed intertwining property $R_\infty\Delta(e_a)=\Delta^{\mathrm{op}}(e_a)R_\infty$ for $1\le a<s$.

• Proof of $R_\infty\Delta(e_s)=\Delta^{\mathrm{op}}(e_s)R_\infty$.

As before, there are three cases to consider: odd m, even m with $\overline{s}=\bar{0}$, even m with $\overline{s}=\bar{1}$. The computations are very similar to those used above to establish (4.20) for $a<s$. Thus, we shall only present the relevant changes in the third case (m is even and $\overline{s}=\bar{1}$) that differs the most.

$$
R_1\Delta(e_s) = (q^{1/2}-q^{-1/2})\Big\{(-1)^{\overline{s}}q^{(-1)^{\overline{s}}\cdot 3/2}\cdot E_{ss}\otimes E_{ss'} + (-1)^{\overline{s}}q^{-(-1)^{\overline{s}}/2}\cdot E_{ss'}\otimes E_{ss}\Big\}\,,
$$

$$
\Delta^{\mathrm{op}}(e_s)R_1 = (q^{1/2}-q^{-1/2})\Big\{(-1)^{\overline{s}}q^{(-1)^{\overline{s}}\cdot 3/2}\cdot E_{s's'}\otimes E_{ss'} + (-1)^{\overline{s}}q^{-(-1)^{\overline{s}}/2}\cdot E_{ss'}\otimes E_{s's'}\Big\}\,,
$$

$$
R_2\Delta(e_s) = -(q^{1/2}-q^{-1/2})\Big\{(-1)^{\overline{s}}q^{-(-1)^{\overline{s}}\cdot 3/2}\cdot E_{s's'}\otimes E_{ss'} + (-1)^{\overline{s}}q^{(-1)^{\overline{s}}/2}\cdot E_{ss'}\otimes E_{s's'}\Big\}\,,
$$

$$
\Delta^{\mathrm{op}}(e_s)R_2 = -(q^{1/2}-q^{-1/2})\Big\{(-1)^{\overline{s}}q^{-(-1)^{\overline{s}}\cdot 3/2}\cdot E_{ss}\otimes E_{ss'} + (-1)^{\overline{s}}q^{(-1)^{\overline{s}}/2}\cdot E_{ss'}\otimes E_{ss}\Big\}\,,
$$

$$
R_3\Delta(e_s) = (q-q^{-1})\left\{\sum_{j<s}(-1)^{\overline{j}}\cdot(E_{sj}q^{\mathsf{h}_s/2})\otimes E_{js'} + \sum_{i>s}(-1)^{\overline{s}}\cdot E_{is'}\otimes(E_{si}q^{-\mathsf{h}_s/2})\right\}\,,
$$

$$
\Delta^{\mathrm{op}}(e_s)R_3 = (q-q^{-1})\left\{\sum_{i>s'}(-1)^{\overline{s}}\cdot(q^{-\mathsf{h}_s/2}E_{is'})\otimes E_{si} + \sum_{j<s'}(-1)^{\overline{j}}\cdot E_{sj}\otimes(q^{\mathsf{h}_s/2}E_{js'})\right\}\,.
$$

For the last two terms, we note that (4.11) implies:

$$
(\rho,2\varepsilon_s) = \frac{(2\varepsilon_s,2\varepsilon_s)}{2} = (-1)^{\overline{s}}\cdot 2\,,
$$

so that we get:

$$\begin{aligned} R_4\Delta(e_s) = &- (q-q^{-1}) \Bigg\{ \sum_{i>s'} (-1)^{\bar{i}\,\bar{s}} (-1)^{\bar{s}} q^{(\rho,\varepsilon_i+\varepsilon_s)} q^{-(-1)^{\bar{s}}} \vartheta_i\vartheta_s \cdot E_{is'} \otimes E_{i's'} \\ &\qquad + \sum_{i>s} (-1)^{\bar{i}\,\bar{s}} q^{(\rho,\varepsilon_i-\varepsilon_s)} q^{(-1)^{\bar{s}}} \vartheta_i\vartheta_s \cdot E_{is'} \otimes E_{i's'} \Bigg\} \\ = &\,(1-q^{-(-1)^{\bar{s}}\cdot 2}) \cdot E_{s's'} \otimes E_{ss'} \end{aligned}$$

as well as

$$\begin{aligned} \Delta^{\mathrm{op}}(e_s) R_4 = &- (q-q^{-1}) \Bigg\{ \sum_{j<s} (-1)^{\bar{s}\,\bar{j}} q^{(\rho,\varepsilon_s-\varepsilon_j)} q^{-(-1)^{\bar{s}}} \vartheta_s\vartheta_j \cdot E_{sj} \otimes E_{sj'} \\ &\qquad + \sum_{j<s'} (-1)^{\bar{s}\,\bar{j}} (-1)^{\bar{s}} q^{-(\rho,\varepsilon_s+\varepsilon_j)} q^{(-1)^{\bar{s}}} \vartheta_s\vartheta_j \cdot E_{sj} \otimes E_{sj'} \Bigg\} \\ = &\,(1-q^{-(-1)^{\bar{s}}\cdot 2}) \cdot E_{ss} \otimes E_{ss'}\,. \end{aligned}$$

Assembling all the terms, we thus obtain:

$$\begin{aligned} &R_1\Delta(e_s) - \Delta^{\mathrm{op}}(e_s)R_1 = \\ &\qquad \Big(q^{(-1)^{\bar{s}}\cdot 2} - q^{(-1)^{\bar{s}}}\Big) \cdot (E_{ss}-E_{s's'}) \otimes E_{ss'} + \Big(1-q^{-(-1)^{\bar{s}}}\Big) \cdot E_{ss'} \otimes (E_{ss}-E_{s's'})\,, \\ &R_2\Delta(e_s) - \Delta^{\mathrm{op}}(e_s)R_2 = \\ &\qquad \Big(q^{-(-1)^{\bar{s}}} - q^{-(-1)^{\bar{s}}\cdot 2}\Big) \cdot (E_{ss}-E_{s's'}) \otimes E_{ss'} + \Big(q^{(-1)^{\bar{s}}}-1\Big) \cdot E_{ss'} \otimes (E_{ss}-E_{s's'})\,, \\ &R_3\Delta(e_s) - \Delta^{\mathrm{op}}(e_s)R_3 = -\Big(q^{(-1)^{\bar{s}}\cdot 2}-1\Big) \cdot (E_{ss}-E_{s's'}) \otimes E_{ss'}\,, \\ &R_4\Delta(e_s) - \Delta^{\mathrm{op}}(e_s)R_4 = -\Big(1-q^{-(-1)^{\bar{s}}\cdot 2}\Big) \cdot (E_{ss}-E_{s's'}) \otimes E_{ss'}\,. \end{aligned}$$

Collecting the similar terms together, we finally get:

$$\begin{aligned} \sum_{k=1}^{4} \Big(R_k\Delta(e_s) - \Delta^{\mathrm{op}}(e_s)R_k\Big) &= - (q^{\mathsf{h}_s/2} - q^{-\mathsf{h}_s/2}) \otimes \mathsf{e}_s + \mathsf{e}_s \otimes (q^{\mathsf{h}_s/2} - q^{-\mathsf{h}_s/2}) \\ &= - \Delta(e_s) + \Delta^{\mathrm{op}}(e_s)\,. \end{aligned}$$

This establishes the claimed intertwining property $R_\infty\Delta(e_s) = \Delta^{\mathrm{op}}(e_s)R_\infty$.

- Proof of $R_\infty\Delta(q^{h_a/2}) = \Delta^{\mathrm{op}}(q^{h_a/2})R_\infty$ for $1 \leq a \leq s$.

Since $\varrho(q^{h_a/2}) = q^{\mathsf{h}_a/2}$ is a diagonal matrix, we can write $\varrho(q^{h_a/2}) = q^{\mathsf{h}_a/2} = \mathrm{diag}(\mathsf{t}_1, \ldots, \mathsf{t}_{1'})$. Furthermore, we note that $\mathsf{t}_i\mathsf{t}_{i'} = 1$ for all i. Therefore, $\Delta(q^{h_a/2}) = \Delta^{\mathrm{op}}(q^{h_a/2}) = q^{\mathsf{h}_a/2} \otimes q^{\mathsf{h}_a/2}$ commutes with all the terms of the form $E_{ii} \otimes E_{jj}$, $E_{ii} \otimes E_{ii}$, $E_{ii} \otimes E_{i'i'}$, $E_{ij} \otimes E_{ji}$, and $E_{ij} \otimes E_{i'j'}$. This implies the desired intertwining property for $q^{h_a/2}$.

- Proof of $R_\infty\Delta(f_a) = \Delta^{\mathrm{op}}(f_a)R_\infty$ for $1 \leq a \leq s$.

We first recall some basic properties of the supertransposition (2.5). For any $X \otimes Y \in \mathrm{End}(V)^{\otimes 2}$, let $(X \otimes Y)^{\mathrm{st}_1} = X^{\mathrm{st}} \otimes Y$ and $(X \otimes Y)^{\mathrm{st}_2} = X \otimes Y^{\mathrm{st}}$ denote the supertransposition applied to the first and the second component, respectively. Then, we have:

$$(XY)^{\mathrm{st}} = (-1)^{|X||Y|} Y^{\mathrm{st}} X^{\mathrm{st}}$$

as well as

$$\Big((X \otimes Y)(X' \otimes Y')\Big)^{\mathrm{st}_1\mathrm{st}_2} = (-1)^{(|X|+|Y|)(|X'|+|Y'|)} (X' \otimes Y')^{\mathrm{st}_1\mathrm{st}_2} (X \otimes Y)^{\mathrm{st}_1\mathrm{st}_2}$$

for any homogeneous $X, X', Y, Y' \in \mathrm{End}(V)$.

We note that $(q^{\mathsf{h}_i/2})^{\mathrm{st}} = q^{\mathsf{h}_i/2}$ and $(\mathsf{e}_i)^{\mathrm{st}}$ is always a nonzero scalar multiple of f_i, due to formulas (2.12)–(2.14). Furthermore, (4.14) also implies

$$R_\infty = \tau_{VV} \circ (R_\infty)^{\mathrm{st}_1\mathrm{st}_2} \circ \tau_{VV}^{-1} \,. \tag{4.21}$$

Thus, applying $\mathrm{st}_1 \circ \mathrm{st}_2$ to the equation $R_\infty \Delta(e_a) = \Delta^{\mathrm{op}}(e_a) R_\infty$ and evoking (4.19), we obtain

$$\Delta(f_a)(R_\infty)^{\mathrm{st}_1\mathrm{st}_2} = (R_\infty)^{\mathrm{st}_1\mathrm{st}_2} \Delta^{\mathrm{op}}(f_a) \,.$$

Conjugating this equality by τ_{VV} and evoking (4.21), we get the desired intertwining property

$$R_\infty \Delta(f_a) = \Delta^{\mathrm{op}}(f_a) R_\infty \,.$$

This completes the proof of the second equality of (4.15).

The intertwining property $R_0 \Delta(x) = \Delta^{\mathrm{op}}(x) R_0$ is actually directly implied by the one for R_∞, which we just proved. To this end, let us first note the following equality:

$$\begin{split}\tau_{VV} \circ R_0 \circ \tau_{VV}^{-1} = \mathrm{I} + (q^{-1/2} - q^{1/2}) \sum_{1\leq i\leq N} (-1)^{\bar{i}} E_{ii} \otimes \Big(q^{-(\varepsilon_i,\varepsilon_i)/2} E_{ii} - q^{(\varepsilon_i,\varepsilon_i)/2} E_{i'i'} \Big) + \\ (q^{-1} - q) \sum_{i>j} (-1)^{\bar{j}} E_{ij} \otimes \Big(E_{ji} - (-1)^{\bar{j}(\bar{i}+\bar{j})} \vartheta_i \vartheta_j q^{-(\rho, \varepsilon_i - \varepsilon_j)} E_{i'j'} \Big) = R_\infty|_{q\mapsto q^{-1}} \,.\end{split} \tag{4.22}$$

As the notation suggests, $R_\infty|_{q\mapsto q^{-1}}$ is the $\mathbb{C}(q)$-valued $N^2 \times N^2$ matrix obtained from R_∞ by applying to all matrix coefficients the $\mathbb{C}$-algebra automorphism

$$\bar{\sigma} \colon \mathbb{C}(q) \to \mathbb{C}(q) \qquad \text{determined by} \quad q \mapsto q^{-1} \,. \tag{4.23}$$

We claim that the assignment

$$\sigma \colon \quad e_a \mapsto e_a \,, \quad f_a \mapsto f_a \,, \quad q^{\pm h_a/2} \mapsto q^{\mp h_a/2} \,, \quad q \mapsto q^{-1} \tag{4.24}$$

gives rise to a $\mathbb{C}$-algebra involution $\sigma \colon U_q(\mathfrak{osp}(V)) \to U_q(\mathfrak{osp}(V))$. To prove this, we note that relations (2.18)–(2.20) are clearly preserved by (4.24), as well as the ideal of $U_q^+(\mathfrak{osp}(V))$ (respectively of $U_q^-(\mathfrak{osp}(V))$) generated by all Serre relations in $\{e_i\}$ (respectively $\{f_i\}$) as follows from [43, Lemma 6.3.1][2]. We also define $\Delta^\sigma, (\Delta^{\mathrm{op}})^\sigma \colon U_q(\mathfrak{osp}(V)) \to U_q(\mathfrak{osp}(V)) \otimes U_q(\mathfrak{osp}(V))$ via

$$\Delta^\sigma = (\sigma \otimes \sigma) \circ \Delta \circ \sigma^{-1} \,, \qquad (\Delta^{\mathrm{op}})^\sigma = (\sigma \otimes \sigma) \circ \Delta^{\mathrm{op}} \circ \sigma^{-1} \,. \tag{4.25}$$

Then, applying $\bar{\sigma}$ to all matrix coefficients in the equality $R_\infty \circ \Delta(x) = \Delta^{\mathrm{op}}(x) \circ R_\infty$ and using the obvious equality $\rho \circ \sigma = \bar{\sigma} \circ \rho$, we obtain

$$R_\infty|_{q\mapsto q^{-1}} \circ \Delta^\sigma(\sigma(x)) = (\Delta^{\mathrm{op}})^\sigma(\sigma(x)) \circ R_\infty|_{q\mapsto q^{-1}} \qquad \forall\, x \in U_q(\mathfrak{osp}(V)) \,. \tag{4.26}$$

However, direct computation of Δ^σ on the generators $e_a, f_a, q^{\pm h_a/2}$ $(1 \leq a \leq s)$ shows that

$$\Delta^\sigma = \Delta^{\mathrm{op}} \,, \qquad (\Delta^{\mathrm{op}})^\sigma = \Delta \,. \tag{4.27}$$

Combining (4.22, 4.26, 4.27) with (4.19), we obtain

$$R_0 \circ \Delta(\sigma(x)) = \Delta^{\mathrm{op}}(\sigma(x)) \circ R_0 \qquad \forall\, x \in U_q(\mathfrak{osp}(V)) \,.$$

As σ is invertible, this implies $R_0 \Delta(x) = \Delta^{\mathrm{op}}(x) R_0$ for any x, thus establishing Proposition 4.8.

5. Factorization of finite R-Matrices

In this Section, we present the factorization formula for Θ and use it to re-derive the formulas for R_{VV} from Theorem 4.6. To this end, we use a combinatorial construction of orthogonal dual bases of $U_q^+(\mathfrak{osp}(V))$ and $U_q^-(\mathfrak{osp}(V))$, based on the combinatorics of dominant Lyndon words.

[2] We note that some of the individual higher order Serre relations of [42] are actually not preserved under (4.24).

5.1. **Shuffle superalgebra.**

Let F be the free $\mathbb{C}(q)$-superalgebra generated by the finite alphabet $I=\{1,2,\dots,s\}$, and let W be the set of words in I, i.e. the monoid generated by I. Thus, F has a basis consisting of finite length words $[i_1\dots i_d]=i_1i_2\dots i_d$, where $i_1,\dots,i_d\in I$. Note that F is $Q^+\times\mathbb{Z}_2$-graded via

$$\deg([i_1\dots i_d])=\alpha_{i_1}+\dots+\alpha_{i_d}\in Q^+\,,\qquad p([i_1\dots i_d])=|\mathsf{e}_{i_1}|+\dots+|\mathsf{e}_{i_d}|\in\mathbb{Z}_2\,,$$

cf. (2.17). Following [7, §3.1], we define the *q-quantum shuffle product* $\diamond\colon \mathsf{F}\times\mathsf{F}\to\mathsf{F}$ inductively via

$$(xi)\diamond(yj)=(x\diamond(yj))i+(-1)^{p(xi)p(j)}q^{-(\deg(xi),\alpha_j)}((xi)\diamond y)j\,,\qquad \emptyset\diamond x=x\diamond\emptyset=x\,,$$

for all $i,j\in I$ and $x,y\in\mathsf{F}$ homogeneous with respect to the $Q^+\times\mathbb{Z}_2$-grading. By iterating this definition, one obtains (a correction of [7, (3.4)]):

$$[i_1\dots i_a]\diamond[i_{a+1}i_{a+2}\dots i_{a+b}]=\sum_\sigma e_{a,b}(\sigma)[i_{\sigma^{-1}(1)}i_{\sigma^{-1}(2)}\dots i_{\sigma^{-1}(a+b)}]\,,$$

where

$$e_{a,b}(\sigma)=\prod_{\substack{k\le a<l,\\ \sigma(k)<\sigma(l)}}\Big((-1)^{p(\mathsf{e}_{i_k})p(\mathsf{e}_{i_l})}q^{-(\alpha_{i_k},\alpha_{i_l})}\Big)$$

and the sum runs over all (a,b)-*shuffles* of $\{1,2,\dots,a+b\}$, i.e. the permutations $\sigma\in S_{a+b}$ such that $\sigma(1)<\sigma(2)<\dots<\sigma(a)$ and $\sigma(a+1)<\dots<\sigma(a+b)$.

The q-shuffle algebra provides a combinatorial model for $U_q^+(\mathfrak{osp}(V))$ via [7, Corollary 3.4]:

Proposition 5.1. *There is a unique superalgebra embedding* $\Psi\colon U_q^+(\mathfrak{osp}(V))\to\mathsf{F}$ *with* $\Psi(e_i)=[i]$.

Let $\mathsf{U}=\Psi(U_q^+(\mathfrak{osp}(V)))$ be the image of this embedding, so that $\Psi\colon U_q^+(\mathfrak{osp}(V))\xrightarrow{\sim}\mathsf{U}$.

5.2. **Combinatorics of words.**

From now on, we fix an order $\le$ on the alphabet I, which induces a lexicographical order on the monoid W. For a nonzero $x\in\mathsf{F}$, its *leading term*, denoted $\max(x)$, is a word $w\in\mathsf{W}$ such that

$$x=\sum_{u\le w}t_u\cdot u\qquad\text{with}\quad t_u\in\mathbb{C}(q)\quad\text{and}\quad t_w\ne 0\,.$$

Following the terminology of [7, §4.1], we call a word $w\in\mathsf{W}$ *dominant* if it appears as a leading term of some element from U, and let W^+ denote the subset of all dominant words in W.

Remark 5.2. It turns out that W^+ can be used to construct various bases of $U_q^+(\mathfrak{osp}(V))$. For example, the set $\{e_w=e_{i_1}\dots e_{i_d}\,|\,w=[i_1\dots i_d]\in\mathsf{W}^+\}$ is a basis of $U_q^+(\mathfrak{osp}(V))$, according to [7, Proposition 4.1]. However, we shall rather work with more sophisticated *Lyndon bases* below.

A word $w=[i_1\dots i_d]$ is called *Lyndon* if it is smaller than any of its proper right factors:

$$w<[i_k\dots i_d]\qquad\forall\,1<k\le d\,.$$

We use L to denote the set of all Lyndon words. It is well-known that any word w admits a unique *canonical factorization* (see [28, Proposition 5.1.5]) as a product of non-increasing Lyndon words:

$$w=\ell_1\ell_2\dots\ell_k\,,\quad \ell_1\ge\ell_2\ge\dots\ge\ell_k\,,\quad \ell_1,\dots,\ell_k\in\mathsf{L}\,. \tag{5.1}$$

Furthermore, any Lyndon word $\ell\in\mathsf{L}$ admits a unique *costandard factorization* $\ell=\ell_1\ell_2$ such that $\ell_1,\ell_2\ne\emptyset$, $\ell_1\in\mathsf{L}$ is the longest possible, in which case also $\ell_2\in\mathsf{L}$ (see [28, Proposition 5.1.3]).

Let $\mathsf{L}^+=\mathsf{W}^+\cap\mathsf{L}$ be the set of all dominant Lyndon words. We also recall the reduced root system $\bar\Phi$ from (2.10). The following result is proved in [7, Theorem 4.8]:

Proposition 5.3. *(a) The map* $\ell\mapsto\deg(\ell)$ *defines a bijection* $\mathrm{l}\colon\mathsf{L}^+\xrightarrow{\sim}\bar\Phi^+$.

(b) A word $w\in\mathsf{W}$ *is dominant if and only if its canonical factorization is of the form*

$$w=\ell_1\ell_2\dots\ell_k\,,\quad \ell_1\ge\ell_2\ge\dots\ge\ell_k\,,\quad \ell_1,\ell_2,\dots,\ell_k\in\mathsf{L}^+ \tag{5.2}$$

where ℓ_p *appears only once if* $\deg(\ell_p)$ *is an isotropic odd root (see Subsection 2.3).*

We note that the above bijection l gives rise to a *lexicographical* ordering on $\bar\Phi^+$:

$$\alpha<\beta\quad\Longleftrightarrow\quad \mathrm{l}^{-1}(\alpha)<\mathrm{l}^{-1}(\beta)\ \text{lexicographically}\,. \tag{5.3}$$

5.3. **Lyndon basis.**

For $x, y \in \mathsf{F}$ homogeneous with respect to $Q^+ \times \mathbb{Z}_2$-grading, their *q-commutator* is defined as

$$[x,y]_q = xy - (-1)^{p(x)p(y)} q^{(\deg(x),\deg(y))} yx \,, \tag{5.4}$$

cf. (2.22). Following [7, §4.3], we define the *q-bracketing* $[\ell] \in \mathsf{F}$ of a Lyndon word $\ell \in \mathsf{L}$ via:

- $[\ell] = \ell$ if $\ell \in I$,
- $[\ell] = [[\ell_1],[\ell_2]]_q$ if $\ell = \ell_1\ell_2$ is the costandard factorization of ℓ.

Evoking the canonical factorization (5.1), we define the q-bracketing of any word $w \in \mathsf{W}$ via:

$$[w] = [\ell_1][\ell_2]\ldots[\ell_k] \,.$$

According to [7, Proposition 4.10], the set $\{[w] \,|\, w \in \mathsf{W}\}$ is a basis for F. Finally, we also define $\Xi\colon (\mathsf{F}, \cdot) \to (\mathsf{F}, \diamond)$ as the algebra homomorphism given by $\Xi([i_1 \ldots i_d]) = i_1 \diamond \cdots \diamond i_d$. Then, we have the following equivalent description of dominant words, see [7, Lemma 4.11]:

Lemma 5.4. *A word $w \in \mathsf{W}$ is dominant if and only if it cannot be expressed modulo* $\ker(\Xi)$ *as a linear combination of words $v > w$.*

For any dominant word $w \in \mathsf{W}^+$, we define

$$R_w = \Xi([w]) \,.$$

For any homogeneous $x, y \in \mathsf{F}$, we introduce $x \diamond_{q,q^{-1}} y$ similarly to (5.4):

$$x \diamond_{q,q^{-1}} y = x \diamond y - (-1)^{p(x)p(y)} q^{(\deg(x),\deg(y))} y \diamond x \,.$$

This formula implies that if $\ell \in \mathsf{L}^+$ has a costandard factorization $\ell = \ell_1\ell_2$, then $R_\ell = R_{\ell_1} \diamond_{q,q^{-1}} R_{\ell_2}$. According to [7, Proposition 4.13], the set $\{R_w \,|\, w \in \mathsf{W}^+\}$ is a basis for U, referred to as the *Lyndon basis* of U. Evoking Proposition 5.3, it has the form:

$$\left\{ R_{\ell_1} \diamond \cdots \diamond R_{\ell_k} \,\middle|\, {}^{k \in \mathbb{Z}_{\geq 0},\ \ell_1,\ldots,\ell_k \in \mathsf{L}^+,\ \ell_1 \geq \cdots \geq \ell_k,}_{\ell_{p-1} > \ell_p > \ell_{p+1} \text{ if } \deg(\ell_p) \in \bar{\Phi}_{\bar{1}} \text{ is isotropic}} \right\} . \tag{5.5}$$

5.4. **Explicit computations.**

In this Subsection, we specify explicitly the set L^+ of dominant Lyndon words, the lexicographical order (5.3) on $\bar{\Phi}^+$, and the map $\bar{\Phi}^+ \to \bar{\Phi}^+ \times \bar{\Phi}^+$ which assigns to a root $\gamma \in \bar{\Phi}^+$ a pair of roots $\alpha = \mathrm{l}(\ell_1), \beta = \mathrm{l}(\ell_2)$, where $\ell = \ell_1\ell_2$ is the costandard factorization of $\ell = \mathrm{l}^{-1}(\gamma)$, see Proposition 5.3. To this end, we choose a specific ordering $1 < 2 < \cdots < s$ on the alphabet I, as in [7, §6].

- Case 1: m is odd. In this case, according to [7, Proposition 6.5]:

$$\mathsf{L}^+ = \{[i \ldots j] \,|\, 1 \leq i \leq j \leq s\} \cup \{[i \ldots ss \ldots j] \,|\, 1 \leq i < j \leq s\} \,. \tag{5.6}$$

This results in the following lexicographical order on the reduced root system:

$$\begin{aligned} &\alpha_1 < \alpha_1 + \alpha_2 < \cdots < \alpha_1 + \cdots + \alpha_s \\ &\quad < \alpha_1 + \cdots + \alpha_{s-1} + 2\alpha_s < \alpha_1 + \cdots + 2\alpha_{s-1} + 2\alpha_s < \cdots < \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_s \\ &\quad < \alpha_2 < \cdots < \alpha_{s-1} < \alpha_{s-1} + \alpha_s < \alpha_{s-1} + 2\alpha_s < \alpha_s \,. \end{aligned} \tag{5.7}$$

Let $\gamma_{ij} = \alpha_i + \cdots + \alpha_j$ for $1 \leq i \leq j \leq s$, and let $\beta_{ij} = \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_s$ for $1 \leq i < j \leq s$. Then, the aforementioned assignment $\gamma \mapsto (\alpha, \beta)$ is explicitly given by:

  - for the roots $\gamma = \gamma_{ij}$ with $i < j$, we have $(\alpha, \beta) = (\gamma_{i,j-1}, \alpha_j)$;
  - for the roots $\gamma = \beta_{is}$ with $1 \leq i < s$, we have $(\alpha, \beta) = (\gamma_{is}, \alpha_s)$;
  - for the roots $\gamma = \beta_{ij}$ with $i < j < s$, we have $(\alpha, \beta) = (\beta_{i,j+1}, \alpha_j)$.

- Case 2: m is even and $\bar{s} = \bar{0}$. In this case, according to [7, Proposition 6.12]:

$$\begin{aligned} \mathsf{L}^+ = &\{[i \ldots j] \,|\, 1 \leq i \leq j \leq s-1\} \cup \{[i \ldots (s-2)s] \,|\, 1 \leq i \leq s-2\} \cup \\ &\{[i \ldots (s-2)s(s-1) \ldots j] \,|\, 1 \leq i < j \leq s-1\} \cup \\ &\{[i \ldots (s-2)(s-1)i \ldots (s-2)s] \,|\, 1 \leq i < s-1 \text{ and } p([i \ldots (s-1)]) = \bar{1}\} \,. \end{aligned} \tag{5.8}$$

This results in the following lexicographical order on the reduced root system:

$$\begin{aligned}\alpha_1 &< \alpha_1+\alpha_2 < \cdots < \alpha_1+\cdots+\alpha_{s-2}+\alpha_{s-1} < \underline{2\alpha_1+\cdots+2\alpha_{s-2}+\alpha_{s-1}+\alpha_s}\\ &< \alpha_1+\cdots+\alpha_{s-2}+\alpha_s < \alpha_1+\cdots+\alpha_s < \alpha_1+\cdots+\alpha_{s-3}+2\alpha_{s-2}+\alpha_{s-1}+\alpha_s\\ &< \cdots < \alpha_1+2\alpha_2+\cdots+2\alpha_{s-2}+\alpha_{s-1}+\alpha_s < \alpha_2 < \cdots < \alpha_{s-2}\\ &< \alpha_{s-2}+\alpha_{s-1} < \underline{2\alpha_{s-2}+\alpha_{s-1}+\alpha_s} < \alpha_{s-2}+\alpha_s < \alpha_{s-2}+\alpha_{s-1}+\alpha_s < \alpha_{s-1} < \alpha_s\,,\end{aligned} \tag{5.9}$$

where the underlined $2\alpha_i+\cdots+2\alpha_{s-2}+\alpha_{s-1}+\alpha_s$ means that it is omitted unless $p([i\ldots(s-1)])=\bar{1}$. Let $\gamma_{ij}=\alpha_i+\cdots+\alpha_j$ for $1\le i\le j<s$, $\beta_{is}=\alpha_i+\cdots+\alpha_{s-2}+\alpha_s$, $\beta_{i,s-1}=\alpha_i+\cdots+\alpha_s$, and finally $\beta_{ij}=\alpha_i+\cdots+\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_{s-2}+\alpha_{s-1}+\alpha_s$ for $i\le j<s-1$ (β_{ii} is omitted unless $p([i\ldots(s-1)])=\bar{1}$). Then, the aforementioned assignment $\gamma\mapsto(\alpha,\beta)$ is explicitly given by:

- for the roots $\gamma=\gamma_{ij}$ with $i<j$, we have $(\alpha,\beta)=(\gamma_{i,j-1},\alpha_j)$;
- for the roots $\gamma=\beta_{is}$ with $i\le s-2$, we have $(\alpha,\beta)=(\gamma_{i,s-2},\alpha_s)$;
- for the roots $\gamma=\beta_{ij}$ with $i<j<s$, we have $(\alpha,\beta)=(\beta_{i,j+1},\alpha_j)$;
- for the underlined roots $\gamma=\beta_{ii}$, we have $(\alpha,\beta)=(\gamma_{i,s-1},\beta_{is})$.

• Case 3: m is even and $\bar{s}=\bar{1}$. In this case, according to [7, Proposition 6.9]:

$$\begin{aligned}\mathsf{L}^+ =&\{[i\ldots j]\,|\,1\le i\le j\le s\}\cup\{[i\ldots(s-1)s(s-1)\ldots j]\,|\,1\le i<j\le s\}\cup\\ &\{[i\ldots(s-1)i\ldots(s-1)s]\,|\,1\le i<s \text{ and } p([i\ldots(s-1)])=\bar{0}\}\,.\end{aligned} \tag{5.10}$$

This results in the following lexicographical order on the reduced root system:

$$\begin{aligned}&\alpha_1 < \alpha_1+\alpha_2 < \cdots < \alpha_1+\cdots+\alpha_{s-1} < \underline{2\alpha_1+\cdots+2\alpha_{s-1}+\alpha_s} < \alpha_1+\cdots+\alpha_s\\ &\quad< \alpha_1+\cdots+\alpha_{s-2}+2\alpha_{s-1}+\alpha_s < \cdots < \alpha_1+2\alpha_2+\cdots+2\alpha_{s-1}+\alpha_s <\\ &\quad\alpha_2 < \cdots < \alpha_{s-1} < \underline{2\alpha_{s-1}+\alpha_s} < \alpha_{s-1}+\alpha_s < \alpha_s\,,\end{aligned} \tag{5.11}$$

where the underlined $2\alpha_i+\cdots+2\alpha_{s-1}+\alpha_s$ means that it is omitted unless $p([i\ldots(s-1)])=\bar{0}$. Let $\gamma_{ij}=\alpha_i+\cdots+\alpha_j$ for $1\le i\le j\le s$ and $\beta_{ij}=\alpha_i+\cdots+\alpha_{j-1}+2\alpha_j+\cdots+2\alpha_{s-1}+\alpha_s$ for $1\le i\le j<s$ (β_{ii} is omitted unless $p([i\ldots(s-1)])=\bar{0}$). Then, the aforementioned assignment $\gamma\mapsto(\alpha,\beta)$ is explicitly given by:

- for the roots $\gamma=\gamma_{ij}$ with $i<j$, we have $(\alpha,\beta)=(\gamma_{i,j-1},\alpha_j)$;
- for the roots $\gamma=\beta_{i,s-1}$ with $1\le i<s-1$, we have $(\alpha,\beta)=(\gamma_{is},\alpha_{s-1})$;
- for the roots $\gamma=\beta_{ij}$ with $i<j<s-1$, we have $(\alpha,\beta)=(\beta_{i,j+1},\alpha_j)$;
- for the underlined roots $\gamma=\beta_{ii}$ with $1\le i<s$, we have $(\alpha,\beta)=(\gamma_{i,s-1},\gamma_{is})$.

5.5. Symmetric pairing.

In this Subsection we shall endow $U_q^+(\mathfrak{osp}(V))^{\otimes 2}$ with a standard *twisted* multiplication:

$$(a\otimes b)(c\otimes d)=(-1)^{|b||c|}q^{-(\deg(b),\deg(c))}(ac)\otimes(bd)$$

for $a,b,c,d\in U_q^+(\mathfrak{osp}(V))$ homogeneous with respect to the $Q^+\times\mathbb{Z}_2$-grading. Following [7, §2.2], we equip $U_q^+(\mathfrak{osp}(V))$ with a *twisted* coproduct $\Delta^{\mathrm{tw}}\colon U_q^+(\mathfrak{osp}(V))\to U_q^+(\mathfrak{osp}(V))^{\otimes 2}$ defined by

$$\Delta^{\mathrm{tw}}(e_i)=e_i\otimes 1+1\otimes e_i \qquad \forall\, i\in I\,.$$

Furthermore, we have the following result of [7, Proposition 2.4]:

Proposition 5.5. *There exists a unique non-degenerate symmetric bilinear pairing*

$$(\cdot,\cdot)^{\mathrm{tw}}\colon U_q^+(\mathfrak{osp}(V))\times U_q^+(\mathfrak{osp}(V))\longrightarrow\mathbb{C}(q)$$

satisfying

$$(1,1)^{\mathrm{tw}}=1,\quad (e_i,e_j)^{\mathrm{tw}}=\delta_{ij},\quad (x,yy')^{\mathrm{tw}}=(\Delta^{\mathrm{tw}}(x),y\otimes y')^{\mathrm{tw}}$$

for any $i,j\in I$ and $x,y,y'\in U_q^+(\mathfrak{osp}(V))$, where $(x'\otimes x'',y'\otimes y'')^{\mathrm{tw}}=(x',y')^{\mathrm{tw}}(x'',y'')^{\mathrm{tw}}$.

Evoking the isomorphism $U_q^+(\mathfrak{osp}(V))\simeq\mathsf{U}$, see Proposition 5.1, we shall use the same notation for the symmetric bilinear pairing on U satisfying similar properties:

$$(\cdot,\cdot)^{\mathrm{tw}}\colon \mathsf{U}\times\mathsf{U}\longrightarrow\mathbb{C}(q)\,.$$

5.6. **Pairing of Lyndon basis.**

We shall now summarize the key results of [7, §5–6] in the form relevant to us. For any $w \in \mathsf{W}^+$, consider its canonical factorization $w = w_1 w_2 \dots w_d$ into dominant Lyndon words (5.2) and define

$$\widetilde{R}_w = R_{w_d} \diamond R_{w_{d-1}} \diamond \dots \diamond R_{w_1} .$$

The following orthogonality result is established in [7, Theorem 5.7] (we note that while the authors work with E_w in [7], they are just multiples of $\widetilde{R}_w$, as follows from [7, §6]):

Proposition 5.6. *Let $\ell, w \in \mathsf{W}^+$. Then $(\widetilde{R}_\ell, \widetilde{R}_w)^{\mathrm{tw}} = 0$ unless $\ell = w$. Moreover, if $\ell = \ell_1^{n_1} \ell_2^{n_2} \dots \ell_d^{n_d}$ with $\ell_1 > \ell_2 > \dots > \ell_d$ is the canonical factorization of ℓ into dominant Lyndon words, then:*

$$(\widetilde{R}_\ell, \widetilde{R}_\ell)^{\mathrm{tw}} = \prod_{t=1}^{d} \Big(C_{\ell_t, n_t} \cdot ((R_{\ell_t}, R_{\ell_t})^{\mathrm{tw}})^{n_t} \Big) \tag{5.12}$$

with

$$C_{\ell,p} = \prod_{k=1}^{p} \frac{1 - \big((-1)^{p(\ell)} q^{-(\deg(\ell),\deg(\ell))}\big)^k}{1 - (-1)^{p(\ell)} q^{-(\deg(\ell),\deg(\ell))}} \qquad \text{for any} \quad \ell \in \mathsf{L}^+,\, p \in \mathbb{Z}_{\geq 0} .$$

The explicit computation of the pairings $(R_\ell, R_\ell)^{\mathrm{tw}}$ for $\ell \in \mathsf{L}^+$ has been carried out in [7, §6], while for $\ell \in I$ we trivially have $(R_\ell, R_\ell)^{\mathrm{tw}} = 1$ by Proposition 5.5. Thus, we shall summarize these formulas for words ℓ of length > 1 in three lemmas below (also correcting several typos from [7]). Following [7, (3.13)], we shall use the following notation for $\alpha = \alpha_{i_1} + \dots + \alpha_{i_r} \in Q^+$:

$$P(\alpha) = \sum_{p<t} |\mathsf{e}_{i_p}||\mathsf{e}_{i_t}| \qquad \text{and} \qquad N(\alpha) = \sum_{p<t} (\alpha_{i_p}, \alpha_{i_t}) . \tag{5.13}$$

As in Subsection 5.4, we shall work with a specific ordering $1 < 2 < \dots < s$ on the alphabet I.

Lemma 5.7. *Let m be odd. Evoking the description of L^+ from (5.6), we have:*

- *If $\ell = [i \dots j]$ with $1 \leq i < j \leq s$, then*

$$(R_\ell, R_\ell)^{\mathrm{tw}} = \prod_{k=i}^{j-1} (\alpha_k, \alpha_{k+1}) \cdot (q - q^{-1})^{j-i} \cdot q^{N(\deg \ell)} .$$

- *If $\ell = [i \dots ss \dots j]$ with $1 \leq i < j \leq s$, then*

$$(R_\ell, R_\ell)^{\mathrm{tw}} = (-1)^{p([j \dots s])} \cdot \prod_{k=i}^{j-2} (\alpha_k, \alpha_{k+1}) \cdot (q - q^{-1})^{2s+1-i-j} \cdot q^{N(\deg \ell)} .$$

Lemma 5.8. *Let m be even and $\bar{s} = \bar{0}$. Evoking the description of L^+ from (5.8), we have:*

- *If $\ell = [i \dots j]$ with $1 \leq i < j \leq s-1$, then*

$$(R_\ell, R_\ell)^{\mathrm{tw}} = \prod_{k=i}^{j-1} (\alpha_k, \alpha_{k+1}) \cdot (q - q^{-1})^{j-i} \cdot q^{N(\deg \ell)} .$$

- *If $\ell = [i \dots (s-2)s]$ with $1 \leq i \leq s-2$, then*

$$(R_\ell, R_\ell)^{\mathrm{tw}} = \prod_{k=i}^{s-2} (\alpha_k, \alpha_{k+1}) \cdot (q - q^{-1})^{s-i-1} \cdot q^{N(\deg \ell)} .$$

- *If $\ell = [i \dots (s-2)s(s-1) \dots j]$ with $1 \leq i < j \leq s-1$, then*

$$(R_\ell, R_\ell)^{\mathrm{tw}} = -\prod_{k=i}^{j-1} (\alpha_k, \alpha_{k+1}) \cdot (q - q^{-1})^{2s-1-i-j} \cdot q^{N(\deg \ell)} .$$

- *If $\ell = [i \dots (s-1) i \dots (s-2)s]$ with $1 \leq i \leq s-1$ and $p([i \dots (s-1)]) = \bar{1}$, then*

$$(R_\ell, R_\ell)^{\mathrm{tw}} = -(q - q^{-1})^{2s-2i-2} (q^2 - q^{-2}) \cdot q^{N(\deg \ell)} .$$

Lemma 5.9. *Let m be even and $\bar{s} = \bar{1}$. Evoking the description of L^+ from (5.10), we have:*

- *If* $\ell = [i \dots j]$ *with* $1 \leq i < j \leq s-1$*, then*
$$(R_\ell, R_\ell)^{\mathrm{tw}} = \prod_{k=i}^{j-1} (\alpha_k, \alpha_{k+1}) \cdot (q-q^{-1})^{j-i} \cdot q^{N(\deg \ell)} \,.$$
- *If* $\ell = [i \dots s]$ *with* $1 \leq i \leq s-1$*, then*
$$(R_\ell, R_\ell)^{\mathrm{tw}} = \frac{1}{2} \prod_{k=i}^{s-1} (\alpha_k, \alpha_{k+1}) \cdot (q-q^{-1})^{s-i-1} (q^2-q^{-2}) \cdot q^{N(\deg \ell)} \,.$$
- *If* $\ell = [i \dots s \dots j]$ *with* $1 \leq i < j \leq s-1$*, then*
$$(R_\ell, R_\ell)^{\mathrm{tw}} = \prod_{k=i}^{j-1} (\alpha_k, \alpha_{k+1}) \cdot (q-q^{-1})^{2s-1-i-j} (q^2-q^{-2}) \cdot q^{N(\deg \ell)} \,.$$
- *If* $\ell = [i \dots (s-1) i \dots (s-1) s]$ *with* $1 \leq i \leq s-1$ *and* $p([i \dots (s-1)]) = \bar{0}$*, then*
$$(R_\ell, R_\ell)^{\mathrm{tw}} = (q-q^{-1})^{2s-2i-2} (q^2-q^{-2})^2 \cdot q^{N(\deg \ell)} \,.$$

Remark 5.10. We warn the reader that [7, §6] contains various small errors in the constants featuring in their elements $R_{\mathbf{i}}, E_{\mathbf{i}}, E^*_{\mathbf{i}}$ and respectively in the pairing $(E_{\mathbf{i}}, E_{\mathbf{i}})$. In particular:

- the second bullet of Lemma 5.7 corrects a sign error in [7, Corollary 6.7(2)] for the dominant Lyndon word $\mathbf{i} = (i, \dots, M, M, \dots, j+1)$,
- the last two bullets of Lemma 5.8 correct a sign error in [7, Corollary 6.14(2)] for $\mathbf{i} = (i, \dots, M-2, M, M-1, \dots, j+1)$ and $\mathbf{i} = (i, \dots, M-1, i, \dots, M-2, M)$,
- the last three bullets of Lemma 5.9 correct various errors in [7, Corollary 6.11(2)], by adding the missing factors $q^2 - q^{-2}$, or $(q-q^{-1})^{-1}$, or a power of q.

5.7. Comparison of pairings.

In this Subsection, we establish the exact relation between the bialgebra pairing
$$(\cdot,\cdot)_J \colon U_q^{\leq}(\mathfrak{osp}(V)) \times U_q^{\geq}(\mathfrak{osp}(V)) \longrightarrow \mathbb{C}(q^{\pm 1/4})$$
from Proposition 4.1 and the symmetric pairing
$$(\cdot,\cdot)^{\mathrm{tw}} \colon U_q^{+}(\mathfrak{osp}(V)) \times U_q^{+}(\mathfrak{osp}(V)) \longrightarrow \mathbb{C}(q)$$
from Proposition 5.5. To this end, we first define a new pairing
$$\{\cdot,\cdot\} \colon U_q^{\geq}(\mathfrak{osp}(V)) \times U_q^{\geq}(\mathfrak{osp}(V)) \longrightarrow \mathbb{C}(q^{\pm 1/4}) \quad \text{via} \quad \{y,x\} = (-1)^{P(\deg(x))} (\omega(y), x)_J \,, \qquad (5.14)$$
cf. (5.13), where $\omega \colon U_q(\mathfrak{osp}(V)) \to U_q(\mathfrak{osp}(V))$ is the $\mathbb{C}(q)$-superalgebra automorphism mapping
$$e_i \mapsto (-1)^{|e_i|} f_i \,, \quad f_i \mapsto e_i \,, \quad q^{\pm h_i/2} \mapsto q^{\mp h_i/2} \qquad \forall\, i \in I \,. \qquad (5.15)$$
We note that
$$\Delta^{J,\mathrm{op}}(\omega(x)) = (\omega \otimes \omega)(\Delta^J(x)) \qquad \forall\, x \in U_q(\mathfrak{osp}(V)) \,. \qquad (5.16)$$
Combining Proposition 4.1 with (5.16), one can easily check the following properties:
$$\{1,1\} = 1 \,, \ \{e_i, e_j\} = \delta_{ij}/(q^{-1}-q) \,, \ \{q^{h_i/2}, q^{h_j/2}\} = q^{a_{ij}/4} \,, \ \{e_i, q^{h_j/2}\} = \{q^{h_j/2}, e_i\} = 0 \,,$$
$$\{yy', x\} = \{y \otimes y', \Delta^J(x)\} \,, \qquad \{y, xx'\} = \{\Delta^J(y), x \otimes x'\}$$
for any $i, j \in I$ and $x, x', y, y' \in U_q^{\geq}(\mathfrak{osp}(V))$, where we set $\{y \otimes y', x \otimes x'\} = \{y,x\}\{y',x'\}$.

Proposition 5.11. *For any* $x, y \in U_q^{+}(\mathfrak{osp}(V))$ *homogeneous with respect to the* $Q \times \mathbb{Z}_2$*-grading, the following equality holds:*
$$\{y,x\} = \bar{\sigma}\Big((\sigma(y), \sigma(x))^{\mathrm{tw}} \Big) / (q^{-1}-q)^{\mathrm{ht}(\deg(x))} \,, \qquad (5.17)$$
cf. (4.23, 4.24), where $\mathrm{ht}(\cdot)$ *is the height function defined via* $\mathrm{ht}(k_1\alpha_1 + \dots + k_s\alpha_s) = k_1 + \dots + k_s$.

Proof. We shall first evaluate explicitly both $(y,x)^{\mathrm{tw}}$ and $\{y,x\}$ for the case when $x = e_{j_1}\cdots e_{j_{d'}}$ and $y = e_{i_1}\cdots e_{i_d}$ are monomials. For degree reasons, we obviously have $(y,x)^{\mathrm{tw}} = 0 = \{y,x\}$ unless $d = d'$, hence we shall assume now that $d = d'$. Direct computation then shows that

$$\begin{aligned}
&(e_{i_1}\cdots e_{i_d}, e_{j_1}\cdots e_{j_d})^{\mathrm{tw}} \\
&= \Big(e_{i_1}\otimes\cdots\otimes e_{i_d}, (\Delta^{\mathrm{tw}})^{(n-1)}(e_{j_1})\cdots(\Delta^{\mathrm{tw}})^{(n-1)}(e_{j_d})\Big)^{\mathrm{tw}} \\
&= \left(e_{i_1}\otimes\cdots\otimes e_{i_d}, \sum_{\sigma\in S_d}\overrightarrow{\prod_{1\le k\le d}}\Big(1^{\otimes(\sigma(k)-1)}\otimes e_{j_k}\otimes 1^{\otimes(n-\sigma(k))}\Big)\right)^{\mathrm{tw}} \\
&= \left(e_{i_1}\otimes\cdots\otimes e_{i_d}, \sum_{\sigma\in S_d}\prod_{k<l}^{\sigma(k)>\sigma(l)}\Big((-1)^{|e_{j_k}||e_{j_l}|}q^{-(\alpha_{j_k},\alpha_{j_l})}\Big)\, e_{j_{\sigma^{-1}(1)}}\otimes\cdots\otimes e_{j_{\sigma^{-1}(d)}}\right)^{\mathrm{tw}} \qquad (5.18)\\
&= \sum_{\sigma\in S_d}\prod_{k<l}^{\sigma(k)>\sigma(l)}\Big((-1)^{|e_{j_k}||e_{j_l}|}q^{-(\alpha_{j_k},\alpha_{j_l})}\Big)\cdot(e_{i_1},e_{j_{\sigma^{-1}(1)}})^{\mathrm{tw}}\cdots(e_{i_d},e_{j_{\sigma^{-1}(d)}})^{\mathrm{tw}} \\
&= \sum_{\sigma\in S_d}\prod_{k<l}^{\sigma(k)>\sigma(l)}\Big((-1)^{|e_{j_k}||e_{j_l}|}q^{-(\alpha_{j_k},\alpha_{j_l})}\Big)\cdot\delta_{i_1,j_{\sigma^{-1}(1)}}\cdots\delta_{i_d,j_{\sigma^{-1}(d)}}\,,
\end{aligned}$$

where $(\Delta^{\mathrm{tw}})^{(n-1)}\colon U_q^+(\mathfrak{osp}(V))\to U_q^+(\mathfrak{osp}(V))^{\otimes n}$ is the map obtained by applying coproduct Δ^{tw} iteratively $n-1$ times, the definition of which is well-defined by coassociativity. Also, the arrow $\to$ over the product sign implies that the multiplication is done in the increasing order of the index.

Analogously, we obtain

$$\begin{aligned}
&\{e_{i_1}\cdots e_{i_d}, e_{j_1}\cdots e_{j_d}\} \\
&= \Big\{e_{i_1}\otimes\cdots\otimes e_{i_d}, (\Delta^{J})^{(n-1)}(e_{j_1})\cdots(\Delta^{J})^{(n-1)}(e_{j_d})\Big\} \\
&= \left\{e_{i_1}\otimes\cdots\otimes e_{i_d}, \sum_{\sigma\in S_d}\overrightarrow{\prod_{1\le k\le d}}\Big((q^{h_{j_k}})^{\otimes(\sigma(k)-1)}\otimes e_{j_k}\otimes 1^{\otimes(n-\sigma(k))}\Big)\right\} \\
&= \left\{e_{i_1}\otimes\cdots\otimes e_{i_d}, \sum_{\sigma\in S_d}\prod_{k<l}^{\sigma(k)>\sigma(l)}\Big((-1)^{|e_{j_k}||e_{j_l}|}q^{(\alpha_{j_k},\alpha_{j_l})}\Big)\cdot\bigotimes_{1\le l\le d}\Big(e_{j_{\sigma^{-1}(l)}}\prod_{\sigma(k)>l}q^{h_{j_k}}\Big)\right\} \qquad (5.19)\\
&= \sum_{\sigma\in S_d}\prod_{k<l}^{\sigma(k)>\sigma(l)}\Big((-1)^{|e_{j_k}||e_{j_l}|}q^{(\alpha_{j_k},\alpha_{j_l})}\Big)\cdot\prod_{1\le l\le d}\Big\{e_{i_l}, e_{j_{\sigma^{-1}(l)}}\prod_{\sigma(k)>l}q^{h_{j_k}}\Big\} \\
&= \sum_{\sigma\in S_d}\prod_{k<l}^{\sigma(k)>\sigma(l)}\Big((-1)^{|e_{j_k}||e_{j_l}|}q^{(\alpha_{j_k},\alpha_{j_l})}\Big)\cdot\delta_{i_1,j_{\sigma^{-1}(1)}}\cdots\delta_{i_d,j_{\sigma^{-1}(d)}}\cdot(q^{-1}-q)^{-d}\,,
\end{aligned}$$

where the map $(\Delta^J)^{(n-1)}\colon U_q^{\ge}(\mathfrak{osp}(V))\to U_q^{\ge}(\mathfrak{osp}(V))^{\otimes n}$ is defined similarly to $(\Delta^{\mathrm{tw}})^{(n-1)}$.

Comparing the above two formulas (5.18) and (5.19), we obtain the validity of (5.17) in the case when both x and y are monomials. The generalization to the $\mathbb{C}(q)$-linear combinations of monomials is now a consequence of our definitions (4.23) and (4.24). This completes the proof. $\square$

Combining Proposition 5.11 and formula (5.14), we thus obtain:

Corollary 5.12. *For any $x\in U_q^+(\mathfrak{osp}(V))$ and $y\in U_q^-(\mathfrak{osp}(V))$ homogeneous with respect to the $Q\times\mathbb{Z}_2$-grading, the following equality holds:*

$$(y,x)_J = (-1)^{P(\deg(x))}\cdot\bar{\sigma}\Big((\sigma(\omega^{-1}(y)),\sigma(x))^{\mathrm{tw}}\Big)/(q^{-1}-q)^{\mathrm{ht}(\deg(x))}\,.$$

5.8. **Factorization formula.**

For $\gamma\in\bar{\Phi}^+$, we define the (quantum) *root vectors* e_γ, f_γ via

$$e_{\alpha_i} = e_i\,,\qquad f_{\alpha_i} = f_i\,,$$

while for $\gamma \in \bar{\Phi}^+ \setminus \{\alpha_i\}_{i=1}^s$ we set

$$\begin{aligned} e_\gamma &= e_\alpha e_\beta - (-1)^{|e_\alpha||e_\beta|} q^{(\alpha,\beta)} e_\beta e_\alpha \,, \\ f_\gamma &= f_\beta f_\alpha - (-1)^{|f_\alpha||f_\beta|} q^{-(\alpha,\beta)} f_\alpha f_\beta \,, \end{aligned} \tag{5.20}$$

with the roots $\alpha, \beta \in \bar{\Phi}^+$ defined via $\alpha = \mathrm{l}(\ell_1)$ and $\beta = \mathrm{l}(\ell_2)$, where $\ell = \ell_1\ell_2$ is the costandard factorization of the dominant Lyndon word $\ell = \mathrm{l}^{-1}(\gamma)$, see Proposition 5.3.

The explicit formulas for the pairing $(f_\gamma, e_\gamma)_J$ are derived in the following lemmas:

Lemma 5.13. *For odd m, we have:*

- *If $\gamma = \varepsilon_i - \varepsilon_j$ with $1 \le i < j \le s+1$, then*
$$(f_\gamma, e_\gamma)_J = (-1)^{\bar{i}+\cdots+\bar{j}} \cdot (q^{-1} - q)^{-1} \,.$$
- *If $\gamma = \varepsilon_i + \varepsilon_j$ with $1 \le i < j \le s$, then*
$$(f_\gamma, e_\gamma)_J = (-1)^{\bar{i}+\cdots+\overline{j-1}} \cdot (q^{-1} - q)^{-1} \,.$$

Lemma 5.14. *For even m and $\overline{s} = \bar{0}$, we have:*

- *If $\gamma = \varepsilon_i - \varepsilon_j$ with $1 \le i < j \le s$, then*
$$(f_\gamma, e_\gamma)_J = (-1)^{\bar{i}+\cdots+\bar{j}} \cdot (q^{-1} - q)^{-1} \,.$$
- *If $\gamma = \varepsilon_i + \varepsilon_j$ with $1 \le i < j \le s$, then*
$$(f_\gamma, e_\gamma)_J = (-1)^{\bar{i}+\cdots+\overline{j-1}} \cdot (q^{-1} - q)^{-1} \,.$$
- *If $\gamma = 2\varepsilon_i$ with $1 \le i \le s-1$ and $\bar{i} = \bar{1}$, then*
$$(f_\gamma, e_\gamma)_J = \frac{q^{-2} - q^2}{(q^{-1} - q)^2} \,.$$

Lemma 5.15. *For even m and $\overline{s} = \bar{1}$, we have:*

- *If $\gamma = \varepsilon_i - \varepsilon_j$ with $1 \le i < j \le s$, then*
$$(f_\gamma, e_\gamma)_J = (-1)^{\bar{i}+\cdots+\bar{j}} \cdot (q^{-1} - q)^{-1} \,.$$
- *If $\gamma = \varepsilon_i + \varepsilon_j$ with $1 \le i < j \le s$, then*
$$(f_\gamma, e_\gamma)_J = (-1)^{\bar{i}+\cdots+\overline{j-1}} \cdot \frac{q^{-2} - q^2}{(q^{-1} - q)^2} \,.$$
- *If $\gamma = 2\varepsilon_i$ with $1 \le i \le s-1$ and $\bar{i} = \bar{1}$, then*
$$(f_\gamma, e_\gamma)_J = \frac{(q^{-2} - q^2)^2}{(q^{-1} - q)^3} \,.$$

Proof of Lemmas 5.13–5.15. According to (5.13), we have:

$$P(\alpha+\beta) = P(\alpha) + P(\beta) + |e_\alpha||e_\beta| \qquad \text{and} \qquad N(\alpha+\beta) = N(\alpha) + N(\beta) + (e_\alpha, e_\beta) \,. \tag{5.21}$$

Combining these equalities with formulas (5.15, 5.20), one easily verifies the formula

$$\omega^{-1}(f_\gamma) = (-1)^{\mathrm{ht}(\gamma)-1} (-1)^{|e_\gamma|} (-1)^{P(\gamma)} q^{-N(\gamma)} e_\gamma \qquad \forall\, \gamma \in \bar{\Phi}^+ \tag{5.22}$$

by an induction on the height $\mathrm{ht}(\gamma)$. Combining this result with Corollary 5.12, we obtain:

$$(f_\gamma, e_\gamma)_J = (-1)^{\mathrm{ht}(\gamma)-1} (-1)^{|e_\gamma|} q^{-N(\gamma)} \cdot \bar{\sigma}\Big(\big(\sigma(e_\gamma), \sigma(e_\gamma)\big)^{\mathrm{tw}} \Big) / (q^{-1} - q)^{\mathrm{ht}(\gamma)} \,. \tag{5.23}$$

To evaluate the pairing $(\sigma(e_\gamma), \sigma(e_\gamma))^{\mathrm{tw}}$, we recall the $\mathbb{C}(q)$-linear endomorphism $\mathcal{T}$ of $U_q^+(\mathfrak{osp}(V))$ from [7, Proposition 2.2(1)] defined by

$$\mathcal{T}(e_i) = e_i \quad \forall\, i \in I \qquad \text{and} \qquad \mathcal{T}(xy) = \mathcal{T}(y)\mathcal{T}(x) \quad \forall\, x, y \in U_q^+(\mathfrak{osp}(V)) \,.$$

Arguing by an induction on $\mathrm{ht}(\gamma)$ again, let us now prove the following formula:

$$\sigma(e_\gamma) = (-1)^{\mathrm{ht}(\gamma)-1} (-1)^{P(\gamma)} q^{-N(\gamma)} \cdot \mathcal{T}(e_\gamma) \qquad \text{for any} \quad \gamma \in \bar{\Phi}^+ \,. \tag{5.24}$$

This equality is clear when $\mathrm{ht}(\gamma)=1$. For any root γ with $\mathrm{ht}(\gamma)>1$, we consider the pair of roots $\alpha,\beta\in\bar{\Phi}^+$ satisfying $e_\gamma=[\![e_\alpha,e_\beta]\!]$, cf. (2.22) and (5.20). Since $\mathrm{ht}(\alpha),\mathrm{ht}(\beta)<\mathrm{ht}(\gamma)$, we may assume by the induction hypothesis that (5.24) holds for the roots α and β, so that:

$$\begin{aligned}\sigma(e_\gamma)&=\sigma(e_\alpha)\sigma(e_\beta)-(-1)^{|e_\alpha||e_\beta|}q^{-(\alpha,\beta)}\cdot\sigma(e_\beta)\sigma(e_\alpha)\\&=(-1)^{\mathrm{ht}(\gamma)-2}(-1)^{P(\alpha)+P(\beta)}q^{-N(\alpha)-N(\beta)}\cdot\Big(\mathcal{T}(e_\beta e_\alpha)-(-1)^{|e_\alpha||e_\beta|}q^{-(\alpha,\beta)}\cdot\mathcal{T}(e_\alpha e_\beta)\Big)\\&\overset{(5.21)}{=}(-1)^{\mathrm{ht}(\gamma)-1}(-1)^{P(\gamma)}q^{-N(\gamma)}\cdot\mathcal{T}(e_\gamma).\end{aligned}$$

This proves the induction step, hence completes the proof of (5.24).

Furthermore, the direct formula (5.18) shows that

$$(\mathcal{T}(x),\mathcal{T}(y))^{\mathrm{tw}}=(x,y)^{\mathrm{tw}} \tag{5.25}$$

for any monomials $x=e_{i_1}\cdots e_{i_d}, y=e_{j_1}\cdots e_{j_{d'}}$, and hence (5.25) holds for any $x,y\in U_q^+(\mathfrak{osp}(V))$, as $\mathcal{T}$ is $\mathbb{C}(q)$-linear. Combining (5.23)–(5.25), we finally obtain:

$$(f_\gamma,e_\gamma)_J=(-1)^{\mathrm{ht}(\gamma)-1}(-1)^{|e_\gamma|}q^{N(\gamma)}\cdot\bar{\sigma}\Big((e_\gamma,e_\gamma)^{\mathrm{tw}}\Big)/(q^{-1}-q)^{\mathrm{ht}(\gamma)}\,. \tag{5.26}$$

In view of this equality, Lemmas 5.13–5.15 are just direct consequences of Lemmas 5.7–5.9. $\square$

We are now ready to construct dual bases of $U_q^\pm(\mathfrak{osp}(V))$ with respect to the bialgebra pairing (4.3) (which relies on the orthogonality result of Proposition 5.6, proved in [7, Theorem 5.7]):

Theorem 5.16. *(a) The ordered products*

$$\left\{\overleftarrow{\prod_{\gamma\in\bar{\Phi}^+}} e_\gamma^{m_\gamma}\,\Big|\,{}^{m_\gamma\geq 0}_{m_\gamma\leq 1\ \text{if}\ \gamma\in\bar{\Phi}_{\bar{1}}\ \text{is isotropic}}\right\}\quad\text{and}\quad\left\{\overleftarrow{\prod_{\gamma\in\bar{\Phi}^+}} f_\gamma^{m_\gamma}\,\Big|\,{}^{m_\gamma\geq 0}_{m_\gamma\leq 1\ \text{if}\ \gamma\in\bar{\Phi}_{\bar{1}}\ \text{is isotropic}}\right\}$$

are bases for $U_q^+(\mathfrak{osp}(V))$ and $U_q^-(\mathfrak{osp}(V))$, respectively. Henceforth, the arrow $\leftarrow$ over the product signs refers to the total order (5.3) on $\bar{\Phi}^+$, thus ordering the elements of $\bar{\Phi}^+$ in decreasing order.

(b) The bialgebra pairing (4.3) is orthogonal with respect to these bases. More explicitly, we have:

$$\left(\overleftarrow{\prod_{\gamma\in\bar{\Phi}^+}} f_\gamma^{n_\gamma},\overleftarrow{\prod_{\gamma\in\bar{\Phi}^+}} e_\gamma^{m_\gamma}\right)_J=(-1)^{\sum_{\gamma<\gamma'}m_\gamma m_{\gamma'}|e_\gamma||e_{\gamma'}|}\cdot\prod_{\gamma\in\bar{\Phi}^+}\Big(\delta_{n_\gamma,m_\gamma}(f_\gamma^{m_\gamma},e_\gamma^{m_\gamma})_J\Big) \tag{5.27}$$

and

$$(f_\gamma^k,e_\gamma^k)_J=(-1)^{\frac{k(k-1)}{2}|e_\gamma|}\cdot\bar{\sigma}(\mathsf{C}_{\gamma,k})\cdot(f_\gamma,e_\gamma)_J^k\,, \tag{5.28}$$

where

$$\mathsf{C}_{\gamma,k}=C_{\mathrm{l}^{-1}(\gamma),k}=\prod_{t=1}^{k}\frac{1-\big((-1)^{|e_\gamma|}q^{-(\gamma,\gamma)}\big)^t}{1-(-1)^{|e_\gamma|}q^{-(\gamma,\gamma)}}\,,$$

cf. Proposition 5.6.

Remark 5.17. This result is known in classical BCD-types where it follows from Lusztig's orthogonal bases (see [19, §8.30]) associated with the reduced decomposition of the longest element $w_0\in W$ that matches (see [35]) the lexicographical convex order (5.3) on the set of positive roots Φ^+. In this context, Lusztig's root vectors (defined via the braid group action) match the above q-commutator construction of (5.20), up to constants computed explicitly in [5, Theorem 4.2].

Proof. (a) First, we note that $e_\gamma=\Psi^{-1}(R_{\mathrm{l}^{-1}(\gamma)})$ for all $\gamma\in\bar{\Phi}^+$. Therefore, the preimage of the Lyndon basis (5.5) of U under the isomorphism $\Psi\colon U_q^+(\mathfrak{osp}(V))\xrightarrow{\sim}\mathsf{U}$ provides (up to rescaling) the claimed basis of $U_q^+(\mathfrak{osp}(V))$. Evoking (5.22), the result for $U_q^-(\mathfrak{osp}(V))$ can be carried out from that for $U_q^+(\mathfrak{osp}(V))$ through the isomorphism $\omega\colon U_q^-(\mathfrak{osp}(V))\to U_q^+(\mathfrak{osp}(V))$ of (5.15).

(b) Let us first compute $(f_\gamma^k,e_\gamma^k)_J$. Following the above proof of Lemmas 5.13–5.15, we obtain:

$$\begin{aligned}(f_\gamma^k,e_\gamma^k)_J&=(-1)^{P(k\gamma)}\cdot\Big((-1)^{\mathrm{ht}(\gamma)-1}(-1)^{|e_\gamma|}(-1)^{P(\gamma)}q^{-N(\gamma)}(q^{-1}-q)^{-\mathrm{ht}(\gamma)}\Big)^k\cdot\bar{\sigma}\left((\sigma(e_\gamma)^k,\sigma(e_\gamma)^k)^{\mathrm{tw}}\right)\\&=(-1)^{P(k\gamma)}\cdot\Big((-1)^{\mathrm{ht}(\gamma)-1}(-1)^{|e_\gamma|}(-1)^{P(\gamma)}q^{N(\gamma)}(q^{-1}-q)^{-\mathrm{ht}(\gamma)}\Big)^k\cdot\bar{\sigma}\left((e_\gamma^k,e_\gamma^k)^{\mathrm{tw}}\right).\end{aligned} \tag{5.29}$$

But evoking (5.12), we note that

$$(e_\gamma^k, e_\gamma^k)^{\mathrm{tw}} = (R^k_{\mathrm{l}^{-1}(\gamma)}, R^k_{\mathrm{l}^{-1}(\gamma)})^{\mathrm{tw}} = C_{\mathrm{l}^{-1}(\gamma),k} \cdot \left((R_{\mathrm{l}^{-1}(\gamma)}, R_{\mathrm{l}^{-1}(\gamma)})^{\mathrm{tw}}\right)^k = \mathsf{C}_{\gamma,k} \cdot \left((e_\gamma, e_\gamma)^{\mathrm{tw}}\right)^k. \quad (5.30)$$

Combining the above two formulas with (5.26), we get:

$$(f_\gamma^k, e_\gamma^k)_J = (-1)^{P(k\gamma)}(-1)^{k\cdot P(\gamma)} \cdot \bar\sigma(\mathsf{C}_{\gamma,k}) \cdot (f_\gamma, e_\gamma)_J^k = (-1)^{\frac{k(k-1)}{2}|e_\gamma|} \cdot \bar\sigma(\mathsf{C}_{\gamma,k}) \cdot (f_\gamma, e_\gamma)_J^k \,,$$

which establishes (5.28).

The proof of (5.27) is analogous. To this end, we first note:

$$\left(\overleftarrow{\prod_{\gamma\in\bar\Phi^+}} f_\gamma^{n_\gamma}, \overleftarrow{\prod_{\gamma\in\bar\Phi^+}} e_\gamma^{m_\gamma}\right)_J = (-1)^{P\left(\sum_\gamma m_\gamma\gamma\right)} \cdot \left(\prod_{\gamma\in\bar\Phi^+} \left((-1)^{\mathrm{ht}(\gamma)-1}(-1)^{|e_\gamma|}(-1)^{P(\gamma)}q^{-N(\gamma)}(q^{-1}-q)^{-\mathrm{ht}(\gamma)}\right)^{n_\gamma}\right)$$
$$\cdot\, \bar\sigma\left(\left(\sigma\left(\overleftarrow{\prod_{\gamma\in\bar\Phi^+}} e_\gamma^{n_\gamma}\right), \sigma\left(\overleftarrow{\prod_{\gamma\in\bar\Phi^+}} e_\gamma^{m_\gamma}\right)\right)^{\mathrm{tw}}\right).$$

Moreover, according to Proposition 5.6, we have:

$$\left(\overrightarrow{\prod_{\gamma\in\bar\Phi^+}} e_\gamma^{n_\gamma}, \overrightarrow{\prod_{\gamma\in\bar\Phi^+}} e_\gamma^{m_\gamma}\right)^{\mathrm{tw}} = \prod_{\gamma\in\bar\Phi^+} \left(\delta_{n_\gamma,m_\gamma} \cdot \mathsf{C}_{\gamma,m_\gamma} \cdot ((e_\gamma, e_\gamma)^{\mathrm{tw}})^{m_\gamma}\right)$$

(we note that the products in the left hand side are taken in increasing order!), so that:

$$\left(\sigma\left(\overleftarrow{\prod_{\gamma\in\bar\Phi^+}} e_\gamma^{n_\gamma}\right), \sigma\left(\overleftarrow{\prod_{\gamma\in\bar\Phi^+}} e_\gamma^{m_\gamma}\right)\right)^{\mathrm{tw}} \overset{(5.25)}{=} \left(\mathcal{T}\sigma\left(\overleftarrow{\prod_{\gamma\in\bar\Phi^+}} e_\gamma^{n_\gamma}\right), \mathcal{T}\sigma\left(\overleftarrow{\prod_{\gamma\in\bar\Phi^+}} e_\gamma^{m_\gamma}\right)\right)^{\mathrm{tw}}$$
$$= \left(\overrightarrow{\prod_{\gamma\in\bar\Phi^+}} (\mathcal{T}\sigma(e_\gamma))^{n_\gamma}, \overrightarrow{\prod_{\gamma\in\bar\Phi^+}} (\mathcal{T}\sigma(e_\gamma))^{m_\gamma}\right)^{\mathrm{tw}}$$
$$\overset{(5.24)}{=} \left(\overrightarrow{\prod_{\gamma\in\bar\Phi^+}} e_\gamma^{n_\gamma}, \overrightarrow{\prod_{\gamma\in\bar\Phi^+}} e_\gamma^{m_\gamma}\right)^{\mathrm{tw}} \cdot \prod_{\gamma\in\bar\Phi^+} \left((-1)^{\mathrm{ht}(\gamma)-1}(-1)^{P(\gamma)}q^{-N(\gamma)}\right)^{n_\gamma+m_\gamma}$$
$$= \prod_{\gamma\in\bar\Phi^+} \left(\delta_{n_\gamma,m_\gamma} \cdot \mathsf{C}_{\gamma,m_\gamma} \cdot q^{-2m_\gamma N(\gamma)} \cdot ((e_\gamma, e_\gamma)^{\mathrm{tw}})^{m_\gamma}\right)$$
$$\overset{(5.30)}{=} \prod_{\gamma\in\bar\Phi^+} \left(\delta_{n_\gamma,m_\gamma} \cdot q^{-2m_\gamma N(\gamma)} \cdot (e_\gamma^{m_\gamma}, e_\gamma^{m_\gamma})^{\mathrm{tw}}\right).$$

Combining the formulas above with (5.29), we get:

$$\left(\overleftarrow{\prod_{\gamma\in\bar\Phi^+}} f_\gamma^{n_\gamma}, \overleftarrow{\prod_{\gamma\in\bar\Phi^+}} e_\gamma^{m_\gamma}\right)_J = (-1)^{P\left(\sum_\gamma m_\gamma\gamma\right)} \cdot \prod_{\gamma\in\bar\Phi^+} \left(\delta_{n_\gamma,m_\gamma} \cdot (-1)^{P(m_\gamma\gamma)} \cdot (f_\gamma^{m_\gamma}, e_\gamma^{m_\gamma})_J\right)$$
$$= (-1)^{\sum_{\gamma<\gamma'} m_\gamma m_{\gamma'}|e_\gamma||e_{\gamma'}|} \cdot \prod_{\gamma\in\bar\Phi^+} \left(\delta_{n_\gamma,m_\gamma} \cdot (f_\gamma^{m_\gamma}, e_\gamma^{m_\gamma})_J\right).$$

This completes the proof of formula (5.27), and hence also of the theorem. □

As an immediate corollary, we obtain the following factorization formula:

Theorem 5.18. *The operator* Θ *of* (4.6) *can be factorized as follows:*

$$\Theta = \overleftarrow{\prod_{\gamma\in\bar\Phi^+}} \left(\sum_{k\geq 0} \frac{e_\gamma^k \otimes f_\gamma^k}{(f_\gamma^k, e_\gamma^k)_J}\right). \quad (5.31)$$

We note that $f_\gamma^k = e_\gamma^k = 0$ if $\gamma \in \bar\Phi_{\bar 1}^+$ is isotropic and $k \geq 2$, according to [7, Corollary 5.2].

Remark 5.19. One can express Θ in an even more compact form. Recall the notion of a q-*exponent*:

$$\exp_{\mathsf{q}}(z) = \sum_{k\geq 0} \frac{z^k}{(k)_{\mathsf{q}}!}\,,$$

where $(k)_{\mathsf{q}}! = (k)_{\mathsf{q}}\dots(1)_{\mathsf{q}}$ with $(k)_{\mathsf{q}} = \frac{1-\mathsf{q}^k}{1-\mathsf{q}}$. We thus have:

$$\sum_{k\geq 0} \frac{e_\gamma^k \otimes f_\gamma^k}{(f_\gamma^k, e_\gamma^k)_J} = \sum_{k\geq 0} \frac{(-1)^{\frac{k(k-1)}{2}|e_\gamma|}\cdot (e_\gamma\otimes f_\gamma)^k}{(-1)^{\frac{k(k-1)}{2}|e_\gamma|}\cdot \bar\sigma(\mathsf{C}_{\gamma,k})\cdot (f_\gamma,e_\gamma)_J^k} = \exp_{q_\gamma}\left(\frac{e_\gamma\otimes f_\gamma}{(f_\gamma,e_\gamma)_J}\right),$$

where $q_\gamma = (-1)^{|e_\gamma|}q^{(\gamma,\gamma)}$. Therefore, the factorization formula (5.31) simplifies to

$$\Theta = \prod_{\gamma\in\bar\Phi^+}^{\longleftarrow} \exp_{q_\gamma}\left(\frac{e_\gamma\otimes f_\gamma}{(f_\gamma,e_\gamma)_J}\right).$$

5.9. **R-Matrix computation.**

We shall now use the factorization formula (5.31) to compute the action of Θ on $V\otimes V$ and then re-derive R_{VV}. Throughout this Subsection, we will use the more convenient notation

$$\Theta_\gamma = \sum_{k\geq 0} \frac{e_\gamma^k\otimes f_\gamma^k}{(f_\gamma^k, e_\gamma^k)_J} \qquad \text{for any} \quad \gamma\in\bar\Phi^+, \tag{5.32}$$

so that equation (5.31) becomes

$$\Theta = \prod_{\gamma\in\bar\Phi^+}^{\longleftarrow} \Theta_\gamma\,.$$

For any $1\leq i,j\leq N$, we also define the following q-deformation of the elements (2.6) of $\mathfrak{osp}(V)$:

$$\mathsf{e}_{ij} = E_{ij} - (-1)^{\bar i(\bar i+\bar j)} q^{-(\rho,\varepsilon_i-\varepsilon_j)} q^{(\varepsilon_i,\varepsilon_i)/2} q^{(\varepsilon_j,\varepsilon_j)/2} \vartheta_i\vartheta_j E_{j'i'}\,,$$
$$\mathsf{f}_{ij} = E_{ji} - (-1)^{\bar j(\bar i+\bar j)} q^{(\rho,\varepsilon_i-\varepsilon_j)} q^{-(\varepsilon_i,\varepsilon_i)/2} q^{-(\varepsilon_j,\varepsilon_j)/2} \vartheta_i\vartheta_j E_{i'j'}\,.$$

5.9.1. *Factorized formula for odd m.*

We start by evaluating the action of the root vectors $\{e_\gamma, f_\gamma\}_{\gamma\in\bar\Phi^+}$ of (5.20) on the $U_q(\mathfrak{osp}(V))$-module V from Proposition 3.1:

Lemma 5.20. *(a) For $\gamma=\varepsilon_i-\varepsilon_j$ with $1\leq i<j\leq s+1$, we have:*

$$\varrho(e_\gamma) = \mathsf{e}_{ij}\,, \qquad \varrho(f_\gamma) = (-1)^{\bar i+\dots+\overline{j-1}}\cdot \mathsf{f}_{ij}\,.$$

(b) For $\gamma=\varepsilon_i+\varepsilon_j$ with $1\leq i<j\leq s$, we have:

$$\varrho(e_\gamma) = \vartheta_j\vartheta_{s+1}\cdot \prod_{k=j}^{s}\Big(-(-1)^{\bar k(\bar k+\overline{k+1})}\Big)\cdot \mathsf{e}_{ij'}\,,$$

$$\varrho(f_\gamma) = (-1)^{\bar i+\dots+\overline{j-1}}\vartheta_j\vartheta_{s+1}\cdot \prod_{k=j}^{s}\Big(-(-1)^{\overline{k+1}(\bar k+\overline{k+1})}\Big)\cdot \mathsf{f}_{ij'}\,.$$

Proof. The proof is done by a straightforward induction. To this end, we proceed by an increasing induction on j for $\gamma=\varepsilon_i-\varepsilon_j$ with $1\leq i<j\leq s+1$, and then by a decreasing induction on j for $\gamma=\varepsilon_i+\varepsilon_j$ with $1\leq i<j\leq s$. □

Combining the result above with Lemma 5.13 and formula (5.28), we obtain:

Lemma 5.21. *The operators Θ_γ of (5.32) act on the $U_q(\mathfrak{osp}(V))$-module $V\otimes V$ as follows:*

- *If $\gamma=\varepsilon_i-\varepsilon_j$ with $i<j<i'$ and $j\neq s+1$, then*

$$\Theta_\gamma = \mathrm{I} - (-1)^{\bar j}(q-q^{-1})\cdot \mathsf{e}_{ij}\otimes \mathsf{f}_{ij}\,.$$

- *If $\gamma=\varepsilon_i$ with $1\leq i\leq s$, then*

$$\Theta_\gamma = \mathrm{I} - (q-q^{-1})\cdot \mathsf{e}_{i,s+1}\otimes \mathsf{f}_{i,s+1} + (q-q^{-1})\Big(1-(-1)^{\bar i}q^{-(\varepsilon_i,\varepsilon_i)}\Big)\cdot E_{ii'}\otimes E_{i'i}\,.$$

Evoking the explicit order (5.7) on $\bar{\Phi}^+$, one immediately obtains the factorization formula

$$\Theta = \Theta_s \Theta_{s-1} \cdots \Theta_1 \tag{5.33}$$

with

$$\Theta_i = \Theta_{\varepsilon_i+\varepsilon_{i+1}} \cdots \Theta_{\varepsilon_i+\varepsilon_s} \Theta_{\varepsilon_i} \Theta_{\varepsilon_i-\varepsilon_s} \cdots \Theta_{\varepsilon_i-\varepsilon_{i+1}} \qquad \text{for any} \quad 1 \leq i \leq s\,.$$

Therefore, it is essential to evaluate each such factor, which is the subject of the next result:

Lemma 5.22. *For $1 \leq i \leq s$, we have:*

$$\Theta_i = \mathrm{I} - (q - q^{-1}) \sum_{i<j<i'} (-1)^{\bar{j}} \mathsf{e}_{ij} \otimes \mathsf{f}_{ij} + (q - q^{-1}) q^{-(\varepsilon_i,\varepsilon_i)} \Big(q^{(2\rho,\varepsilon_i)} - (-1)^{\bar{i}} \Big) \cdot E_{ii'} \otimes E_{i'i}\,.$$

Proof. First we note that $\Theta_{\varepsilon_i+\varepsilon_j}$ commutes with $\Theta_{\varepsilon_i+\varepsilon_k}$ for $k \neq j, j'$, so that

$$\Theta_i = \Theta_{\varepsilon_i+\varepsilon_{i+1}} \cdots \Theta_{\varepsilon_i+\varepsilon_s} \Theta_{\varepsilon_i} \Theta_{\varepsilon_i-\varepsilon_s} \cdots \Theta_{\varepsilon_i-\varepsilon_{i+1}} = \Theta_{\varepsilon_i} \prod_{j=i+1}^{s} \big(\Theta_{\varepsilon_i+\varepsilon_j} \Theta_{\varepsilon_i-\varepsilon_j} \big)\,.$$

According to Lemma 5.21, for $i < j \leq s$ we have:

$$\Theta_{\varepsilon_i+\varepsilon_j} \Theta_{\varepsilon_i-\varepsilon_j} = \mathrm{I} - (-1)^{\bar{j}} (q - q^{-1}) \cdot (\mathsf{e}_{ij} \otimes \mathsf{f}_{ij} + \mathsf{e}_{ij'} \otimes \mathsf{f}_{ij'}) + (-1)^{\bar{j}} (q - q^{-1})^2 q^{(\rho,2\varepsilon_j)} \cdot E_{ii'} \otimes E_{i'i}\,.$$

Combining the two formulas above, we obtain:

$$\begin{aligned}\Theta_i = {} & \mathrm{I} - (q - q^{-1}) \sum_{i<j<i'} (-1)^{\bar{j}} \mathsf{e}_{ij} \otimes \mathsf{f}_{ij} \\ & + (q - q^{-1}) \Big(1 - (-1)^{\bar{i}} q^{-(\varepsilon_i,\varepsilon_i)} \Big) E_{ii'} \otimes E_{i'i} + (q - q^{-1})^2 \sum_{j=i+1}^{s} (-1)^{\bar{j}} q^{(\rho,2\varepsilon_j)} E_{ii'} \otimes E_{i'i}\,. \end{aligned} \tag{5.34}$$

The last sum can be simplified using (4.11) as follows:

$$\begin{aligned} (q - q^{-1}) \sum_{j=i+1}^{s} (-1)^{\bar{j}} q^{(\rho,2\varepsilon_j)} &= \sum_{j=i+1}^{s} \big(q^{(\varepsilon_j,\varepsilon_j)} - q^{-(\varepsilon_j,\varepsilon_j)} \big) q^{(2\rho,\varepsilon_j)} \\ &= \sum_{j=i+1}^{s} \Big(q^{-(\varepsilon_{j-1},\varepsilon_{j-1})} q^{(2\rho,\varepsilon_{j-1})} - q^{-(\varepsilon_j,\varepsilon_j)} q^{(2\rho,\varepsilon_j)} \Big) = q^{-(\varepsilon_i,\varepsilon_i)} q^{(2\rho,\varepsilon_i)} - 1\,. \end{aligned} \tag{5.35}$$

Combining formulas (5.34, 5.35), we get

$$\Theta_i = \mathrm{I} - (q - q^{-1}) \sum_{i<j<i'} (-1)^{\bar{j}} \mathsf{e}_{ij} \otimes \mathsf{f}_{ij} + (q - q^{-1}) q^{-(\varepsilon_i,\varepsilon_i)} \Big(q^{(2\rho,\varepsilon_i)} - (-1)^{\bar{i}} \Big) \cdot E_{ii'} \otimes E_{i'i}\,,$$

which completes the proof. □

The result above together with the factorization (5.33) finally allows us to evaluate Θ:

Proposition 5.23. *The action of the operator Θ on the $U_q(\mathfrak{osp}(V))$-module $V \otimes V$ is given by*

$$\Theta = \mathrm{I} - (q - q^{-1}) \sum_{1 \leq i < j \leq 1'} (-1)^{\bar{j}} E_{ij} \otimes \Big(q^{(\varepsilon_i,\varepsilon_j)} E_{ji} - (-1)^{\bar{j}(\bar{i}+\bar{j})} \vartheta_i \vartheta_j q^{(\rho,\varepsilon_i-\varepsilon_j)} q^{-(\varepsilon_i,\varepsilon_i)/2} q^{-(\varepsilon_j,\varepsilon_j)/2} E_{i'j'} \Big)\,. \tag{5.36}$$

Proof. We shall prove formula (5.36) by induction on s. The base case $s = 1$ follows from the direct evaluation of $\Theta = \Theta_1$ in Lemma 5.22 and (4.11). As per the induction step, let us consider the subspace V° of V spanned by the vectors $\{v_i\}_{2 \leq i \leq 2'}$; we shall likewise use the symbol $^\circ$ to denote any object corresponding to V° instead of V. Then, we have an algebra homomorphism $\iota\colon U_q(\mathfrak{osp}(V^\circ)) \to U_q(\mathfrak{osp}(V))$ mapping each generator $\{e_i, f_i, q^{\pm h_i/2}\}_{i=2}^{s}$ in $U_q(\mathfrak{osp}(V^\circ))$ to the same-named generator in $U_q(\mathfrak{osp}(V))$. Furthermore, let $\varrho^\circ\colon U_q(\mathfrak{osp}(V^\circ)) \to \mathrm{End}(V^\circ)$ and $\varrho\colon U_q(\mathfrak{osp}(V)) \to \mathrm{End}(V)$ be the representations of the corresponding algebras, as defined in Proposition 3.1. It is clear that the representation $\varrho \circ \iota$ of $U_q(\mathfrak{osp}(V^\circ))$ on V preserves V°, and its restriction onto V° coincides with ϱ°. Moreso, the generators $\{e_i, f_i\}_{i=2}^{s}$ act trivially on v_1 and $v_{1'}$. Likewise, the bialgebra pairings from Proposition 4.1 are related via

$$(y, x)^\circ_J = (\iota(y), \iota(x))_J \qquad \text{for any} \quad x \in U_q^{\geq}(\mathfrak{osp}(V^\circ)),\, y \in U_q^{\leq}(\mathfrak{osp}(V^\circ))\,,$$

which follows from the defining properties. Therefore, combining the observations above with the induction hypothesis, we see that the canonical tensor of $U_q(\mathfrak{osp}(V^\circ))$ associated with ϱ equals

$$\Theta^\circ = \mathrm{I}-(q-q^{-1}) \sum_{1<i<j<1'} (-1)^{\bar{j}} E_{ij}\otimes\Big(q^{(\varepsilon_i,\varepsilon_j)}E_{ji}-(-1)^{\bar{j}(\bar{i}+\bar{j})}\vartheta_i\vartheta_j q^{(\rho^\circ,\varepsilon_i-\varepsilon_j)}q^{-(\varepsilon_i,\varepsilon_i)/2}q^{-(\varepsilon_j,\varepsilon_j)/2}E_{i'j'}\Big)$$

where ρ° denotes the Weyl vector corresponding to $U_q(\mathfrak{osp}(V^\circ))$, cf. (4.10). Though the Weyl vectors ρ and ρ° are different, the equality $(\rho^\circ,\alpha_i)=\frac{1}{2}(\alpha_i,\alpha_i)=(\rho,\alpha_i)$ still holds for all $2\leq i\leq s$, cf. (4.11), so that $(\rho^\circ,\gamma)=(\rho,\gamma)$ for any root γ in the root system Φ° of $U_q(\mathfrak{osp}(V^\circ))$. Thus:

$$\Theta^\circ = \mathrm{I}-(q-q^{-1}) \sum_{1<i<j<1'} (-1)^{\bar{j}} E_{ij}\otimes\Big(q^{(\varepsilon_i,\varepsilon_j)}E_{ji}-(-1)^{\bar{j}(\bar{i}+\bar{j})}\vartheta_i\vartheta_j q^{(\rho,\varepsilon_i-\varepsilon_j)}q^{-(\varepsilon_i,\varepsilon_i)/2}q^{-(\varepsilon_j,\varepsilon_j)/2}E_{i'j'}\Big)\,. \tag{5.37}$$

Combining this formula with $\Theta=\Theta^\circ\Theta_1$ due to (5.33), we finally obtain:

$$\begin{aligned}\Theta = \Theta^\circ-(q-q^{-1}) \sum_{1<a<1'} \Big\{&(-1)^{\bar{a}}E_{1a}\otimes E_{a1}-(-1)^{\bar{1}\bar{a}}q^{-(\rho,\varepsilon_1-\varepsilon_a)}q^{(\varepsilon_1,\varepsilon_1)/2}q^{(\varepsilon_a,\varepsilon_a)/2}\vartheta_{1'}\vartheta_{a'}\cdot E_{a'1'}\otimes E_{a1}\\
&+(-1)^{\bar{1}}E_{a'1'}\otimes E_{1'a'}-(-1)^{\bar{1}\bar{a}}q^{(\rho,\varepsilon_1-\varepsilon_a)}q^{-(\varepsilon_1,\varepsilon_1)/2}q^{-(\varepsilon_a,\varepsilon_a)/2}\vartheta_1\vartheta_a\cdot E_{1a}\otimes E_{1'a'}\Big\}\\
&+(q-q^{-1})q^{-(\varepsilon_1,\varepsilon_1)}\Big(q^{(2\rho,\varepsilon_1)}-(-1)^{\bar{1}}\Big)E_{11'}\otimes E_{1'1}\\
&-(q-q^{-1})^2\sum_{2\leq a\leq s}(-1)^{\bar{1}}(\bar{1}+\bar{a})q^{-(\rho,\varepsilon_1-\varepsilon_a)}q^{(\varepsilon_1,\varepsilon_1)/2}q^{-(\varepsilon_a,\varepsilon_a)/2}\vartheta_1\vartheta_a\cdot E_{a1'}\otimes E_{a'1}\\
+(q-q^{-1})^2\sum_{1<i<1'}&\left(\sum_{a=2}^{i-1}(-1)^{\bar{a}}q^{(\rho,2\varepsilon_a)}\right)(-1)^{\bar{1}+\bar{i}}(-1)^{\bar{1}\bar{i}}q^{-(\rho,\varepsilon_1+\varepsilon_i)}q^{(\varepsilon_1,\varepsilon_1)/2}q^{-(\varepsilon_i,\varepsilon_i)/2}\vartheta_1\vartheta_i\cdot E_{i'1'}\otimes E_{i1}\,.\end{aligned} \tag{5.38}$$

The coefficient of each term in this formula coincides with the one from the right-hand side of (5.36), except possibly only for the coefficients of $\{E_{i'1'}\otimes E_{i1}\}_{1<i<1'}$. Let us treat the latter ones:

- Case 1: $2\leq i\leq s+1$.
  First, we note that a computation analogous to (5.35) gives

$$(q-q^{-1})\sum_{a=2}^{i-1}(-1)^{\bar{a}}q^{(\rho,2\varepsilon_a)} = q^{-(\varepsilon_1,\varepsilon_1)}q^{(\rho,2\varepsilon_1)}-q^{-(\varepsilon_{i-1},\varepsilon_{i-1})}q^{(\rho,2\varepsilon_{i-1})}\,. \tag{5.39}$$

  Therefore, the coefficient of $E_{i'1'}\otimes E_{i1}$ in the right-hand side of (5.38) equals:

$$\begin{aligned}&(q-q^{-1})(-1)^{\bar{1}\bar{i}}q^{-(\rho,\varepsilon_1-\varepsilon_i)}q^{(\varepsilon_1,\varepsilon_1)/2}q^{(\varepsilon_i,\varepsilon_i)/2}\vartheta_{1'}\vartheta_{i'}\\
&\quad+(q-q^{-1})^2\Big(\sum_{a=2}^{i-1}(-1)^{\bar{a}}q^{(\rho,2\varepsilon_a)}\Big)(-1)^{\bar{1}+\bar{i}}(-1)^{\bar{1}\bar{i}}q^{-(\rho,\varepsilon_1+\varepsilon_i)}q^{(\varepsilon_1,\varepsilon_1)/2}q^{-(\varepsilon_i,\varepsilon_i)/2}\vartheta_1\vartheta_i\\
&=(q-q^{-1})(-1)^{\bar{1}\bar{i}}q^{-(\rho,\varepsilon_1-\varepsilon_i)}q^{(\varepsilon_1,\varepsilon_1)/2}q^{(\varepsilon_i,\varepsilon_i)/2}\vartheta_{1'}\vartheta_{i'}\times\\
&\quad\left(1+q^{-(\rho,2\varepsilon_i)}q^{-(\varepsilon_i,\varepsilon_i)}\cdot(q-q^{-1})\sum_{a=2}^{i-1}(-1)^{\bar{a}}q^{(\rho,2\varepsilon_a)}\right)\\
&\overset{(4.11),(5.39)}{=}(q-q^{-1})(-1)^{\bar{1}\bar{i}}q^{(\rho,\varepsilon_1-\varepsilon_i)}q^{-(\varepsilon_1,\varepsilon_1)/2}q^{-(\varepsilon_i,\varepsilon_i)/2}\vartheta_{1'}\vartheta_{i'}\,,\end{aligned}$$

  which coincides with the coefficient of $E_{i'1'}\otimes E_{i1}$ in the right-hand side of (5.36).

- Case 2: $s'\leq i\leq 2'$.
  Similarly to the previous case, we have:

$$(q-q^{-1})\sum_{a=2}^{i-1}(-1)^{\bar{a}}q^{(\rho,2\varepsilon_a)} = \Big(q^{-(\varepsilon_1,\varepsilon_1)}q^{(\rho,2\varepsilon_1)}-q^{-(\varepsilon_{i-1},\varepsilon_{i-1})}q^{(\rho,2\varepsilon_{i-1})}\Big)+(q-q^{-1})\,. \tag{5.40}$$

Indeed, this formula is obvious for $i = s' = s+2$, while for $i > s'$ it follows from

$$
\begin{aligned}
(q-q^{-1}) \sum_{a=s+2}^{i-1} (-1)^{\overline{a}} q^{(\rho,2\varepsilon_a)} &= (q-q^{-1}) \sum_{a=i'+1}^{s} (q^{(\varepsilon_a,\varepsilon_a)} - q^{-(\varepsilon_a,\varepsilon_a)}) q^{-(\rho,2\varepsilon_a)} \\
&\overset{(4.11)}{=} \sum_{a=i'+1}^{s} \Big(q^{-(\varepsilon_{a+1},\varepsilon_{a+1})} q^{-(\rho,2\varepsilon_{a+1})} - q^{-(\varepsilon_a,\varepsilon_a)} q^{-(\rho,2\varepsilon_a)} \Big) \\
&= q^{-(\varepsilon_{s+1},\varepsilon_{s+1})} q^{-(\rho,2\varepsilon_{s+1})} - q^{-(\varepsilon_{i'+1},\varepsilon_{i'+1})} q^{-(\rho,2\varepsilon_{i'+1})} \overset{(2.7)}{=} 1 - q^{-(\varepsilon_{i-1},\varepsilon_{i-1})} q^{(\rho,2\varepsilon_{i-1})} .
\end{aligned}
$$

Therefore, the coefficient of $E_{i'1'} \otimes E_{i1}$ in the right-hand side of (5.38) equals:

$$
\begin{aligned}
& (q-q^{-1})(-1)^{\overline{1}\,\overline{i}} q^{-(\rho,\varepsilon_1-\varepsilon_i)} q^{(\varepsilon_1,\varepsilon_1)/2} q^{(\varepsilon_i,\varepsilon_i)/2} \vartheta_{1'} \vartheta_{i'} \\
& \quad + (q-q^{-1})^2 \Big(\sum_{a=2}^{i-1} (-1)^{\overline{a}} q^{(\rho,2\varepsilon_a)} \Big) (-1)^{\overline{1}+\overline{i}} (-1)^{\overline{1}\,\overline{i}} q^{-(\rho,\varepsilon_1+\varepsilon_i)} q^{(\varepsilon_1,\varepsilon_1)/2} q^{-(\varepsilon_i,\varepsilon_i)/2} \vartheta_1 \vartheta_i \\
& \quad - (q-q^{-1})^2 (-1)^{\overline{1}\,\overline{i}} q^{-(\rho,\varepsilon_1+\varepsilon_i)} q^{(\varepsilon_1,\varepsilon_1)/2} q^{-(\varepsilon_i,\varepsilon_i)/2} \vartheta_{1'} \vartheta_{i'} \\
& = (q-q^{-1})(-1)^{\overline{1}\,\overline{i}} q^{-(\rho,\varepsilon_1-\varepsilon_i)} q^{(\varepsilon_1,\varepsilon_1)/2} q^{(\varepsilon_i,\varepsilon_i)/2} \vartheta_{1'} \vartheta_{i'} \times \\
& \quad \Big(1 - (q-q^{-1}) q^{-(\rho,2\varepsilon_i)} q^{-(\varepsilon_i,\varepsilon_i)} + q^{-(\rho,2\varepsilon_i)} q^{-(\varepsilon_i,\varepsilon_i)} \cdot (q-q^{-1}) \sum_{a=2}^{i-1} (-1)^{\overline{a}} q^{(\rho,2\varepsilon_a)} \Big) \\
& \overset{(4.11),(5.40)}{=} (q-q^{-1})(-1)^{\overline{1}\,\overline{i}} q^{(\rho,\varepsilon_1-\varepsilon_i)} q^{-(\varepsilon_1,\varepsilon_1)/2} q^{-(\varepsilon_i,\varepsilon_i)/2} \vartheta_{1'} \vartheta_{i'} ,
\end{aligned}
\tag{5.41}
$$

which coincides with the coefficient of $E_{i'1'} \otimes E_{i1}$ in the right-hand side of (5.36). This completes the proof of the induction step, thus establishing formula (5.36). □

Finally, we can re-derive our formula (4.13) for $R_{VV} = \tau_{VV} \circ \hat{R}_{VV} = \tilde{f}^{1/2} \circ \Theta \circ \tilde{f}^{1/2}$ from above result. Since the action of $\tilde{f}^{1/2}$ on $V \otimes V$ is given by

$$
\tilde{f}^{1/2} = \sum_{i,j} q^{-(\varepsilon_i,\varepsilon_j)/2} E_{ii} \otimes E_{jj} , \tag{5.42}
$$

the explicit formula (5.36) for Θ implies

$$
\begin{aligned}
R_{VV} = \ & \tilde{f}^{1/2} \circ \Theta \circ \tilde{f}^{1/2} \\
& = \sum_{i,j} q^{-(\varepsilon_i,\varepsilon_j)} E_{ii} \otimes E_{jj} + (q^{-1}-q) \sum_{i<j} (-1)^{\overline{j}} E_{ij} \otimes \Big(E_{ji} - (-1)^{\overline{j}(\overline{i}+\overline{j})} \vartheta_i \vartheta_j q^{(\rho,\varepsilon_i-\varepsilon_j)} E_{i'j'} \Big) \\
& = \mathrm{I} + (q^{-1/2} - q^{1/2}) \sum_{i=1}^{N} (-1)^{\overline{i}} E_{ii} \otimes \Big(q^{-(\varepsilon_i,\varepsilon_i)/2} E_{ii} - q^{(\varepsilon_i,\varepsilon_i)/2} E_{i'i'} \Big) \\
& \qquad + (q^{-1}-q) \sum_{i<j} (-1)^{\overline{j}} E_{ij} \otimes \Big(E_{ji} - (-1)^{\overline{j}(\overline{i}+\overline{j})} \vartheta_i \vartheta_j q^{(\rho,\varepsilon_i-\varepsilon_j)} E_{i'j'} \Big) ,
\end{aligned}
$$

which precisely recovers R_0 from (4.13). This provides an alternative proof of Theorem 4.6.

5.9.2. *Factorized formula for even m.*
Similarly to Lemma 5.20, we start by evaluating the action of the root vectors $\{e_\gamma, f_\gamma\}_{\gamma \in \bar{\Phi}^+}$ of (5.20) on the $U_q(\mathfrak{osp}(V))$-module V from Proposition 3.1:

Lemma 5.24. *If $\overline{s} = \bar{0}$, then the action of the root generators is as follows.*

(a) For $\gamma = \varepsilon_i - \varepsilon_j$ with $1 \le i < j \le s$, we have:

$$
\varrho(e_\gamma) = \mathsf{e}_{ij} , \qquad \varrho(f_\gamma) = (-1)^{\overline{i}+\dots+\overline{j-1}} \cdot \mathsf{f}_{ij} .
$$

(b) For $\gamma=\varepsilon_i+\varepsilon_j$ with $1\leq i<j\leq s$, we have:

$$\varrho(e_\gamma)=-\vartheta_j\vartheta_s\cdot\prod_{k=j}^{s}\Big(-(-1)^{\overline{k}(\overline{k}+\overline{k+1})}\Big)\cdot \mathsf{e}_{ij'}\,,$$

$$\varrho(f_\gamma)=-(-1)^{\overline{i}+\cdots+\overline{j-1}}\vartheta_j\vartheta_s\cdot\prod_{k=j}^{s}\Big(-(-1)^{\overline{k+1}(\overline{k}+\overline{k+1})}\Big)\cdot \mathsf{f}_{ij'}\,.$$

(c) For $\gamma=2\varepsilon_i$ with $1\leq i\leq s$ and $\overline{i}=\overline{1}$, we have:

$$\varrho(e_\gamma)=q^{-(\rho,\varepsilon_i)}(1+q^{-2})\vartheta_i\vartheta_s\cdot E_{ii'}\,,$$
$$\varrho(f_\gamma)=-q^{(\rho,\varepsilon_i)}(1+q^{2})\vartheta_i\vartheta_s\cdot E_{i'i}\,.$$

Lemma 5.25. *If $\overline{s}=\overline{1}$, then the action of the root generators is as follows.*

(a) For $\gamma=\varepsilon_i-\varepsilon_j$ with $1\leq i<j\leq s$, we have:

$$\varrho(e_\gamma)=\mathsf{e}_{ij}\,,\qquad \varrho(f_\gamma)=(-1)^{\overline{i}+\cdots+\overline{j-1}}\cdot\mathsf{f}_{ij}\,.$$

(b) For $\gamma=\varepsilon_i+\varepsilon_j$ with $1\leq i<j\leq s$, we have:

$$\varrho(e_\gamma)=-\vartheta_j\vartheta_s\cdot\prod_{k=j}^{s}\Big(-(-1)^{\overline{k}(\overline{k}+\overline{k+1})}\Big)\cdot \mathsf{e}_{ij'}\,,$$

$$\varrho(f_\gamma)=(-1)^{\overline{i}+\cdots+\overline{j-1}}\vartheta_j\vartheta_s\cdot(q+q^{-1})\cdot\prod_{k=j}^{s}\Big(-(-1)^{\overline{k+1}(\overline{k}+\overline{k+1})}\Big)\cdot \mathsf{f}_{ij'}\,.$$

(c) For $\gamma=2\varepsilon_i$ with $1\leq i\leq s$ and $\overline{i}=\overline{1}$, we have

$$\varrho(e_\gamma)=q^{-(\rho,\varepsilon_i)}(1+q^{-2})\vartheta_i\vartheta_s\cdot E_{ii'}\,,$$
$$\varrho(f_\gamma)=-q^{(\rho,\varepsilon_i)}(1+q^{2})\vartheta_i\vartheta_s\cdot(q+q^{-1})\cdot E_{i'i}\,.$$

Combining the results above with Lemmas 5.14–5.15, we obtain the following counterpart of Lemma 5.21 which can be written in a uniform way (independent of the parity of $\overline{s}=|v_s|$):

Lemma 5.26. *The operators Θ_γ of (5.32) act on the $U_q(\mathfrak{osp}(V))$-module $V\otimes V$ as follows:*

- *If $\gamma=\varepsilon_i-\varepsilon_j$ with $i<j<i'$, then*

$$\Theta_\gamma=\mathrm{I}-(-1)^{\overline{j}}(q-q^{-1})\cdot\mathsf{e}_{ij}\otimes\mathsf{f}_{ij}\,.$$

- *If $\gamma=2\varepsilon_i$ with $1\leq i\leq s$ and $\overline{i}=\overline{1}$, then*

$$\Theta_\gamma=\mathrm{I}+(q-q^{-1})\Big(q^{-1}-(-1)^{\overline{i}}q^{-(\varepsilon_i,\varepsilon_i)}\Big)\cdot E_{ii'}\otimes E_{i'i}\,.$$

We note that the way we wrote the last formula above allows to define $\Theta_{2\varepsilon_i}=\mathrm{I}$ when $\overline{i}=\overline{0}$. With this extension of the notation $\Theta_{2\varepsilon_i}$ to all indices i, let us define

$$\Theta_i=\Theta_{\varepsilon_i+\varepsilon_{i+1}}\cdots\Theta_{\varepsilon_i+\varepsilon_s}\Theta_{2\varepsilon_i}\Theta_{\varepsilon_i-\varepsilon_s}\cdots\Theta_{\varepsilon_i-\varepsilon_{i+1}}\qquad\text{for any}\quad 1\leq i\leq s\,.$$

Evoking the explicit orders (5.9, 5.11) on $\bar{\Phi}^+$, one immediately obtains the factorization

$$\Theta=\Theta_s\Theta_{s-1}\cdots\Theta_1\,. \tag{5.43}$$

The following result is an analogue of Lemma 5.22:

Lemma 5.27. *For $1\leq i\leq s$, we have*

$$\Theta_i=\mathrm{I}-(q-q^{-1})\sum_{i<j<i'}(-1)^{\overline{j}}\mathsf{e}_{ij}\otimes\mathsf{f}_{ij}+(q-q^{-1})q^{-(\varepsilon_i,\varepsilon_i)}\Big(q^{(2\rho,\varepsilon_i)}-(-1)^{\overline{i}}\Big)\cdot E_{ii'}\otimes E_{i'i}\,.$$

Proof. Arguing exactly as in the proof of Lemma 5.22, we obtain the following analogue of (5.34):

$$\Theta_i = \mathrm{I} - (q-q^{-1}) \sum_{i<j<i'} (-1)^{\bar{j}} \mathsf{e}_{ij} \otimes \mathsf{f}_{ij} + (q-q^{-1})\Big(q^{-1} - (-1)^{\bar{i}} q^{-(\varepsilon_i,\varepsilon_i)}\Big) E_{ii'} \otimes E_{i'i} + (q-q^{-1})^2 \sum_{j=i+1}^{s} (-1)^{\bar{j}} q^{(\rho,2\varepsilon_j)} E_{ii'} \otimes E_{i'i}\,.$$

The last sum can be simplified similarly to (5.35) as follows:

$$(q-q^{-1}) \sum_{j=i+1}^{s} (-1)^{\bar{j}} q^{(\rho,2\varepsilon_j)} = \sum_{j=i+1}^{s} (q^{(\varepsilon_j,\varepsilon_j)} - q^{-(\varepsilon_j,\varepsilon_j)}) q^{(2\rho,\varepsilon_j)} = q^{-(\varepsilon_i,\varepsilon_i)} q^{(2\rho,\varepsilon_i)} - q^{-(\varepsilon_s,\varepsilon_s)} q^{(2\rho,\varepsilon_s)}\,.$$

Thus, the claimed formula for Θ_i follows from the two equalities above and the identity

$$(2\rho,\varepsilon_s) - (\varepsilon_s,\varepsilon_s) = -1\,. \tag{5.44}$$

The latter is a simple consequence of (4.11):

- if $\bar{s}=\bar{1}$, then $\alpha_s = 2\varepsilon_s$ and so $(2\rho,\varepsilon_s) = \frac{1}{2}(2\rho,\alpha_s) = \frac{1}{2}(\alpha_s,\alpha_s) = -2 = (\varepsilon_s,\varepsilon_s) - 1$;
- if $\bar{s}=\bar{0}$, then $\alpha_{s-1} = \varepsilon_{s-1} - \varepsilon_s$, $\alpha_s = \varepsilon_{s-1} + \varepsilon_s$, so that
$$(2\rho,\varepsilon_s) = \tfrac{1}{2}\big((2\rho,\alpha_s) - (2\rho,\alpha_{s-1})\big) = \tfrac{1}{2}\big((\alpha_s,\alpha_s) - (\alpha_{s-1},\alpha_{s-1})\big) = 0 = (\varepsilon_s,\varepsilon_s) - 1\,.$$

This establishes (5.44) and thus completes the proof of the lemma. ☐

Combining this result with the factorization (5.43), we can finally evaluate Θ:

Proposition 5.28. *The action of the operator Θ on the $U_q(\mathfrak{osp}(V))$-module $V \otimes V$ is given by*

$$\Theta = \mathrm{I} - (q-q^{-1}) \sum_{1\le i<j\le 1} (-1)^{\bar{j}} E_{ij} \otimes \Big(q^{(\varepsilon_i,\varepsilon_j)} E_{ji} - (-1)^{\bar{j}(\bar{i}+\bar{j})} \vartheta_i \vartheta_j q^{(\rho,\varepsilon_i-\varepsilon_j)} q^{-(\varepsilon_i,\varepsilon_i)/2} q^{-(\varepsilon_j,\varepsilon_j)/2} E_{i'j'}\Big)\,.$$

Let us note right away that the formula above is identical to (5.36).

Proof. The proof of this result is completely analogous to that of Proposition 5.23, and proceeds by induction on s. The base case $s=1$ follows from the evaluation of $\Theta = \Theta_1$ in Lemma 5.27.

As per the induction step, we obtain precisely formula (5.38) expressing Θ through Θ°, the latter been given by the same formula (5.37), due to the observation preceding the proof. The rest of the proof proceeds without any changes. ☐

Analogously to odd m, we can use the result above to re-derive formula (4.13) for R_{VV}. Indeed, since the formula in Proposition 5.28 is identical to (5.36), the same computation can be applied without any changes, thus providing an alternative proof of Theorem 4.6 in that case.

6. R-Matrices with a spectral parameter

6.1. Orthosymplectic quantum affine groups.

Let θ be the highest root of $\mathfrak{osp}(V)$ with respect to the fixed polarization of (2.9), and let $\{\mathsf{k}_i\}_{i=1}^s$ be the corresponding coefficients in the decomposition $\theta = \sum_{i=1}^s \mathsf{k}_i\alpha_i$. Explicitly, we have:

$$\theta = \begin{cases} \varepsilon_1+\varepsilon_2 & \text{if } |v_1| = \bar{0} \\ 2\varepsilon_1 & \text{if } |v_1| = \bar{1} \end{cases}.$$

We define the lattice $\widehat{P} = \mathbb{Z}\delta \oplus P = \mathbb{Z}\delta \oplus \bigoplus_{i=1}^s \mathbb{Z}\varepsilon_i$, with P introduced right before (2.21). Then $\alpha_1,\ldots,\alpha_s$ as well as $\alpha_0 = \delta-\theta$ can be viewed as elements of $\widehat{P}$. We extend the bilinear pairing $(\cdot,\cdot)$ on P, defined via (2.11), to that on $\widehat{P}$ by setting $(\delta,\delta) = (\delta,\varepsilon_i) = (\varepsilon_i,\delta) = 0$ for all i. We define the *symmetrized extended Cartan matrix* of $\mathfrak{osp}(V)$ as $(a_{ij})_{i,j=0}^s$ with $a_{ij} = (\alpha_i,\alpha_j)$. It extends the Cartan matrix of (2.15) through $a_{00} = (\theta,\theta)$ and $a_{0i} = a_{i0} = -(\theta,\alpha_i)$ for $1 \le i \le s$.

The *orthosymplectic quantum affine supergroup* $U_q(\widehat{\mathfrak{osp}}(V))$ is a $\mathbb{C}(q^{\pm 1/2})$-superalgebra generated by $\{e_i, f_i, q^{\pm h_i/2}\}_{i=0}^s \cup \{\gamma^{\pm 1}, D^{\pm 1}\}$, with the $\mathbb{Z}_2$-grading

$$|e_0| = |f_0| = \begin{cases} \bar{0} & \text{if } \theta \in \Phi_{\bar{0}} \\ \bar{1} & \text{if } \theta \in \Phi_{\bar{1}} \end{cases}, \qquad |e_i| = |f_i| = \begin{cases} \bar{0} & \text{if } \alpha_i \in \Phi_{\bar{0}} \\ \bar{1} & \text{if } \alpha_i \in \Phi_{\bar{1}} \end{cases} \quad \text{for} \quad 1 \le i \le s\,,$$

$$|\gamma^{\pm 1}| = |D^{\pm 1}| = |h_i| = \bar{0} \qquad \text{for} \quad 0 \le i \le s\,,$$

subject to the following defining relations:

$$D^{\pm 1} \cdot D^{\mp 1} = 1\,, \quad [D, q^{h_i/2}] = 0\,, \quad D e_i D^{-1} = q^{\delta_{0i}} e_i\,, \quad D f_i D^{-1} = q^{-\delta_{0i}} f_i\,, \tag{6.1}$$

$$\gamma^{\pm 1} \cdot \gamma^{\mp 1} = 1\,, \quad \gamma = q^{h_0/2} \cdot \prod_{i=1}^{s} (q^{h_i/2})^{\mathsf{k}_i}\,, \quad \gamma - \text{central element}\,, \tag{6.2}$$

the counterpart of (2.18)–(2.20) but now with $0 \le i, j \le s$:

$$[q^{h_i/2}, q^{h_j/2}] = 0\,, \quad q^{\pm h_i/2} q^{\mp h_i/2} = 1\,, \tag{6.3}$$

$$q^{h_i/2} e_j q^{-h_i/2} = q^{a_{ij}/2} e_j, \quad q^{h_i/2} f_j q^{-h_i/2} = q^{-a_{ij}/2} f_j\,, \tag{6.4}$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}\,, \tag{6.5}$$

together with the *standard* and the *higher order q-Serre relations*, which the interested reader may find in [43, relations (QS4, QS5), cf. Theorem 6.8.2]. We note that $U_q(\widehat{\mathfrak{osp}}(V))$ is equipped with a Hopf superalgebra structure, with the coproduct Δ, the counit ϵ, and the antipode S defined on the generators $\{e_i, f_i, q^{\pm h_i/2}\}_{i=0}^{s}$ by the same formulas as in the end of Subsection 2.4, while also

$$\Delta(D) = D \otimes D\,, \quad S(D) = D^{-1}\,, \quad \epsilon(D) = 1\,, \quad \Delta(\gamma) = \gamma \otimes \gamma\,, \quad S(\gamma) = \gamma^{-1}\,, \quad \epsilon(\gamma) = 1\,.$$

It is often more convenient to work with a version of $U_q(\widehat{\mathfrak{osp}}(V))$ without the degree generators $D^{\pm 1}$. Explicitly, $U'_q(\widehat{\mathfrak{osp}}(V))$ is the $\mathbb{C}(q^{\pm 1/2})$-superalgebra generated by $\{e_i, f_i, q^{\pm h_i/2}\}_{i=0}^{s} \cup \{\gamma^{\pm 1}\}$, with the same $\mathbb{Z}_2$-grading, the same defining relations excluding (6.1), and the same Hopf structure.

6.2. Evaluation modules and affine R-matrices.

Proposition 6.1. *For any $u \in \mathbb{C}^\times$ and $a, b \in \mathbb{C}^\times$ specified below, the $U_q(\mathfrak{osp}(V))$-action ϱ on V from Proposition 3.1 can be extended to a $U'_q(\widehat{\mathfrak{osp}}(V))$-action $\varrho_u^{a,b}$ on $V(u) = V$ by setting*

$$\varrho_u^{a,b}(x) = \varrho(x) \qquad \text{for all} \quad x \in \{e_i, f_i, q^{\pm h_i/2}\}_{i=1}^{s}$$

and defining the action of the remaining generators $e_0, f_0, q^{\pm h_0/2}, \gamma^{\pm 1}$ via (6.6) or (6.7) below:

- *Case 1: $|v_1| = \bar{1}$.*

$$\begin{aligned} &\varrho_u^{a,b}(e_0) = au \cdot E_{1'1}\,, \qquad \varrho_u^{a,b}(f_0) = bu^{-1} \cdot E_{11'}\,, \\ &\varrho_u^{a,b}(q^{\pm h_0/2}) = q^{\pm X_{11}}\,, \quad \varrho_u^{a,b}(\gamma^{\pm 1}) = \mathrm{I} \end{aligned} \tag{6.6}$$

  with parameters a, b subject to $ab = -(q + q^{-1})$.
- *Case 2: $|v_1| = \bar{0}$.*

$$\begin{aligned} &\varrho_u^{a,b}(e_0) = au \cdot X_{2'1}\,, \qquad \varrho_u^{a,b}(f_0) = bu^{-1} \cdot X_{12'}\,, \\ &\varrho_u^{a,b}(q^{\pm h_0/2}) = q^{\mp((-1)^{\bar{1}} X_{11} + (-1)^{\bar{2}} X_{22})/2}\,, \qquad \varrho_u^{a,b}(\gamma^{\pm 1}) = \mathrm{I} \end{aligned} \tag{6.7}$$

  with parameters a, b subject to $ab = (-1)^{\bar{2}}$.

Proof. We need to show that the operators defined above satisfy the defining relations (6.2)–(6.5) together with all q-Serre relations. This verification is straightforward and proceeds similarly to our proof of Proposition 3.1.

• Case 1: $|v_1| = \bar{1}$.

The second relation of (6.2) is verified by direct calculations, treating three cases as before: m is odd, m is even and $\bar{s} = \bar{0}$, or m is even and $\bar{s} = \bar{1}$. The relations (6.3, 6.4) then immediately follow from their validity for $i, j \ne 0$, due to Proposition 3.1. It remains to verify (6.5) for $i = 0$ or $j = 0$. The relations $[\varrho_u^{a,b}(e_0), \varrho_u^{a,b}(f_i)] = 0 = [\varrho_u^{a,b}(f_0), \varrho_u^{a,b}(e_i)]$ for $i \ne 0$ are obvious, since all four operators $\varrho_u^{a,b}(e_0)\varrho_u^{a,b}(f_i), \varrho_u^{a,b}(f_i)\varrho_u^{a,b}(e_0), \varrho_u^{a,b}(f_0)\varrho_u^{a,b}(e_i), \varrho_u^{a,b}(e_i)\varrho_u^{a,b}(f_0)$ act by 0. Finally, we have:

$$[\varrho_u^{a,b}(e_0), \varrho_u^{a,b}(f_0)] = (q + q^{-1})(E_{11} - E_{1'1'}) = \frac{q^{2X_{11}} - q^{-2X_{11}}}{q - q^{-1}} = \frac{\varrho_u^{a,b}(q^{h_0}) - \varrho_u^{a,b}(q^{-h_0})}{q - q^{-1}}\,.$$

• Case 2: $|v_1| = \bar{0}$.

The verification of (6.2)–(6.4) is similar to that in Case 1. We also note that $[\varrho_u^{a,b}(e_0), \varrho_u^{a,b}(f_i)] = 0$ and $[\varrho_u^{a,b}(f_0), \varrho_u^{a,b}(e_i)] = 0$ for $i \neq 0, 1$ by the same reason as in Case 1. Finally, we have:

$$[\varrho_u^{a,b}(e_0), \varrho_u^{a,b}(f_1)] = au[X_{2'1}, X_{21}] = 0\,, \qquad [\varrho_u^{a,b}(f_0), \varrho_u^{a,b}(e_1)] = bu^{-1}[X_{12'}, X_{12}] = 0$$

as well as

$$[\varrho_u^{a,b}(e_0), \varrho_u^{a,b}(f_0)] = -(-1)^{\bar{1}}X_{11} - (-1)^{\bar{2}}X_{22} = \frac{\varrho_u^{a,b}(q^{h_0}) - \varrho_u^{a,b}(q^{-h_0})}{q - q^{-1}}\,,$$

where we used (3.2) in the last equality.

The verification of q-Serre relations proceeds as in our proof of Proposition 3.1. To this end, we note that the algebra $U'_q(\widehat{\mathfrak{osp}}(V))$ is P-graded via (2.21) combined with $\deg(e_0) = -\theta, \deg(f_0) = \theta$, $\deg(q^{\pm h_0/2}) = \deg(\gamma^{\pm 1}) = 0$, and the above assignment preserves this P-grading, cf. (3.3). Referring to the explicit form of all q-Serre relations, left-hand sides of which are presented in [43, (QS4, QS5)], one can easily see that all of them, besides the cases (7, 8, 11), are homogeneous whose degrees are <u>not</u> in the set $\{\varepsilon_i - \varepsilon_j \,|\, 1 \leq i, j \leq N\}$. Hence, they act trivially on the superspace V. We shall now directly check the cases (7, 8, 11) of [43, (QS4)], while [43, (QS5)] are analogous.

- *Serre relation [43, (QS4)(8)].* The corresponding relation reads (cf. notation (2.22))

$$[\![[\![[\![e_j, e_i]\!], [\![e_j, e_k]\!]]\!], [\![e_j, e_l]\!]]\!] = [\![[\![[\![e_j, e_i]\!], [\![e_j, e_l]\!]]\!], [\![e_j, e_k]\!]]\!]$$

and it only occurs for $\mathfrak{osp}(V) = \mathfrak{osp}(4|2)$ in either of the following two cases:

(1) parity sequence $\gamma_V = (\bar{1}, \bar{0}, \bar{0})$ and indices $i = 0, j = 1, k = 2, l = 3$;
(2) parity sequence $\gamma_V = (\bar{0}, \bar{0}, \bar{1})$ and indices $i = 3, j = 2, k = 0, l = 1$.

In case (1), both sides of this equality (LHS and RHS) have P-degrees equal to $\varepsilon_1 - \varepsilon_2$ and thus act trivially on v_p for $p \notin \{2, 1'\}$. By direct calculations, we find: $\mathrm{LHS}(v_2) = au\vartheta_1 \cdot v_1 = \mathrm{RHS}(v_2)$ as well as $\mathrm{LHS}(v_{1'}) = q^{-1}au \cdot v_{2'} = \mathrm{RHS}(v_{1'})$. In case (2), both sides of this equality (LHS and RHS) have P-degrees equal to $\varepsilon_2 - \varepsilon_3$ and thus act trivially on v_p for $p \notin \{3, 2'\}$. By direct calculations, we find: $\mathrm{LHS}(v_3) = 2q^{-1}au\vartheta_3 \cdot v_2 = \mathrm{RHS}(v_3)$ as well as $\mathrm{LHS}(v_{2'}) = -2qau \cdot v_{3'} = \mathrm{RHS}(v_{2'})$. This completes our verification of [43, (QS4)(8)].

- *Serre relation [43, (QS4)(11)].* The corresponding relation reads

$$[\![[\![e_k, e_j]\!], [\![[\![e_k, e_j]\!], [\![[\![e_k, e_j]\!], e_i]\!]]\!]]\!] = (1 - [2]_q)[\![[\![[\![e_k, e_j]\!], [\![e_k, [\![e_k, [\![e_j, e_i]\!]]\!]]\!]]\!], e_j]\!]$$

and it only occurs for $\mathfrak{osp}(V) = \mathfrak{osp}(3|2)$ with the parity sequence $\gamma_V = (\bar{1}, \bar{0})$ and $i = 0, j = 1, k = 2$. Both sides of this equality (LHS and RHS) have P-degrees equal to ε_1 and thus act trivially on v_p for $p \notin \{3, 1'\}$. By direct calculations, we find: $\mathrm{LHS}(v_3) = (1 - q^{-1} + q^{-2})au\vartheta_1 \cdot v_1 = \mathrm{RHS}(v_3)$ as well as $\mathrm{LHS}(v_{1'}) = (1 - q + q^2)au \cdot v_3 = \mathrm{RHS}(v_{1'})$. This completes our verification of [43, (QS4)(11)].

- *Serre relation [43, (QS4)(7)].* The corresponding relation reads

$$(-1)^{|\alpha_i||\alpha_k|}[(\alpha_i, \alpha_k)]_q[\![[\![e_i, e_j]\!], e_k]\!] = (-1)^{|\alpha_i||\alpha_j|}[(\alpha_i, \alpha_j)]_q[\![[\![e_i, e_k]\!], e_j]\!]$$

whenever $(\alpha_i, \alpha_j) \neq 0, (\alpha_i, \alpha_k) \neq 0, (\alpha_j, \alpha_k) \neq 0$, $(\alpha_i, \alpha_j) + (\alpha_i, \alpha_k) + (\alpha_j, \alpha_k) = 0$, and $|\alpha_i||\alpha_j| + |\alpha_i||\alpha_k| + |\alpha_j||\alpha_k| = \bar{1}$. We can further assume that $\{i, j, k\} = \{0, 1, 2\}$. The above parity condition implies that $\theta = \varepsilon_1 + \varepsilon_2, \alpha_1 = \varepsilon_1 - \varepsilon_2$, and $|v_1| = \bar{0}, |v_2| = \bar{1}$. Due to the symmetry $j \leftrightarrow k$ of the above relation, there are three cases to consider:

(1) $i = 0, j = 1, k = 2$;
(2) $i = 2, j = 0, k = 1$;
(3) $i = 1, j = 0, k = 2$.

In case (1), both sides of this equality (LHS and RHS) have P-degrees equal to $-\varepsilon_2 - \varepsilon_3$ and thus act trivially on v_p for $p \notin \{2, 3\}$. By direct calculations, we find: $\mathrm{LHS}(v_2) = (1 + q^2)au\vartheta_2\vartheta_3 \cdot v_{3'} = \mathrm{RHS}(v_2)$ and $\mathrm{LHS}(v_3) = -(-1)^{|v_3|}(1 + q^{-2})au \cdot v_{2'} = \mathrm{RHS}(v_3)$. In case (2), both sides of this equality (LHS and RHS) have P-degrees equal to $-\varepsilon_2 - \varepsilon_3$ and thus act trivially on v_p for $p \notin \{2, 3\}$. By direct calculations, we find: $\mathrm{LHS}(v_2) = -au\vartheta_2\vartheta_3 \cdot v_{3'} = \mathrm{RHS}(v_2)$ and $\mathrm{LHS}(v_3) = (-1)^{|v_3|}au \cdot v_{2'} = \mathrm{RHS}(v_3)$. In case (3), both sides of this equality (LHS and RHS) have P-degrees equal to $-\varepsilon_2 - \varepsilon_3$ and thus act trivially on v_p for $p \notin \{2, 3\}$. By direct calculations, we find:

$\mathrm{LHS}(v_2) = (1+q^2)au\vartheta_2\vartheta_3 \cdot v_{3'} = \mathrm{RHS}(v_2)$ and $\mathrm{LHS}(v_3) = -(-1)^{|v_3|}(1+q^{-2})au \cdot v_{2'} = \mathrm{RHS}(v_3)$. This completes our verification of [43, (QS4)(7)]. □

These evaluation $U'_q(\widehat{\mathfrak{osp}}(V))$-modules $\varrho^{a,b}_u$ can be naturally upgraded to $U_q(\widehat{\mathfrak{osp}}(V))$-modules:

Proposition 6.2. *Let u be an indeterminate and redefine $V(u)$ via $V(u) = V \otimes_{\mathbb{C}} \mathbb{C}[u,u^{-1}]$. Then, the formulas defining $\varrho^{a,b}_u$ on the generators from Proposition 6.1 together with*

$$\varrho^{a,b}_u(D^{\pm 1})(v \otimes u^k) = q^{\pm k} \cdot v \otimes u^k \qquad \forall\, v \in V\,, k \in \mathbb{Z}$$

give rise to the same-named action $\varrho^{a,b}_u$ of $U_q(\widehat{\mathfrak{osp}}(V))$ on $V(u)$.

Let $U^+_q(\widehat{\mathfrak{osp}}(V))$ and $U^-_q(\widehat{\mathfrak{osp}}(V))$ be the subalgebras of $U_q(\widehat{\mathfrak{osp}}(V))$ generated by $\{e_i\}_{i=0}^s$ and $\{f_i\}_{i=0}^s$, respectively. We also define $U^{\geq}_q(\widehat{\mathfrak{osp}}(V))$ and $U^{\leq}_q(\widehat{\mathfrak{osp}}(V))$ as subalgebras of $U_q(\widehat{\mathfrak{osp}}(V))$ generated by $\{e_i, q^{\pm h_i/2}, \gamma^{\pm 1}, D^{\pm 1}\}_{i=0}^s$ and $\{f_i, q^{\pm h_i/2}, \gamma^{\pm 1}, D^{\pm 1}\}_{i=0}^s$. We likewise define the subalgebras $U'^{,+}_q(\widehat{\mathfrak{osp}}(V))$, $U'^{,-}_q(\widehat{\mathfrak{osp}}(V))$, $U'^{,\geq}_q(\widehat{\mathfrak{osp}}(V))$, $U'^{,\leq}_q(\widehat{\mathfrak{osp}}(V))$ of $U'_q(\widehat{\mathfrak{osp}}(V))$. We note that $U^{\geq}_q(\widehat{\mathfrak{osp}}(V)), U^{\leq}_q(\widehat{\mathfrak{osp}}(V)), U'^{,\geq}_q(\widehat{\mathfrak{osp}}(V)), U'^{,\leq}_q(\widehat{\mathfrak{osp}}(V))$ are actually Hopf subalgebras, and moreover

$$U^+_q(\widehat{\mathfrak{osp}}(V)) \simeq U'^{,+}_q(\widehat{\mathfrak{osp}}(V))\,, \qquad U^-_q(\widehat{\mathfrak{osp}}(V)) \simeq U'^{,-}_q(\widehat{\mathfrak{osp}}(V))\,. \tag{6.8}$$

Finally, similarly to Proposition 4.1, one has bilinear pairings

$$\begin{aligned} &(\cdot,\cdot)_J \colon U^{\leq}_q(\widehat{\mathfrak{osp}}(V)) \times U^{\geq}_q(\widehat{\mathfrak{osp}}(V)) \longrightarrow \mathbb{C}(q^{1/4})\,, \\ &(\cdot,\cdot)_J \colon U'^{,\leq}_q(\widehat{\mathfrak{osp}}(V)) \times U'^{,\geq}_q(\widehat{\mathfrak{osp}}(V)) \longrightarrow \mathbb{C}(q^{1/4})\,. \end{aligned} \tag{6.9}$$

The restrictions of both pairings to $U'^{,-}_q(\widehat{\mathfrak{osp}}(V)) \times U'^{,+}_q(\widehat{\mathfrak{osp}}(V))$ coincide, cf. (6.8), and are non-degenerate by [42, 43], cf. Remark 4.2. However, the second pairing in (6.9) is degenerate as $\gamma - 1$ is in its kernel. On the other hand (which is the key reason to add the generators $D^{\pm 1}$), the first pairing in (6.9) is non-degenerate, and hence allows to realize $U_q(\widehat{\mathfrak{osp}}(V))$ as a Drinfeld double of its Hopf subalgebras $U^{\leq}_q(\widehat{\mathfrak{osp}}(V))$ and $U^{\geq}_q(\widehat{\mathfrak{osp}}(V))$ with respect to the pairing above.

The above discussion yields the universal R-matrix for $U_q(\widehat{\mathfrak{osp}}(V))$, which induces intertwiners $V \otimes W \xrightarrow{\sim} W \otimes V$ for suitable $U_q(\widehat{\mathfrak{osp}}(V))$-modules V, W, akin to Subsection 4.1. In order to not overburden the exposition, we choose to skip the detailed presentation on this standard but rather technical discussion. Instead, we shall now proceed directly to the main goal of this paper–the evaluation of such intertwiners when $V = \varrho^{a,b}_u$ and $W = \varrho^{a,b}_v$ are the above evaluation modules. In this context, we are looking for $U_q(\widehat{\mathfrak{osp}}(V))$-module intertwiners $\hat{R}(u/v)$ satisfying

$$\hat{R}(u/v) \circ (\varrho^{a,b}_u \otimes \varrho^{a,b}_v)(x) = (\varrho^{a,b}_v \otimes \varrho^{a,b}_u)(x) \circ \hat{R}(u/v) \tag{6.10}$$

for all $x \in U_q(\widehat{\mathfrak{osp}}(V))$ (equivalently, for all $x \in U'_q(\widehat{\mathfrak{osp}}(V))$ in the context of $U'_q(\widehat{\mathfrak{osp}}(V))$-modules). In fact, the space of such solutions is one-dimensional due to the irreducibility of the tensor product $\varrho^{a,b}_u \otimes \varrho^{a,b}_v$ (which still holds when viewing them as $U'_q(\widehat{\mathfrak{osp}}(V))$-modules as long as u, v are *generic*), in contrast to Proposition 3.2. As an immediate corollary, see [21, Proposition 3], the operator $R(u/v) = \tau \circ \hat{R}(u/v)$ satisfies the Yang-Baxter relation with a spectral parameter:

$$\begin{aligned} &R_{12}(v/w)R_{13}(u/w)R_{23}(u/v) = R_{23}(u/v)R_{13}(u/w)R_{12}(v/w)\,, \\ &\hat{R}_{12}(v/w)\hat{R}_{23}(u/w)\hat{R}_{12}(u/v) = \hat{R}_{23}(u/v)\hat{R}_{12}(u/w)\hat{R}_{23}(v/w)\,. \end{aligned} \tag{6.11}$$

We shall now present the explicit formula for such $\hat{R}(z)$, which is the main result of this note:

Theorem 6.3. *For any u, v, set $z = u/v$. For $U_q(\widehat{\mathfrak{osp}}(V))$-modules $\varrho_u^{a,b}, \varrho_v^{a,b}$ from Proposition 6.2 (with the specified value of ab), the operator $\hat{R}(z) = \tau \circ R(z)$ satisfies (6.10), where*

$$R(z) = (z - q^{-m+n+2}) \Big\{ \mathrm{I} + (q^{1/2} - q^{-1/2}) \sum_{i=1}^{N} (-1)^{\bar{i}} E_{ii} \otimes \Big(q^{(\varepsilon_i,\varepsilon_i)/2} E_{ii} - q^{-(\varepsilon_i,\varepsilon_i)/2} E_{i'i'} \Big) + (q - q^{-1}) \sum_{i>j} (-1)^{\bar{j}} E_{ij} \otimes \Big(E_{ji} - (-1)^{\bar{j}(\bar{i}+\bar{j})} \vartheta_i \vartheta_j q^{(\rho,\varepsilon_i - \varepsilon_j)} E_{i'j'} \Big) \Big\} + (q - q^{-1}) \frac{z - q^{-m+n+2}}{z - 1} \tau - (q - q^{-1}) q^{-m+n+2} \sum_{i,j=1}^{N} (-1)^{\bar{i}\bar{j}} \vartheta_i \vartheta_j q^{(\rho,\varepsilon_i - \varepsilon_j)} \cdot E_{ij} \otimes E_{i'j'} \,. \quad (6.12)$$

Combining this result with the preceding paragraph, we conclude that $\hat{R}(z)$ coincides, up to a prefactor, with the action of the universal R-matrix, and thus $R(z)$ of (6.12) does satisfy (6.11).

Remark 6.4. We note that rescaling $R(z)$ of (6.12) by the factor $\frac{1}{z-q^{-m+n+2}}$ and further specializing at $z = 0$ and ∞, we recover our finite R-matrices R_0 and R_∞ from (4.13) and (4.14), respectively.

Remark 6.5. We note that rescaling $R(z)$ of (6.12) by $\frac{1}{z-q^{-m+n+2}}$, setting $q = e^{-\hbar/2}$, $z = e^{\hbar u}$, and further taking the limit $\hbar \to 0$ recovers the rational R-matrix of [15, (3.4)] (first considered in [1] for the standard parity sequence) used to define the orthosymplectic superYangian $Y(\mathfrak{osp}(V))$:

$$\lim_{\hbar \to 0} \left\{ \frac{R(z)}{z - q^{-m+n+2}} \Big|_{q = e^{-\hbar/2}, z = e^{\hbar u}} \right\} = \mathrm{I} - \frac{\tau}{u} + \frac{1}{u - \frac{m-n-2}{2}} \sum_{i,j=1}^{N} (-1)^{\bar{i}\bar{j}} \vartheta_i \vartheta_j \cdot E_{ij} \otimes E_{i'j'} \,.$$

Remark 6.6. For the standard parity sequence $\gamma_V = (\bar{1}, \ldots, \bar{1}, \bar{0}, \ldots, \bar{0})$, the exact relation between our formula (6.12) and the R-matrix $R^{[\mathrm{MDGL}]}(z)$ of [31] is given by:

$$R(z) = \frac{(qz - q^{-1})(z - q^{-m+n+2})}{z - 1} R^{[\mathrm{MDGL}]}(1/z) \,.$$

We note that the change of the spectral parameter $z = u/v \mapsto v/u = 1/z$ above is simply due to the order of the tensorands $V(u)$ and $V(v)$.

The proof of Theorem 6.3 is straightforward and crucially relies on the expression of $R(z)$ from (6.12) through R_0, R_∞ of (4.13, 4.14), which is a special case of the *Yang-Baxterization* from [17].

6.3. Yang-Baxterization.

In this Subsection, we express $R(z)$ via R_0 and R_∞ through the Yang-Baxterization procedure of [17]. This formal procedure produces $\hat{R}(z)$ satisfying (6.11) from $\hat{R}$ satisfying (4.8) when the latter has at most 3 eigenvalues. In our setup, the R-matrices $\hat{R}_{VV} = \hat{R} = \tau_{VV} R_0$ have only eigenvalues $\lambda_1, \lambda_2, \lambda_3$, in accordance with Proposition 4.11 combined with Appendix B.2.[3] In that setup, the Yang-Baxterization of [17, (3.29), (3.31)] produces the following two solutions to (6.11):

$$\hat{R}^{(1)}(z) = \lambda_1 z(z-1)\hat{R}^{-1} + \left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_1^2}{\lambda_2\lambda_3}\right) z\mathrm{I} - \frac{\lambda_1}{\lambda_2\lambda_3}(z-1)\hat{R}$$

and

$$\hat{R}^{(2)}(z) = \lambda_1 z(z-1)\hat{R}^{-1} + \left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_2}{\lambda_3}\right) z\mathrm{I} - \frac{1}{\lambda_3}(z-1)\hat{R}$$

provided that $\hat{R}$ satisfies the additional relations of [17, (3.27)] (cf. correction [17, (A.9)]), which, in particular, hold whenever $\hat{R}$ is a representation of a *Birman-Wenzl algebra*.

Remark 6.7. For our purpose, we shall not really need to verify these additional relations, since according to Theorem 6.3 the constructed $\hat{R}(z)$ do manifestly satisfy the relation (6.11).

[3] According to Propositions B.2, B.3, B.4 and Proposition 4.11, $\hat{R}_{VV}$ acts with two eigenvalues λ_1, λ_2 on the codimension 1 submodule $W^+ \oplus W^-$ of $V \otimes V$, where W^+, W^- are $U_q(\mathfrak{osp}(V))$-submodules generated by w_1, w_2, respectively. Finally, $\hat{R}_{VV}$ acts on the 1-dimensional quotient space $V \otimes V/(W^+ \oplus W^-)$ via multiplication by $\lambda_3 = q^{m-n-1}$, due to (B.3, B.8, B.19) and Proposition 4.11(b).

Proposition 6.8. *The affine R-matrix (6.12) coincides (up to τ and a rational function in z) with the Yang-Baxterization of $\hat{R}_{VV} = \tau \circ R_0$, cf. (4.12). To be more specific, for $\hat{R}(z) = \tau \circ R(z)$:*

$$\lambda_1(z-1)\hat{R}(z) = \lambda_1 z(z-1)\hat{R}_{VV}^{-1} + \left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_1^2}{\lambda_2\lambda_3}\right) z\mathrm{I} - \frac{\lambda_1}{\lambda_2\lambda_3}(z-1)\hat{R}_{VV} \tag{6.13}$$

if $|v_1| = \bar{1}$ and

$$\lambda_1(z-1)\hat{R}(z) = \lambda_1 z(z-1)\hat{R}_{VV}^{-1} + \left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_2}{\lambda_3}\right) z\mathrm{I} - \frac{1}{\lambda_3}(z-1)\hat{R}_{VV} \tag{6.14}$$

if $|v_1| = \bar{0}$, with $\lambda_1, \lambda_2, \lambda_3$ precisely as in (4.18).

Proof. By straightforward tedious computations, based on (4.13, 4.14, 6.12), one verifies that

$$\lambda_1(z-1)R(z) = \lambda_1 z(z-1)R_\infty + \left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_1^2}{\lambda_2\lambda_3}\right) z\tau - \frac{\lambda_1}{\lambda_2\lambda_3}(z-1)R_0 \tag{6.15}$$

if $|v_1| = \bar{1}$ and

$$\lambda_1(z-1)R(z) = \lambda_1 z(z-1)R_\infty + \left(1 + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1}{\lambda_3} + \frac{\lambda_2}{\lambda_3}\right) z\tau - \frac{1}{\lambda_3}(z-1)R_0 \tag{6.16}$$

if $|v_1| = \bar{0}$. Composing with τ on the left, and using (4.12), we obtain (6.13, 6.14). □

6.4. Proof of the main result.

Due to Proposition 6.8 and Theorem 4.6, it only remains to verify (6.10) for $x = e_0$ and $x = f_0$. We shall now present the direct verification for $x = e_0$, while $x = f_0$ can be treated analogously to the finite case using the supertransposition (2.5). Since both sides of (6.10) for $x = e_0$ depend linearly on a, without loss of generality, we shall now assume that $a = 1$.

For the latter purpose, let us first evaluate (ρ, ε_1). Since

$$\begin{aligned} 2\varepsilon_1 &= (\varepsilon_1 - \varepsilon_2) + (\varepsilon_2 - \varepsilon_3) + \cdots + (\varepsilon_{2'} - \varepsilon_{1'}) \\ &= \begin{cases} 2\alpha_1 + \cdots + 2\alpha_s & \text{if } m \text{ is odd} \\ 2\alpha_1 + \cdots + 2\alpha_{s-2} + \alpha_{s-1} + \alpha_s & \text{if } m \text{ is even and } \bar{s} = \bar{0} \\ 2\alpha_1 + \cdots + 2\alpha_{s-1} + \alpha_s & \text{if } m \text{ is even and } \bar{s} = \bar{1} \end{cases}, \end{aligned}$$

a direct application of (4.11) implies that

$$\begin{aligned} 2(\rho, \varepsilon_1) &= \begin{cases} (-1)^{\bar{1}} + (-1)^{\bar{2}} \cdot 2 + \cdots + (-1)^{\bar{s}} \cdot 2 & \text{if } m \text{ is odd} \\ (-1)^{\bar{1}} + (-1)^{\bar{2}} \cdot 2 + \cdots + (-1)^{\overline{s-1}} \cdot 2 + (-1)^{\bar{s}} & \text{if } m \text{ is even and } \bar{s} = \bar{0} \\ (-1)^{\bar{1}} + (-1)^{\bar{2}} \cdot 2 + \cdots + (-1)^{\overline{s-1}} \cdot 2 + (-1)^{\bar{s}} \cdot 3 & \text{if } m \text{ is even and } \bar{s} = \bar{1} \end{cases} \\ &= -(-1)^{\bar{1}} - 1 + (m-n)\,. \end{aligned}$$

Thus, we have the following uniform formula:

$$(\rho, \varepsilon_1) = \tfrac{1}{2}\big(m - n - 1 - (-1)^{\bar{1}}\big)\,. \tag{6.17}$$

• Case 1: $|v_1| = \bar{1}$.

Since $\varrho_u^{a,b}(q^{h_0/2})$ is a diagonal matrix, we shall write it as $\varrho_u^{a,b}(q^{h_0/2}) = \mathrm{diag}(\mathtt{t}_1, \ldots, \mathtt{t}_{1'})$. We shall also use the same decomposition $R_\infty = \mathrm{I} + R_1 + R_2 + R_3 + R_4$ as in Subsection 4.3. By direct computation, we get:

$$R_1\Delta(e_0) = (q^{-1} - 1)\Big(q^{-1}E_{1'1'} \otimes vE_{1'1} + uE_{1'1} \otimes qE_{1'1'}\Big)\,,$$

$$\Delta^{\mathrm{op}}(e_0)R_1 = (q^{-1} - 1)\Big(q^{-1}E_{11} \otimes vE_{1'1} + uE_{1'1} \otimes qE_{11}\Big)\,,$$

$$R_2\Delta(e_0) = -(1 - q)\Big(qE_{11} \otimes vE_{1'1} + uE_{1'1} \otimes q^{-1}E_{11}\Big)\,,$$

$$\Delta^{\mathrm{op}}(e_0)R_2 = -(1 - q)\Big(qE_{1'1'} \otimes vE_{1'1} + uE_{1'1} \otimes q^{-1}E_{1'1'}\Big)\,,$$

$$R_3\Delta(e_0) = (q - q^{-1}) \sum_{1 \le j \le N} (-1)^{\bar{j}}(\mathtt{t}_j E_{1'j} \otimes vE_{j1}) + (q - q^{-1})(q^{-1}E_{1'1'} \otimes vE_{1'1})\,,$$

$$\Delta^{\mathrm{op}}(e_0)R_3 = (q-q^{-1}) \sum_{1\le i\le N} (-1)^{\bar 1}(\mathsf{t}_i^{-1}E_{i1}\otimes vE_{1'i}) + (q-q^{-1})(q^{-1}E_{11}\otimes vE_{1'1})\,,$$

$$R_4\Delta(e_0) = -(q-q^{-1}) \sum_{1\le i\le N} (-1)^{\bar i}\vartheta_i\vartheta_1 q^{(\rho,\varepsilon_i-\varepsilon_1)}(qE_{i1}\otimes vE_{i'1}) - (q-q^{-1})(qE_{11}\otimes vE_{1'1})\,,$$

$$\Delta^{\mathrm{op}}(e_0)R_4 = -(q-q^{-1}) \sum_{1\le j\le N} (-1)^{\bar j}\vartheta_{1'}\vartheta_j q^{(\rho,\varepsilon_{1'}-\varepsilon_j)}(qE_{1'j}\otimes vE_{1'j'}) - (q-q^{-1})(qE_{1'1'}\otimes vE_{1'1})\,.$$

Assembling all the terms (and using (6.17) for the last two equalities), we get:

$$\Delta(e_0)-\Delta^{\mathrm{op}}(e_0) = (q-q^{-1})v\cdot(E_{11}-E_{1'1'})\otimes E_{1'1} - (q-q^{-1})u\cdot E_{1'1}\otimes(E_{11}-E_{1'1'})\,,$$

$$R_1\Delta(e_0)-\Delta^{\mathrm{op}}(e_0)R_1 = (q^{-1}-q^{-2})v\cdot(E_{11}-E_{1'1'})\otimes E_{1'1} + (q-1)u\cdot E_{1'1}\otimes(E_{11}-E_{1'1'})\,,$$

$$R_2\Delta(e_0)-\Delta^{\mathrm{op}}(e_0)R_2 = (q^{2}-q)v\cdot(E_{11}-E_{1'1'})\otimes E_{1'1} + (1-q^{-1})u\cdot E_{1'1}\otimes(E_{11}-E_{1'1'})\,,$$

$$\begin{aligned} R_3\Delta(e_0)-\Delta^{\mathrm{op}}(e_0)R_3 = &-(1-q^{-2})v\cdot(E_{11}-E_{1'1'})\otimes E_{1'1} \\ &+ (q-q^{-1})v\cdot \sum_{1\le j\le N}(-1)^{\bar j}\mathsf{t}_j\cdot E_{1'j}\otimes E_{j1} + (q-q^{-1})v\cdot\sum_{1\le i\le N}\mathsf{t}_i^{-1}\cdot E_{i1}\otimes E_{1'i}\,, \end{aligned}$$

$$\begin{aligned} R_4\Delta(e_0)-\Delta^{\mathrm{op}}(e_0)R_4 = &-(q^{2}-1)v\cdot(E_{11}-E_{1'1'})\otimes E_{1'1} \\ &-(q-q^{-1})v\cdot q^{-(m-n-2)/2}\sum_{1\le i\le N}(-1)^{\bar i}\vartheta_i\vartheta_1 q^{(\rho,\varepsilon_i)}\cdot E_{i1}\otimes E_{i'1} \\ &+(q-q^{-1})v\cdot q^{-(m-n-2)/2}\sum_{1\le j\le N}(-1)^{\bar j}\vartheta_{1'}\vartheta_j q^{-(\rho,\varepsilon_j)}\cdot E_{1'j}\otimes E_{1'j'}\,. \end{aligned}$$

Collecting the terms together, we obtain:

$$\begin{aligned} R_\infty\Delta(e_0)-\Delta^{\mathrm{op}}(e_0)R_\infty = &-(q-q^{-1})v\cdot q^{-(m-n-2)/2}\sum_{1\le i\le N}(-1)^{\bar i}\vartheta_i\vartheta_1 q^{(\rho,\varepsilon_i)}\cdot E_{i1}\otimes E_{i'1} \\ &+(q-q^{-1})v\cdot q^{-(m-n-2)/2}\sum_{1\le j\le N}(-1)^{\bar j}\vartheta_{1'}\vartheta_j q^{-(\rho,\varepsilon_j)}\cdot E_{1'j}\otimes E_{1'j'} \\ &+(q-q^{-1})v\cdot\sum_{1\le j\le N}(-1)^{\bar j}\mathsf{t}_j\cdot E_{1'j}\otimes E_{j1} + (q-q^{-1})v\cdot\sum_{1\le i\le N}\mathsf{t}_i^{-1}\cdot E_{i1}\otimes E_{1'i}\,. \end{aligned} \tag{6.18}$$

Though one can evaluate $R_0\Delta(e_0)-\Delta^{\mathrm{op}}(e_0)R_0$ in a similar way, we shall rather present a simple derivation of the resulting formula by utilizing the automorphism σ of $U_q(\mathfrak{osp}(V))$ from (4.24). To this end, we note that σ can be extended to a $\mathbb{C}$-algebra automorphism of $U_q(\widehat{\mathfrak{osp}}(V))$ by assigning

$$\sigma\colon\quad e_0\mapsto e_0\,,\quad f_0\mapsto f_0\,,\quad q^{\pm h_0/2}\mapsto q^{\mp h_0/2}\,,\quad \gamma^{\pm1}\mapsto\gamma^{\mp1}\,,\quad D^{\pm1}\mapsto D^{\mp1}\,.$$

Then, equalities (4.27) still hold, cf. (4.25). Therefore, applying $\bar\sigma$ of (4.23) to all matrix coefficients in the equality (6.18), conjugating with τ, and using (4.27) together with (4.22), we get:

$$\begin{aligned} R_0(\tau\Delta^{\mathrm{op}}(e_0)\tau^{-1})-(\tau\Delta(e_0)\tau^{-1})R_0 = &-(q-q^{-1})v\cdot q^{(m-n-2)/2}\sum_{1\le i\le N}(-1)^{\bar i}\vartheta_i\vartheta_1 q^{(\rho,\varepsilon_i)}\cdot E_{i1}\otimes E_{i'1} \\ &+(q-q^{-1})v\cdot q^{(m-n-2)/2}\sum_{1\le j\le N}(-1)^{\bar j}\vartheta_{1'}\vartheta_j q^{-(\rho,\varepsilon_j)}\cdot E_{1'j}\otimes E_{1'j'} \\ &+(q-q^{-1})v\cdot\sum_{1\le j\le N}\mathsf{t}_j^{-1}\cdot E_{j1}\otimes E_{1'j} + (q-q^{-1})v\cdot\sum_{1\le i\le N}(-1)^{\bar i}\mathsf{t}_i\cdot E_{1'i}\otimes E_{i1}\,. \end{aligned} \tag{6.19}$$

We also note the following equality of endomorphisms of $V(v)\otimes V(u)$:

$$\tau\circ(\varrho_u^{a,b}\otimes\varrho_v^{a,b})(\Delta(x))\circ\tau^{-1} = (\varrho_v^{a,b}\otimes\varrho_u^{a,b})(\Delta^{\mathrm{op}}(x))\qquad\text{for any}\quad x\in U_q(\widehat{\mathfrak{osp}}(V))\,.$$

Hence, switching the roles of the spectral variables u and v in (6.19), we obtain:

$$\begin{aligned} R_0\Delta(e_0) - \Delta^{\mathrm{op}}(e_0)R_0 = &-(q-q^{-1})u\cdot q^{(m-n-2)/2}\sum_{1\le i\le N}(-1)^{\bar{i}}\vartheta_i\vartheta_1 q^{(\rho,\varepsilon_i)}\cdot E_{i1}\otimes E_{i'1}\\ &+(q-q^{-1})u\cdot q^{(m-n-2)/2}\sum_{1\le j\le N}(-1)^{\bar{j}}\vartheta_{1'}\vartheta_j q^{-(\rho,\varepsilon_j)}\cdot E_{1'j}\otimes E_{1'j'}\\ &+(q-q^{-1})u\cdot\sum_{1\le j\le N}\mathsf{t}_j^{-1}\cdot E_{j1}\otimes E_{1'j}+(q-q^{-1})u\cdot\sum_{1\le i\le N}(-1)^{\bar{i}}\mathsf{t}_i\cdot E_{1'i}\otimes E_{i1}\,. \end{aligned} \tag{6.20}$$

Combining (6.18) and (6.20) with formula (6.15) and the equality

$$\tau\Delta(e_0)-\Delta^{\mathrm{op}}(e_0)\tau=(v-u)\sum_{1\le j\le N}(-1)^{\bar{j}}\mathsf{t}_j\cdot E_{1'j}\otimes E_{j1}+(v-u)\sum_{1\le i\le N}\mathsf{t}_i^{-1}\cdot E_{i1}\otimes E_{1'i}\,,$$

we ultimately get the desired result:

$$R(z)\Delta(e_0)-\Delta^{\mathrm{op}}(e_0)R(z)=0\,.$$

- Case 2: $|v_1|=\bar{0}$.

We use the same notations as above. By direct computation, we obtain:

$$\begin{aligned} R_1\Delta(e_0)=(q^{1/2}-q^{-1/2})\Big\{&(-1)^{\bar{2}}q^{(-1)^{\bar{2}}}v\cdot E_{2'2'}\otimes E_{2'1}-(-1)^{\bar{1}}q^{(-1)^{\bar{1}}}\vartheta_2 v\cdot E_{1'1'}\otimes E_{1'2}\\ &+(-1)^{\bar{2}}u\cdot E_{2'1}\otimes E_{2'2'}-(-1)^{\bar{1}}\vartheta_2 u\cdot E_{1'2}\otimes E_{1'1'}\Big\}, \end{aligned}$$

$$\begin{aligned} \Delta^{\mathrm{op}}(e_0)R_1=(q^{1/2}-q^{-1/2})\Big\{&(-1)^{\bar{1}}q^{(-1)^{\bar{1}}}v\cdot E_{11}\otimes E_{2'1}-(-1)^{\bar{2}}q^{(-1)^{\bar{2}}}\vartheta_2 v\cdot E_{22}\otimes E_{1'2}\\ &+(-1)^{\bar{1}}u\cdot E_{2'1}\otimes E_{11}-(-1)^{\bar{2}}\vartheta_2 u\cdot E_{1'2}\otimes E_{22}\Big\}, \end{aligned}$$

$$\begin{aligned} R_2\Delta(e_0)=-\,(q^{1/2}-q^{-1/2})\Big\{&(-1)^{\bar{2}}q^{-(-1)^{\bar{2}}}v\cdot E_{22}\otimes E_{2'1}-(-1)^{\bar{1}}q^{-(-1)^{\bar{1}}}\vartheta_2 v\cdot E_{11}\otimes E_{1'2}\\ &+(-1)^{\bar{2}}u\cdot E_{2'1}\otimes E_{22}-(-1)^{\bar{1}}\vartheta_2 u\cdot E_{1'2}\otimes E_{11}\Big\}, \end{aligned}$$

$$\begin{aligned} \Delta^{\mathrm{op}}(e_0)R_2=-\,(q^{1/2}-q^{-1/2})\Big\{&(-1)^{\bar{1}}q^{-(-1)^{\bar{1}}}v\cdot E_{1'1'}\otimes E_{2'1}-(-1)^{\bar{2}}q^{-(-1)^{\bar{2}}}\vartheta_2 v\cdot E_{2'2'}\otimes E_{1'2}\\ &+(-1)^{\bar{1}}u\cdot E_{2'1}\otimes E_{1'1'}-(-1)^{\bar{2}}\vartheta_2 u\cdot E_{1'2}\otimes E_{2'2'}\Big\}, \end{aligned}$$

$$\begin{aligned} R_3\Delta(e_0)=(q-q^{-1})\Big\{&\sum_{1\le j\le N}(-1)^{\bar{j}}\mathsf{t}_j v\cdot E_{2'j}\otimes E_{j1}-(-1)^{\bar{2}}q^{(-1)^{\bar{2}}/2}v\cdot E_{2'2'}\otimes E_{2'1}\\ &-(-1)^{\bar{1}}q^{(-1)^{\bar{1}}/2}v\cdot E_{2'1'}\otimes E_{1'1}-\sum_{1\le j\le N}(-1)^{\bar{j}}\vartheta_2\mathsf{t}_j v\cdot E_{1'j}\otimes E_{j2}\\ &+(-1)^{\bar{1}}q^{(-1)^{\bar{1}}/2}\vartheta_2 v\cdot E_{1'1'}\otimes E_{1'2}+(-1)^{\bar{1}}q^{-(-1)^{\bar{1}}/2}u\cdot E_{1'1}\otimes E_{2'1'}\Big\}, \end{aligned}$$

$$\begin{aligned} \Delta^{\mathrm{op}}(e_0)R_3=(q-q^{-1})\Big\{&\sum_{1\le i\le N}(-1)^{\bar{2}\,\bar{i}}\mathsf{t}_i^{-1}v\cdot E_{i1}\otimes E_{2'i}-q^{(-1)^{\bar{1}}/2}v\cdot E_{11}\otimes E_{2'1}\\ &-\sum_{1\le i\le N}(-1)^{\bar{2}\,\bar{i}}\vartheta_2\mathsf{t}_i^{-1}v\cdot E_{i2}\otimes E_{1'i}+q^{(-1)^{\bar{1}}/2}\vartheta_2 v\cdot E_{12}\otimes E_{1'1}\\ &+(-1)^{\bar{2}}q^{(-1)^{\bar{2}}/2}\vartheta_2 v\cdot E_{22}\otimes E_{1'2}-(-1)^{\bar{1}}q^{-(-1)^{\bar{1}}/2}\vartheta_2 u\cdot E_{1'1}\otimes E_{12}\Big\}, \end{aligned}$$

$$
\begin{aligned}
&R_4\Delta(e_0) = \\
&-(q-q^{-1})\Big\{\sum_{1\le i\le N}(-1)^{\bar{i}\,\bar{2}}\vartheta_i\vartheta_2 q^{(\rho,\varepsilon_i)}q^{(-1)^{\bar{1}}/2}q^{-(m-n-2)/2}v\cdot E_{i2}\otimes E_{i'1} - q^{(-1)^{\bar{1}}/2}\vartheta_2 v\cdot E_{12}\otimes E_{1'1}\\
&\qquad -(-1)^{\bar{2}}q^{-(-1)^{\bar{2}}/2}v\cdot E_{22}\otimes E_{2'1} - \sum_{1\le i\le N}\vartheta_i\vartheta_2 q^{(\rho,\varepsilon_i)}q^{-(-1)^{\bar{1}}/2}q^{-(m-n-2)/2}v\cdot E_{i1}\otimes E_{i'2}\\
&\qquad + q^{-(-1)^{\bar{1}}/2}\vartheta_2 v\cdot E_{11}\otimes E_{1'2} + q^{-(-1)^{\bar{1}}/2}\vartheta_2 u\cdot E_{1'1}\otimes E_{12}\Big\},
\end{aligned}
$$

$$
\begin{aligned}
\Delta^{\mathrm{op}}(e_0)R_4 = -(q-q^{-1})\Big\{&\sum_{1\le j\le N}(-1)^{\bar{2}\,\bar{j}}\vartheta_j q^{-(\rho,\varepsilon_j)}q^{-(-1)^{\bar{1}}/2}q^{-(m-n-2)/2}v\cdot E_{1'j}\otimes E_{2'j'}\\
&- q^{-(-1)^{\bar{1}}/2}v\cdot E_{1'1'}\otimes E_{2'1} - \sum_{1\le j\le N}\vartheta_j q^{-(\rho,\varepsilon_j)}q^{(-1)^{\bar{1}}/2}q^{-(m-n-2)/2}v\cdot E_{2'j}\otimes E_{1'j'}\\
&+ q^{(-1)^{\bar{1}}/2}v\cdot E_{2'1'}\otimes E_{1'1} + q^{-(-1)^{\bar{2}}/2}\vartheta_{2'}v\cdot E_{2'2'}\otimes E_{1'2} - q^{-(-1)^{\bar{1}}/2}u\cdot E_{1'1}\otimes E_{2'1'}\Big\},
\end{aligned}
$$

where we used (4.11, 6.17) in the last two equalities. Combining the above eight formulas, we get:

$$
\begin{aligned}
&R_\infty\Delta(e_0) - \Delta^{\mathrm{op}}(e_0)R_\infty = \\
&\quad (q-q^{-1})v\cdot\sum_{1\le j\le N}(-1)^{\bar{j}}\mathsf{t}_j\cdot E_{2'j}\otimes E_{j1} - (q-q^{-1})\vartheta_2 v\cdot\sum_{1\le j\le N}(-1)^{\bar{j}}\mathsf{t}_j\cdot E_{1'j}\otimes E_{j2}\\
&\quad -(q-q^{-1})v\cdot\sum_{1\le i\le N}(-1)^{\bar{2}\,\bar{i}}\mathsf{t}_i^{-1}\cdot E_{i1}\otimes E_{2'i} + (q-q^{-1})\vartheta_2 v\cdot\sum_{1\le i\le N}(-1)^{\bar{2}\,\bar{i}}\mathsf{t}_i^{-1}\cdot E_{i2}\otimes E_{1'i}\\
&\quad -(q-q^{-1})\cdot q^{-(m-n-2)/2}\vartheta_2 v\cdot\sum_{1\le i\le N}(-1)^{\bar{i}\,\bar{2}}\vartheta_i q^{(\rho,\varepsilon_i)}q^{1/2}\cdot E_{i2}\otimes E_{i'1}\\
&\quad +(q-q^{-1})\cdot q^{-(m-n-2)/2}\vartheta_2 v\cdot\sum_{1\le i\le N}\vartheta_i q^{(\rho,\varepsilon_i)}q^{-1/2}\cdot E_{i1}\otimes E_{i'2}\\
&\quad +(q-q^{-1})\cdot q^{-(m-n-2)/2}v\cdot\sum_{1\le j\le N}(-1)^{\bar{2}\,\bar{j}}\vartheta_j q^{-(\rho,\varepsilon_j)}q^{-1/2}\cdot E_{1'j}\otimes E_{2'j'}\\
&\quad -(q-q^{-1})\cdot q^{-(m-n-2)/2}v\cdot\sum_{1\le j\le N}\vartheta_j q^{-(\rho,\varepsilon_j)}q^{1/2}\cdot E_{2'j}\otimes E_{1'j'}\,.
\end{aligned}
$$

Evoking the paragraph after (6.18), we immediately obtain (similarly to Case 1):

$$
\begin{aligned}
&R_0\Delta(e_0) - \Delta^{\mathrm{op}}(e_0)R_0 = \\
&\quad -(q-q^{-1})u\cdot\sum_{1\le j\le N}(-1)^{\bar{2}\,\bar{j}}\mathsf{t}_j^{-1}\cdot E_{j1}\otimes E_{2'j} + (q-q^{-1})\vartheta_2 u\cdot\sum_{1\le j\le N}(-1)^{\bar{2}\,\bar{j}}\mathsf{t}_j^{-1}\cdot E_{j2}\otimes E_{1'j}\\
&\quad +(q-q^{-1})u\cdot\sum_{1\le i\le N}(-1)^{\bar{i}}\mathsf{t}_i\cdot E_{2'i}\otimes E_{i1} - (q-q^{-1})\vartheta_2 u\cdot\sum_{1\le i\le N}(-1)^{\bar{i}}\mathsf{t}_i\cdot E_{1'i}\otimes E_{i2}\\
&\quad +(q-q^{-1})\cdot q^{(m-n-2)/2}\vartheta_2 u\cdot\sum_{1\le i\le N}\vartheta_i q^{(\rho,\varepsilon_i)}q^{-1/2}\cdot E_{i1}\otimes E_{i'2}\\
&\quad -(q-q^{-1})\cdot q^{(m-n-2)/2}\vartheta_2 u\cdot\sum_{1\le i\le N}(-1)^{\bar{i}\,\bar{2}}\vartheta_i q^{(\rho,\varepsilon_i)}q^{1/2}\cdot E_{i2}\otimes E_{i'1}\\
&\quad -(q-q^{-1})\cdot q^{(m-n-2)/2}u\cdot\sum_{1\le j\le N}\vartheta_j q^{-(\rho,\varepsilon_j)}q^{1/2}\cdot E_{2'j}\otimes E_{1'j'}\\
&\quad +(q-q^{-1})\cdot q^{(m-n-2)/2}u\cdot\sum_{1\le j\le N}(-1)^{\bar{2}\,\bar{j}}\vartheta_j q^{-(\rho,\varepsilon_j)}q^{-1/2}\cdot E_{1'j}\otimes E_{2'j'}\,.
\end{aligned}
$$

Likewise, we also obtain:

$$\tau\Delta(e_0) - \Delta^{\mathrm{op}}(e_0)\tau = (v-u)\cdot\Bigg\{\sum_{1\le j\le N}(-1)^{\bar{j}}\mathsf{t}_j E_{2'j}\otimes E_{j1}$$

$$-\sum_{1\le j\le N}(-1)^{\bar{j}}\vartheta_2\mathsf{t}_j E_{1'j}\otimes E_{j2} - \sum_{1\le i\le N}(-1)^{\bar{i}\,\bar{2}}\mathsf{t}_i^{-1}E_{i1}\otimes E_{2'i} + \sum_{1\le i\le N}(-1)^{\bar{i}\,\bar{2}}\vartheta_2\mathsf{t}_i^{-1}E_{i2}\otimes E_{1'i}\Bigg\}.$$

Combining the above three equalities with formula (6.16), we ultimately get the desired result:

$$R(z)\Delta(e_0) - \Delta^{\mathrm{op}}(e_0)R(z) = 0\,.$$

This completes the proof of Theorem 6.3.

7. Conflicts of interests

The authors state that there is no conflict of interest.

8. Data availability statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Appendix A. A-type counterpart

In this Appendix, we present an analogous (though simpler) derivation of both finite and affine R-matrices associated with the first fundamental representation of A-type quantum supergroups. While these R-matrices are well-known to experts, type A served as a prototype for our treatment of orthosymplectic type. We note that solutions (A.19) of the Yang-Baxter equation with a spectral parameter go back to the physics paper [37], thus preceding the development of quantum groups.

A.1. **A-type Lie superalgebras.**

We shall follow the notations of Section 2.1 with the exception that we do not assume n to be even and we do not assume (2.1). Recall the Lie superalgebra $\mathfrak{gl}(V)$ of Section 2.2. The elements $\{E_{ij}\}_{i,j=1}^N$ form a basis of $\mathfrak{gl}(V)$. We choose the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(V)$ to consist of all diagonal matrices. Thus, $\{E_{ii}\}_{i=1}^N$ is a basis of $\mathfrak{h}$ and $\{\varepsilon_i\}_{i=1}^N$ is a dual basis of $\mathfrak{h}^*$. The computation $[E_{ii},E_{ab}] = (\varepsilon_a-\varepsilon_b)(E_{ii})E_{ab}$ shows that E_{ab} is a root vector corresponding to the root $\varepsilon_a-\varepsilon_b$. Hence, we get the *root space decomposition* $\mathfrak{gl}(V) = \mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi}\mathfrak{gl}(V)_\alpha$ with the root system

$$\Phi = \{\varepsilon_a-\varepsilon_b \,|\, a\ne b\}\,. \tag{A.1}$$

It decomposes $\Phi = \Phi_{\bar 0}\cup\Phi_{\bar 1}$ into *even* and *odd* roots. We also choose the following polarization:

$$\Phi^+ = \{\varepsilon_a-\varepsilon_b \,|\, a<b\}\,, \qquad \Phi^- = \{\varepsilon_a-\varepsilon_b \,|\, a>b\}\,. \tag{A.2}$$

We note that $\bar\Phi = \Phi$, cf. (2.10), and all odd roots are isotropic in the present setup.

Consider the non-degenerate supertrace bilinear form $(\cdot,\cdot)\colon \mathfrak{gl}(V)\times\mathfrak{gl}(V)\to\mathbb{C}$ defined by $(X,Y) = \mathrm{sTr}(XY)$. Its restriction to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(V)$ is non-degenerate, giving rise to an identification $\mathfrak{h}\simeq\mathfrak{h}^*$ via $\varepsilon_i\leftrightarrow(-1)^{\bar i}E_{ii}$ and inducing a bilinear form $(\cdot,\cdot)$ on $\mathfrak{h}^*$ such that

$$(\varepsilon_i,\varepsilon_j) = \delta_{ij}(-1)^{\bar i} \qquad \text{for any} \qquad 1\le i,j\le N\,.$$

Following the above choice of polarization (A.2) of the root system (A.1), the simple roots are $\alpha_i = \varepsilon_i-\varepsilon_{i+1}$ $(1\le i<N)$ and the corresponding root vectors are given by:

$$\mathsf{e}_i = E_{i,i+1}\,, \quad \mathsf{f}_i = (-1)^{\bar i}E_{i+1,i}\,, \quad \mathsf{h}_i = (-1)^{\bar i}E_{ii} - (-1)^{\overline{i+1}}E_{i+1,i+1} \qquad \forall\, 1\le i<N\,. \tag{A.3}$$

As before, we define the *symmetrized Cartan matrix* $(a_{ij})_{i,j=1}^{N-1}$ via $a_{ij} = (\alpha_i,\alpha_j)$. Then, the above elements $\{\mathsf{e}_i,\mathsf{f}_i,\mathsf{h}_i\}_{i=1}^{N-1}$ are easily seen to satisfy the Chevalley-type relations:

$$[\mathsf{h}_i,\mathsf{h}_j]=0\,, \quad [\mathsf{h}_i,\mathsf{e}_j] = a_{ij}\mathsf{e}_j\,, \quad [\mathsf{h}_i,\mathsf{f}_j] = -a_{ij}\mathsf{f}_j\,, \quad [\mathsf{e}_i,\mathsf{f}_j] = \delta_{ij}\mathsf{h}_i\,. \tag{A.4}$$

Define a Lie subalgebra $\mathfrak{sl}(V)$ of $\mathfrak{gl}(V)$ via $\mathfrak{sl}(V) = \{x\in\mathfrak{gl}(V)\,|\,\mathrm{sTr}(X)=0\}$. For $m=n$, we note that the identity map I belongs to $\mathfrak{sl}(V)$. This basic A-type Lie superalgebra $\mathfrak{sl}(V)$ admits a generators-and-relations presentation, due to [44, Main Theorem]. Explicitly, it is generated by $\{\mathsf{e}_i, \mathsf{f}_i, \mathsf{h}_i\}_{i=1}^{N-1}$, with the $\mathbb{Z}_2$-grading

$$|\mathsf{e}_i| = |\mathsf{f}_i| = \begin{cases} \bar{0} & \text{if } \alpha_i \in \Phi_{\bar{0}} \\ \bar{1} & \text{if } \alpha_i \in \Phi_{\bar{1}} \end{cases}, \qquad |\mathsf{h}_i| = \bar{0}, \tag{A.5}$$

with the defining relations (A.4) as well as the following *Serre relations*:

$$[\mathsf{e}_i, \mathsf{e}_j] = 0, \quad [\mathsf{f}_i, \mathsf{f}_j] = 0 \quad \text{if} \quad a_{ij} = 0, \tag{A.6}$$

$$[\mathsf{e}_i, [\mathsf{e}_i, \mathsf{e}_j]] = 0, \quad [\mathsf{f}_i, [\mathsf{f}_i, \mathsf{f}_j]] = 0 \quad \text{if} \quad j = i \pm 1 \quad \text{and} \quad \alpha_i \in \Phi_{\bar{0}}, \tag{A.7}$$

$$[[[\mathsf{e}_{i-1}, \mathsf{e}_i], \mathsf{e}_{i+1}], \mathsf{e}_i] = 0, \quad [[[\mathsf{f}_{i-1}, \mathsf{f}_i], \mathsf{f}_{i+1}], \mathsf{f}_i] = 0 \quad \text{if} \quad \alpha_i \in \Phi_{\bar{1}}. \tag{A.8}$$

A.2. **A-type quantum supergroups.**

The *A-type quantum supergroup* $U_q(\mathfrak{sl}(V))$ is a natural quantization of the universal enveloping superalgebra $U(\mathfrak{sl}(V))$. Explicitly, $U_q(\mathfrak{sl}(V))$ is a $\mathbb{C}(q^{\pm 1/2})$-superalgebra generated by $\{e_i, f_i, q^{\pm h_i/2}\}_{i=1}^{N-1}$, with the $\mathbb{Z}_2$-grading as in (A.5), subject to the analogues of (2.18)–(2.20):

$$[q^{h_i/2}, q^{h_j/2}] = 0, \qquad q^{\pm h_i/2} q^{\mp h_i/2} = 1,$$
$$q^{h_i/2} e_j q^{-h_i/2} = q^{a_{ij}/2} e_j, \quad q^{h_i/2} f_j q^{-h_i/2} = q^{-a_{ij}/2} f_j,$$
$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}},$$

as well as the following q-*Serre relations* (cf. [42, Proposition 10.4.1]):

$$[\![e_i, e_j]\!] = 0, \quad [\![f_i, f_j]\!] = 0 \quad \text{if} \quad a_{ij} = 0, \tag{A.9}$$

$$[\![e_i, [\![e_i, e_j]\!]]\!] = 0, \quad [\![f_i, [\![f_i, f_j]\!]]\!] = 0 \quad \text{if} \quad j = i \pm 1 \quad \text{and} \quad \alpha_i \in \Phi_{\bar{0}}, \tag{A.10}$$

$$[\![[\![[\![e_{i-1}, e_i]\!], e_{i+1}]\!], e_i]\!] = 0, \quad [\![[\![[\![f_{i-1}, f_i]\!], f_{i+1}]\!], f_i]\!] = 0 \quad \text{if} \quad \alpha_i \in \Phi_{\bar{1}}. \tag{A.11}$$

Here, we use the notation $[\![\cdot, \cdot]\!]$ from (2.22), which relies on the natural Q-grading of $U_q(\mathfrak{sl}(V))$ defined analogously to (2.21), see (A.12) below. Moreover, $U_q(\mathfrak{sl}(V))$ is equipped with a Hopf superalgebra structure via the same formulas as in Subsection 2.4.

A.3. **First fundamental representations.**

Using the notation of Section 3, we have the following analogue of Proposition 3.1:

Proposition A.1. *The following defines a representation* $\varrho\colon U_q(\mathfrak{sl}(V)) \to \mathrm{End}(V)$*:*

$$\varrho(e_i) = \mathsf{e}_i, \qquad \varrho(f_i) = \mathsf{f}_i, \qquad \varrho(q^{\pm h_i/2}) = q^{\pm \mathsf{h}_i/2} \qquad \textit{for} \quad 1 \leq i < N,$$

where $\{\mathsf{e}_i, \mathsf{f}_i, \mathsf{h}_i\}_{i=1}^{N-1}$ *denote the action of Chevalley-type generators of* $\mathfrak{sl}(V)$ *given by* (A.3).

The proof of this result is analogous (but simpler) to that of Proposition 3.1. In particular, all q-Serre relations hold for degree reasons. Here, we note that both the algebra $U_q(\mathfrak{sl}(V))$ and the vector space V have compatible (cf. (3.3)) grading by $P = \bigoplus_{i=1}^N \mathbb{Z}\varepsilon_i$ via (for any $a < N, i \leq N$):

$$\deg(e_a) = \varepsilon_a - \varepsilon_{a+1}, \quad \deg(f_a) = -\varepsilon_a + \varepsilon_{a+1}, \quad \deg(q^{h_a/2}) = 0, \quad \deg(v_i) = \varepsilon_i. \tag{A.12}$$

A.4. **Tensor square of the first fundamental representation.**

The following result is an A-type analogue of Proposition 3.2:

Proposition A.2. *(a) The following are highest weight vectors in* $U_q(\mathfrak{sl}(V))$*-module* $V \otimes V$*:*

$$w_1 = v_1 \otimes v_1, \qquad w_2 = v_1 \otimes v_2 - (-1)^{\bar{1}(\bar{1}+\bar{2})} q^{(-1)^{\bar{1}}} \cdot v_2 \otimes v_1. \tag{A.13}$$

(b) The $U_q(\mathfrak{sl}(V))$*-representation* $V \otimes V$ *is generated by these vectors* w_1, w_2 *of* (A.13).

Proof. (a) Let us show that the vectors w_1 and w_2 are indeed highest weight vectors for the action $\varrho^{\otimes 2}$ of $U_q(\mathfrak{sl}(V))$ on $V \otimes V$. First, we note that these vectors are eigenvectors with respect to $q^{h_i/2}$:

$$\varrho^{\otimes 2}(q^{h_i/2}) w_1 = q^{2\varepsilon_1(\mathsf{h}_i/2)} w_1, \quad \varrho^{\otimes 2}(q^{h_i/2}) w_2 = q^{(\varepsilon_1+\varepsilon_2)(\mathsf{h}_i/2)} w_2 \qquad \forall\, 1 \leq i < N.$$

It remains to verify that w_1 and w_2 are annihilated by all $\varrho^{\otimes 2}(e_i)$. The equality $\varrho^{\otimes 2}(e_i)(w_1)=0$ follows from $\varrho(e_i)(v_1)=0$. Likewise, $\varrho^{\otimes 2}(e_i)(w_2)=0$ for $i>1$ follows from $\varrho(e_i)v_1=\varrho(e_i)v_2=0$. Meanwhile, combining $\varrho(e_1)v_2=v_1$, $\varrho(e_1)v_1=0$, $\varrho(q^{h_1/2})v_1=q^{(-1)^{\overline{1}}/2}v_1$, and (2.23), we also get:

$$\begin{aligned}\varrho^{\otimes 2}(e_1)w_2 &= (\varrho(q^{h_1/2})\otimes\varrho(e_1))(v_1\otimes v_2)-(-1)^{\overline{1}(\overline{1}+\overline{2})}q^{(-1)^{\overline{1}}}(\varrho(e_1)\otimes\varrho(q^{-h_1/2}))(v_2\otimes v_1)\\ &= \Big((-1)^{(\overline{1}+\overline{2})\overline{1}}\cdot q^{(-1)^{\overline{1}}/2}-(-1)^{\overline{1}(\overline{1}+\overline{2})}q^{(-1)^{\overline{1}}}\cdot q^{-(-1)^{\overline{1}}/2}\Big)\cdot v_1\otimes v_1=0\,.\end{aligned}$$

(b) Part (b) is established in Proposition B.1 from Appendix B. □

A.5. Explicit finite R-matrices.

Let ρ be the Weyl vector of Φ, defined by the same formula (4.10). We note that it still satisfies (4.11). We also define the $U_q(\mathfrak{sl}(V))$-module isomorphism $\hat{R}_{VV}\colon V\otimes V\xrightarrow{\sim}V\otimes V$ precisely as in Proposition 4.3. The following is an A-type counterpart of Theorem 4.6:

Theorem A.3. *The $U_q(\mathfrak{sl}(V))$-module isomorphism $\hat{R}_{VV}\colon V\otimes V\xrightarrow{\sim}V\otimes V$ and its inverse $\hat{R}^{-1}_{VV}$ for the $U_q(\mathfrak{sl}(V))$-module V constructed in Proposition A.1 are given by*

$$\hat{R}_{VV}=\tau_{VV}\circ R_0 \qquad\text{and}\qquad \hat{R}^{-1}_{VV}=\tau_{VV}\circ R_\infty \tag{A.14}$$

with the following explicit operators

$$R_0=\mathrm{I}+(q^{-1/2}-q^{1/2})\sum_{i=1}^N(-1)^{\overline{i}}q^{-(\varepsilon_i,\varepsilon_i)/2}E_{ii}\otimes E_{ii}+(q^{-1}-q)\sum_{i<j}(-1)^{\overline{j}}E_{ij}\otimes E_{ji}\,, \tag{A.15}$$

$$R_\infty=\mathrm{I}+(q^{1/2}-q^{-1/2})\sum_{i=1}^N(-1)^{\overline{i}}q^{(\varepsilon_i,\varepsilon_i)/2}E_{ii}\otimes E_{ii}+(q-q^{-1})\sum_{i>j}(-1)^{\overline{j}}E_{ij}\otimes E_{ji}\,. \tag{A.16}$$

Remark A.4. We note that all the summands in (A.15, A.16) already featured in (4.13, 4.14).

Remark A.5. In analogy to Remark 4.7, let us also present here the formula for the operator $R=\Theta\circ\widetilde{f}$ and its inverse R^{-1}, corresponding to the usual coproduct Δ^J of (2.24), as follows:

$$\begin{aligned}R&=\widetilde{f}^{-1/2}\circ R_0\circ\widetilde{f}^{1/2}\\ &=\mathrm{I}+(q^{-1/2}-q^{1/2})\sum_{i=1}^N(-1)^{\overline{i}}q^{-(\varepsilon_i,\varepsilon_i)/2}E_{ii}\otimes E_{ii}+(q^{-1}-q)\sum_{i<j}(-1)^{\overline{j}}E_{ij}\otimes E_{ji}=R_0\,,\\ R^{-1}&=\tau\circ\widetilde{f}^{-1/2}\circ R_\infty\circ\widetilde{f}^{1/2}\circ\tau\\ &=\mathrm{I}+(q^{1/2}-q^{-1/2})\sum_{i=1}^N(-1)^{\overline{i}}q^{(\varepsilon_i,\varepsilon_i)/2}E_{ii}\otimes E_{ii}+(q-q^{-1})\sum_{i<j}(-1)^{\overline{j}}E_{ij}\otimes E_{ji}\,.\end{aligned}$$

The proof is analogous to that of Theorem 4.6 and follows from the next four Propositions.

Proposition A.6. *For any element $x\in U_q(\mathfrak{sl}(V))$, the following equalities hold (cf. (4.2)):*

$$R_0\Delta(x)=\Delta^{\mathrm{op}}(x)R_0 \qquad\textit{and}\qquad R_\infty\Delta(x)=\Delta^{\mathrm{op}}(x)R_\infty\,.$$

Proof. We shall only verify $R_\infty\Delta(x)=\Delta^{\mathrm{op}}(x)R_\infty$ when $x=e_a$ (the proof for the other generators $x=q^{\pm h_a/2}$, f_a as well as for R_0 instead of R_∞ is completely analogous to our treatment in the proof of Proposition 4.8). Since $q^{\mathsf{h}_a/2}$ is a diagonal matrix, we shall write it as $q^{\mathsf{h}_a/2}=\mathrm{diag}(\mathsf{t}_1,\ldots,\mathsf{t}_N)$. By direct computation, we get:

$$\begin{aligned}R_\infty\Delta(e_a)=(\varrho\otimes\varrho)(\Delta(e_a))+(q^{(-1)^{\overline{a}}}-1)\Big\{q^{(-1)^{\overline{a}}/2}\cdot E_{aa}\otimes E_{a,a+1}+q^{-(-1)^{\overline{a}}/2}\cdot E_{a,a+1}\otimes E_{aa}\Big\}\\ +(q-q^{-1})\sum_{j<a}(-1)^{\overline{j}}\mathsf{t}_j\cdot E_{aj}\otimes E_{j,a+1}\\ +(q-q^{-1})\sum_{i>a}(-1)^{\overline{a}}(-1)^{(\overline{a}+\overline{i})(\overline{a}+\overline{a+1})}\mathsf{t}_i^{-1}\cdot E_{i,a+1}\otimes E_{ai}\,,\end{aligned}$$

$$
\begin{aligned}
\Delta^{\mathrm{op}}(e_a)R_\infty = (\varrho\otimes\varrho)(\Delta^{\mathrm{op}}(e_a))& \\
&+ (q^{(-1)^{\overline{a+1}}}-1)\Big\{q^{(-1)^{\overline{a+1}}/2}\cdot E_{a+1,a+1}\otimes E_{a,a+1} + q^{-(-1)^{\overline{a+1}}/2}\cdot E_{a,a+1}\otimes E_{a+1,a+1}\Big\} \\
&+ (q-q^{-1})\sum_{i>a+1}(-1)^{\overline{a+1}}(-1)^{(\overline{a}+\overline{a+1})(\overline{i}+\overline{a+1})}\mathsf{t}_i^{-1}\cdot E_{i,a+1}\otimes E_{ai} \\
&+ (q-q^{-1})\sum_{j<a+1}(-1)^{\overline{j}}\mathsf{t}_j\cdot E_{aj}\otimes E_{j,a+1}\,.
\end{aligned}
$$

Combining these two formulas with

$$
\begin{aligned}
(\varrho\otimes\varrho)(\Delta(e_a)-\Delta^{\mathrm{op}}(e_a)) = (q^{1/2}-q^{-1/2})\Big\{&\Big((-1)^{\overline{a}}E_{aa}-(-1)^{\overline{a+1}}E_{a+1,a+1}\Big)\otimes E_{a,a+1} \\
&- E_{a,a+1}\otimes\Big((-1)^{\overline{a}}E_{aa}-(-1)^{\overline{a+1}}E_{a+1,a+1}\Big)\Big\},
\end{aligned}
$$

we obtain the desired equality $R_\infty\Delta(e_a)-\Delta^{\mathrm{op}}(e_a)R_\infty=0$ for all $1\le a<N$. ☐

Next, we evaluate the eigenvalues of $\tau R_0, \tau R_\infty, \hat{R}_{VV}$ on the highest weight vectors from (A.13).

Proposition A.7. *The highest weight vectors w_1 and w_2 from (A.13) are eigenvectors of $\tau_{VV}\circ R_0$ with the eigenvalues $\mu_1^0=(-1)^{\overline{1}}q^{-(-1)^{\overline{1}}}$ and $\mu_2^0=-(-1)^{\overline{1}}q^{(-1)^{\overline{1}}}$, respectively (cf. (4.16)).*

Proposition A.8. *The highest weight vectors w_1 and w_2 are eigenvectors of $\tau_{VV}\circ R_\infty$ with the eigenvalues $\mu_1^\infty=(-1)^{\overline{1}}q^{(-1)^{\overline{1}}}=1/\mu_1^0$ and $\mu_2^\infty=-(-1)^{\overline{1}}q^{-(-1)^{\overline{1}}}=1/\mu_2^0$, respectively.*

Proposition A.9. *The highest weight vectors w_1 and w_2 are eigenvectors of $\hat{R}_{VV}$ with the eigenvalues $\lambda_1=(-1)^{\overline{1}}q^{-(-1)^{\overline{1}}}=\mu_1^0$ and $\lambda_2=-(-1)^{\overline{1}}q^{(-1)^{\overline{1}}}=\mu_2^0$, respectively (cf. (4.18)).*

The above three results are proved completely analogously to Propositions 4.9, 4.10, and 4.11.

A.6. **Factorization of finite R-matrices.**

For the order $1<2<\cdots<N-1$ on the alphabet $I=\{1,2,\ldots,N-1\}$, the dominant Lyndon words were computed in [7, Proposition 6.1]:

$$\mathsf{L}^+=\{[i\ldots j]\,|\,1\le i\le j\le N-1\}.$$

This results in the following lexicographical order on the (reduced) root system:

$$\alpha_1<\alpha_1+\alpha_2<\cdots<\alpha_1+\cdots+\alpha_{N-1}<\alpha_2<\cdots<\alpha_{N-2}<\alpha_{N-2}+\alpha_{N-1}<\alpha_{N-1}\,.$$

Let $\gamma_{ij}=\alpha_i+\cdots+\alpha_j$ for $1\le i\le j\le N-1$. Then, the assignment $\gamma\mapsto(\alpha,\beta)$ corresponding to the costandard factorization of Lyndon words is explicitly given by $\gamma_{ij}\mapsto(\gamma_{i,j-1},\alpha_j)$ for $i<j$.

The following is the counterpart of Lemmas 5.7–5.9, which has been carried out in [7, §6.1].

Lemma A.10. *For $\ell=[i\ldots j]$ with $1\le i<j\le N-1$, we have:*

$$(R_\ell,R_\ell)^{\mathrm{tw}}=\prod_{k=i}^{j-1}(\alpha_k,\alpha_{k+1})\cdot(q-q^{-1})^{j-i}\cdot q^{N(\deg\ell)}.$$

Combining this lemma with (5.26), we obtain the following counterpart of Lemmas 5.13–5.15:

Lemma A.11. *If $\gamma=\varepsilon_i-\varepsilon_j$ with $1\le i<j\le N$, then*

$$(f_\gamma,e_\gamma)_J=(-1)^{\overline{i}+\cdots+\overline{j}}\cdot(q^{-1}-q)^{-1}\,.$$

The rest of this Subsection proceeds completely analogously to Subsection 5.9, hence we shall only state the results (skipping identical proofs). First, we compute the action of the root vectors:

Lemma A.12. *For $\gamma=\varepsilon_i-\varepsilon_j$ with $1\le i<j\le N$, we have:*

$$\varrho(e_\gamma)=E_{ij}\,,\qquad \varrho(f_\gamma)=(-1)^{\overline{i}+\cdots+\overline{j-1}}\cdot E_{ji}\,.$$

Combining the above two lemmas, we obtain:

Lemma A.13. *The operators Θ_γ of (5.32) act on the $U_q(\mathfrak{sl}(V))$-module $V\otimes V$ as follows:*

$$\Theta_{\varepsilon_i-\varepsilon_j}=\mathrm{I}-(-1)^{\bar{j}}(q-q^{-1})\cdot E_{ij}\otimes E_{ji} \qquad \forall\, 1\leq i<j\leq N\,.$$

For any $1\leq i\leq N-1$, we now evaluate explicitly the product $\Theta_i=\Theta_{\varepsilon_i-\varepsilon_N}\cdots\Theta_{\varepsilon_i-\varepsilon_{i+1}}$:

Lemma A.14. *For $1\leq i\leq N-1$, we have:*

$$\Theta_i=\mathrm{I}-(q-q^{-1})\sum_{j=i+1}^{N}(-1)^{\bar{j}}E_{ij}\otimes E_{ji}\,.$$

Combining the above result with the factorization $\Theta=\Theta_{N-1}\cdots\Theta_1$, we finally get:

Proposition A.15. *The action of the operator Θ on the $U_q(\mathfrak{sl}(V))$-module $V\otimes V$ is given by*

$$\Theta=\mathrm{I}-(q-q^{-1})\sum_{i<j}(-1)^{\bar{j}}E_{ij}\otimes E_{ji}\,. \tag{A.17}$$

Combining the above formula with (5.42), we obtain

$$R_{VV}=\tau_{VV}\circ\hat{R}_{VV}=\tilde{f}^{1/2}\circ\Theta\circ\tilde{f}^{1/2}=\sum_{i,j}q^{-(\varepsilon_i,\varepsilon_j)}E_{ii}\otimes E_{jj}+(q^{-1}-q)\sum_{i<j}(-1)^{\bar{j}}E_{ij}\otimes E_{ji}$$
$$=\mathrm{I}+(q^{-1/2}-q^{1/2})\sum_{i=1}^{N}(-1)^{\bar{i}}q^{-(\varepsilon_i,\varepsilon_i)/2}E_{ii}\otimes E_{ii}+(q^{-1}-q)\sum_{i<j}(-1)^{\bar{j}}E_{ij}\otimes E_{ji}\,,$$

which precisely recovers R_0 from (A.15). This provides an alternative proof of Theorem A.3.

A.7. **Explicit affine R-matrices.**

Let $\theta=\varepsilon_1-\varepsilon_N=\alpha_1+\cdots+\alpha_{N-1}$ be the highest root of $\mathfrak{sl}(V)$ with respect to the polarization (A.2). Define the *symmetrized extended Cartan matrix* $(a_{ij})_{i,j=0}^{N-1}$ of $\mathfrak{sl}(V)$ as in Subsection 6.1. The *A-type quantum affine supergroup*, denoted by $U_q(\widehat{\mathfrak{sl}}(V))$, is a $\mathbb{C}(q^{\pm 1/2})$-superalgebra generated by $\{e_i,f_i,q^{\pm h_i/2}\}_{i=0}^{N-1}\cup\{\gamma^{\pm 1},D^{\pm 1}\}$, with the $\mathbb{Z}_2$-grading

$$|\gamma^{\pm1}|=|D^{\pm1}|=|h_i|=\bar{0}\,,\quad |e_i|=|f_i|=\begin{cases}\bar{0} & \text{if } \alpha_i \text{ is even}\\ \bar{1} & \text{if } \alpha_i \text{ is odd}\end{cases}\quad \text{for}\quad 0\leq i<N\,,$$

where α_0 is a root of the same parity as θ, subject to the analogues of (6.1)–(6.5):

$$D^{\pm1}\cdot D^{\mp1}=1\,,\quad [D,q^{h_i/2}]=0\,,\quad De_iD^{-1}=q^{\delta_{0i}}e_i\,,\quad Df_iD^{-1}=q^{-\delta_{0i}}f_i\,,$$
$$\gamma^{\pm1}\cdot\gamma^{\mp1}=1\,,\quad \gamma=q^{h_0/2}q^{h_1/2}\cdots q^{h_{N-1}/2}\,,\quad \gamma-\text{central element}\,,$$

$$[q^{h_i/2},q^{h_j/2}]=0\,,\quad q^{\pm h_i/2}q^{\mp h_i/2}=1\,,$$
$$q^{h_i/2}e_jq^{-h_i/2}=q^{a_{ij}/2}e_j\,,\quad q^{h_i/2}f_jq^{-h_i/2}=q^{-a_{ij}/2}f_j\,,$$
$$[e_i,f_j]=\delta_{ij}\frac{q^{h_i}-q^{-h_i}}{q-q^{-1}}\,,$$

together with the *q-Serre relations* specified in (A.9)–(A.11), whereas the indices $i\pm 1$ are now understood modulo N, and the following relations if $mn=2$:

$$\begin{gathered}[\![e_j,[\![e_k,[\![e_j,[\![e_k,e_i]\!]]\!]]\!]]\!]=[\![e_k,[\![e_j,[\![e_k,[\![e_j,e_i]\!]]\!]]\!]]\!]\,,\\ [\![f_j,[\![f_k,[\![f_j,[\![f_k,f_i]\!]]\!]]\!]]\!]=[\![f_k,[\![f_j,[\![f_k,[\![f_j,f_i]\!]]\!]]\!]]\!]\,,\end{gathered} \tag{A.18}$$

with $\{i,j,k\}=\{0,1,2\}$, α_i being even, and α_j,α_k being odd. The Hopf superalgebra structure on $U_q(\widehat{\mathfrak{sl}}(V))$ is given by the same formulas as in Subsection 6.1. Similarly to the last paragraph of Subsection 6.1, we also define the superalgebra $U'_q(\widehat{\mathfrak{sl}}(V))$ by ignoring the degree generators $D^{\pm1}$.

Proposition A.16. *For any $u\in\mathbb{C}^\times$ and $a,b\in\mathbb{C}^\times$ satisfying $ab=(-1)^{\overline{N}}$, the $U_q(\mathfrak{sl}(V))$-action ϱ on V from Proposition A.1 can be extended to a $U'_q(\widehat{\mathfrak{sl}}(V))$-action $\varrho^{a,b}_u$ on $V(u)=V$ by setting*

$$\varrho^{a,b}_u(x)=\varrho(x)\qquad \text{for all}\quad x\in\{e_i,f_i,q^{\pm h_i/2}\}_{i=1}^{N-1}$$

and defining the action of the remaining generators $e_0, f_0, q^{\pm h_0/2}, \gamma^{\pm 1}$ as follows:

$$\varrho_u^{a,b}(e_0) = au \cdot E_{N1}\,, \qquad \varrho_u^{a,b}(f_0) = bu^{-1} \cdot E_{1N}\,,$$
$$\varrho_u^{a,b}(q^{\pm h_0/2}) = q^{\mp((-1)^{\overline{1}}E_{11} - (-1)^{\overline{N}}E_{NN})/2}\,, \quad \varrho_u^{a,b}(\gamma^{\pm 1}) = \mathrm{I}\,.$$

Proof. The proof is analogous (though much simpler) to that of Proposition 6.1. We extend the P-grading (A.12) on $U_q(\mathfrak{sl}(V))$ to that on $U_q'(\widehat{\mathfrak{sl}}(V))$ via $\deg(e_0) = \varepsilon_N - \varepsilon_1$, $\deg(f_0) = \varepsilon_1 - \varepsilon_N$, and $\deg(q^{\pm h_0/2}) = \deg(\gamma^{\pm 1}) = 0$. With respect to this grading, all q-Serre relations except for (A.18) hold for degree reasons. Meanwhile, the remaining relation (A.18), which occurs only if $mn = 2$, is straightforwardly verified in the basis vectors v_1, v_2, v_3 of V. □

We also note the following analogue of Proposition 6.2:

Proposition A.17. *Let u be an indeterminate and redefine $V(u)$ via $V(u) = V \otimes_{\mathbb{C}} \mathbb{C}[u, u^{-1}]$. Then, the formulas defining $\varrho_u^{a,b}$ on the generators from Proposition A.16 together with*

$$\varrho_u^{a,b}(D^{\pm 1})(v \otimes u^k) = q^{\pm k} \cdot v \otimes u^k \qquad \forall\, v \in V\,,\ k \in \mathbb{Z}$$

give rise to the same-named action $\varrho_u^{a,b}$ of $U_q(\widehat{\mathfrak{sl}}(V))$ on $V(u)$.

We shall now present the explicit formula for $\hat{R}(z)$, cf. Theorem 6.3:

Theorem A.18. *For any u, v, set $z = u/v$. For $U_q(\widehat{\mathfrak{sl}}(V))$-modules $\varrho_u^{a,b}, \varrho_v^{a,b}$ from Proposition A.17 (with $ab = (-1)^{\overline{N}}$), the operator $\hat{R}(z) = \tau \circ R(z)$ satisfies (6.10), where*

$$R(z) = (z-1)\Big\{\mathrm{I} + (q^{1/2} - q^{-1/2}) \sum_{1 \le i \le N} (-1)^{\overline{i}} q^{(\varepsilon_i, \varepsilon_i)/2} \cdot E_{ii} \otimes E_{ii} + (q - q^{-1}) \sum_{i > j} (-1)^{\overline{j}} E_{ij} \otimes E_{ji}\Big\} + (q - q^{-1})\tau\,. \qquad \text{(A.19)}$$

Similarly to the observation made after Theorem 6.3, we conclude that $\hat{R}(z)$ coincides, up to a prefactor, with the action of the universal R-matrix, and thus $R(z)$ of (A.19) does satisfy (6.11).

Remark A.19. We note that rescaling $R(z)$ of (A.19) by $\frac{1}{z-1}$, setting $q = e^{-\hbar/2}$, $z = e^{\hbar u}$, and further taking the limit $\hbar \to 0$ recovers the rational R-matrix (super-analogue of the Yang's R-matrix):

$$\lim_{\hbar \to 0} \left\{ \frac{R(z)}{z-1} \Big|_{q = e^{-\hbar/2}, z = e^{\hbar u}} \right\} = \mathrm{I} - \frac{\tau}{u}\,.$$

The proof of Theorem A.18 is straightforward and crucially relies on the expression of $R(z)$ from (A.19) through R_0, R_∞ of (A.15, A.16), which is a special case of the *Yang-Baxterization* from [17]. Recall that the R-matrix $\hat{R}_{VV} = \hat{R} = \tau_{VV} R_0$ has two distinct eigenvalues λ_1 and λ_2, in accordance with Propositions A.2, A.7, A.9. In that setup, the Yang-Baxterization of [17, (3.15)] produces the following solution to (6.11): $\hat{R}(z) = \lambda_2^{-1}\hat{R} + z\lambda_1\hat{R}^{-1}$.

Proposition A.20. *The affine R-matrix $R(z)$ of (A.19) coincides (up to τ and a scalar multiple) with the Yang-Baxterization of $\hat{R}_{VV} = \tau \circ R_0$. To be more specific, for $\hat{R}(z) = \tau \circ R(z)$:*

$$\lambda_1 \hat{R}(z) = \lambda_2^{-1} \hat{R}_{VV} + z\lambda_1 \hat{R}_{VV}^{-1}\,. \qquad \text{(A.20)}$$

with λ_1, λ_2 precisely as in Proposition A.9.

Proof. By straightforward computation, based on (A.15, A.16, A.19), one verifies that

$$\lambda_1 R(z) = \lambda_2^{-1} R_0 + z\lambda_1 R_\infty\,. \qquad \text{(A.21)}$$

Composing with τ on the left, and using (A.14), we obtain (A.20). □

Remark A.21. Due to (A.21), one recovers R_0, R_∞ of (A.15, A.16) as renormalized limits of $R(z)$:

$$R_0 = -R(z)|_{z=0}\,, \qquad R_\infty = \lim_{z \to \infty} \{R(z)/z\}\,.$$

A.8. **Proof of the main result in A-type.**

Due to Proposition A.20 and Theorem A.3, it only remains to verify (6.10) for $x = e_0$ and $x = f_0$. We shall now present the direct verification for $x = e_0$, while $x = f_0$ can be treated analogously to the finite case. We shall also assume that $a = 1$, as in the orthosymplectic case.

Since $\varrho^{a,b}_u(q^{h_0/2})$ is a diagonal matrix, we shall write it as $\varrho^{a,b}_u(q^{h_0/2}) = \mathrm{diag}(\mathtt{t}_1, \ldots, \mathtt{t}_N)$. By direct computation, we get:

$$R_\infty \Delta(e_0) = \Delta(e_0) + (q^{(-1)^{\overline{N}}} - 1)\Big\{ q^{(-1)^{\overline{N}}/2} v \cdot E_{NN} \otimes E_{N1} + q^{-(-1)^{\overline{N}}/2} u \cdot E_{N1} \otimes E_{NN} \Big\}$$
$$+ (q - q^{-1}) \sum_{1 \le j \le N} (-1)^{\overline{j}} \mathtt{t}_j v \cdot E_{Nj} \otimes E_{j1} - (q - q^{-1})(-1)^{\overline{N}} q^{(-1)^{\overline{N}}/2} v \cdot E_{NN} \otimes E_{N1} \, ,$$
$$\Delta^{\mathrm{op}}(e_0) R_\infty = \Delta^{\mathrm{op}}(e_0) + (q^{(-1)^{\overline{1}}} - 1)\Big\{ q^{(-1)^{\overline{1}}/2} v \cdot E_{11} \otimes E_{N1} + q^{-(-1)^{\overline{1}}/2} u \cdot E_{N1} \otimes E_{11} \Big\}$$
$$+ (q - q^{-1}) \sum_{1 \le i \le N} (-1)^{\overline{N}\,\overline{i} + \overline{1}\,\overline{N} + \overline{1}\,\overline{i}} \mathtt{t}_i^{-1} v \cdot E_{i1} \otimes E_{Ni} - (q - q^{-1})(-1)^{\overline{1}} q^{(-1)^{\overline{1}}/2} v \cdot E_{11} \otimes E_{N1} \, .$$

Collecting the terms together, we thus obtain:

$$R_\infty \Delta(e_0) - \Delta^{\mathrm{op}}(e_0) R_\infty =$$
$$(q - q^{-1}) v \cdot \sum_{1 \le j \le N} (-1)^{\overline{j}} \mathtt{t}_j \cdot E_{Nj} \otimes E_{j1} - (q - q^{-1}) v \cdot \sum_{1 \le i \le N} (-1)^{\overline{N}\,\overline{i} + \overline{1}\,\overline{N} + \overline{1}\,\overline{i}} \mathtt{t}_i^{-1} \cdot E_{i1} \otimes E_{Ni} \, .$$

Evoking the paragraph after (6.18), we immediately obtain:

$$R_0 \Delta(e_0) - \Delta^{\mathrm{op}}(e_0) R_0 =$$
$$(q - q^{-1}) u \cdot \sum_{1 \le j \le N} (-1)^{\overline{j}} \mathtt{t}_j \cdot E_{Nj} \otimes E_{j1} - (q - q^{-1}) u \cdot \sum_{1 \le i \le N} (-1)^{\overline{N}\,\overline{i} + \overline{1}\,\overline{N} + \overline{1}\,\overline{i}} \mathtt{t}_i^{-1} \cdot E_{i1} \otimes E_{Ni} \, .$$

Combining the above two equalities with formula (A.21), we get the desired result:

$$R(z)\Delta(e_0) - \Delta^{\mathrm{op}}(e_0) R(z) = 0 \, .$$

Appendix B. Generating vectors for tensor square

B.1. **Two vectors in A-type.**

We first define the following elements $u^{\pm}_{ij}$ in $V \otimes V$ for $1 \le i \le j \le N$:

$$u^+_{ij} = v_i \otimes v_j + (-1)^{\overline{1}} (-1)^{\overline{i}\,\overline{j}} q^{-(-1)^{\overline{1}}} q^{-(\varepsilon_i, \varepsilon_j)} v_j \otimes v_i \, ,$$
$$u^-_{ij} = v_i \otimes v_j - (-1)^{\overline{1}} (-1)^{\overline{i}\,\overline{j}} q^{(-1)^{\overline{1}}} q^{-(\varepsilon_i, \varepsilon_j)} v_j \otimes v_i \, .$$

In particular, we note that $u^+_{ii} = 0$ iff $\overline{i} \ne \overline{1}$ and $u^-_{ii} = 0$ iff $\overline{i} = \overline{1}$. We define subspaces $W^{\pm}$ of $V \otimes V$ to be spanned by the corresponding (nonzero) vectors:

$$\begin{aligned} W^+ &= \mathrm{Span}\Big(\{ u^+_{ij} \,|\, 1 \le i < j \le N \} \cup \{ u^+_{ii} \,|\, \overline{i} = \overline{1} \} \Big) \, , \\ W^- &= \mathrm{Span}\Big(\{ u^-_{ij} \,|\, 1 \le i < j \le N \} \cup \{ u^-_{ii} \,|\, \overline{i} \ne \overline{1} \} \Big) \, . \end{aligned} \tag{B.1}$$

Proposition B.1. *(a) The subspaces W^+ and W^- are $U_q(\mathfrak{sl}(V))$-subrepresentations of $V \otimes V$.*

(b) The $U_q(\mathfrak{sl}(V))$-representation $V \otimes V$ decomposes as a direct sum of these subrepresentations

$$V \otimes V \simeq W^+ \oplus W^- \, .$$

(c) W^+ and W^- are irreducible $U_q(\mathfrak{sl}(V))$-representations generated by the corresponding highest weight vectors w_1 and w_2 of (A.13), respectively.

Proof. (a) We first verify that W^+ is stable under the $U_q(\mathfrak{sl}(V))$-action via direct computations. The generators $\{e_a, f_a\}_{a=1}^{N-1}$ act on the above vectors u^+_{ij} as follows:

- Case 1: u^+_{ii} for $\bar{i}=\bar{1}$.

$$\Big(1+q^{-(-1)^{\bar{i}}\cdot 2}\Big)^{-1}\varrho^{\otimes 2}(e_a)u^+_{ii}=\delta_{a+1,i}\cdot q^{(-1)^{\bar{i}}/2}\cdot u^+_{a,a+1}\,,$$
$$(-1)^{\bar{a}}\Big(1+q^{-(-1)^{\bar{i}}\cdot 2}\Big)^{-1}\varrho^{\otimes 2}(f_a)u^+_{ii}=\delta_{ai}\cdot(-1)^{\bar{a}(\bar{a}+\overline{a+1})}q^{(-1)^{\bar{i}}/2}\cdot u^+_{a,a+1}\,.$$

- Case 2: u^+_{ij} for $i<j$.

$$\varrho^{\otimes 2}(e_a)u^+_{ij}=\delta_{a+1,i}\cdot u^+_{a,j}+\delta_{a+1,j}\cdot(-1)^{\bar{i}(\bar{a}+\overline{a+1})}\cdot q^{(\varepsilon_a,\varepsilon_i)/2}\cdot u^+_{i,a}\,,$$
$$(-1)^{\bar{a}}\varrho^{\otimes 2}(f_a)u^+_{ij}=\delta_{ai}\cdot q^{(\varepsilon_{a+1},\varepsilon_j)/2}\cdot u^+_{a+1,j}+\delta_{aj}\cdot(-1)^{\bar{i}(\bar{a}+\overline{a+1})}\cdot u^+_{i,a+1}\,,$$

where we recall that $u^+_{jj}=0$ if $\bar{j}\neq\bar{1}$.

This shows that W^+ is stable under the action of $\{e_a,f_a\}_{a=1}^{N-1}$. As each u^+_{ij} is homogeneous of degree $\varepsilon_i+\varepsilon_j$ with respect to the P-grading defined by $\deg(v_a\otimes v_b)=\varepsilon_a+\varepsilon_b$ for all $1\le a,b\le N$, cf. (A.12), we also have $\varrho^{\otimes 2}(q^{h_a/2})u^+_{ij}=q^{(\varepsilon_i+\varepsilon_j)(\mathsf{h}_a)/2}\cdot u^+_{ij}$. This establishes part (a) for W^+.

The proof of (a) for W^- is analogous (the above formulas hold with each u^+_{ab} replaced by u^-_{ab}).

(b) It is enough to show that the vectors entering (B.1) form a basis for $V\otimes V$. As there are precisely $N^2=\dim(V\otimes V)$ of such vectors, it suffices to show their linear independence. Moreover, since each such vector is homogeneous with respect to the above P-grading, it suffices to verify the linear independence in each weight spaces, which can be easily seen.

(c) Let us first prove the "generating" property of W^+. To this end, it suffices to show that each u^+_{ij} is contained in the $U_q(\mathfrak{sl}(V))$-submodule generated by u^+_{11} (a nonzero scalar multiple of w_1), which can be done by acting with the generators $\{f_a\}_{a=1}^{N-1}$ iteratively on u^+_{11}. According to the explicit formulas in part (a): $\varrho^{\otimes 2}(f_{j-1}\ldots f_2f_1)u^+_{11}$ is a nonzero scalar multiple of u^+_{1j} and further $\varrho^{\otimes 2}(f_{i-1}\ldots f_2f_1)u^+_{1j}$ is a nonzero scalar multiple of u^+_{ij}. This shows that indeed $W^+=U_q(\mathfrak{sl}(V))w_1$.

The proof of the irreducibility of W^+ is similar. Since any nonzero submodule of $V\otimes V$ has a weight space decomposition, it is enough to show that for any nonzero P-homogeneous element $w\in W^+$, i.e. a nonzero scalar multiple of some u^+_{ij}, we can obtain a nonzero scalar multiple of w_1 by acting with the generators $\{e_a\}_{a=1}^{N-1}$ iteratively on w. Due to the explicit formulas in part (a), we have: $\varrho^{\otimes 2}(e_1e_2\ldots e_{i-1})u^+_{ij}$ is a nonzero scalar multiple of u^+_{1j}, and $\varrho^{\otimes 2}(e_1e_2\ldots e_{j-1})u^+_{1j}$ is a nonzero scalar multiple of w_1. This establishes (c) for W^+. The proof of (c) for W^- is analogous. $\square$

B.2. **Three vectors in orthosymplectic type.**

Motivated by Subsection B.1, we shall now present a similar analysis for the structure of $U_q(\mathfrak{osp}(V))$-representation $V\otimes V$. To this end, we consider three cases separately: m is odd, m is even and $\bar{s}=\bar{0}$, m is even and $\bar{s}=\bar{1}$. Besides lengthier calculations, the orthosymplectic setup is considerably harder due to the higher dimensional degree 0 component of this tensor square. In particular, our presentation emphasizes in full details the importance of the special case $n=m$ when $V\otimes V$ is not semisimple. We note that it is this major difference that forced us to work with the vectors $\widetilde{w}_3,\hat{w}_3$ instead of the highest weight vector w_3 in Section 4.

B.2.1. *Generating property for odd m.*

We first define the following elements $u^\pm_{ij}$ in $V\otimes V$ for $1\le i\le j\le N$ and $(i,j)\neq(s+1,s+1)$:

$$u^+_{ij}=\begin{cases}v_i\otimes v_j+(-1)^{\bar{1}}(-1)^{\bar{i}\bar{j}}q^{-(-1)^{\bar{1}}}q^{-(\varepsilon_i,\varepsilon_j)}v_j\otimes v_i & \text{for}\quad j\neq i'\\ \Big\{v_i\otimes v_{i'}+(-1)^{\bar{1}+\bar{i}}q^{-(-1)^{\bar{1}}}q^{-(\varepsilon_i,\varepsilon_i)}v_{i'}\otimes v_i\Big\}-(-1)^{\bar{i}+\overline{i+1}}q^{-(\varepsilon_i,\varepsilon_i)/2}q^{-(\varepsilon_{i+1},\varepsilon_{i+1})/2} & \\ \quad\cdot\vartheta_i\vartheta_{i+1}\Big\{v_{i+1}\otimes v_{(i+1)'}+(-1)^{\bar{1}+\overline{i+1}}q^{-(-1)^{\bar{1}}}q^{(\varepsilon_{i+1},\varepsilon_{i+1})}v_{(i+1)'}\otimes v_{i+1}\Big\} & \text{for}\quad j=i'\end{cases}$$

and

$$u^-_{ij}=\begin{cases}v_i\otimes v_j-(-1)^{\bar{1}}(-1)^{\bar{i}\bar{j}}q^{(-1)^{\bar{1}}}q^{-(\varepsilon_i,\varepsilon_j)}v_j\otimes v_i & \text{for}\quad j\neq i'\\ \Big\{v_i\otimes v_{i'}-(-1)^{\bar{1}+\bar{i}}q^{(-1)^{\bar{1}}}q^{-(\varepsilon_i,\varepsilon_i)}v_{i'}\otimes v_i\Big\}-(-1)^{\bar{i}+\overline{i+1}}q^{-(\varepsilon_i,\varepsilon_i)/2}q^{-(\varepsilon_{i+1},\varepsilon_{i+1})/2} & \\ \quad\cdot\vartheta_i\vartheta_{i+1}\Big\{v_{i+1}\otimes v_{(i+1)'}-(-1)^{\bar{1}+\overline{i+1}}q^{(-1)^{\bar{1}}}q^{(\varepsilon_{i+1},\varepsilon_{i+1})}v_{(i+1)'}\otimes v_{i+1}\Big\} & \text{for}\quad j=i'\end{cases}.$$

In particular, we note that $u^+_{ii}=0$ iff $\bar{i}\neq\bar{1}$ and $u^-_{ii}=0$ iff $\bar{i}=\bar{1}$. We define subspaces $W^{\pm}$ of $V\otimes V$ to be spanned by the corresponding (nonzero) vectors:

$$\begin{aligned} W^+ &= \mathrm{Span}\Big(\{u^+_{ij}\,|\,1\leq i<j\leq N\}\cup\{u^+_{ii}\,|\,i\neq s+1\,,\,\bar{i}=\bar{1}\}\Big)\,,\\ W^- &= \mathrm{Span}\Big(\{u^-_{ij}\,|\,1\leq i<j\leq N\}\cup\{u^-_{ii}\,|\,i\neq s+1\,,\,\bar{i}\neq\bar{1}\}\Big)\,. \end{aligned} \tag{B.2}$$

We also consider a one-dimensional subspace $W_3=\mathrm{Span}(w_3)$ of $V\otimes V$, cf. (3.5).

Proposition B.2. *(a) The subspaces W^+,W^-,W_3 are $U_q(\mathfrak{osp}(V))$-subrepresentations of $V\otimes V$.*

(b) The $U_q(\mathfrak{osp}(V))$-representation $V\otimes V$ decomposes as a direct sum of these subrepresentations

$$V\otimes V\simeq W^+\oplus W^-\oplus W_3\,.$$

(c) Both $\widetilde{w}_3=v_1\otimes v_{1'}$ and $\hat{w}_3=v_{1'}\otimes v_1$ do not belong to $W^+\oplus W^-$, while

$$\widetilde{w}_3-(-1)^{\bar{1}}q^{(-1)^{\bar{1}}}q^{n-m+1}\hat{w}_3\in W^+\oplus W^-\,. \tag{B.3}$$

(d) W^+, W^-, W_3 are irreducible $U_q(\mathfrak{osp}(V))$-representations generated by the corresponding highest weight vectors w_1, w_2, and w_3 of (3.5), respectively.

Proof. (a) Let us first show that W^+ is stable under the $U_q(\mathfrak{osp}(V))$-action through direct but rather tedious computations. The action of the generators $\{f_a\}_{a=1}^s$ on the above vectors u^+_{ij} is summarized in the following formulas (split into five cases):

- Case 1: u^+_{ii} for $\bar{i}=\bar{1}$.

$$\begin{aligned} &(-1)^{\bar{a}}\Big(1+q^{-(-1)^{\bar{i}}\cdot 2}\Big)^{-1}\varrho^{\otimes 2}(f_a)u^+_{ii}\\ &\quad=\delta_{ai}\cdot(-1)^{\bar{a}(\bar{a}+\overline{a+1})}q^{(-1)^{\bar{i}}/2}\cdot u^+_{a,a+1}-\delta_{(a+1)',i}\cdot\vartheta_a\vartheta_{a+1}q^{(-1)^{\bar{i}}/2}\cdot u^+_{(a+1)',a'}\,. \end{aligned}$$

- Case 2: u^+_{ij} for $i<j$ and $i+j\neq N,N+1$.

$$\begin{aligned} &(-1)^{\bar{a}}\varrho^{\otimes 2}(f_a)u^+_{ij}\\ &=\delta_{ai}\cdot q^{(\varepsilon_{a+1},\varepsilon_j)/2}\cdot u^+_{a+1,j}+\delta_{aj}\cdot(-1)^{\bar{i}(\bar{a}+\overline{a+1})}\cdot u^+_{i,a+1}\\ &\quad-\delta_{(a+1)',i}\cdot(-1)^{\overline{a+1}(\bar{a}+\overline{a+1})}\vartheta_a\vartheta_{a+1}q^{-(\varepsilon_a,\varepsilon_j)/2}\cdot u^+_{a'j}-\delta_{(a+1)',j}\cdot(-1)^{(\bar{i}+\overline{a+1})(\bar{a}+\overline{a+1})}\vartheta_a\vartheta_{a+1}\cdot u^+_{ia'}\,, \end{aligned}$$

  where we recall that $u^+_{jj}=0$ if $\bar{j}\neq\bar{1}$.

- Case 3: u^+_{ij} for $i<j$ and $i+j=N$.

$$(-1)^{\bar{a}}\varrho^{\otimes 2}(f_a)u^+_{i,(i+1)'}=-\delta_{ai}\cdot(-1)^{\bar{a}+\overline{a+1}}\vartheta_a\vartheta_{a+1}q^{(\varepsilon_a,\varepsilon_a)/2}\cdot u^+_{aa'}\,.$$

- Case 4: u^+_{ij} for $i<j$, $i+j=N+1$ and $i\neq s$.

$$\begin{aligned} &(-1)^{\bar{a}}\varrho^{\otimes 2}(f_a)u^+_{ii'}\\ &=\delta_{ai}\delta_{\bar{a},\overline{a+1}}\cdot q^{(\varepsilon_a,\varepsilon_a)/2}\Big(1+q^{-(\varepsilon_a,\varepsilon_a)\cdot 2}\Big)u^+_{a+1,a'}\\ &\quad-\delta_{a,i+1}\cdot(-1)^{\overline{a-1}+\bar{a}}\vartheta_{a-1}\vartheta_a q^{-(\varepsilon_{a-1},\varepsilon_{a-1})/2}\cdot u^+_{a+1,a'}-\delta_{a+1,i}\cdot\vartheta_a\vartheta_{a+1}q^{-(\varepsilon_{a+1},\varepsilon_{a+1})/2}\cdot u^+_{a+1,a'}\,. \end{aligned}$$

- Case 5: u^+_{ij} for $(i,j)=(s,s')$.

$$\begin{aligned} &(-1)^{\bar{a}}\varrho^{\otimes 2}(f_a)u^+_{ss'}\\ &=\delta_{as}\Big\{(-1)^{\bar{s}+\overline{s+1}}q^{-(\varepsilon_s,\varepsilon_s)/2}+\delta_{\bar{1}\bar{s}}\cdot q^{(\varepsilon_s,\varepsilon_s)/2}\Big(1+q^{-(-1)^{\bar{1}}\cdot 2}\Big)\Big\}u^+_{s+1,s'}\\ &\quad-\delta_{a,s-1}\cdot\vartheta_{s-1}\vartheta_s q^{-(\varepsilon_s,\varepsilon_s)/2}\cdot u^+_{s,(s-1)'}\,. \end{aligned}$$

The above computations show that W^+ is stable under the action of $\{f_a\}_{a=1}^s$. To check that W^+ is also stable under the action of $\{e_a\}_{a=1}^s$, we consider a vector space isomorphism

$$\phi\colon V\xrightarrow{\sim}V\quad\text{given by}\quad v_i\mapsto \mathsf{c}_{i'}v_{i'}\quad\text{for all}\quad 1\leq i\leq N\,, \tag{B.4}$$

where the coefficients c_i's are determined by $\mathsf{c}_1 = 1$ and the following relations:

$$\mathsf{c}_{a+1} = -(-1)^{\overline{a}+\overline{a+1}+\overline{a}\,\overline{a+1}}\vartheta_a\vartheta_{a+1}\cdot \mathsf{c}_a \qquad \text{for } \ 1 \leq a \leq s\,,$$
$$\mathsf{c}_{a'} = -(-1)^{\overline{a}\,\overline{a+1}}\vartheta_a\vartheta_{a+1}\cdot \mathsf{c}_{(a+1)'} \qquad \text{for } \ 1 \leq a \leq s\,.$$

In particular, we note that

$$(-1)^{\overline{a}}\mathsf{c}_a\mathsf{c}_{a'} = (-1)^{\overline{1}}\mathsf{c}_1\mathsf{c}_{1'} \qquad \forall\, 1 \leq a \leq N\,. \tag{B.5}$$

Evoking ω from (5.15), it is straightforward to check that

$$\varrho(x)v = (\phi^{-1}\circ \varrho(\omega(x))\circ \phi)(v) \tag{B.6}$$

holds for any $v = v_i$ and $x \in \{e_a, f_a, q^{\pm h_a/2}\}_{a=1}^s$. Thus (B.6) holds for all $v \in V, x \in U_q(\mathfrak{osp}(V))$. It is also easy to check (verifying on the generators) that $\Delta^\omega = \Delta^{\mathrm{op}}$, cf. (4.25, 4.27), where

$$\Delta^\omega = (\omega\otimes\omega)\circ\Delta\circ\omega^{-1}\,.$$

Furthermore, we note that $W^\pm$ are invariant under $\tau\circ\phi^{\otimes 2}$ due to the following equality:

$$(\tau_{VV}\circ(\phi\otimes\phi))(u^\pm_{ij}) = (-1)^{\overline{i}\,\overline{j}}\mathsf{c}_{i'}\mathsf{c}_{j'}\cdot u^\pm_{j'i'}\,.$$

Combining all these results, we finally obtain:

$$\begin{aligned}
\varrho^{\otimes 2}(e_a)u^+_{ij} = \Delta(e_a)u^+_{ij} &= \Big((\phi^{\otimes 2})^{-1}\circ\Delta^\omega(\omega(e_a))\circ\phi^{\otimes 2}\Big)\,(u^+_{ij})\\
&= (-1)^{\overline{a}+\overline{a+1}}\cdot\Big((\phi^{\otimes 2})^{-1}\circ\Delta^\omega(f_a)\circ\phi^{\otimes 2}\Big)\,(u^+_{ij})\\
&\overset{(4.19)}{=} (-1)^{\overline{a}+\overline{a+1}}\cdot\Big((\tau_{VV}\circ\phi^{\otimes 2})^{-1}\circ\Delta(f_a)\circ(\tau_{VV}\circ\phi^{\otimes 2})\Big)\,(u^+_{ij})\,,
\end{aligned} \tag{B.7}$$

which proves that W^+ is stable under the action of $\{e_a\}_{a=1}^s$. Finally, each u^+_{ij} is homogeneous of degree $\varepsilon_i+\varepsilon_j$ with respect to the P-grading defined by $\deg(v_a\otimes v_b) = \varepsilon_a+\varepsilon_b$ for all $1\leq a,b\leq N$, so that $\varrho^{\otimes 2}(q^{h_a/2})u^+_{ij} = q^{(\varepsilon_i+\varepsilon_j)(\mathsf{h}_a)/2}\cdot u^+_{ij}$. This completes the proof of part (a) for W^+.

The proof of part (a) for W^- is completely analogous. Therefore, we shall only present the explicit formulas for the action of the generators $\{f_a\}_{a=1}^s$ on the vectors u^-_{ij}:

- Case 1: u^-_{ii} for $\overline{i}\neq\overline{1}$.

$$(-1)^{\overline{a}}\Big(1+q^{-(-1)^{\overline{i}}\cdot 2}\Big)^{-1}\varrho^{\otimes 2}(f_a)u^-_{ii}$$
$$= \delta_{ai}\cdot(-1)^{\overline{a}(\overline{a}+\overline{a+1})}q^{(-1)^{\overline{i}}/2}\cdot u^-_{a,a+1} - \delta_{(a+1)',i}\cdot\vartheta_a\vartheta_{a+1}q^{(-1)^{\overline{i}}/2}\cdot u^-_{(a+1)',a'}\,.$$

- Case 2: u^-_{ij} for $i<j$ and $i+j\neq N, N+1$.

$$(-1)^{\overline{a}}\varrho^{\otimes 2}(f_a)u^-_{ij}$$
$$= \delta_{ai}\cdot q^{(\varepsilon_{a+1},\varepsilon_j)/2}\cdot u^-_{a+1,j} + \delta_{aj}\cdot(-1)^{\overline{i}(\overline{a}+\overline{a+1})}\cdot u^-_{i,a+1}$$
$$- \delta_{(a+1)',i}\cdot(-1)^{\overline{a+1}(\overline{a}+\overline{a+1})}\vartheta_a\vartheta_{a+1}q^{-(\varepsilon_a,\varepsilon_j)/2}\cdot u^-_{a'j} - \delta_{(a+1)',j}\cdot(-1)^{(\overline{i}+\overline{a+1})(\overline{a}+\overline{a+1})}\vartheta_a\vartheta_{a+1}\cdot u^-_{ia'}\,,$$

where we recall that $u^-_{jj}=0$ if $\overline{j}=\overline{1}$.

- Case 3: u^-_{ij} for $i<j$ and $i+j=N$.

$$(-1)^{\overline{a}}\varrho^{\otimes 2}(f_a)u^-_{i(i+1)'} = -\delta_{ai}\cdot(-1)^{\overline{a}+\overline{a+1}}\vartheta_a\vartheta_{a+1}q^{(\varepsilon_a,\varepsilon_a)/2}\cdot u^-_{aa'}\,.$$

- Case 4: u^-_{ij} for $i<j$, $i+j=N+1$ and $i\neq s$.

$$(-1)^{\overline{a}}\varrho^{\otimes 2}(f_a)u^-_{ii'}$$
$$= \delta_{ai}\delta_{\overline{a},\overline{a+1}}\cdot q^{(\varepsilon_a,\varepsilon_a)/2}\Big(1+q^{-(\varepsilon_a,\varepsilon_a)\cdot 2}\Big)u^-_{a+1,a'}$$
$$- \delta_{a,i+1}\cdot(-1)^{\overline{a-1}+\overline{a}}\vartheta_{a-1}\vartheta_a q^{-(\varepsilon_{a-1},\varepsilon_{a-1})/2}\cdot u^-_{a+1,a'} - \delta_{a+1,i}\cdot\vartheta_a\vartheta_{a+1}q^{-(\varepsilon_{a+1},\varepsilon_{a+1})/2}\cdot u^-_{a+1,a'}\,.$$

- Case 5: u^-_{ij} for $(i,j)=(s,s')$.
$$\begin{aligned}&(-1)^{\overline{a}}\varrho^{\otimes 2}(f_a)u^-_{ss'}\\ &=\delta_{as}\Big\{(-1)^{\overline{s}+\overline{s+1}}q^{-(\varepsilon_s,\varepsilon_s)/2}+\delta_{\overline{1}\neq\overline{s}}\cdot q^{(\varepsilon_s,\varepsilon_s)/2}\Big(1+q^{(-1)^{\overline{1}}\cdot 2}\Big)\Big\}u^-_{s+1,s'}\\ &\quad-\delta_{a,s-1}\cdot\vartheta_{s-1}\vartheta_s q^{-(\varepsilon_s,\varepsilon_s)/2}\cdot u^-_{s,(s-1)'}\,,\end{aligned}$$
where $\delta_{\overline{1}\neq\overline{s}}$ equals 1 if $\overline{1}\neq\overline{s}$ and is 0 otherwise.

To prove part (a) for W_3, we recall that $\varrho^{\otimes 2}(e_i)w_3=0$, $\varrho^{\otimes 2}(q^{h_i/2})w_3=w_3$ for any $1\leq i\leq s$, as established in our proof of Proposition 3.2(a). The remaining vanishing $\varrho^{\otimes 2}(f_i)w_3=0$ for $1\leq i\leq s$ follow from $\varrho^{\otimes 2}(e_i)w_3=0$ via (B.7) and $(\tau_{VV}\circ\phi^{\otimes 2})(w_3)=(-1)^{\overline{1}}\mathsf{c}_1\mathsf{c}_{1'}\cdot w_3$, due to (B.5).

(b) It is enough to show that the vectors entering (B.2) together with w_3 form a basis for $V\otimes V$. As there are precisely $N^2=\dim(V\otimes V)$ of such vectors, it suffices to show their linear independence. Moreover, since each such vector is homogeneous with respect to the above P-grading, it suffices to verify the linear independence in each weight spaces. This is clear for nonzero weight spaces. Finally, for the zero weight space the proof is done by straightforward computation, which is analogous to the even m case treated in full details below, cf. Remark B.5.

(c) The proof of $\widetilde{w}_3,\hat{w}_3\notin W^+\oplus W^-$ is completely analogous to that of part (b) with w_3 replaced by either $\widetilde{w}_3$ or $\hat{w}_3$. Meanwhile, the proof of (B.3) is completely analogous to that of (B.8) below. Here, we shall only state the explicit linear dependence:
$$\sum_{i=1}^{s}(b^+_i u^+_{ii'}+b^-_i u^-_{ii'})=v_1\otimes v_{1'}-(-1)^{\overline{1}}q^{(-1)^{\overline{1}}}q^{n-m+1}v_{1'}\otimes v_1$$
where
$$b^+_i=\frac{(-1)^{\overline{1}+\overline{i}}\vartheta_1\vartheta_i}{q+q^{-1}}q^{-\sum_{k=1}^{s-1}(\rho,\alpha_k)}\left(q^{(\varepsilon_1,\varepsilon_1)}q^{\sum_{k=i}^{s-1}(\rho,\alpha_k)}-(-1)^{\overline{1}}q^{(\varepsilon_i,\varepsilon_i)-(\varepsilon_s,\varepsilon_s)}q^{-\sum_{k=i}^{s-1}(\rho,\alpha_k)}\right),$$
$$b^-_i=\frac{(-1)^{\overline{1}+\overline{i}}\vartheta_1\vartheta_i}{q+q^{-1}}q^{-\sum_{k=1}^{s-1}(\rho,\alpha_k)}\left(q^{-(\varepsilon_1,\varepsilon_1)}q^{\sum_{k=i}^{s-1}(\rho,\alpha_k)}+(-1)^{\overline{1}}q^{(\varepsilon_i,\varepsilon_i)-(\varepsilon_s,\varepsilon_s)}q^{-\sum_{k=i}^{s-1}(\rho,\alpha_k)}\right)$$
for $1\leq i\leq s$.

(d) Let us first prove the "generating" property of W^+. To this end, we need to show that each u^+_{ij} is contained in the $U_q(\mathfrak{osp}(V))$-submodule generated by u^+_{11} (a nonzero scalar multiple of w_1), which can be done by acting with the generators $\{f_a\}_{a=1}^s$ iteratively on u^+_{11}:

- Case 1: u^+_{ij} for $i+j<N+1$.
According to the explicit formulas (Cases 1, 2) in part (a): $\varrho^{\otimes 2}(f_{j-1}\dots f_2f_1)u^+_{11}$ is a nonzero scalar multiple of u^+_{1j} for $j\leq s+1$, $\varrho^{\otimes 2}(f_{j'}\dots f_{s-1}f_s)u^+_{1,s+1}$ is a nonzero scalar multiple of u^+_{1j} for $j>s+1$, and $\varrho^{\otimes 2}(f_{i-1}\dots f_2f_1)u^+_{1j}$ is a nonzero scalar multiple of u^+_{ij}.
- Case 2: u^+_{ij} for $i+j=N+1$.
We note that $\varrho^{\otimes 2}(f_i)u^+_{i,(i+1)'}$ is a nonzero scalar multiple of $u^+_{ii'}$, due to Case 3 from (a).
- Case 3: u^+_{ij} for $i+j=N+2$.
According to the explicit formula (Case 4) in part (a): $\varrho^{\otimes 2}(f_1)u^+_{22'}$ is a nonzero scalar multiple of $u^+_{21'}$ and $\varrho^{\otimes 2}(f_i)u^+_{i-1,(i-1)'}$ is a nonzero scalar multiple of $u^+_{i+1,i'}$ for $2\leq i\leq s$.
- Case 4: u^+_{ij} for $i+j>N+2$.
According to the explicit formula (Case 2) in part (a): $\varrho^{\otimes 2}(f_{i-1}\dots f_{j'+2}f_{j'+1})u^+_{j'+1,j}$ is a nonzero scalar multiple of u^+_{ij} for $i\leq s+1$ and likewise $\varrho^{\otimes 2}(f_{i'}\dots f_{s-1}f_s)u^+_{s+1,j}$ is a nonzero scalar multiple of u^+_{ij} for $i>s+1$.

This proves that w_1 generates the entire W^+ under the action of $U_q(\mathfrak{osp}(V))$. The proof of the irreducibility of W^+ is similar. Since any nonzero submodule of $V\otimes V$ has a weight space decomposition, it is enough to show that for any nonzero P-homogeneous element $w\in W^+$, we can obtain a nonzero scalar multiple of w_1 by acting with the generators $\{e_a\}_{a=1}^s$ iteratively on w (the corresponding formulas can be deduced from those of part (a) through (B.7)):

- Case 1: $w = u^+_{ij}$ for $i+j < N+1$.

  First, we note that $\varrho^{\otimes 2}(e_1e_2\ldots e_{i-1})u^+_{ij}$ is a nonzero scalar multiple of u^+_{1j}. For $j > s+1$, we further note that $\varrho^{\otimes 2}(e_se_{s-1}\ldots e_{j'})u^+_{1j}$ is a nonzero scalar multiple of $u^+_{1,s+1}$. Finally, we likewise note that $\varrho^{\otimes 2}(e_1e_2\ldots e_{j-1})u^+_{1j}$ is a nonzero scalar multiple of u^+_{11} for $j \leq s+1$. This shows that we can get a nonzero multiple of w_1 by acting with e_a's on w.
- Case 2: $\deg(w) = 0$.

  Since w_3 is the unique (up to scaling) highest weight vector in $V \otimes V$ of degree zero (see Remark 3.3) and $w_3 \notin W^+$ according to part (b), we have $w' = \varrho^{\otimes 2}(e_a)w \neq 0$ for some $1 \leq a \leq s$. Replacing w by w', we thus reduced the setup to Case 1 treated above.
- Case 3: $w = u^+_{ij}$ for $i+j > N+1$.

  For $i > s+1$, we first note that $\varrho^{\otimes 2}(e_se_{s-1}\ldots e_{i'})u^+_{ij}$ is a nonzero scalar multiple of $u^+_{s+1,j}$ and likewise $\varrho^{\otimes 2}(e_{j'}e_{j'+1}\ldots e_{i-1})u^+_{ij}$ is a nonzero scalar multiple of $u^+_{j'j}$ for $i \leq s+1$. Thus, $w' = u^+_{j'j} \in W^+$ and we reduced this setup to Case 2 treated above.

This proves the irreducibility of W^+, thus establishing part (d) for W^+.

The proof of part (d) for W^- is completely analogous. □

B.2.2. *Generating property for even m with $\overline{s} = \bar{0}$.*

Similarly to the odd m case, we define the following elements $u^\pm_{ij}$ in $V \otimes V$ for $1 \leq i \leq j \leq N$:

$$u^+_{ij} = \begin{cases} v_i \otimes v_j + (-1)^{\bar{1}}(-1)^{\bar{i}\bar{j}} q^{-(-1)^{\bar{1}}} q^{-(\varepsilon_i,\varepsilon_j)} v_j \otimes v_i & \text{if } j' \neq i \\ \Big\{ v_i \otimes v_{i'} + (-1)^{\bar{1}+\bar{i}} q^{-(-1)^{\bar{1}}} q^{-(\varepsilon_i,\varepsilon_i)} v_{i'} \otimes v_i \Big\} - (-1)^{\bar{i}+\overline{i+1}} q^{-(\varepsilon_i,\varepsilon_i)/2} q^{-(\varepsilon_{i+1},\varepsilon_{i+1})/2} & \\ \quad \cdot \vartheta_i\vartheta_{i+1} \Big\{ v_{i+1} \otimes v_{(i+1)'} + (-1)^{\bar{1}+\overline{i+1}} q^{-(-1)^{\bar{1}}} q^{(\varepsilon_{i+1},\varepsilon_{i+1})} v_{(i+1)'} \otimes v_{i+1} \Big\} & \text{if } j' = i \neq s \\ \Big\{ v_{s-1} \otimes v_{(s-1)'} + (-1)^{\bar{1}+\overline{s-1}} q^{-(-1)^{\bar{1}}} q^{-(\varepsilon_{s-1},\varepsilon_{s-1})} v_{(s-1)'} \otimes v_{s-1} \Big\} - (-1)^{\overline{s-1}+\bar{s}} & \\ \quad \cdot q^{-(\varepsilon_{s-1},\varepsilon_{s-1})/2} q^{-(\varepsilon_s,\varepsilon_s)/2} \vartheta_{s-1}\vartheta_s \Big\{ v_{s'} \otimes v_s + (-1)^{\bar{1}+\bar{s}} q^{-(-1)^{\bar{1}}} q^{(\varepsilon_s,\varepsilon_s)} v_s \otimes v_{s'} \Big\} & \text{if } j' = i = s \end{cases}$$

and

$$u^-_{ij} = \begin{cases} v_i \otimes v_j - (-1)^{\bar{1}}(-1)^{\bar{i}\bar{j}} q^{(-1)^{\bar{1}}} q^{-(\varepsilon_i,\varepsilon_j)} v_j \otimes v_i & \text{if } j' \neq i \\ \Big\{ v_i \otimes v_{i'} - (-1)^{\bar{1}+\bar{i}} q^{(-1)^{\bar{1}}} q^{-(\varepsilon_i,\varepsilon_i)} v_{i'} \otimes v_i \Big\} - (-1)^{\bar{i}+\overline{i+1}} q^{-(\varepsilon_i,\varepsilon_i)/2} q^{-(\varepsilon_{i+1},\varepsilon_{i+1})/2} & \\ \quad \cdot \vartheta_i\vartheta_{i+1} \Big\{ v_{i+1} \otimes v_{(i+1)'} - (-1)^{\bar{1}+\overline{i+1}} q^{(-1)^{\bar{1}}} q^{(\varepsilon_{i+1},\varepsilon_{i+1})} v_{(i+1)'} \otimes v_{i+1} \Big\} & \text{if } j' = i \neq s \\ \Big\{ v_{s-1} \otimes v_{(s-1)'} - (-1)^{\bar{1}+\overline{s-1}} q^{(-1)^{\bar{1}}} q^{-(\varepsilon_{s-1},\varepsilon_{s-1})} v_{(s-1)'} \otimes v_{s-1} \Big\} - (-1)^{\overline{s-1}+\bar{s}} & \\ \quad \cdot q^{-(\varepsilon_{s-1},\varepsilon_{s-1})/2} q^{-(\varepsilon_s,\varepsilon_s)/2} \vartheta_{s-1}\vartheta_{s'} \Big\{ v_{s'} \otimes v_s - (-1)^{\bar{1}+\bar{s}} q^{(-1)^{\bar{1}}} q^{(\varepsilon_s,\varepsilon_s)} v_s \otimes v_{s'} \Big\} & \text{if } j' = i = s \end{cases}.$$

Again, we note that $u^+_{ii} = 0$ iff $\bar{i} \neq \bar{1}$ and $u^-_{ii} = 0$ iff $\bar{i} = \bar{1}$. For convenience, let us define

$$\begin{aligned} u_{ss'} = &\Big\{ v_{s-1} \otimes v_{(s-1)'} - (-1)^{\overline{s-1}} \cdot q \cdot q^{-(\varepsilon_{s-1},\varepsilon_{s-1})} v_{(s-1)'} \otimes v_{s-1} \Big\} \\ &- (-1)^{\overline{s-1}+\bar{s}} q^{-(\varepsilon_{s-1},\varepsilon_{s-1})/2} q^{-(\varepsilon_s,\varepsilon_s)/2} \vartheta_{s-1}\vartheta_{s'} \Big\{ v_{s'} \otimes v_s - (-1)^{\bar{s}} \cdot q \cdot q^{(\varepsilon_s,\varepsilon_s)} v_s \otimes v_{s'} \Big\}. \end{aligned}$$

Then $u^+_{ss'} = u^+_{s-1,(s-1)'}$ and $u^-_{ss'} = u_{ss'}$ if $\bar{1} = \bar{s}$, and $u^+_{ss'} = u_{ss'}$ and $u^-_{ss'} = u^-_{s-1,(s-1)'}$ if $\bar{1} \neq \bar{s}$. We define subspaces $W^\pm$ of $V \otimes V$ to be spanned by the corresponding (nonzero) vectors:

$$W^+ = \mathrm{Span}\left(\{u^+_{ij} \,|\, 1 \leq i < j \leq N\} \cup \{u^+_{ii} \,|\, \bar{i} = \bar{1}\} \right),$$
$$W^- = \mathrm{Span}\left(\{u^-_{ij} \,|\, 1 \leq i < j \leq N\} \cup \{u^-_{ii} \,|\, \bar{i} \neq \bar{1}\} \right).$$

We also consider a one-dimensional subspace $W_3 = \mathrm{Span}(w_3)$ of $V \otimes V$, cf. (3.5).

Proposition B.3. *(a) The subspaces W^+, W^-, W_3 are $U_q(\mathfrak{osp}(V))$-subrepresentations of $V \otimes V$.*

(b) For $n \neq m$, the $U_q(\mathfrak{osp}(V))$-representation $V \otimes V$ decomposes into the direct sum of those:

$$V \otimes V \simeq W^+ \oplus W^- \oplus W_3\,.$$

(c) For $n = m$, $W_3 \subset W^+$ if $\bar{1} = \bar{0} = \bar{s}$ and $W_3 \subset W^-$ if $\bar{1} = \bar{1} \neq \bar{s}$, and $W^+ \oplus W^-$ is a codimension 1 subspace of $V \otimes V$.

(d) Both $\widetilde{w}_3 = v_1 \otimes v_{1'}$ and $\hat{w}_3 = v_{1'} \otimes v_1$ do not belong to $W^+ \oplus W^-$, while

$$\widetilde{w}_3 - (-1)^{\bar{1}} q^{(-1)^{\bar{1}}} q^{n-m+1} \hat{w}_3 \in W^+ \oplus W^- . \tag{B.8}$$

(e) W^+, W^-, W_3 are $U_q(\mathfrak{osp}(V))$-representations generated by the corresponding highest weight vectors w_1, w_2, w_3. Moreover, these representations are irreducible if $n \neq m$.

Proof. (a) The proof is analogous to the odd m case, so we only present the key difference in formulas. The action of the generators $\{f_a\}_{a=1}^s$ on the above vectors u^+_{ij} is given by the exact same formula unless $f_a = f_s$ or $(i,j) = (s,s')$. The action in the remaining cases is given by:

- Case 1: u^+_{ii} for $\bar{i} = \bar{1}$ and $f_a = f_s$.

$$(-1)^{\overline{s-1}}\Big(1 + q^{-(-1)^{\bar{i}}\cdot 2}\Big)^{-1} \varrho^{\otimes 2}(f_s) u^+_{ii}$$
$$= \delta_{s-1,i} \cdot (-1)^{\overline{s-1}(\overline{s-1}+\bar{s})} q^{(-1)^{\bar{i}}/2} \cdot u^+_{s-1,s'} - \delta_{si} \cdot \vartheta_{s-1}\vartheta_{s'} q^{(-1)^{\bar{i}}/2} \cdot u^+_{s,(s-1)'} .$$

- Case 2: u^+_{ij} for $i < j$, $i + j \neq N-1, N+1$ and $f_a = f_s$.

$$(-1)^{\overline{s-1}} \varrho^{\otimes 2}(f_s) u^+_{ij}$$
$$= \delta_{s-1,i} \cdot q^{-(\varepsilon_s,\varepsilon_j)/2} \cdot u^+_{s'j} + \delta_{s-1,j} \cdot (-1)^{\bar{i}(\overline{s-1}+\bar{s})} \cdot u^+_{is'}$$
$$- \delta_{si} \cdot (-1)^{\bar{s}(\overline{s-1}+\bar{s})} \vartheta_{s-1}\vartheta_{s'} q^{-(\varepsilon_{s-1},\varepsilon_j)/2} \cdot u^+_{(s-1)',j} - \delta_{sj} \cdot (-1)^{(\bar{i}+\bar{s})(\overline{s-1}+\bar{s})} \vartheta_{s-1}\vartheta_{s'} \cdot u^+_{i,(s-1)'} .$$

- Case 3: u^+_{ij} for $i < j$, $i + j = N-1$ and $f_a = f_s$.

$$(-1)^{\overline{s-1}} \varrho^{\otimes 2}(f_s) u^+_{i,(i+2)'} = -\delta_{s-1,i} \cdot (-1)^{\overline{s-1}+\bar{s}} \vartheta_{s-1}\vartheta_{s'} q^{(\varepsilon_{s-1},\varepsilon_{s-1})/2} \cdot u^+_{ss'} .$$

- Case 4: u^+_{ij} for $i < j$, $i + j = N+1$, $i \neq s$ and $f_a = f_s$.

$$(-1)^{\overline{s-1}} \varrho^{\otimes 2}(f_s) u^+_{ii'}$$
$$= \delta_{s-1,i}\delta_{\bar{1}\,\overline{s-1}} \cdot q^{(-1)^{\bar{1}}/2}\Big(1 + q^{-(-1)^{\bar{1}}\cdot 2}\Big) u^+_{s',(s-1)'}$$
$$- \delta_{s-1,i+1} \cdot (-1)^{\overline{s-2}+\overline{s-1}} \vartheta_{s-2}\vartheta_{s-1} q^{-(\varepsilon_{s-2},\varepsilon_{s-2})/2} \cdot u^+_{s',(s-1)'} .$$

- Case 5: u^+_{ij} for $(i,j) = (s,s')$ and $f_a \neq f_s$.

$$(-1)^{\bar{a}} \varrho^{\otimes 2}(f_a) u^+_{ss'} = \delta_{a,s-1}\delta_{\bar{1}\,\overline{s-1}} \cdot q^{(-1)^{\bar{1}}/2}\Big(1 + q^{-(-1)^{\bar{1}}\cdot 2}\Big) u^+_{s,(s-1)'} .$$

- Case 6: u^+_{ij} for $(i,j) = (s,s')$ and $f_a = f_s$.

$$(-1)^{\overline{s-1}} \varrho^{\otimes 2}(f_s) u^+_{ss'} = \delta_{\overline{s-1}\,\bar{s}} \cdot q^{(\varepsilon_s,\varepsilon_s)/2}\Big(1 + q^{-(\varepsilon_s,\varepsilon_s)\cdot 2}\Big) u^+_{s',(s-1)'} .$$

We also have the following counterparts for the vectors u^-_{ij}.

- Case 1: u^-_{ii} for $\bar{i} \neq \bar{1}$ and $f_a = f_s$.

$$(-1)^{\overline{s-1}}\Big(1 + q^{-(-1)^{\bar{i}}\cdot 2}\Big)^{-1} \varrho^{\otimes 2}(f_s) u^-_{ii}$$
$$= \delta_{s-1,i} \cdot (-1)^{\overline{s-1}(\overline{s-1}+\bar{s})} q^{(-1)^{\bar{i}}/2} \cdot u^-_{s-1,s'} - \delta_{si} \cdot \vartheta_{s-1}\vartheta_{s'} q^{(-1)^{\bar{i}}/2} \cdot u^-_{s,(s-1)'} .$$

- Case 2: u^-_{ij} for $i < j$, $i + j \neq N-1, N+1$ and $f_a = f_s$.

$$(-1)^{\overline{s-1}} \varrho^{\otimes 2}(f_s) u^-_{ij}$$
$$= \delta_{s-1,i} \cdot q^{-(\varepsilon_s,\varepsilon_j)/2} \cdot u^-_{s'j} + \delta_{s-1,j} \cdot (-1)^{\bar{i}(\overline{s-1}+\bar{s})} \cdot u^-_{is'}$$
$$- \delta_{si} \cdot (-1)^{\bar{s}(\overline{s-1}+\bar{s})} \vartheta_{s-1}\vartheta_{s'} q^{-(\varepsilon_{s-1},\varepsilon_j)/2} \cdot u^-_{(s-1)',j} - \delta_{sj} \cdot (-1)^{(\bar{i}+\bar{s})(\overline{s-1}+\bar{s})} \vartheta_{s-1}\vartheta_{s'} \cdot u^-_{i,(s-1)'} .$$

- Case 3: u^-_{ij} for $i<j$, $i+j=N-1$ and $f_a=f_s$.

$$(-1)^{\overline{s-1}}\varrho^{\otimes 2}(f_s)u^-_{i,(i+2)'}=-\delta_{s-1,i}\cdot(-1)^{\overline{s-1}+\bar{s}}\vartheta_{s-1}\vartheta_{s'}q^{(\varepsilon_{s-1},\varepsilon_{s-1})/2}\cdot u^-_{ss'}\,.$$

- Case 4: u^-_{ij} for $i<j$, $i+j=N+1$, $i\neq s$ and $f_a=f_s$.

$$\begin{aligned}&(-1)^{\overline{s-1}}\varrho^{\otimes 2}(f_s)u^-_{ii'}\\&=\delta_{s-1,i}\delta_{\bar{1}\neq\overline{s-1}}\cdot q^{(-1)^{\bar{1}}/2}\Big(1+q^{(-1)^{\bar{1}}\cdot 2}\Big)u^-_{s',(s-1)'}\\&\quad-\delta_{s-1,i+1}\cdot(-1)^{\overline{s-2}+\overline{s-1}}\vartheta_{s-2}\vartheta_{s-1}q^{-(\varepsilon_{s-2},\varepsilon_{s-2})/2}\cdot u^-_{s',(s-1)'}\,.\end{aligned}$$

- Case 5: u^-_{ij} for $(i,j)=(s,s')$ and $f_a\neq f_s$.

$$(-1)^{\bar{a}}\varrho^{\otimes 2}(f_a)u^-_{ss'}=\delta_{a,s-1}\delta_{\bar{1}\neq\overline{s-1}}\cdot q^{-(-1)^{\bar{1}}/2}\Big(1+q^{(-1)^{\bar{1}}\cdot 2}\Big)u^-_{s,(s-1)'}\,.$$

- Case 6: u^-_{ij} for $(i,j)=(s,s')$ and $f_a=f_s$.

$$(-1)^{\overline{s-1}}\varrho^{\otimes 2}(f_s)u^-_{ss'}=\delta_{\overline{s-1}\,\bar{s}}\cdot q^{(\varepsilon_s,\varepsilon_s)/2}\Big(1+q^{-(\varepsilon_s,\varepsilon_s)\cdot 2}\Big)u^-_{s',(s-1)'}\,.$$

We also have an analogue of the isomorphism $\phi\colon V\to V$ from (B.4) satisfying (B.5, B.6) and consecutively (B.7), but the coefficients c_i's determining ϕ should be rather chosen to satisfy:

$$\begin{aligned}\mathsf{c}_{a+1}&=-(-1)^{\bar{a}+\overline{a+1}+\bar{a}\,\overline{a+1}}\vartheta_a\vartheta_{a+1}\cdot\mathsf{c}_a\qquad\text{for}\ \ 1\leq a\leq s-1\,,\\\mathsf{c}_{a'}&=-(-1)^{\bar{a}\,\overline{a+1}}\vartheta_a\vartheta_{a+1}\cdot\mathsf{c}_{(a+1)'}\qquad\text{for}\ \ 1\leq a\leq s-1\,,\\\mathsf{c}_{s'}&=-(-1)^{\overline{s-1}+\bar{s}+\overline{s-1}\,\bar{s}}\vartheta_{s-1}\vartheta_{s'}\cdot\mathsf{c}_{s-1}\,.\end{aligned}$$

This allows to show that $W^{\pm}$ is also stable under the action of $\{e_a\}_{a=1}^s$.

(b) Analogously to the odd m case, it is enough to show that the following set of vectors

$$\{u^{\pm}_{ij}\,|\,1\leq i<j\leq N,\,(i,j)\neq(s,s')\}\cup\{u^+_{ii}\,|\,\bar{i}=\bar{1}\}\cup\{u^-_{ii}\,|\,\bar{i}\neq\bar{1}\}\cup\{u_{ss'}\}\cup\{w_3\}\tag{B.9}$$

is linearly independent in each weight space, with the only nontrivial verification in degree $0\in P$. For convenience, we consider the following multiple of w_3:

$$w_3^\circ=q^{-\sum_{k=1}^{s-1}(\rho,\alpha_k)}\vartheta_1\cdot w_3=\sum_{i=1}^{s}\vartheta_i\left\{q^{-\sum_{k=i}^{s-1}(\rho,\alpha_k)}v_i\otimes v_{i'}+(-1)^{\bar{i}+\bar{s}}\cdot q^{\sum_{k=i}^{s-1}(\rho,\alpha_k)}v_{i'}\otimes v_i\right\}.$$

Let us now assume that

$$\sum_{i=1}^{s-1}(b_i^+u^+_{ii'}+b_i^-u^-_{ii'})+b_su_{ss'}=bw_3^\circ\tag{B.10}$$

for some constants $b_i^{\pm},b_s,b\in\mathbb{C}(q^{1/2})$. Comparing the coefficients of $v_a\otimes v_{a'}$ in (B.10), we obtain:

- For $v_1\otimes v_{1'}$ and $v_{1'}\otimes v_1$, we have

$$\begin{aligned}b_1^++b_1^-&=\vartheta_1q^{-\sum_{k=1}^{s-1}(\rho,\alpha_k)}\cdot b\,,\\q^{-(-1)^{\bar{1}}\cdot 2}\cdot b_1^+-b_1^-&=(-1)^{\bar{1}+\bar{s}}\vartheta_1q^{\sum_{k=1}^{s-1}(\rho,\alpha_k)}\cdot b\,.\end{aligned}$$

- For $v_i\otimes v_{i'}$ and $v_{i'}\otimes v_i$ with $2\leq i\leq s-2$, we have

$$(b_i^++b_i^-)-(-1)^{\overline{i-1}+\bar{i}}\vartheta_{i-1}\vartheta_iq^{-(\varepsilon_{i-1},\varepsilon_{i-1})/2}q^{-(\varepsilon_i,\varepsilon_i)/2}\cdot(b_{i-1}^++b_{i-1}^-)=\vartheta_iq^{-\sum_{k=i}^{s-1}(\rho,\alpha_k)}\cdot b\,,$$

$$\begin{aligned}&(-1)^{\bar{1}+\bar{i}}q^{-(\epsilon_i,\epsilon_i)}\Big(q^{-(-1)^{\bar{1}}}\cdot b_i^+-q^{(-1)^{\bar{1}}}\cdot b_i^-\Big)-(-1)^{\bar{1}+\overline{i-1}}\vartheta_{i-1}\vartheta_i\\&\quad\cdot q^{-(\varepsilon_{i-1},\varepsilon_{i-1})/2}q^{(\varepsilon_i,\varepsilon_i)/2}\cdot\Big(q^{-(-1)^{\bar{1}}}\cdot b_{i-1}^+-q^{(-1)^{\bar{1}}}\cdot b_{i-1}^-\Big)=(-1)^{\bar{i}+\bar{s}}\vartheta_iq^{\sum_{k=i}^{s-1}(\rho,\alpha_k)}\cdot b\,.\end{aligned}$$

- For $v_{s-1}\otimes v_{(s-1)'}$ and $v_{(s-1)'}\otimes v_{s-1}$, we have

$$(b^+_{s-1}+b^-_{s-1}+b_s)-(-1)^{\overline{s-2}+\overline{s-1}}\vartheta_{s-2}\vartheta_{s-1}q^{-(\varepsilon_{s-2},\varepsilon_{s-2})/2}q^{-(\varepsilon_{s-1},\varepsilon_{s-1})/2}\cdot(b^+_{s-2}+b^-_{s-2}) \\ =\vartheta_{s-1}q^{-(\rho,\alpha_{s-1})}\cdot b\,,$$

$$(-1)^{\overline{1}+\overline{s-1}}q^{-(\epsilon_{s-1},\epsilon_{s-1})}\Big(q^{-(-1)^{\overline{1}}}\cdot b^+_{s-1}-q^{(-1)^{\overline{1}}}\cdot b^-_{s-1}\Big)-(-1)^{\overline{s-1}}\cdot q\cdot q^{-(\varepsilon_{s-1},\varepsilon_{s-1})}\cdot b_s \\ -(-1)^{\overline{1}+\overline{s-2}}\vartheta_{s-2}\vartheta_{s-1}q^{-(\varepsilon_{s-2},\varepsilon_{s-2})/2}q^{(\varepsilon_{s-1},\varepsilon_{s-1})/2}\cdot\Big(q^{-(-1)^{\overline{1}}}\cdot b^+_{s-2}-q^{(-1)^{\overline{1}}}\cdot b^-_{s-2}\Big) \\ =(-1)^{\overline{s-1}+\overline{s}}\vartheta_{s-1}q^{(\rho,\alpha_{s-1})}\cdot b\,.$$

- For $v_s\otimes v_{s'}$ and $v_{s'}\otimes v_s$, we have

$$-(-1)^{\overline{s-1}+\overline{s}}\vartheta_{s-1}\vartheta_s q^{-(\varepsilon_{s-1},\varepsilon_{s-1})/2}q^{-(\varepsilon_s,\varepsilon_s)/2}\cdot(b^+_{s-1}+b^-_{s-1}-(-1)^{\overline{s}}\cdot q\cdot q^{(\varepsilon_s,\varepsilon_s)}\cdot b_s)=\vartheta_s\cdot b\,,$$

$$-(-1)^{\overline{1}+\overline{s-1}}\vartheta_{s-1}\vartheta_s q^{-(\varepsilon_{s-1},\varepsilon_{s-1})/2}q^{(\varepsilon_s,\varepsilon_s)/2}\cdot\Big(q^{-(-1)^{\overline{1}}}\cdot b^+_{s-1}-q^{(-1)^{\overline{1}}}\cdot b^-_{s-1}+(-1)^{\overline{1}+\overline{s}}\cdot q^{-(\varepsilon_s,\varepsilon_s)}\cdot b_s\Big) \\ =\vartheta_s\cdot b\,.$$

Evoking (4.11), one can inductively deduce:

$$\begin{gathered} b^+_i+b^-_i=(-1)^{\overline{i}}q^{-\sum_{k=i}^{s-1}(\rho,\alpha_k)}\left(\sum_{j=1}^{i}(-1)^{\overline{j}}q^{-\sum_{k=j}^{i-1}(2\rho,\alpha_k)}\right)\vartheta_i\cdot b\,, \\ q^{-(-1)^{\overline{1}}}\cdot b^+_i-q^{(-1)^{\overline{1}}}\cdot b^-_i=(-1)^{\overline{1}+\overline{s}+\overline{i}}q^{(\varepsilon_i,\varepsilon_i)}q^{\sum_{k=i}^{s-1}(\rho,\alpha_k)}\left(\sum_{j=1}^{i}(-1)^{\overline{j}}q^{\sum_{k=j}^{i-1}(2\rho,\alpha_k)}\right)\vartheta_i\cdot b \end{gathered} \tag{B.11}$$

for any $1\le i\le s-2$, as well as the following four equalities:

$$b^+_{s-1}+b^-_{s-1}+b_s=(-1)^{\overline{s-1}}q^{-(\rho,\alpha_{s-1})}\left(\sum_{j=1}^{s-1}(-1)^{\overline{j}}q^{-\sum_{k=j}^{s-2}(2\rho,\alpha_k)}\right)\vartheta_{s-1}\cdot b\,, \tag{B.12}$$

$$q^{-(-1)^{\overline{1}}}\cdot b^+_{s-1}-q^{(-1)^{\overline{1}}}\cdot b^-_{s-1}-(-1)^{\overline{1}}q\cdot b_s \\ =(-1)^{\overline{1}+\overline{s}+\overline{s-1}}q^{(\varepsilon_{s-1},\varepsilon_{s-1})}q^{(\rho,\alpha_{s-1})}\left(\sum_{j=1}^{s-1}(-1)^{\overline{j}}q^{\sum_{k=j}^{s-2}(2\rho,\alpha_k)}\right)\vartheta_{s-1}\cdot b\,, \tag{B.13}$$

$$b^+_{s-1}+b^-_{s-1}-q^2\cdot b_s=-(-1)^{\overline{s-1}+\overline{s}}q^{(\varepsilon_{s-1},\varepsilon_{s-1})/2}q^{(\varepsilon_s,\varepsilon_s)/2}\vartheta_{s-1}\cdot b\,, \tag{B.14}$$

$$q^{-(-1)^{\overline{1}}}\cdot b^+_{s-1}-q^{(-1)^{\overline{1}}}\cdot b^-_{s-1}+(-1)^{\overline{1}}q^{-1}\cdot b_s=-(-1)^{\overline{1}+\overline{s-1}}q^{(\varepsilon_{s-1},\varepsilon_{s-1})/2}q^{-(\varepsilon_s,\varepsilon_s)/2}\vartheta_{s-1}\cdot b\,. \tag{B.15}$$

Subtracting (B.14) from (B.12) and evoking (4.11) implies

$$(1+q^2)b_s=(-1)^{\overline{s-1}}q^{(\rho,\alpha_{s-1})}\left(\sum_{j=1}^{s}(-1)^{\overline{j}}q^{-\sum_{k=j}^{s-1}(2\rho,\alpha_k)}\right)\vartheta_{s-1}b\,.$$

Likewise, subtracting (B.15) from (B.13) and further multiplying by $(-1)^{\overline{1}}q$, we get

$$-(1+q^2)b_s=(-1)^{\overline{s-1}}q^{(\rho,\alpha_{s-1})}\left(\sum_{j=1}^{s}(-1)^{\overline{j}}q^{\sum_{k=j}^{s-1}(2\rho,\alpha_k)}\right)\vartheta_{s-1}b\,.$$

Adding the above two equations, we obtain:

$$\sum_{j=1}^{s}(-1)^{\overline{j}}\left(q^{\sum_{k=j}^{s-1}(2\rho,\alpha_k)}+q^{-\sum_{k=j}^{s-1}(2\rho,\alpha_k)}\right)b=0\,. \tag{B.16}$$

However, similarly to (5.39), we have:

$$(q-q^{-1})\sum_{j=1}^{s-1}(-1)^{\bar{j}}q^{\sum_{k=j}^{s-1}(2\rho,\alpha_k)} = \sum_{j=1}^{s-1}\Big(q^{(\varepsilon_j,\varepsilon_j)}-q^{-(\varepsilon_j,\varepsilon_j)}\Big)q^{\sum_{k=j}^{s-1}(2\rho,\alpha_k)}$$

$$\overset{(4.11)}{=}\sum_{j=1}^{s-1}\Big(q^{(\varepsilon_j,\varepsilon_j)}q^{\sum_{k=j}^{s-1}(2\rho,\alpha_k)}-q^{(\varepsilon_{j+1},\varepsilon_{j+1})}q^{\sum_{k=j+1}^{s-1}(2\rho,\alpha_k)}\Big) = q^{(\varepsilon_1,\varepsilon_1)}q^{\sum_{k=1}^{s-1}(2\rho,\alpha_k)}-q^{(\varepsilon_s,\varepsilon_s)}$$

and therefore

$$(q-q^{-1})\sum_{j=1}^{s}(-1)^{\bar{j}}q^{\sum_{k=j}^{s-1}(2\rho,\alpha_k)} = q^{(\varepsilon_1,\varepsilon_1)}q^{\sum_{k=1}^{s-1}(2\rho,\alpha_k)}-q^{-(\varepsilon_s,\varepsilon_s)}$$
$$= q^{-(\varepsilon_s,\varepsilon_s)}\Big(q^{(-1)^{\bar{1}}\cdot 2+\dots+(-1)^{\bar{s}}\cdot 2}-1\Big) = q^{-(\varepsilon_s,\varepsilon_s)}(q^{m-n}-1)\,. \qquad \text{(B.17)}$$

Combining (B.17) and its version with q replaced by q^{-1} allows us to rewrite (B.16) as:

$$\left(\frac{q^{m-n-1}-q^{-(m-n-1)}}{q-q^{-1}}+1\right)b=0\,.$$

Since $n\neq m$, we get $b=0$. Then, the equations (B.12)–(B.15) imply that $b^+_{s-1}=b^-_{s-1}=b_s=0$, and analogously $b^+_i=b^-_i=0$ for all $1\leq i\leq s-2$. This proves the linear independence of (B.9).

(c) For $n=m$, iterating the argument from the above proof of part (b), we obtain the following linear dependence

$$\sum_{i=1}^{s-1}(b^+_iu^+_{ii'}+b^-_iu^-_{ii'})+b_su_{ss'}=w^\circ_3$$

where $b_s=0$ and the coefficients $\{b^\pm_i\}_{i=1}^{s-1}$ satisfy the following relations (arising from (B.11)):

$$\begin{aligned}
b^+_i+b^-_i &= (-1)^{\bar{i}}\vartheta_iq^{-\sum_{k=i}^{s-1}(\rho,\alpha_k)}\left(\sum_{j=1}^{i}(-1)^{\bar{j}}q^{-\sum_{k=j}^{i-1}(2\rho,\alpha_k)}\right)\\
&= \frac{(-1)^{\bar{i}}\vartheta_iq^{-\sum_{k=i}^{s-1}(\rho,\alpha_k)}}{q-q^{-1}}\Big(q^{(\varepsilon_i,\varepsilon_i)}-q^{-(\varepsilon_1,\varepsilon_1)}q^{-\sum_{k=1}^{i-1}(2\rho,\alpha_k)}\Big)\\
&= \frac{(-1)^{\bar{i}}\vartheta_iq^{-\sum_{k=1}^{s-1}(\rho,\alpha_k)}}{q-q^{-1}}\Big(q^{(\varepsilon_i,\varepsilon_i)}q^{\sum_{k=1}^{i-1}(\rho,\alpha_k)}-q^{-(\varepsilon_1,\varepsilon_1)}q^{-\sum_{k=1}^{i-1}(\rho,\alpha_k)}\Big)\\
&= \frac{(-1)^{\bar{i}}\vartheta_iq^{(\varepsilon_1,\varepsilon_1)/2+(\varepsilon_s,\varepsilon_s)/2}}{q-q^{-1}}\Big(q^{(\varepsilon_i,\varepsilon_i)}q^{\sum_{k=1}^{i-1}(\rho,\alpha_k)}-q^{-(\varepsilon_1,\varepsilon_1)}q^{-\sum_{k=1}^{i-1}(\rho,\alpha_k)}\Big)
\end{aligned}$$

and

$$\begin{aligned}
&q^{-(-1)^{\bar{1}}}\cdot b^+_i-q^{(-1)^{\bar{1}}}\cdot b^-_i\\
&\quad= (-1)^{\bar{1}+\bar{s}+\bar{i}}\vartheta_iq^{(\varepsilon_i,\varepsilon_i)}q^{\sum_{k=i}^{s-1}(\rho,\alpha_k)}\left(\sum_{j=1}^{i}(-1)^{\bar{j}}q^{\sum_{k=j}^{i-1}(2\rho,\alpha_k)}\right)\\
&\quad= \frac{(-1)^{\bar{1}+\bar{s}+\bar{i}}\vartheta_iq^{(\varepsilon_i,\varepsilon_i)}q^{\sum_{k=i}^{s-1}(\rho,\alpha_k)}}{q-q^{-1}}\Big(q^{(\varepsilon_1,\varepsilon_1)}q^{\sum_{k=1}^{i-1}(2\rho,\alpha_k)}-q^{-(\varepsilon_i,\varepsilon_i)}\Big)\\
&\quad= \frac{(-1)^{\bar{1}+\bar{s}+\bar{i}}\vartheta_iq^{(\varepsilon_i,\varepsilon_i)}q^{\sum_{k=1}^{s-1}(\rho,\alpha_k)}}{q-q^{-1}}\Big(q^{(\varepsilon_1,\varepsilon_1)}q^{\sum_{k=1}^{i-1}(\rho,\alpha_k)}-q^{-(\varepsilon_i,\varepsilon_i)}q^{-\sum_{k=1}^{i-1}(\rho,\alpha_k)}\Big)\\
&\quad= \frac{(-1)^{\bar{1}+\bar{s}+\bar{i}}\vartheta_iq^{(\varepsilon_i,\varepsilon_i)}q^{-(\varepsilon_1,\varepsilon_1)/2-(\varepsilon_s,\varepsilon_s)/2}}{q-q^{-1}}\Big(q^{(\varepsilon_1,\varepsilon_1)}q^{\sum_{k=1}^{i-1}(\rho,\alpha_k)}-q^{-(\varepsilon_i,\varepsilon_i)}q^{-\sum_{k=1}^{i-1}(\rho,\alpha_k)}\Big)\,,
\end{aligned}$$

derived in analogy with (B.17), where we used

$$\sum_{k=1}^{s-1}(\rho,\alpha_k)=\frac{m-n-(\varepsilon_1,\varepsilon_1)-(\varepsilon_s,\varepsilon_s)}{2}=-\frac{(\varepsilon_1,\varepsilon_1)+(\varepsilon_s,\varepsilon_s)}{2}\,.$$

Solving the above two equations, we obtain:

$$\begin{aligned}
b_i^+ &= \frac{(-1)^{\bar{i}}\vartheta_i}{q^2-q^{-2}}\left(q^{(\varepsilon_1,\varepsilon_1)/2+(\varepsilon_s,\varepsilon_s)/2}+(-1)^{\bar{1}+\bar{s}}q^{-(\varepsilon_1,\varepsilon_1)/2-(\varepsilon_s,\varepsilon_s)/2}\right)\\
&\qquad\cdot\left(q^{(\varepsilon_1,\varepsilon_1)+(\varepsilon_i,\varepsilon_i)}q^{\sum_{k=1}^{i-1}(\rho,\alpha_k)}-q^{-\sum_{k=1}^{i-1}(\rho,\alpha_k)}\right)\\
&=\delta_{\bar{1}\bar{s}}\cdot\frac{(-1)^{\bar{i}}\vartheta_i}{q-q^{-1}}\left(q^{(\varepsilon_1,\varepsilon_1)+(\varepsilon_i,\varepsilon_i)}q^{\sum_{k=1}^{i-1}(\rho,\alpha_k)}-q^{-\sum_{k=1}^{i-1}(\rho,\alpha_k)}\right),\\
b_i^- &= \frac{(-1)^{\bar{i}}\vartheta_i}{q^2-q^{-2}}\left(q^{-(\varepsilon_1,\varepsilon_1)/2+(\varepsilon_s,\varepsilon_s)/2}-(-1)^{\bar{1}+\bar{s}}q^{(\varepsilon_1,\varepsilon_1)/2-(\varepsilon_s,\varepsilon_s)/2}\right)\\
&\qquad\cdot\left(q^{(\varepsilon_i,\varepsilon_i)}q^{\sum_{k=1}^{i-1}(\rho,\alpha_k)}-q^{-(\varepsilon_1,\varepsilon_1)}q^{-\sum_{k=1}^{i-1}(\rho,\alpha_k)}\right)\\
&=\delta_{\bar{1}\neq\bar{s}}\cdot\frac{(-1)^{\bar{i}}\vartheta_i}{q-q^{-1}}\left(q^{(\varepsilon_i,\varepsilon_i)}q^{\sum_{k=1}^{i-1}(\rho,\alpha_k)}-q^{-(\varepsilon_1,\varepsilon_1)}q^{-\sum_{k=1}^{i-1}(\rho,\alpha_k)}\right).
\end{aligned}$$

This implies that $w_3\in W^+$ if $\bar{1}=\bar{s}$ and $w_3\in W^-$ if $\bar{1}\neq\bar{s}$. Furthermore, any solution of (B.10) is clearly a multiple of the above one, which implies the codimension 1 property of $W^+\oplus W^-\subset V\otimes V$.

(d) The proof of $\widetilde{w}_3,\hat{w}_3\notin W^+\oplus W^-$ is completely analogous to the above proof of part (b), with a simpler computation when w_3 in (B.9) is rather replaced by either $\widetilde{w}_3$ or $\hat{w}_3$.

On the other hand, iterating the argument from the above proof of part (b), we establish (B.8) by explicitly presenting the linear dependence:

$$\sum_{i=1}^{s-1}(b_i^+u_{ii'}^+ + b_i^-u_{ii'}^-)+b_su_{ss'}=v_1\otimes v_{1'}-(-1)^{\bar{1}+\bar{s}}q^{(-1)^{\bar{1}}+(-1)^{\bar{s}}}q^{n-m}v_{1'}\otimes v_1$$

where

$$\begin{aligned}
b_i^+ &= \frac{(-1)^{\bar{1}+\bar{i}}\vartheta_1\vartheta_i}{q+q^{-1}}q^{-\sum_{k=1}^{s-1}(\rho,\alpha_k)}\left(q^{(\varepsilon_1,\varepsilon_1)}q^{\sum_{k=i}^{s-1}(\rho,\alpha_k)}-(-1)^{\bar{1}+\bar{s}}q^{(\varepsilon_i,\varepsilon_i)}q^{-\sum_{k=i}^{s-1}(\rho,\alpha_k)}\right),\\
b_i^- &= \frac{(-1)^{\bar{1}+\bar{i}}\vartheta_1\vartheta_i}{q+q^{-1}}q^{-\sum_{k=1}^{s-1}(\rho,\alpha_k)}\left(q^{-(\varepsilon_1,\varepsilon_1)}q^{\sum_{k=i}^{s-1}(\rho,\alpha_k)}+(-1)^{\bar{1}+\bar{s}}q^{(\varepsilon_i,\varepsilon_i)}q^{-\sum_{k=i}^{s-1}(\rho,\alpha_k)}\right)
\end{aligned}$$

for $1\leq i\leq s-2$,

$$\begin{aligned}
b_{s-1}^+ &= \frac{(-1)^{\bar{1}+\overline{s-1}}\vartheta_1\vartheta_{s-1}}{q+q^{-1}}q^{-\sum_{k=1}^{s-1}(\rho,\alpha_k)}\\
&\qquad\cdot\left(q^{(\varepsilon_1,\varepsilon_1)}q^{(\rho,\alpha_{s-1})}-(-1)^{\bar{1}+\bar{s}}q^{(\varepsilon_{s-1},\varepsilon_{s-1})}q^{-(\rho,\alpha_{s-1})}(1-\delta_{\bar{1}\neq\bar{s}}\vartheta_{s-1}\vartheta_s)\right),\\
b_{s-1}^- &= \frac{(-1)^{\bar{1}+\overline{s-1}}\vartheta_1\vartheta_{s-1}}{q+q^{-1}}q^{-\sum_{k=1}^{s-1}(\rho,\alpha_k)}\\
&\qquad\cdot\left(q^{-(\varepsilon_1,\varepsilon_1)}q^{(\rho,\alpha_{s-1})}+(-1)^{\bar{1}+\bar{s}}q^{(\varepsilon_{s-1},\varepsilon_{s-1})}q^{-(\rho,\alpha_{s-1})}(1-\delta_{\bar{1}\bar{s}}\vartheta_{s-1}\vartheta_s)\right)
\end{aligned}$$

and

$$b_s=\frac{(-1)^{\bar{1}+\overline{s-1}}\vartheta_1\vartheta_s}{q+q^{-1}}q^{(\varepsilon_{s-1},\varepsilon_{s-1})/2-(\varepsilon_s,\varepsilon_s)/2}q^{-\sum_{k=1}^{s-1}(\rho,\alpha_k)}.$$

(e) The proof of part (e) is completely analogous to that of Proposition B.2(d). □

B.2.3. *Generating property for even m with $\bar{s}=\bar{1}$.*

Similarly to the previous cases, we define the following elements $u_{ij}^{\pm}$ in $V\otimes V$ for $1\leq i\leq j\leq N$:

$$u_{ij}^+=\begin{cases}
v_i\otimes v_j+(-1)^{\bar{1}}(-1)^{\bar{i}\bar{j}}q^{-(-1)^{\bar{1}}}q^{-(\varepsilon_i,\varepsilon_j)}v_j\otimes v_i & \text{if } j'\neq i\\
\left\{v_i\otimes v_{i'}+(-1)^{\bar{1}+\bar{i}}q^{-(-1)^{\bar{1}}}q^{-(\varepsilon_i,\varepsilon_i)}v_{i'}\otimes v_i\right\}-(-1)^{\bar{i}+\overline{i+1}}q^{-(\varepsilon_i,\varepsilon_i)/2}q^{-(\varepsilon_{i+1},\varepsilon_{i+1})/2} & \\
\quad\cdot\vartheta_i\vartheta_{i+1}\left\{v_{i+1}\otimes v_{(i+1)'}+(-1)^{\bar{1}+\overline{i+1}}q^{-(-1)^{\bar{1}}}q^{(\varepsilon_{i+1},\varepsilon_{i+1})}v_{(i+1)'}\otimes v_{i+1}\right\} & \text{if } j'=i\neq s\\
v_s\otimes v_{s'}+(-1)^{\bar{1}+\bar{s}}q^{-(-1)^{\bar{1}}}q^{-(\varepsilon_s,\varepsilon_s)}v_{s'}\otimes v_s & \text{if } j'=i=s
\end{cases}$$

and

$$u^-_{ij} = \begin{cases} v_i \otimes v_j - (-1)^{\overline{1}}(-1)^{\overline{i}\,\overline{j}} q^{(-1)^{\overline{1}}} q^{-(\varepsilon_i,\varepsilon_j)} v_j \otimes v_i & \text{if } j' \neq i \\ \Big\{ v_i \otimes v_{i'} - (-1)^{\overline{1}+\overline{i}} q^{(-1)^{\overline{1}}} q^{-(\varepsilon_i,\varepsilon_i)} v_{i'} \otimes v_i \Big\} - (-1)^{\overline{i}+\overline{i+1}} q^{-(\varepsilon_i,\varepsilon_i)/2} q^{-(\varepsilon_{i+1},\varepsilon_{i+1})/2} & \\ \quad \cdot \vartheta_i \vartheta_{i+1} \Big\{ v_{i+1} \otimes v_{(i+1)'} - (-1)^{\overline{1}+\overline{i+1}} q^{(-1)^{\overline{1}}} q^{(\varepsilon_{i+1},\varepsilon_{i+1})} v_{(i+1)'} \otimes v_{i+1} \Big\} & \text{if } j' = i \neq s \\ v_s \otimes v_{s'} - (-1)^{\overline{1}+\overline{s}} q^{(-1)^{\overline{1}}} q^{-(\varepsilon_s,\varepsilon_s)} v_{s'} \otimes v_s & \text{if } j' = i = s \end{cases}.$$

Again, we note that $u^+_{ii} = 0$ iff $\overline{i} \neq \overline{1}$ and $u^-_{ii} = 0$ iff $\overline{i} = \overline{1}$. For convenience, let us define

$$u_{ss'} = v_s \otimes v_{s'} + q \cdot q^{-(\varepsilon_s,\varepsilon_s)} v_{s'} \otimes v_s \,.$$

Then $u^+_{ss'} = u_{ss'}$ if $\overline{1} = \overline{s}$, and $u^-_{ss'} = u_{ss'}$ if $\overline{1} \neq \overline{s}$. We define subspaces $W^\pm$ of $V \otimes V$ to be spanned by the corresponding (nonzero) vectors:

$$\begin{aligned} W^+ &= \mathrm{Span}\Big(\{u^+_{ij} \,|\, 1 \le i < j \le N,\ (i,j) \neq (s,s')\} \cup \{u^+_{ii} \,|\, \overline{i} = \overline{1}\} \cup \{u^+_{ss'} \text{ if } \ \overline{1} = \overline{s}\} \Big), \\ W^- &= \mathrm{Span}\Big(\{u^-_{ij} \,|\, 1 \le i < j \le N,\ (i,j) \neq (s,s')\} \cup \{u^-_{ii} \,|\, \overline{i} \neq \overline{1}\} \cup \{u^-_{ss'} \text{ if } \ \overline{1} \neq \overline{s}\} \Big). \end{aligned} \tag{B.18}$$

We also consider a one-dimensional subspace $W_3 = \mathrm{Span}(w_3)$ of $V \otimes V$, cf. (3.5).

Proposition B.4. *(a) The subspaces W^+, W^-, W_3 are $U_q(\mathfrak{osp}(V))$-subrepresentations of $V \otimes V$.*

(b) For $n \neq m$, the $U_q(\mathfrak{osp}(V))$-representation $V \otimes V$ decomposes into the direct sum of those:

$$V \otimes V \simeq W^+ \oplus W^- \oplus W_3 \,.$$

(c) For $n = m$, $W_3 \subset W^+$ if $\overline{1} = \overline{0} \neq \overline{s}$ and $W_3 \subset W^-$ if $\overline{1} = \overline{1} = \overline{s}$, and $W^+ \oplus W^-$ is a codimension 1 subspace of $V \otimes V$.

(d) Both $\widetilde{w}_3 = v_1 \otimes v_{1'}$ and $\hat{w}_3 = v_{1'} \otimes v_1$ do not belong to $W^+ \oplus W^-$, while

$$\widetilde{w}_3 - (-1)^{\overline{1}} q^{(-1)^{\overline{1}}} q^{n-m+1} \hat{w}_3 \in W^+ \oplus W^- \,. \tag{B.19}$$

(e) W^+, W^-, W_3 are $U_q(\mathfrak{osp}(V))$-representations generated by the corresponding highest weight vectors w_1, w_2, w_3. Moreover, these representations are irreducible if $n \neq m$.

Proof. (a) The proof is analogous to the previous cases, so we only present the key difference in formulas. The action of the generators $\{f_a\}_{a=1}^s$ on the above vectors u^+_{ij} is given by the exact same formula unless $f_a = f_s$ or $(i,j) = (s,s')$. The action in the remaining cases is given by:

- Case 1: u^+_{ii} for $\overline{i} = \overline{1}$ and $f_a = f_s$.

$$(-1)^{\overline{s}} (q+q^{-1})^{-1} \Big(1 + q^{-(-1)^{\overline{1}} \cdot 2}\Big)^{-1} \varrho^{\otimes 2}(f_s) u^+_{ii} = \delta_{si} \cdot q^{(-1)^{\overline{1}}} \cdot u^+_{ss'} \,.$$

  This is the only case when $u^+_{ss'}$ arises in the RHS, explaining when to include $u^+_{ss'}$ in (B.18).
- Case 2: u^+_{ij} for $i < j$, $i + j \neq N+1$ and $f_a = f_s$.

$$(-1)^{\overline{s}} (q+q^{-1})^{-1} \varrho^{\otimes 2}(f_s) u^+_{ij} = \delta_{si} \cdot u^+_{s'j} + \delta_{sj} \cdot u^+_{is'} \,.$$

- Case 3: u^+_{ij} for $i < j$, $i + j = N+1$, $i \neq s$ and $f_a = f_s$.

$$(-1)^{\overline{s}} (q+q^{-1})^{-1} \varrho^{\otimes 2}(f_s) u^+_{ii'} = -\delta_{s,i+1} \cdot (-1)^{\overline{s-1}+\overline{s}} \vartheta_{s-1} \vartheta_s q^{-(\varepsilon_{s-1},\varepsilon_{s-1})/2} q^{(\varepsilon_s,\varepsilon_s)/2} \cdot u^+_{s's'} \,.$$

- Case 4: u^+_{ij} for $(i,j) = (s,s')$ and $f_a \neq f_s$.

$$(-1)^{\overline{a}} \varrho^{\otimes 2}(f_a) u^+_{ss'} = \delta_{a,s-1} \cdot (-1)^{\overline{s}(\overline{s-1}+\overline{s})} q^{-(\varepsilon_s,\varepsilon_s)/2} \cdot u^+_{s,(s-1)'} \,.$$

- Case 5: u^+_{ij} for $(i,j) = (s,s')$ and $f_a = f_s$.

$$(-1)^{\overline{s}} (q+q^{-1})^{-1} \varrho^{\otimes 2}(f_s) u^+_{ss'} = q^{(-1)^{\overline{1}}} \Big(1 + q^{-(-1)^{\overline{1}} \cdot 4}\Big) \Big(1 + q^{-(-1)^{\overline{1}} \cdot 2}\Big)^{-1} u^+_{s's'} \,.$$

We also have the following counterparts for the vectors u^-_{ij}.

- Case 1: u^-_{ii} for $\bar{i} \neq \bar{1}$ and $f_a = f_s$.
$$(-1)^{\bar{s}}(q+q^{-1})^{-1}\Big(1+q^{(-1)^{\bar{1}}\cdot 2}\Big)^{-1} \varrho^{\otimes 2}(f_s)u^-_{ii} = \delta_{si}\cdot q^{-(-1)^{\bar{1}}}\cdot u^-_{ss'}\,.$$
This is the only case when $u^-_{ss'}$ arises in the RHS, explaining when to include $u^-_{ss'}$ in (B.18).
- Case 2: u^-_{ij} for $i<j$, $i+j \neq N+1$ and $f_a = f_s$.
$$(-1)^{\bar{s}}(q+q^{-1})^{-1}\varrho^{\otimes 2}(f_s)u^-_{ij} = \delta_{si}\cdot u^-_{s'j} + \delta_{sj}\cdot u^-_{is'}\,.$$
- Case 3: u^-_{ij} for $i<j$, $i+j = N+1$, $i\neq s$ and $f_a = f_s$.
$$(-1)^{\bar{s}}(q+q^{-1})^{-1}\varrho^{\otimes 2}(f_s)u^-_{ii'} = -\delta_{s,i+1}\cdot(-1)^{\overline{s-1}+\bar{s}}\vartheta_{s-1}\vartheta_s q^{-(\varepsilon_{s-1},\varepsilon_{s-1})/2}q^{(\varepsilon_s,\varepsilon_s)/2}\cdot u^-_{s's'}\,.$$
- Case 4: u^-_{ij} for $(i,j)=(s,s')$ and $f_a \neq f_s$.
$$(-1)^{\bar{a}}\varrho^{\otimes 2}(f_a)u^-_{ss'} = \delta_{a,s-1}\cdot(-1)^{\bar{s}(\overline{s-1}+\bar{s})}q^{-(\varepsilon_s,\varepsilon_s)/2}\cdot u^-_{s,(s-1)'}\,.$$
- Case 5: u^-_{ij} for $(i,j)=(s,s')$ and $f_a = f_s$.
$$(-1)^{\bar{s}}(q+q^{-1})^{-1}\varrho^{\otimes 2}(f_s)u^-_{ss'} = q^{-(-1)^{\bar{1}}}\Big(1+q^{(-1)^{\bar{1}}\cdot 4}\Big)\Big(1+q^{(-1)^{\bar{1}}\cdot 2}\Big)^{-1}u^-_{s's'}\,.$$

We also have an analogue of the isomorphism $\phi\colon V\to V$ from (B.4) satisfying (B.5, B.6) and consecutively (B.7), but the coefficients c_i's determining ϕ should be rather chosen to satisfy:
$$\begin{aligned}
&\mathsf{c}_{a+1} = -(-1)^{\bar{a}+\overline{a+1}+\bar{a}\,\overline{a+1}}\vartheta_a\vartheta_{a+1}\cdot\mathsf{c}_a \qquad \text{for } 1\leq a\leq s-1\,,\\
&\mathsf{c}_{a'} = -(-1)^{\bar{a}\,\overline{a+1}}\vartheta_a\vartheta_{a+1}\cdot\mathsf{c}_{(a+1)'} \qquad \text{for } 1\leq a\leq s-1\,,\\
&\mathsf{c}_{s'} = -(q+q^{-1})\cdot\mathsf{c}_s\,.
\end{aligned}$$
This allows to show that $W^{\pm}$ is also stable under the action of $\{e_a\}_{a=1}^s$.

(b) Analogously to the previous cases, it is enough to show that the following set of vectors
$$\{u^{\pm}_{ij}\,|\,1\leq i<j\leq N,\,(i,j)\neq(s,s')\}\cup\{u^+_{ii}\,|\,\bar{i}=\bar{1}\}\cup\{u^-_{ii}\,|\,\bar{i}\neq\bar{1}\}\cup\{u_{ss'}\}\cup\{w_3\} \tag{B.20}$$
is linearly independent in each weight space, with the only nontrivial verification in degree $0\in P$. For convenience, we consider the following multiple of w_3:
$$w^\circ_3 = q^{1-\sum_{k=1}^{s-1}(\rho,\alpha_k)}\vartheta_1\cdot w_3 = \sum_{i=1}^{s}\vartheta_i\left\{q\cdot q^{-\sum_{k=i}^{s-1}(\rho,\alpha_k)}v_i\otimes v_{i'} - (-1)^{\bar{i}+\bar{s}}\cdot q^{-1}\cdot q^{\sum_{k=i}^{s-1}(\rho,\alpha_k)}v_{i'}\otimes v_i\right\}.$$
Let us now assume that (cf. (B.10))
$$\sum_{i=1}^{s-1}(b^+_i u^+_{ii'} + b^-_i u^-_{ii'}) + b_s u_{ss'} = b w^\circ_3 \tag{B.21}$$
for some constants $b^{\pm}_i, b_s, b\in\mathbb{C}(q^{1/2})$. Since the proof is completely analogous to that of Proposition B.3(b), we only present here the key formulas:
$$b^+_i + b^-_i = (-1)^{\bar{i}}\cdot q\cdot q^{-\sum_{k=i}^{s-1}(\rho,\alpha_k)}\left(\sum_{j=1}^{i}(-1)^{\bar{j}}q^{-\sum_{k=j}^{i-1}(2\rho,\alpha_k)}\right)\vartheta_i\cdot b\,,$$
$$q^{-(-1)^{\bar{1}}}\cdot b^+_i - q^{(-1)^{\bar{1}}}\cdot b^-_i = -(-1)^{\bar{1}+\bar{s}+\bar{i}}\cdot q^{-1}\cdot q^{(\varepsilon_i,\varepsilon_i)}q^{\sum_{k=i}^{s-1}(\rho,\alpha_k)}\left(\sum_{j=1}^{i}(-1)^{\bar{j}}q^{\sum_{k=j}^{i-1}(2\rho,\alpha_k)}\right)\vartheta_i\cdot b$$
for any $1\leq i\leq s-1$, and
$$(-1)^{\bar{s}}b_s = q\cdot\left(\sum_{j=1}^{s}(-1)^{\bar{j}}q^{-\sum_{k=j}^{s-1}(2\rho,\alpha_k)}\right)\vartheta_s\cdot b \overset{(B.17)}{=} \frac{1-q^{-(m-n)}}{q-q^{-1}}\vartheta_s\cdot b\,,$$
$$-(-1)^{\bar{s}}b_s = q^{-1}\cdot\left(\sum_{j=1}^{s}(-1)^{\bar{j}}q^{\sum_{k=j}^{s-1}(2\rho,\alpha_k)}\right)\vartheta_s\cdot b \overset{(B.17)}{=} \frac{q^{m-n}-1}{q-q^{-1}}\vartheta_s\cdot b\,.$$

For $n \neq m$, adding these two formulas we get $b_s = b = 0$ and hence $b_i^+ = b_i^- = 0$ for all $1 \leq i \leq s-1$, so that (B.21) has only the trivial solution, thus establishing the linear independence of (B.20).

(c) The proof of part (c) is completely analogous to that of Proposition B.3(c). For $n = m$, iterating the argument from the above proof of part (b), we obtain the following linear dependence

$$\sum_{i=1}^{s-1} (b_i^+ u_{ii'}^+ + b_i^- u_{ii'}^-) + b_s u_{ss'} = w_3^\circ$$

where $b_s = 0$ and

$$b_i^+ = \delta_{\bar{1}\neq\bar{s}} \cdot \frac{(-1)^{\bar{i}}\vartheta_i}{q-q^{-1}} \left(q^{(\varepsilon_1,\varepsilon_1)+(\varepsilon_i,\varepsilon_i)} q^{\sum_{k=1}^{i-1}(\rho,\alpha_k)} - q^{-\sum_{k=1}^{i-1}(\rho,\alpha_k)} \right),$$

$$b_i^- = \delta_{\bar{1}\bar{s}} \cdot \frac{(-1)^{\bar{i}}\vartheta_i}{q-q^{-1}} \left(q^{(\varepsilon_i,\varepsilon_i)} q^{\sum_{k=1}^{i-1}(\rho,\alpha_k)} - q^{-(\varepsilon_1,\varepsilon_1)} q^{-\sum_{k=1}^{i-1}(\rho,\alpha_k)} \right).$$

This implies that $w_3 \in W^-$ if $\bar{1} = \bar{s}$ and $w_3 \in W^+$ if $\bar{1} \neq \bar{s}$. Furthermore, any solution of (B.21) is clearly a multiple of the above one, which implies the codimension 1 property of $W^+ \oplus W^- \subset V \otimes V$.

(d) The proof of $\tilde{w}_3, \hat{w}_3 \notin W^+ \oplus W^-$ is completely analogous to the above proof of part (b), with a simpler computation when w_3 in (B.9) is rather replaced by either $\tilde{w}_3$ or $\hat{w}_3$.

On the other hand, iterating the argument from the above proof of part (b), we establish (B.19) by explicitly presenting the linear dependence:

$$\sum_{i=1}^{s-1} (b_i^+ u_{ii'}^+ + b_i^- u_{ii'}^-) + b_s u_{ss'} = v_1 \otimes v_{1'} + (-1)^{\bar{1}+\bar{s}} q^{(-1)^{\bar{1}}-(-1)^{\bar{s}}} q^{n-m} v_{1'} \otimes v_1$$

where

$$b_i^+ = \frac{(-1)^{\bar{1}+\bar{i}}\vartheta_1\vartheta_i}{q+q^{-1}} q^{-\sum_{k=1}^{s-1}(\rho,\alpha_k)} \left(q^{(\varepsilon_1,\varepsilon_1)} q^{\sum_{k=i}^{s-1}(\rho,\alpha_k)} + (-1)^{\bar{1}+\bar{s}} q^{(\varepsilon_i,\varepsilon_i)} q^{-(\varepsilon_s,\varepsilon_s)\cdot 2} q^{-\sum_{k=i}^{s-1}(\rho,\alpha_k)} \right),$$

$$b_i^- = \frac{(-1)^{\bar{1}+\bar{i}}\vartheta_1\vartheta_i}{q+q^{-1}} q^{-\sum_{k=1}^{s-1}(\rho,\alpha_k)} \left(q^{-(\varepsilon_1,\varepsilon_1)} q^{\sum_{k=i}^{s-1}(\rho,\alpha_k)} - (-1)^{\bar{1}+\bar{s}} q^{(\varepsilon_i,\varepsilon_i)} q^{-(\varepsilon_s,\varepsilon_s)\cdot 2} q^{-\sum_{k=i}^{s-1}(\rho,\alpha_k)} \right)$$

for $1 \leq i \leq s-1$ and

$$b_s = (-1)^{\bar{1}+\bar{s}} \vartheta_1 \vartheta_s q^{-\sum_{k=1}^{s-1}(\rho,\alpha_k)}.$$

(e) The proof of part (e) is completely analogous to that of Proposition B.2(d). □

Remark B.5. We note that analogues of Propositions B.2, B.3, B.4 already hold at $q = 1$ case. In particular, the respective "generating" properties by $\bar{w}_1 = w_1|_{q=1} = v_1 \otimes v_1$, $\bar{w}_2 = w_2|_{q=1} = v_1 \otimes v_2 - (-1)^{\bar{1}(\bar{1}+\bar{2})} \cdot v_2 \otimes v_1$, and either of $\bar{\tilde{w}}_3 = \tilde{w}_3|_{q=1} = v_1 \otimes v_{1'}$, $\bar{\hat{w}}_3 = \hat{w}_3|_{q=1} = v_{1'} \otimes v_1$, or $\bar{w}_3 = w_3|_{q=1} = \sum_{i=1}^N \vartheta_1^{-1}\vartheta_i v_i \otimes v_{i'}$ if $n \neq m$ can already be observed there (via simpler calculations).

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