ORTHOSYMPLECTIC QUANTUM GROUPS REVISITED

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Abstract. We present the RLL-realization of extended orthosymplectic quantum supergroups for any parity sequence, with R-matrices evaluated in the earlier work [9]. Our isomorphism is compatible with the internal structure of generalized doubles. We also relate different sign conventions through 2-cocycle twists. Furthermore, we establish a factorization of the reduced R-matrix within the RLL-realization.

Contents

1. Introduction

1.1. Summary.

For classical Lie algebras $\mathfrak{g}$, the quantum groups $U_q^{\mathrm{RLL}}(\mathfrak{g})$ first implicitly appeared in the work of Faddeev's school on the *quantum inverse scattering method*, see e.g. [7]. In this *RLL-realization*, the algebra generators are encoded by two square matrices $L^{\pm}$ subject to the famous *RLL-relations*

$$RL_1^{\pm}L_2^{\pm} = L_2^{\pm}L_1^{\pm}R\,, \qquad RL_1^{+}L_2^{-} = L_2^{-}L_1^{+}R$$

(along with central reduction), where R is a solution of the *Yang-Baxter equation*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\,.$$

This is a natural analogue of the matrix realization of classical Lie algebras, and it manifestly exhibits the Hopf algebra structure, with the coproduct Δ, antipode S, and counit ϵ given by

$$\Delta(L^{\pm}) = L^{\pm}\otimes L^{\pm}\,, \qquad S(L^{\pm}) = (L^{\pm})^{-1}\,, \qquad \epsilon(L^{\pm}) = \mathrm{I}\,.$$

The uniform definition of quantum groups $U_q^{\mathrm{DJ}}(\mathfrak{g})$ for simple $\mathfrak{g}$ was provided by Drinfeld [6] and Jimbo [11], and is usually referred to as *Drinfeld–Jimbo-realization*. In this presentation, the generators $e_i, f_i, k_i^{\pm1} = q^{\pm h_i}$ are labeled by simple roots α_i of $\mathfrak{g}$, while the Hopf algebra structure is given formally by the assignment on the generators. In A-type, the corresponding isomorphism

$$U_q^{\mathrm{RLL}}(\mathfrak{gl}_n) \simeq U_q^{\mathrm{DJ}}(\mathfrak{gl}_n)\,, \tag{1.1}$$

and subsequently its $\mathfrak{sl}_n$-version, were constructed in [5, §2] by considering the Gauss decomposition of $L^{\pm}$.

The theory of quantum supergroups associated with Lie superalgebras is still far from a full development. One of the major deficiencies lies in the absence of the uniform (new) Drinfeld realization of such algebras in affine types (except for types A and $D(2,1,x)$), which is crucial for the study of their representation theory and further applications to mathematical physics. A novel feature of Lie superalgebras is that they admit several non-isomorphic Dynkin diagrams, which can be labeled by so-called parity sequences. The isomorphism of the Lie superalgebras corresponding to different Dynkin diagrams of the same finite/affine type was established in [17, Appendix]. Upgrading the latter to the isomorphism of quantum finite/affine superalgebras associated with different Dynkin diagrams is a highly non-trivial technical task that constitutes one of the major results of [23]. We also note that super setup features so-called *higher order* Serre relations, which often cause major technical complications in generalizing classical results to the super setup.

This note is devoted to the generalization of the aforementioned isomorphism (1.1) to the context of classical ($ABCD$-type) finite quantum supergroups. We want to emphasize right away that this is a Hopf superalgebra isomorphism. To this end, it is fundamental that both algebras admit *generalized double* realizations in the context of a pair of Hopf superalgebras endowed with skew pairing (we present full details in Appendix A as we could not find a uniform treatment of that construction in the super setup, besides a somewhat related paper [1]). As the pairing between Borel positive and negative subalgebras of Drinfeld–Jimbo quantum supergroups is known to be non-degenerate (which is actually the key result of [22] and is the source of all higher-order Serre relations), once a surjective map $U_q^{\mathrm{DJ}}(\mathfrak{g}) \to U_q^{\mathrm{RLL}}(\mathfrak{g})$ is constructed, compatible with coproducts and pairing, it must be an isomorphism. This approach is different from the one used in [5] to prove (1.1) as well as the one utilized in [12, 13] to establish BCD-type generalizations, where one rather had to appeal to faithful representations (which is more subtle in the super case) or to the universal R-matrix $\mathfrak{R}$ (which requires one to use $\hbar$-adic completion), respectively.

As another technical aspect of the super version of quantum groups, we note that there have been multiple sign-conventions in the literature. In particular, in pursuit to generalize the foundational work [5] to super-A type (both finite and affine) works [2, 8, 26] used slightly different definitions of the corresponding RLL-realizations. We show that different sign-conventions can be related through the more general construction of 2-cocycle twists. This gives rise to the twisted and untwisted versions, see details in Section 3.5. Though the untwisted algebra is the correct object to work with, it is often more convenient to use the twisted one (in particular, when studying the Gauss decomposition) in order to simplify numerous signs.

We note that evaluation of the R-matrices used in the RLL-realization was carried out in our earlier paper [9] using the combinatorial approach of [4] to the construction of PBW bases of $U_q^{\mathrm{DJ}}(\mathfrak{g})$. This manifestly featured a factorization over the reduced system of positive roots of its unipotent part R_u into "local q-exponents" akin to the well-known factorization of [15]. In the present note, we also establish a similar factorization of the reduced R-matrix into "local q-exponents" working entirely within the RLL-realization, generalizing the super-A type of [24] and classical BCD-types of [19, Appendix B].

1.2. Outline.

The structure of the present paper is the following:

• In Section 2, we recall the basic conventions on superalgebras and evoke the notion of Drinfeld–Jimbo general linear quantum supergroups, see Definitions 2.20 and 2.32. We emphasize the key property of them being Drinfeld doubles in Proposition 2.40 and introduce their twisted version $U_q(\mathfrak{gl}(V))^{\mathcal{F}}$ in Subsection 2.6.

• In Section 3, we recall the RLL-realization of the general linear quantum supergroups, see Definition 3.13, with the R-matrix evoked in (3.9). The key property of them being generalized doubles is established in Proposition 3.36. In Subsection 3.5, we discuss a closely related algebra $\widetilde{U}(R)$ which is often more convenient to work with in order to track various Koszul signs, and establish its relation to $U(R)$ via the 2-cocycle twist in Proposition 3.51 (this addresses a common discrepancy in various definitions of RLL-superalgebras in the existing literature). One of the key results is the identification of Drinfeld–Jimbo and RLL realizations of the general linear quantum supergroups, established in Theorem 3.71. While the inverse map is constructed in Subsection 3.8, we wish to emphasize that the map ξ of Theorem 3.71 is compatible with Drinfeld double structures and is thus easily argued to be an isomorphism. In Subsection 3.9, we factorize the reduced part of the canonical tensor of the double $U(R)$, realized as a Drinfeld double, into "local q-exponents".

• In Section 4, we recall the basic conventions about orthosymplectic Lie superalgebra $\mathfrak{osp}(V)$ and its extended version $\mathfrak{gosp}(V)$. We then evoke the notion of Drinfeld–Jimbo-type (extended) orthosymplectic quantum supergroups, see Definitions 4.15 and 4.20. The key property of $U_q(\mathfrak{gosp}(V))$ being a Drinfeld double and the corresponding Drinfeld twist $U_q(\mathfrak{gosp}(V))^{\mathcal{F}}$ are discussed in Propositions 4.25 and 4.28.

• In Section 5, we study the RLL-realization of extended orthosymplectic quantum supergroups, with the corresponding R-matrix (5.5) evaluated in [9]. Similar to Section 3, we establish their generalized double realization and relate them to the twisted algebra $\widetilde{U}(R)$ in Propositions 5.11 and 5.17. The key (twisted) relations among the entries in the Gauss decomposition of the matrices $L^{\pm}$ are summarized in tables of Subsection 5.3. One of the key results is the identification of Drinfeld–Jimbo and RLL realizations of the extended orthosymplectic quantum supergroups, established in Theorem 5.78. Again, while the inverse map is constructed in Subsection 5.5, we emphasize that the map ξ of Theorem 5.78 is compatible with Drinfeld double structures and is thus easily argued to be an isomorphism. In Subsection 5.6, we factorize the reduced part of the canonical tensor of the double $U(R)$, realized as a Drinfeld double, into "local q-exponents", generalizing [19, 24].

• In Appendix A, we discuss in details the super versions of the usual Drinfeld and generalized doubles.

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2. General linear Lie superalgebras and quantum supergroups

2.1. Superalgebras and Lie superalgebras.

A vector space V over a base field $\mathbb{K}$ is called a *superspace* if it is $\mathbb{Z}_2$-graded, i.e. $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $V_{\bar{i}}$ denotes the component of degree $\bar{i} \in \mathbb{Z}_2$. For a homogeneous element $v \in V$, its $\mathbb{Z}_2$-degree is denoted by $|v| \in \mathbb{Z}_2$. Elements of degree $\bar{0}$ (resp. $\bar{1}$) are called *even* (resp. *odd*). If the underlying vector space of V is finite-dimensional, the pair $(\dim V_{\bar{0}}, \dim V_{\bar{1}})$ is called the *dimension* of the superspace V.

The category $\mathrm{sVect}_{\mathbb{K}}$ of all superspaces over $\mathbb{K}$, together with all degree-preserving linear maps, forms a symmetric tensor category. The tensor product is the standard tensor product of vector spaces equipped with the $\mathbb{Z}_2$-grading:

$$(V \otimes W)_{\bar{k}} = \bigoplus_{\bar{i}+\bar{j}=\bar{k}} V_{\bar{i}} \otimes W_{\bar{j}} \,,$$

and the tensor unit is the base field $\mathbb{K}$, viewed as a superspace concentrated in degree $\bar{0}$. The braiding is given by the graded permutation:

$$\tau_{VW} \colon V \otimes W \to W \otimes V \,, \qquad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v \,.$$

Here, we adopt the convention $(-1)^{\bar{0}} = 1$ and $(-1)^{\bar{1}} = -1$. Unless stated otherwise, $\mathbb{K}$ as an object of $\mathrm{sVect}_{\mathbb{K}}$ will always denote this tensor unit. We frequently omit the subscripts on the tensor product $\otimes$ and the braiding τ whenever the underlying base field or vector spaces are clear from the context.

Remark 2.1. Since the braiding τ is symmetric (that is, $\tau_{WV} \circ \tau_{VW} = \mathrm{Id}_{V\otimes W}$), composite morphisms built from consecutive braidings depend only on the underlying permutation. Thus, for each permutation σ in the symmetric group S_r, there is a uniquely defined natural isomorphism with components

$$\tau^{\sigma}_{V_1\dots V_r} \colon V_1 \otimes \cdots \otimes V_r \xrightarrow{\sim} V_{\sigma^{-1}(1)} \otimes \cdots \otimes V_{\sigma^{-1}(r)} \,. \tag{2.2}$$

Again, the subscripts will frequently be omitted when the underlying superspaces are clear from the context.

Remark 2.3. We note that the forgetful functor from $\mathrm{sVect}_{\mathbb{K}}$ to $\mathrm{Vect}_{\mathbb{K}}$ is strictly monoidal (but not braided monoidal) and faithful. However, it is not full, as an arbitrary linear map between superspaces need not be degree-preserving. In practice, we often regard superspaces as objects in $\mathrm{Vect}_{\mathbb{K}}$ to consider such arbitrary linear maps. To prevent ambiguity, we reserve the term *morphism* for degree-preserving linear maps (i.e. the morphisms in $\mathrm{sVect}_{\mathbb{K}}$), and use the term *linear map* or *map* for arbitrary linear transformations in $\mathrm{Vect}_{\mathbb{K}}$.

Since $\mathrm{sVect}_{\mathbb{K}}$ is monoidal, one can define *superalgebras*, which are monoid objects of $\mathrm{sVect}_{\mathbb{K}}$. Moreover, since $\mathrm{sVect}_{\mathbb{K}}$ is braided, one can define the tensor product of superalgebras A, B with the multiplication given by

$$\mu_{A\otimes B} \colon (A\otimes B)\otimes(A\otimes B) \xrightarrow{\mathrm{Id}\otimes\tau_{BA}\otimes\mathrm{Id}} (A\otimes A)\otimes(B\otimes B) \xrightarrow{\mu_A\otimes\mu_B} A\otimes B\,, \tag{2.4}$$

in which μ_A, μ_B denote the multiplication of A, B respectively. Concretely, for any homogeneous $a, a' \in A$ and $b, b' \in B$, the multiplication can be written as follows:

$$(a\otimes b)(a'\otimes b') = (-1)^{|b||a'|}(aa')\otimes(bb')\,.$$

We note that if both A and B are superalgebras, then $\tau\colon A\otimes B \to B\otimes A$ is a superalgebra homomorphism.

Finally, the symmetric braiding allows one to define Lie algebra objects in $\mathrm{sVect}_{\mathbb{K}}$, called *Lie superalgebras*. Explicitly, a Lie superalgebra is a superspace $\mathfrak{g}$ together with a morphism (hence degree-preserving)

$$[-,-]\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}\,,$$

called a *Lie superbracket*, which will be frequently identified with the corresponding bilinear map $\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, satisfying the graded skew-symmetry:

$$[x,y] = -[\tau(x\otimes y)] = -(-1)^{|x||y|}[y,x] \qquad \text{for any homogeneous } x,y\in\mathfrak{g}\,,$$

and the graded Jacobi identity (equivalent to saying that the superbracket gives an adjoint action of $\mathfrak{g}$):

$$[x,[y,z]] = [[x,y],z] + (-1)^{|x||y|}[y,[x,z]] \qquad \text{for any homogeneous } x,y,z\in\mathfrak{g}\,.$$

We note that any superalgebra A admits a canonical Lie superalgebra structure given by the *supercommutator*:

$$[a,a'] \coloneqq aa' - (-1)^{|a||a'|}a'a \qquad \text{for any homogeneous } a,a'\in A\,. \tag{2.5}$$

This construction provides a forgetful functor from the category of superalgebras to the category of Lie superalgebras, which admits a left adjoint known as the *universal enveloping superalgebra* functor. To be specific, for any Lie superalgebra $\mathfrak{g}$, its universal enveloping superalgebra $U(\mathfrak{g})$ is defined as the quotient of the tensor superalgebra $T(\mathfrak{g}) \coloneqq \bigoplus_{k\geq 0}\mathfrak{g}^{\otimes k}$ by the two-sided ideal generated by $\{x\otimes y - \tau(x\otimes y) - [x,y] \mid x,y\in\mathfrak{g}\}$.

For a superspace V together with a choice of an ordered homogeneous basis $\{v_1, v_2, \dots, v_N\}$, we define the *parity* of an index $i\in\mathbb{I}\coloneqq\{1,2,\dots,N\}$ to be the $\mathbb{Z}_2$-degree of the corresponding basis vector v_i:

$$\bar{i} \coloneqq |v_i| = \begin{cases} \bar{\mathbf{0}} & \text{if } v_i \text{ is even}\,,\\ \bar{\mathbf{1}} & \text{if } v_i \text{ is odd}\,.\end{cases}$$

The grading of this ordered basis is recorded in the *parity sequence* γ_V, defined as the N-tuple:

$$\gamma_V \coloneqq (|v_1|,\dots,|v_N|) = (\bar{1},\dots,\overline{N}) \in \{\bar{\mathbf{0}},\bar{\mathbf{1}}\}^N\,.$$

2.2. General linear Lie superalgebras.

For any pair of objects $V, W\in\mathrm{sVect}_{\mathbb{K}}$, there exists an *internal hom* object, whose underlying vector space consists of all $\mathbb{K}$-linear maps $\mathrm{Hom}_{\mathbb{K}}(V,W) = \bigoplus_{\bar{\mathbf{i}},\bar{\mathbf{j}}\in\mathbb{Z}_2}\mathrm{Hom}_{\mathbb{K}}(V_{\bar{\mathbf{i}}}, W_{\bar{\mathbf{j}}})$, equipped with the $\mathbb{Z}_2$-grading:

$$\mathrm{Hom}_{\mathbb{K}}(V,W)_{\bar{\mathbf{k}}} = \bigoplus_{\bar{\mathbf{j}}-\bar{\mathbf{i}}=\bar{\mathbf{k}}} \mathrm{Hom}_{\mathbb{K}}(V_{\bar{\mathbf{i}}}, W_{\bar{\mathbf{j}}})\,.$$

In other words, $\mathrm{sVect}_{\mathbb{K}}$ is a closed monoidal category.

Thus, for a fixed superspace V, the set of all $\mathbb{K}$-linear maps $\mathrm{End}_{\mathbb{K}}(V)\coloneqq\mathrm{Hom}_{\mathbb{K}}(V,V)$ forms a superalgebra, where the multiplication is given by the composition of linear maps. Moreover, by (2.5), this superalgebra admits a canonical Lie superalgebra structure, called the *general linear Lie superalgebra*. We denote this Lie superalgebra by $\mathfrak{gl}(V)$ to distinguish the Lie structure from the associative structure of $\mathrm{End}_{\mathbb{K}}(V)$.

We also introduce a closely related object, the *special linear Lie superalgebra*. To this end, for a finite-dimensional superspace V, recall the definition of the *supertrace* morphism

$$\mathrm{sTr}\colon \mathfrak{gl}(V) \simeq V \otimes V^* \xrightarrow{\tau} V^* \otimes V \xrightarrow{\mathrm{ev}} \mathbb{C},$$

where we use the canonical identification $\mathfrak{gl}(V) \simeq V \otimes V^*$ and ev is the (left) evaluation morphism. By construction, one can verify that the supertrace satisfies

$$\mathrm{sTr}(XY) = (-1)^{|X||Y|}\mathrm{sTr}(YX) \qquad \text{for any homogeneous } X, Y \in \mathrm{End}_{\mathbb{K}}(V),$$

which immediately implies $\mathrm{sTr}([X,Y]) = 0$ for all $X, Y \in \mathfrak{gl}(V)$. Therefore, the kernel of the supertrace morphism forms an ideal of $\mathfrak{gl}(V)$, called the *special linear Lie superalgebra* and denoted by $\mathfrak{sl}(V)$.

2.3. **Chevalley–Serre-type presentation.**

Henceforth, we fix a superspace V of dimension (m,n) over the base field $\mathbb{K} = \mathbb{C}$. We also make a choice of an ordered homogeneous basis $\{v_1, v_2, \ldots, v_N\}$ of V, where $N := m + n$, which allows the identification of $\mathfrak{gl}(V)$ with the space of all $N \times N$ matrices. Let E_{ij} be the elementary matrix with 1 at the (i,j)-entry and 0 elsewhere. The set $\{E_{ij}\}_{i,j=1}^N$ forms a homogeneous basis of $\mathfrak{gl}(V)$ with $\mathbb{Z}_2$-grading $|E_{ij}| = \bar{i} + \bar{j}$ and the Lie superbracket given by:

$$[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})}\delta_{li}E_{kj}. \tag{2.6}$$

We also note that $\mathrm{sTr}(E_{ij}) = \delta_{ij}(-1)^{\bar{i}}$, and consequently, the following forms a homogeneous basis of $\mathfrak{sl}(V)$:

$$\big\{E_{ij} \,\big|\, 1 \le i \ne j \le N\big\} \cup \big\{(-1)^{\bar{i}}E_{ii} - (-1)^{\overline{i+1}}E_{i+1,i+1} \,\big|\, 1 \le i < N\big\}.$$

We further choose the Borel subalgebra of $\mathfrak{gl}(V)$ consisting of all upper triangular matrices, giving rise to the Borel subalgebra of $\mathfrak{sl}(V)$. The corresponding Cartan subalgebra $\mathfrak{h}_{\mathfrak{gl}}$ of $\mathfrak{gl}(V)$ consists of all diagonal matrices, with a basis $\{E_{ii}\}_{i=1}^N$, while the Cartan subalgebra $\mathfrak{h}_{\mathfrak{sl}}$ of $\mathfrak{sl}(V)$ is a codimension 1 subspace in $\mathfrak{h}_{\mathfrak{gl}}$:

$$\mathfrak{h}_{\mathfrak{sl}} = \big\{X \in \mathfrak{h}_{\mathfrak{gl}} \,\big|\, \mathrm{sTr}(X) = 0\big\}.$$

We define the linear functionals $\{\varepsilon_i\}_{i=1}^N$ in $\mathfrak{h}_{\mathfrak{gl}}^*$ to be the dual basis to $\{E_{ii}\}_{i=1}^N$, i.e. $\varepsilon_i(E_{jj}) = \delta_{ij}$ for all i, j.

A direct computation shows that $[E_{ii}, E_{ab}] = (\varepsilon_a - \varepsilon_b)(E_{ii}) \cdot E_{ab}$, so that E_{ab} is a root vector corresponding to the root $\varepsilon_a - \varepsilon_b$ (for $a \ne b$). Hence, we obtain the *root space decomposition* $\mathfrak{gl}(V) = \mathfrak{h}_{\mathfrak{gl}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{gl}(V)_\alpha$ with the root system given by

$$\Phi = \big\{\varepsilon_a - \varepsilon_b \,\big|\, 1 \le a, b \le N,\ a \ne b\big\}, \tag{2.7}$$

and the corresponding polarization:

$$\Phi^+ = \big\{\varepsilon_a - \varepsilon_b \,\big|\, a < b\big\}, \qquad \Phi^- = \big\{\varepsilon_a - \varepsilon_b \,\big|\, a > b\big\}. \tag{2.8}$$

The root system has a decomposition $\Phi = \Phi_{\bar{0}} \cup \Phi_{\bar{1}}$ into *even* and *odd* parts, where

$$\Phi_{\bar{0}} = \big\{\varepsilon_a - \varepsilon_b \in \Phi \,\big|\, \bar{a} = \bar{b}\big\}, \qquad \Phi_{\bar{1}} = \big\{\varepsilon_a - \varepsilon_b \in \Phi \,\big|\, \bar{a} \ne \bar{b}\big\}.$$

We note that $\mathfrak{sl}(V)$ shares the same root system Φ and admits the corresponding root space decomposition

$$\mathfrak{sl}(V) = \mathfrak{h}_{\mathfrak{sl}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{sl}(V)_\alpha \qquad \text{with} \quad \mathfrak{sl}(V)_\alpha = \mathfrak{gl}(V)_\alpha \quad \forall\, \alpha \in \Phi.$$

Following the polarization (2.8) of the root system (2.7), the simple roots and the corresponding root vectors can be written as follows (with $1 \le i < N$):

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \qquad \mathsf{e}_i = E_{i,i+1}, \qquad \mathsf{f}_i = (-1)^{\bar{i}}E_{i+1,i}. \tag{2.9}$$

We define the *Weyl vector* ρ of Φ by

$$\rho := \frac{1}{2}\sum_{\alpha\in\Phi_{\bar{0}}^+} \alpha - \frac{1}{2}\sum_{\alpha\in\Phi_{\bar{1}}^+} \alpha. \tag{2.10}$$

The following property of the Weyl vector will be used extensively in the sequel (cf. [3, Proposition 1.33]):

$$(\rho, \alpha_i) = \tfrac{1}{2}(\alpha_i, \alpha_i). \tag{2.11}$$

We also define the following elements of the Cartan subalgebra $\mathfrak{h}_{\mathfrak{gl}}$:

$$\mathsf{H}_k = (-1)^{\bar{k}}E_{kk} \quad \text{for } 1 \le k \le N, \qquad \mathsf{h}_i = \mathsf{H}_i - \mathsf{H}_{i+1} \quad \text{for } 1 \le i < N. \tag{2.12}$$

Note that the sets $\{\mathsf{H}_k\}_{k=1}^N$ and $\{\mathsf{h}_i\}_{i=1}^{N-1}$ form bases for $\mathfrak{h}_{\mathfrak{gl}}$ and $\mathfrak{h}_{\mathfrak{sl}}$, respectively. Furthermore, the lattices $P^\vee := \bigoplus_{k=1}^N \mathbb{Z}\mathsf{H}_k$ and $Q^\vee := \bigoplus_{i=1}^{N-1} \mathbb{Z}\mathsf{h}_i$ constitute the coweight and coroot lattices of $\mathfrak{gl}(V)$, respectively.

Dually, we define the weight lattice $P := \bigoplus_{k=1}^N \mathbb{Z}\varepsilon_k$, the root lattice $Q := \bigoplus_{i=1}^{N-1} \mathbb{Z}\alpha_i \subseteq P$, and $Q^+ := \bigoplus_{i=1}^{N-1} \mathbb{Z}_{\geq 0}\alpha_i$, together with the following partial order on P: $\mu \leq \nu \iff \nu - \mu \in Q^+$. Then (2.6) implies that $\mathfrak{gl}(V)$ is graded by Q (and thus also by P) via:

$$\deg(E_{ij}) = \varepsilon_i - \varepsilon_j \,,$$

which restricts to a Q-grading on $\mathfrak{sl}(V)$, and is compatible with the $\mathbb{Z}_2$-grading via the group homomorphism

$$P \to \mathbb{Z}_2 \,, \qquad \varepsilon_i \mapsto \bar{i} \,. \tag{2.13}$$

In particular, we have

$$\deg(\mathsf{e}_i) = \alpha_i \,, \qquad \deg(\mathsf{f}_i) = -\alpha_i \,, \qquad \deg(\mathsf{h}_i) = 0 \,, \qquad \deg(\mathsf{H}_k) = 0 \,. \tag{2.14}$$

We consider the non-degenerate invariant *supertrace* bilinear form $(\cdot,\cdot)\colon \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to \mathbb{C}$ defined by

$$(X,Y) = \mathrm{sTr}(XY) \,, \qquad X,Y \in \mathfrak{gl}(V) \,.$$

Its restriction to the Cartan subalgebra $\mathfrak{h}_{\mathfrak{gl}}$ is non-degenerate, thus giving rise to an identification $\mathfrak{h}^*_{\mathfrak{gl}} \simeq \mathfrak{h}_{\mathfrak{gl}}$ via $\varepsilon_k \leftrightarrow (-1)^{\overline{k}} E_{kk}$ and inducing a bilinear form $(\cdot,\cdot)\colon \mathfrak{h}^*_{\mathfrak{gl}} \times \mathfrak{h}^*_{\mathfrak{gl}} \to \mathbb{C}$ such that

$$(\varepsilon_k, \varepsilon_l) = \delta_{kl}(-1)^{\overline{k}} \qquad \text{for any} \qquad 1 \leq k,l \leq N \,. \tag{2.15}$$

We also recall that an odd root $\alpha \in \Phi_{\bar{1}}$ is called *isotropic* if $(\alpha,\alpha) = 0$.

Define the *symmetrized Cartan matrix* $(a_{ij})_{i,j=1}^{N-1}$ of $\mathfrak{gl}(V)$ via $a_{ij} = (\alpha_i, \alpha_j)$. It is straightforward to verify that the elements $\{\mathsf{e}_i, \mathsf{f}_i\}_{i=1}^{N-1} \cup \{\mathsf{H}_k\}_{k=1}^N$ satisfy the following relations:

$$[\mathsf{H}_k, \mathsf{H}_l] = 0 \,, \quad [\mathsf{H}_k, \mathsf{e}_j] = (\varepsilon_k, \alpha_j)\mathsf{e}_j \,, \quad [\mathsf{H}_k, \mathsf{f}_j] = -(\varepsilon_k, \alpha_j)\mathsf{f}_j \,, \quad [\mathsf{e}_i, \mathsf{f}_j] = \delta_{ij}\mathsf{h}_i \,, \tag{2.16}$$

which imply the following Chevalley-type relations:

$$[\mathsf{h}_i, \mathsf{h}_j] = 0 \,, \qquad [\mathsf{h}_i, \mathsf{e}_j] = a_{ij}\mathsf{e}_j \,, \qquad [\mathsf{h}_i, \mathsf{f}_j] = -a_{ij}\mathsf{f}_j \,. \tag{2.17}$$

Furthermore, it is shown in [25] that $\mathfrak{sl}(V)$ is isomorphic to the Lie superalgebra generated by $\{\mathsf{e}_i, \mathsf{f}_i, \mathsf{h}_i\}_{i=1}^{N-1}$, subject to the $\mathbb{Z}_2$-grading induced from the Q-grading (2.14), the Chevalley-type relations (2.17), together with the *standard* and *higher order Serre relations*. As we shall not need the explicit form of the Serre relations, we refer the interested reader to [25, §3.2.1] for the exact formulas. Similarly, $\mathfrak{gl}(V)$ admits a generator-relation presentation: it is isomorphic to the Lie superalgebra generated by $\{\mathsf{e}_i, \mathsf{f}_i\}_{i=1}^{N-1} \cup \{\mathsf{H}_k\}_{k=1}^N$, subject to the same grading and Serre relations (which depend only on $\mathsf{e}_i, \mathsf{f}_i$), but satisfying the Chevalley-type relations (2.16).

Remark 2.18. Instead of using Cartan generators h_i or H_k, one can use an additive subgroup $\Gamma \subseteq \mathfrak{h}_{\mathfrak{gl}}$ containing $\{\mathsf{h}_i\}_{i=1}^{N-1}$ to provide a uniform Chevalley-Serre type presentation for both $\mathfrak{gl}(V)$ and $\mathfrak{sl}(V)$. Consider the Lie superalgebra $\mathfrak{g}(\Gamma)$ generated by $\{\mathsf{e}_i, \mathsf{f}_i\}_{i=1}^{N-1}$ and Γ, subject to the Serre relations together with the following Chevalley-type relations:

$$[\mathsf{H}, \mathsf{H}'] = 0 \,, \quad [\mathsf{H}, \mathsf{e}_j] = \alpha_j(\mathsf{H})\mathsf{e}_j \,, \quad [\mathsf{H}, \mathsf{f}_j] = -\alpha_j(\mathsf{H})\mathsf{f}_j \,, \quad [\mathsf{e}_i, \mathsf{f}_j] = \delta_{ij}\mathsf{h}_i \quad \forall\, \mathsf{H}, \mathsf{H}' \in \Gamma \,, 1 \leq i,j < N \,.$$

It is clear that $\mathfrak{g}(P^\vee) = \mathfrak{gl}(V)$ and $\mathfrak{g}(Q^\vee) = \mathfrak{sl}(V)$. While various choices of the lattice Γ yield only these two distinct Lie superalgebras, the Γ-dependence will be more significant for quantum supergroups below.

2.4. **Drinfeld–Jimbo-type general linear quantum supergroups.**

The *Drinfeld–Jimbo-type general linear quantum supergroup* $U_q(\mathfrak{gl}(V))$ is a $\mathbb{C}(q)$-superalgebra which is a natural quantization of the universal enveloping $U(\mathfrak{gl}(V))$. To construct it, we first define a more general class of quantum supergroups $U_q(\Gamma)$, parameterized by an additive subgroup $\Gamma \subseteq P^\vee$ containing $Q^\vee$ (cf. Remark 2.18).

Remark 2.19. To avoid confusion, in the quantum supergroup setting, the elements H_k and $\mathsf{h}_i \in \mathfrak{h}_{\mathfrak{gl}}$ will be denoted by H_k and h_i respectively to distinguish them from the classical Lie superalgebra setting. Moreover, following the physics literature, we use formal exponentials $q^{\pm H_k}$ and $q^{\pm h_i}$ instead of the more common $K_k^{\pm 1}$ and $k_i^{\pm 1}$.

Definition 2.20. For a fixed additive subgroup $\Gamma \subseteq P^\vee$ containing $Q^\vee$, the quantum supergroup $U_q(\Gamma)$ is the $\mathbb{C}(q)$-superalgebra generated by $\{e_i, f_i\}_{i=1}^{N-1}$ and the formal symbols $\{q^{\pm H} \,|\, H \in \Gamma\}$. It is equipped with the Q-grading (cf. (2.14)):

$$\deg(e_i) = \alpha_i \,, \qquad \deg(f_i) = -\alpha_i \,, \qquad \deg(q^{\pm H}) = 0 \,, \tag{2.21}$$

and the compatible $\mathbb{Z}_2$-grading, see (2.13). These generators are subject to the Serre relations (2.26)–(2.28) introduced later in this subsection, together with the following Chevalley-type relations:

$$q^H q^{H'} = q^{H+H'}\,, \qquad [q^H, q^{H'}] = 0\,, \qquad q^0 = 1\,, \tag{2.22}$$

$$q^H e_j q^{-H} = q^{\alpha_j(H)} e_j\,, \quad q^H f_j q^{-H} = q^{-\alpha_j(H)} f_j\,, \tag{2.23}$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}\,, \tag{2.24}$$

for all $H, H' \in \Gamma$ and $1 \le i, j < N$, where $[-,-]$ denotes the supercommutator (2.5), and $\alpha_j(H) \in \mathbb{Z}$ represents the evaluation of the root α_j on the Cartan element $H \in P^\vee$.

To express the Serre relations, we introduce the *q-supercommutator* $[\![-,-]\!]$ via

$$[\![a, b]\!] := ab - (-1)^{|a||b|} q^{(\deg(a), \deg(b))} \cdot ba \tag{2.25}$$

for any Q-homogeneous elements $a, b \in U_q(\Gamma)$. The Serre relations, which complement the Chevalley-type relations above, are given as follows (cf. [22, Proposition 10.4.1]):

$$[\![e_i, e_j]\!] = 0\,, \quad [\![f_i, f_j]\!] = 0 \qquad \text{if} \;\; a_{ij} = 0\,, \tag{2.26}$$

$$[\![e_i, [\![e_i, e_j]\!]]\!] = 0\,, \quad [\![f_i, [\![f_i, f_j]\!]]\!] = 0 \qquad \text{if} \;\; |i-j| = 1 \;\; \text{and} \;\; \alpha_i \in \Phi_{\bar{0}}\,, \tag{2.27}$$

$$[\![[\![[\![e_{i-1}, e_i]\!], e_{i+1}]\!], e_i]\!] = 0\,, \quad [\![[\![[\![f_{i-1}, f_i]\!], f_{i+1}]\!], f_i]\!] = 0 \qquad \text{if} \;\; 2 \le i \le N-2 \;\; \text{and} \;\; \alpha_i \in \Phi_{\bar{1}}\,. \tag{2.28}$$

Remark 2.29. In particular, if $\alpha_i \in \Phi_{\bar{1}}$, then (2.26) immediately implies $e_i^2 = f_i^2 = 0$. We also note that the relation (2.27) can be alternatively written as $[\![[\![e_{i-1}, e_i]\!], e_i]\!] = 0$ and $[\![[\![f_{i-1}, f_i]\!], f_i]\!] = 0$.

The quantum supergroup $U_q(\Gamma)$ is naturally endowed with a Hopf superalgebra structure, defined on the generators by the following comultiplication:

$$\Delta(e_i) = q^{h_i} \otimes e_i + e_i \otimes 1\,, \qquad \Delta(f_i) = 1 \otimes f_i + f_i \otimes q^{-h_i}\,, \qquad \Delta(q^H) = q^H \otimes q^H\,, \tag{2.30}$$

and with the counit and antipode given explicitly by

$$\begin{aligned} &\epsilon(e_i) = 0\,, \quad \epsilon(f_i) = 0\,, \quad \epsilon(q^H) = 1\,, \\ &S(e_i) = -q^{-h_i} e_i\,, \quad S(f_i) = -f_i q^{h_i}\,, \quad S(q^H) = q^{-H}\,. \end{aligned} \tag{2.31}$$

The assignment (2.21) indeed gives rise to a Q-grading on $U_q(\Gamma)$, since all the defining relations are homogeneous. Furthermore, the explicit formulas (2.30, 2.31) show that $U_q(\Gamma)$ is actually a Q-graded Hopf superalgebra.

Definition 2.32. (a) The quantum supergroup $U_q(P^\vee)$ is called the *Drinfeld–Jimbo-type general linear quantum supergroup* and is denoted by $U_q(\mathfrak{gl}(V))$. Equivalently, $U_q(\mathfrak{gl}(V))$ admits a generator-relation presentation with only finitely many Cartan generators: it is the $\mathbb{C}(q)$-superalgebra generated by $\{e_i, f_i\}_{i=1}^{N-1}$ and $\{q^{\pm H_k}\}_{k=1}^N$, equipped with the same Q-grading and $\mathbb{Z}_2$-grading as in (2.21). These generators are subject to the Serre relations (2.26)–(2.28), alongside the following explicit Chevalley-type relations:

$$[q^{H_k}, q^{H_l}] = 0\,, \qquad q^{\pm H_k} q^{\mp H_k} = 1\,, \tag{2.33}$$

$$q^{H_k} e_j q^{-H_k} = q^{(\varepsilon_k, \alpha_j)} e_j\,, \quad q^{H_k} f_j q^{-H_k} = q^{-(\varepsilon_k, \alpha_j)} f_j\,, \tag{2.34}$$

$$[e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}\,, \tag{2.35}$$

for all $1 \le k, l \le N$ and $1 \le i, j < N$, where $q^{h_i} := q^{H_i} q^{-H_{i+1}}$, and the pairing in (2.34) is given by (2.15).
(b) Analogously, the *Drinfeld–Jimbo-type special linear quantum supergroup* is defined as $U_q(\mathfrak{sl}(V)) := U_q(Q^\vee)$. Equivalently, it admits a finite generator-relation presentation as the $\mathbb{C}(q)$-superalgebra generated by $\{e_i, f_i\}_{i=1}^{N-1}$ and the Cartan generators $\{q^{\pm h_i}\}_{i=1}^{N-1}$, subject to the same Q-grading and $\mathbb{Z}_2$-grading as in (2.21). These generators are subject to the Serre relations (2.26)–(2.28) and the following explicit Chevalley-type relations:

$$[q^{h_i}, q^{h_j}] = 0\,, \quad q^{\pm h_i} q^{\mp h_i} = 1\,, \quad q^{h_i} e_j q^{-h_i} = q^{(\alpha_i, \alpha_j)} e_j\,, \quad q^{h_i} f_j q^{-h_i} = q^{-(\alpha_i, \alpha_j)} f_j\,, \quad [e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}\,,$$

for all $1 \le i, j < N$. By definition, there is a canonical Hopf superalgebra morphism from $U_q(\mathfrak{sl}(V))$ to $U_q(\mathfrak{gl}(V))$. This morphism can be shown to be an embedding via the triangular decomposition (2.38) discussed below.

Remark 2.36. We note that a slightly different definition of the q-supercommutator is used in [22]:

$$[\![a,b]\!]_{\mathrm{Y}} := ab-(-1)^{|a||b|}q^{-(\deg(a),\deg(b))}\cdot ba = -(-1)^{|a||b|}q^{-(\deg(a),\deg(b))}\cdot [\![b,a]\!]\,.$$

Nevertheless, the superalgebra $U_q(\mathfrak{gl}(V))_{\mathrm{Y}}$ defined via $[\![-,-]\!]_{\mathrm{Y}}$ in [22] is isomorphic to $U_{q^{-1}}(\mathfrak{gl}(V))$ as $\mathbb{C}(q)$-superalgebras via the assignment $e_i\mapsto e_i$, $f_i\mapsto f_i$, $q^{\pm H_k}\mapsto q^{\mp H_k}$.

Remark 2.37. We note that the assignment (see (2.10))

$$\omega_{\mathrm{DJ}}(q^{\pm H_k})=q^{\mp H_k}\,,\quad \omega_{\mathrm{DJ}}(e_i)=-(-1)^{\overline{i}\,\overline{i+1}}q^{3(\rho,\alpha_i)}f_i\,,\quad \omega_{\mathrm{DJ}}(f_i)=-(-1)^{\overline{i}\,\overline{i+1}}(-1)^{\overline{i}+\overline{i+1}}q^{-3(\rho,\alpha_i)}e_i$$

for $1\le k\le N$, $1\le i<N$ gives rise to a Hopf superalgebra isomorphism $\omega_{\mathrm{DJ}}\colon U_q(\mathfrak{gl}(V))\xrightarrow{\sim}U_q(\mathfrak{gl}(V))^{\mathrm{cop}}$.

2.5. Drinfeld double construction of $U_q(\mathfrak{gl}(V))$.

We first introduce the *triangular decomposition* of $U_q(\mathfrak{gl}(V))$. Let $U_q^{>}(\mathfrak{gl}(V))$, $U_q^{<}(\mathfrak{gl}(V))$, and $U_q^{0}(\mathfrak{gl}(V))$ denote the subalgebras of $U_q(\mathfrak{gl}(V))$ generated by $\{e_i\}_{i=1}^{N-1}$, $\{f_i\}_{i=1}^{N-1}$, and $\{q^{\pm H_k}\}_{k=1}^{N}$, respectively. The triangular decomposition states that the multiplication morphism

$$U_q^{<}(\mathfrak{gl}(V))\otimes U_q^{0}(\mathfrak{gl}(V))\otimes U_q^{>}(\mathfrak{gl}(V))\xrightarrow{\sim}U_q(\mathfrak{gl}(V)) \tag{2.38}$$

is an isomorphism of underlying superspaces. We further define the *positive* and *negative Borel subalgebras*, denoted $U_q^{\geq}(\mathfrak{gl}(V))$ and $U_q^{\leq}(\mathfrak{gl}(V))$, as the subalgebras generated by $\{e_i\}_{i=1}^{N-1}\cup\{q^{\pm H_k}\}_{k=1}^{N}$ and $\{f_i\}_{i=1}^{N-1}\cup\{q^{\pm H_k}\}_{k=1}^{N}$, respectively. Note that these are not just subalgebras but are also Hopf subalgebras of $U_q(\mathfrak{gl}(V))$.

Remark 2.39. The triangular decomposition (2.38) implies that the multiplication morphisms

$$U_q^{0}(\mathfrak{gl}(V))\otimes U_q^{>}(\mathfrak{gl}(V))\xrightarrow{\sim}U_q^{\geq}(\mathfrak{gl}(V))\,,\qquad U_q^{<}(\mathfrak{gl}(V))\otimes U_q^{0}(\mathfrak{gl}(V))\xrightarrow{\sim}U_q^{\leq}(\mathfrak{gl}(V))$$

are isomorphisms of underlying superspaces. Invoking the automorphism ω_{DJ} of Remark 2.37, the following multiplication morphisms are also isomorphisms:

$$U_q^{>}(\mathfrak{gl}(V))\otimes U_q^{0}(\mathfrak{gl}(V))\xrightarrow{\sim}U_q^{\geq}(\mathfrak{gl}(V))\,,\qquad U_q^{0}(\mathfrak{gl}(V))\otimes U_q^{<}(\mathfrak{gl}(V))\xrightarrow{\sim}U_q^{\leq}(\mathfrak{gl}(V))\,.$$

A fundamental structural property of quantum (super)groups is their realization via the Drinfeld double construction (cf. Subsection A.4). Recall that the *opposite comultiplication* is defined by $\Delta^{\mathrm{op}}(x):=\tau(\Delta(x))$.

Proposition 2.40. *(a) There exists a unique skew-pairing (cf. (A.21))*

$$(\cdot,\cdot)_{\mathrm{DJ}}\colon U_q^{\leq}(\mathfrak{gl}(V))\times U_q^{\geq}(\mathfrak{gl}(V))\to\mathbb{C}(q) \tag{2.41}$$

satisfying the following properties on the generators for $1\le i,j<N$ and $1\le k,l\le N$:

$$(f_i,q^{\pm H_k})_{\mathrm{DJ}}=0\,,\quad (q^{\pm H_k},e_i)_{\mathrm{DJ}}=0\,,\quad (f_i,e_j)_{\mathrm{DJ}}=\frac{\delta_{ij}(-1)^{|f_i||e_j|}}{q^{-1}-q}\,,\quad (q^{H_k},q^{H_l})_{\mathrm{DJ}}=q^{-(\varepsilon_k,\varepsilon_l)}\,. \tag{2.42}$$

Moreover, this pairing is non-degenerate.
(b) $U_q(\mathfrak{gl}(V))$ is isomorphic to the quotient of the Drinfeld double $\mathcal{D}_{\mathrm{DJ}}:=\mathcal{D}(U_q^{\leq}(\mathfrak{gl}(V)),U_q^{\geq}(\mathfrak{gl}(V)))$ obtained by identifying the Cartan subalgebras $U_q^{0}(\mathfrak{gl}(V))$ from both factors.

Proof. For part (a), the existence of such a pairing is established in [22] (cf. [10, Propositions 6.12, 6.18] for non-super case). This pairing is of Q-degree zero, i.e. for any Q-homogeneous $x\in U_q^{\geq}(\mathfrak{gl}(V))$ and $y\in U_q^{\leq}(\mathfrak{gl}(V))$:

$$\deg(x)+\deg(y)\neq 0\implies (y,x)_{\mathrm{DJ}}=0\,. \tag{2.43}$$

Define the following subspaces, spanned by elements of strictly positive and negative degrees, respectively:

$$\begin{aligned} U_q^{>}(\mathfrak{gl}(V))_{\deg>0} &:= \{x\in U_q^{>}(\mathfrak{gl}(V))\mid \epsilon(x)=0\}\,,\\ U_q^{<}(\mathfrak{gl}(V))_{\deg<0} &:= \{y\in U_q^{<}(\mathfrak{gl}(V))\mid \epsilon(y)=0\}\,. \end{aligned} \tag{2.44}$$

For any Q-homogeneous elements $e\in U_q^{>}(\mathfrak{gl}(V))$ and $f\in U_q^{<}(\mathfrak{gl}(V))$, the structure of (2.30) implies that

$$\Delta(e)\in e\otimes 1+U_q^{\geq}(\mathfrak{gl}(V))\otimes U_q^{>}(\mathfrak{gl}(V))_{\deg>0}\,,\qquad \Delta(f)\in 1\otimes f+U_q^{<}(\mathfrak{gl}(V))_{\deg<0}\otimes U_q^{\leq}(\mathfrak{gl}(V))\,. \tag{2.45}$$

Consequently, for any Q-homogeneous $e\in U_q^{>}(\mathfrak{gl}(V))$, $f\in U_q^{<}(\mathfrak{gl}(V))$ and group-like $k,k'\in U_q^{0}(\mathfrak{gl}(V))$:

$$\begin{aligned} (fk',ek)_{\mathrm{DJ}} &= (f\otimes k',\Delta(e)\Delta(k))_{\mathrm{DJ}} \overset{(2.43,\ 2.45)}{=} (f,ek)_{\mathrm{DJ}}(k',k)_{\mathrm{DJ}}\\ &= (\Delta^{\mathrm{op}}(f),e\otimes k)_{\mathrm{DJ}}(k',k)_{\mathrm{DJ}} \overset{(2.43,\ 2.45)}{=} (f,e)_{\mathrm{DJ}}(1,k)_{\mathrm{DJ}}(k',k)_{\mathrm{DJ}}=(f,e)_{\mathrm{DJ}}(k',k)_{\mathrm{DJ}}\,. \end{aligned} \tag{2.46}$$

As $U_q^{0}(\mathfrak{gl}(V))$ is spanned by its group-like elements, bilinearity ensures that (2.46) holds for all $k,k'\in U_q^{0}(\mathfrak{gl}(V))$. Thus, the non-degeneracy of (2.41) follows from that of its restrictions to $U_q^{<}(\mathfrak{gl}(V))\times U_q^{>}(\mathfrak{gl}(V))$ and $U_q^{0}(\mathfrak{gl}(V))\times U_q^{0}(\mathfrak{gl}(V))$. The former is a non-trivial result established in [22], while the latter is straightforward.

Remark 2.47. We note that the q-Serre relations were introduced in [22] precisely to ensure the non-degeneracy of the skew-pairing (2.41).

Part (b) follows from the fact that the Drinfeld double $\mathcal{D}_{\mathrm{DJ}}$ is the quotient of the free product (super)algebra $U_q^{\leq}(\mathfrak{gl}(V)) * U_q^{\geq}(\mathfrak{gl}(V))$, subject to the following cross-relations (cf. Proposition A.24):

$$\sum_{(x)(y)} (-1)^{|x_2||y|}(-1)^{|y_1||x_1|} \cdot (y_1, x_1)_{\mathrm{DJ}} \cdot y_2 x_2 = \sum_{(x)(y)} (-1)^{|x_2||y|} \cdot x_1 y_1 \cdot (y_2, x_2)_{\mathrm{DJ}}\,, \tag{2.48}$$

for all $\mathbb{Z}_2$-homogeneous $x \in U_q^{\geq}(\mathfrak{gl}(V))$ and $y \in U_q^{\leq}(\mathfrak{gl}(V))$, using Sweedler notation. To avoid confusion, we denote the Cartan generators of $U_q^{\geq}(\mathfrak{gl}(V))$ and $U_q^{\leq}(\mathfrak{gl}(V))$ by $\{q^{\pm H_k^e}\}_{k=1}^N$ and $\{q^{\pm H_k^f}\}_{k=1}^N$, respectively. Analogously, we use $\{q^{\pm h_i^e}\}_{i=1}^{N-1}$ and $\{q^{\pm h_i^f}\}_{i=1}^{N-1}$ to refer to the corresponding composite elements.

Following the notation in Proposition A.24, consider the elements $u_{y,x}$ for $\mathbb{Z}_2$-homogeneous $y \in U_q^{\leq}(\mathfrak{gl}(V))$ and $x \in U_q^{\geq}(\mathfrak{gl}(V))$, and let $\mathcal{I}$ be the two-sided ideal generated by them. Since $U_q^{\geq}(\mathfrak{gl}(V))$ and $U_q^{\leq}(\mathfrak{gl}(V))$ are graded by Q^+ and $-Q^+$ respectively, the explicit comultiplication formula (2.30) together with an induction on the degree via Proposition A.25 implies that $\mathcal{I}$ is generated by the elements $u_{y,x}$ where y and x range over the generators of $U_q^{\leq}(\mathfrak{gl}(V))$ and $U_q^{\geq}(\mathfrak{gl}(V))$, respectively. Applying (2.48) to pairs of generators yields:

$$(-1)^{|e_i||f_j|} \cdot (1, q^{h_i^e})_{\mathrm{DJ}} \cdot f_j e_i + (-1)^{|e_i||f_j|} \cdot (f_j, e_i)_{\mathrm{DJ}} \cdot q^{-h_j^f} = e_i f_j \cdot (q^{-h_j^f}, 1)_{\mathrm{DJ}} + (-1)^{|e_i||f_j|} \cdot q^{h_i^e} \cdot (f_j, e_i)_{\mathrm{DJ}}\,,$$

$$(1, q^{H_k^e})_{\mathrm{DJ}} \cdot f_j q^{H_k^e} = q^{H_k^e} f_j \cdot (q^{-h_j^f}, q^{H_k^e})_{\mathrm{DJ}}\,,$$
$$(q^{H_l^f}, q^{h_i^e})_{\mathrm{DJ}} \cdot q^{H_l^f} e_i = e_i q^{H_l^f} \cdot (q^{H_l^f}, 1)_{\mathrm{DJ}}\,,$$
$$(q^{H_l^f}, q^{H_k^e})_{\mathrm{DJ}} \cdot q^{H_l^f} q^{H_k^e} = q^{H_k^e} q^{H_l^f} \cdot (q^{H_l^f}, q^{H_k^e})_{\mathrm{DJ}}\,,$$

which simplifies as follows:

$$e_i f_j - (-1)^{|e_i||f_j|} f_j e_i = \delta_{ij} \frac{q^{h_i^e} - q^{-h_i^f}}{q - q^{-1}}\,,$$

$$q^{H_k^e} f_j q^{-H_k^e} = q^{-(\alpha_j, \varepsilon_k)} f_j\,, \qquad q^{H_l^f} e_i q^{-H_k^f} = q^{(\varepsilon_l, \alpha_i)} e_i\,, \qquad q^{H_l^f} q^{H_k^e} = q^{H_k^e} q^{H_l^f}\,.$$

To identify the Cartan elements from both factors, we shall quotient the double $\mathcal{D}_{\mathrm{DJ}}$ by the two-sided ideal

$$\mathcal{J} := \left\langle q^{\pm H_k^e} - q^{\pm H_k^f} \mid 1 \leq k \leq N \right\rangle.$$

Passing to the quotient $\mathcal{D}_{\mathrm{DJ}}/\mathcal{J}$ precisely recovers the commutation relations (2.33)–(2.35) between the Borel subalgebras $U_q^{\leq}(\mathfrak{gl}(V))$ and $U_q^{\geq}(\mathfrak{gl}(V))$, which concludes the proof that $\mathcal{D}_{\mathrm{DJ}}/\mathcal{J} \simeq U_q(\mathfrak{gl}(V))$. $\square$

Remark 2.49. By Proposition A.28, one may alternatively use the transposed inverse

$$(-,-)_{\widetilde{\mathrm{DJ}}} \colon U_q^{\geq}(\mathfrak{gl}(V)) \times U_q^{\leq}(\mathfrak{gl}(V)) \to \mathbb{C}(q)$$

of $(-,-)_{\mathrm{DJ}}$ in the double construction, which amounts to taking the Borel subalgebras in the opposite order. The latter order is necessary to align $(-,-)_{\widetilde{\mathrm{DJ}}}$ with the corresponding skew-pairing (3.37) of the RLL-realization $U(R)$ defined below, cf. Remark 3.45. Specifically, the isomorphism ξ from Theorem 3.71 maps the subbialgebras $U_q^{\geq}(\mathfrak{gl}(V))$ and $U_q^{\leq}(\mathfrak{gl}(V))$ onto $U^{\geq}(R)$ and $U^{\leq}(R)$, respectively. Because both Borel subalgebras are Hopf superalgebras with invertible antipodes, (A.10) ensures that the transposed inverse pairing is also non-degenerate, and its evaluation on the generators is given as follows:

$$(e_i, q^{\pm H_k})_{\widetilde{\mathrm{DJ}}} = 0\,, \quad (q^{\pm H_k}, f_i)_{\widetilde{\mathrm{DJ}}} = 0\,, \quad (e_i, f_j)_{\widetilde{\mathrm{DJ}}} = \frac{\delta_{ij}}{q - q^{-1}}\,, \quad (q^{H_k}, q^{H_l})_{\widetilde{\mathrm{DJ}}} = q^{(\varepsilon_k, \varepsilon_l)}\,.$$

Remark 2.50. Instead of the Hopf subalgebras $U_q^{\leq}(\mathfrak{gl}(V))$ and $U_q^{\geq}(\mathfrak{gl}(V))$, the double construction can also be carried out using the subbialgebras $U_q^{-}(\mathfrak{gl}(V))$ and $U_q^{+}(\mathfrak{gl}(V))$ generated by $\{f_i\}_{i=1}^{N-1} \cup \{q^{-H_k}\}_{k=1}^N$ and $\{e_i\}_{i=1}^{N-1} \cup \{q^{H_k}\}_{k=1}^N$ respectively, equipped with the restriction of the exact same skew-pairing.

Remark 2.51. Evoking the Hopf superalgebra isomorphism ω_{DJ} of Remark 2.37, we note that it restricts to superalgebra isomorphisms between $U_q^{\leq}(\mathfrak{gl}(V))$ and $U_q^{\geq}(\mathfrak{gl}(V))$ and also between $U_q^{-}(\mathfrak{gl}(V))$ and $U_q^{+}(\mathfrak{gl}(V))$.

2.6. **Drinfeld twist of $U_q(\mathfrak{gl}(V))$.**

For the later use in Subsection 3.7, we shall now introduce a specific Drinfeld twist of $U_q(\mathfrak{gl}(V))$. Recall that a *Drinfeld twist* (or gauge transformation) of a Hopf superalgebra H is an invertible element $\mathcal{F} \in H \otimes H$ satisfying the 2-cocycle condition (cf. [21, (4.3)])

$$(\mathcal{F} \otimes 1)(\Delta \otimes \mathrm{Id})(\mathcal{F}) = (1 \otimes \mathcal{F})(\mathrm{Id} \otimes \Delta)(\mathcal{F}),$$

and the counitality condition $(\epsilon \otimes \mathrm{Id})(\mathcal{F}) = 1 = (\mathrm{Id} \otimes \epsilon)(\mathcal{F})$. Given such an element, one can construct a new Hopf superalgebra $H^{\mathcal{F}}$ whose underlying superalgebra structure and the counit morphism are identical to those of H, but with the comultiplication conjugated by $\mathcal{F}$:

$$\Delta^{\mathcal{F}}(x) = \mathcal{F}\Delta(x)\mathcal{F}^{-1} \qquad \text{for all} \ \ x \in H,$$

and its antipode $S^{\mathcal{F}}$ modified correspondingly.

We now apply the above construction to $H = U_q(\mathfrak{gl}(V))$ using the element $\mathcal{F} = \mathfrak{R}_s$, which will be defined in (3.7). Since $\mathfrak{R}_s$ is the canonical element of the Drinfeld double $\mathcal{D}(U_q^0(\mathfrak{gl}(V)), U_q^0(\mathfrak{gl}(V)))$ (cf. Remark 3.2), it naturally satisfies the conditions of a Drinfeld twist by Proposition A.31. A direct $\hbar$-adic computation (cf. Remark 3.3) yields the following twisted comultiplication $\Delta^{\mathcal{F}}$ on the generators:

$$\Delta^{\mathcal{F}}(e_i) = 1 \otimes e_i + e_i \otimes q^{-h_i}, \qquad \Delta^{\mathcal{F}}(f_i) = q^{h_i} \otimes f_i + f_i \otimes 1, \qquad \Delta^{\mathcal{F}}(q^{\pm H_k}) = q^{\pm H_k} \otimes q^{\pm H_k},$$

and the corresponding antipode is determined by $S^{\mathcal{F}}(e_i) = -e_i q^{h_i}$, $S^{\mathcal{F}}(f_i) = -q^{-h_i} f_i$, $S^{\mathcal{F}}(q^{\pm H_k}) = q^{\mp H_k}$.

In general, a Hopf superalgebra H and its Drinfeld twist $H^{\mathcal{F}}$ need not be isomorphic as Hopf superalgebras. However, there is a simple Hopf superalgebra isomorphism between $U_q(\mathfrak{gl}(V))$ and its twist $U_q(\mathfrak{gl}(V))^{\mathcal{F}}$. Let $U_q^{\geq}(\mathfrak{gl}(V))^{\mathcal{F}}$ and $U_q^{\leq}(\mathfrak{gl}(V))^{\mathcal{F}}$ denote the positive and negative Borel subalgebras of $U_q(\mathfrak{gl}(V))^{\mathcal{F}}$ generated by $\{e_i\}_{i=1}^{N-1} \cup \{q^{\pm H_k}\}_{k=1}^N$ and $\{f_i\}_{i=1}^{N-1} \cup \{q^{\pm H_k}\}_{k=1}^N$, respectively. Being closed under $\Delta^{\mathcal{F}}$ and $S^{\mathcal{F}}$, they are actually Hopf subalgebras. We emphasize that they share the exact same underlying superalgebra structure as their untwisted counterparts $U_q^{\geq}(\mathfrak{gl}(V))$ and $U_q^{\leq}(\mathfrak{gl}(V))$, differing only in their supercoalgebra structure.

Proposition 2.52. *(a) The assignment*

$$\phi_{\mathcal{F}}(e_i) = e_i q^{-h_i}, \qquad \phi_{\mathcal{F}}(f_i) = q^{h_i} f_i, \qquad \phi_{\mathcal{F}}(q^{\pm H_k}) = q^{\pm H_k} \tag{2.53}$$

uniquely extends to a Q-graded Hopf superalgebra isomorphism $\phi_{\mathcal{F}} \colon U_q(\mathfrak{gl}(V))^{\mathcal{F}} \xrightarrow{\sim} U_q(\mathfrak{gl}(V))$.

(b) The isomorphism $\phi_{\mathcal{F}}$ preserves the Borel subalgebras $U_q^{\geq}(\mathfrak{gl}(V))$ and $U_q^{\leq}(\mathfrak{gl}(V))$.

(c) Define the skew-pairing $(-,-)_{\mathrm{DJ}}^{\mathcal{F}}$ on the Drinfeld twist via $(x,y)_{\mathrm{DJ}}^{\mathcal{F}} := (\phi_{\mathcal{F}}(x), \phi_{\mathcal{F}}(y))_{\mathrm{DJ}}$. The skew-pairing $(-,-)_{\mathrm{DJ}}^{\mathcal{F}}$ is also non-degenerate and evaluates identically to the skew-pairing $(-,-)_{\mathrm{DJ}}$ on the generators (2.42).

Remark 2.54. As an immediate consequence, $U_q(\mathfrak{gl}(V))^{\mathcal{F}}$ can also be realized as the Drinfeld double $\mathcal{D}_{\mathrm{DJ}}^{\mathcal{F}}$ of $U_q^{\leq}(\mathfrak{gl}(V))^{\mathcal{F}}$ and $U_q^{\geq}(\mathfrak{gl}(V))^{\mathcal{F}}$ with respect to the skew-pairing $(-,-)_{\mathrm{DJ}}^{\mathcal{F}}$, modulo the identification of the Cartan subalgebras.

Remark 2.55. The same argument shows that Proposition 2.52(c) remains valid when using the alternate skew-pairing $(-,-)_{\widetilde{\mathrm{DJ}}}$ from Remark 2.49. Hence, the same Drinfeld double $\mathcal{D}_{\mathrm{DJ}}^{\mathcal{F}}$ can be realized with respect to the pullback $(-,-)_{\widetilde{\mathrm{DJ}}}^{\mathcal{F}}$ via $\phi_{\mathcal{F}}$, using the opposite order of the Borel subalgebras $U_q^{\geq}(\mathfrak{gl}(V))^{\mathcal{F}}$ and $U_q^{\leq}(\mathfrak{gl}(V))^{\mathcal{F}}$.

Proof of Proposition 2.52. (a) We first check that the assignment (2.53) gives rise to a superalgebra morphism. As the underlying superalgebra structure of $U_q(\mathfrak{gl}(V))^{\mathcal{F}}$ is identical to that of $U_q(\mathfrak{gl}(V))$, we need to show that (2.53) defines a superalgebra endomorphism on $U_q(\mathfrak{gl}(V))$. The Chevalley-type relations are straightforward, e.g. (2.24) is verified as follows:

$$\begin{aligned}[\phi_{\mathcal{F}}(e_i), \phi_{\mathcal{F}}(f_j)] = e_i q^{-h_i+h_j} f_j - (-1)^{|e_i||f_j|} q^{h_j} f_j e_i q^{-h_i} &= q^{(\alpha_i - \alpha_j, \alpha_j)} [e_i, f_j] q^{-h_i+h_j} \\ &= \delta_{ij} \frac{\phi_{\mathcal{F}}(q^{h_i}) - \phi_{\mathcal{F}}(q^{-h_i})}{q - q^{-1}}.\end{aligned}$$

For the Serre relations, we note that $\phi_{\mathcal{F}}(e_i)\phi_{\mathcal{F}}(e_j) = q^{-(\alpha_i,\alpha_j)} \cdot e_i e_j q^{-h_i-h_j}$, which implies that their q-supercommutator evaluates to $[\![\phi_{\mathcal{F}}(e_i), \phi_{\mathcal{F}}(e_j)]\!] = q^{-(\alpha_i,\alpha_j)} \cdot [\![e_i, e_j]\!] q^{-h_i-h_j}$. By induction, any nested q-supercommutator evaluated on the images $\phi_{\mathcal{F}}(e_i)$ is simply proportional to the same commutator evaluated on the generators e_i, right-multiplied by a Cartan element uniquely determined by its total Q-degree. As the Serre relations vanish on the generators e_i in $U_q(\mathfrak{gl}(V))$, they identically vanish on the images $\phi_{\mathcal{F}}(e_i)$.

The inverse map is given by the assignment $\phi_{\mathcal{F}}^{-1}(e_i) = e_i q^{h_i}$, $\phi_{\mathcal{F}}^{-1}(f_i) = q^{-h_i} f_i$, $\phi_{\mathcal{F}}^{-1}(q^{\pm H_k}) = q^{\pm H_k}$, which can be shown to define a valid superalgebra homomorphism by a completely analogous argument.

Furthermore, the compatibility of the coalgebra structures $(\phi_{\mathcal{F}} \otimes \phi_{\mathcal{F}}) \circ \Delta^{\mathcal{F}} = \Delta \circ \phi_{\mathcal{F}}$ is directly checked on the generators. Since $\phi_{\mathcal{F}}$ clearly preserves the Q-grading, this proves part (a).

(b) This is immediate from the definitions, as $\phi_{\mathcal{F}}(e_i) = e_i q^{-h_i} \in U_q^{\geq}(\mathfrak{gl}(V))$ and $\phi_{\mathcal{F}}(f_i) = q^{h_i} f_i \in U_q^{\leq}(\mathfrak{gl}(V))$.

(c) We first emphasize that $(-,-)_{\mathrm{DJ}}^{\mathcal{F}}$ is indeed a skew-pairing by part (a). The evaluation of $(f_i, e_i)_{\mathrm{DJ}}^{\mathcal{F}}$ is the only nontrivial case among the generators in (2.42), the computation of which is done as follows:

$$\begin{aligned}(f_i, e_i)_{\mathrm{DJ}}^{\mathcal{F}} = (q^{h_i} f_i, e_i q^{-h_i})_{\mathrm{DJ}} = q^{-(\alpha_i,\alpha_i)} (f_i q^{h_i}, e_i q^{-h_i})_{\mathrm{DJ}} \\ \overset{(2.46)}{=} q^{-(\alpha_i,\alpha_i)} (f_i, e_i)_{\mathrm{DJ}} (q^{h_i}, q^{-h_i})_{\mathrm{DJ}} = (f_i, e_i)_{\mathrm{DJ}}\,.\end{aligned}$$

□

3. RLL-realization of general linear quantum supergroups

3.1. The universal R-matrix.

A fundamental property of the Drinfeld double of Hopf superalgebras is that it naturally carries the structure of a (topological) quasi-triangular Hopf superalgebra via its canonical element (A.30), see Remark 3.3 for more details. By Remark 2.39 and the identity (2.46), the skew-pairing (2.41) factors multiplicatively into its restrictions to $U_q^{<}(\mathfrak{gl}(V)) \times U_q^{>}(\mathfrak{gl}(V))$ and $U_q^{0}(\mathfrak{gl}(V)) \times U_q^{0}(\mathfrak{gl}(V))$. Let $\{u_i^f\}_i$ and $\{u_i^e\}_i$ be dual bases for $U_q^{<}(\mathfrak{gl}(V))$ and $U_q^{>}(\mathfrak{gl}(V))$ with respect to this restricted pairing. Analogously, let $\{s_j^f\}_j$ and $\{s_j^e\}_j$ be dual bases for the respective copies of $U_q^0(\mathfrak{gl}(V))$. It then follows that the products $\{u_i^f s_j^f\}_{i,j}$ and $\{u_i^e s_j^e\}_{i,j}$ form dual bases for $U_q^{\leq}(\mathfrak{gl}(V))$ and $U_q^{\geq}(\mathfrak{gl}(V))$. Therefore, the canonical element $\mathfrak{R}$ factors as $\mathfrak{R} = \mathfrak{R}_u \mathfrak{R}_s$, where (note that every element in $U_q^0(\mathfrak{gl}(V))$ has $\mathbb{Z}_2$-degree $\bar{\mathbf{0}}$)

$$\mathfrak{R}_u = \sum_i u_i^e \otimes u_i^f\,, \qquad \mathfrak{R}_s = \sum_j s_j^e \otimes s_j^f\,.$$

Remark 3.1. Motivated by their actions on finite-dimensional weight modules, the above factors $\mathfrak{R}_s$ and $\mathfrak{R}_u$ are often referred to as the *semisimple* and *unipotent* parts of the canonical element $\mathfrak{R}$. Furthermore, the unipotent part $\mathfrak{R}_u$ is also often called the *reduced* R-matrix.

Remark 3.2. Since $U_q^0(\mathfrak{gl}(V))$ is in fact a Hopf subalgebra, the restriction of the skew-pairing (2.41) yields a Drinfeld double $\mathcal{D}(U_q^0(\mathfrak{gl}(V)), U_q^0(\mathfrak{gl}(V)))$ with $\mathfrak{R}_s$ being the corresponding canonical element.

Remark 3.3. (a) To treat the canonical element $\mathfrak{R} = \mathfrak{R}_u \mathfrak{R}_s$ rigorously, one must address the fact that its expansion involves infinite sums of tensor products, which are not well-defined elements in the algebraic tensor product $U_q(\mathfrak{gl}(V)) \otimes U_q(\mathfrak{gl}(V))$. While the infinite sum in $\mathfrak{R}_u$ can be resolved quite simply, $\mathfrak{R}_s$ presents a more fundamental algebraic obstruction: the dual bases for the Cartan subalgebras cannot be constructed in $U_q(\mathfrak{gl}(V))$, as expressing them requires taking formal logarithms of the generators q^H. To resolve this and make sense of these elements, one must pass to the *$\hbar$-adic topological quantum supergroups* $U_\hbar(\mathfrak{gl}(V))$, defined over $\mathbb{C}[[\hbar]]$, and consider the completed tensor product $U_\hbar(\mathfrak{gl}(V)) \hat{\otimes} U_\hbar(\mathfrak{gl}(V))$. In this framework, the indeterminate $q \in \mathbb{C}(q)$ and the Cartan elements q^H (for $H \in \Gamma$) are treated as formal exponentials $q = e^\hbar := \sum_{k \geq 0} \hbar^k/k! \in \mathbb{C}[[\hbar]]$ and $q^H = e^{\hbar H} := \sum_{k\geq 0} \hbar^k H^k/k!$. We refer the reader to [14, §8.3.2] or [16, §17] for a systematic development of this topological setting in the non-super case.

(b) However, $\mathfrak{R}_s$ is rarely necessary in practice: it generally suffices to work with its evaluations on finite-dimensional weight modules. By adapting the approach of [10, §7] to the super setting (as in [9, §4.1]), one can evaluate these matrices directly within the q-adic framework, entirely bypassing topological completions.

(c) In the present paper, the only instance explicitly requiring the universal element $\mathfrak{R}$ is the construction of the isomorphism in the opposite direction in Subsections 3.8 and 5.5. We emphasize that the key isomorphism between the Drinfeld–Jimbo and RLL realizations constructed in Subsections 3.7 and 5.4 is unequivocally established via the generalized double construction. Therefore, if one only requires the existence of the isomorphism without an explicit formula for the inverse map, the issue of $\hbar$-adic completions can be completely circumvented.

To obtain an explicit formula for $\mathfrak{R}$, we first recall the PBW-type bases of $U_q^{>}(\mathfrak{gl}(V))$ and $U_q^{<}(\mathfrak{gl}(V))$ (cf. [9, Theorem 5.16]). For convenience, let $\gamma_{ij} := \varepsilon_i - \varepsilon_j$ for all $1 \leq i < j \leq N$, so that

$$\Phi^+ = \{\gamma_{ij} \mid 1 \leq i < j \leq N\}\,.$$

For each $\gamma_{ij} \in \Phi^+$, we define the corresponding *quantum root vectors* $e_{\gamma_{ij}}$ and $f_{\gamma_{ij}}$ inductively on $j - i$. The base cases are $e_{\gamma_{i,i+1}} := e_i$ and $f_{\gamma_{i,i+1}} := f_i$. For $j > i$, we set:

$$\begin{aligned} e_{\gamma_{i,j+1}} &:= e_{\gamma_{ij}} e_j - (-1)^{|e_{\gamma_{ij}}||e_j|} q^{(\gamma_{ij},\alpha_j)} e_j e_{\gamma_{ij}} = [\![e_{\gamma_{ij}}, e_j]\!]\,, \\ f_{\gamma_{i,j+1}} &:= f_j f_{\gamma_{ij}} - (-1)^{|f_{\gamma_{ij}}||f_j|} q^{-(\gamma_{ij},\alpha_j)} f_{\gamma_{ij}} f_j = -(-1)^{|f_{\gamma_{ij}}||f_j|} q^{-(\gamma_{ij},\alpha_j)} [\![f_{\gamma_{ij}}, f_j]\!]\,. \end{aligned} \tag{3.4}$$

We also consider the *convex order* on Φ^+, corresponding to the lexicographic order on pairs $\{(i,j)\}_{1\leq i<j\leq N}$:

$$\gamma_{12} \prec \gamma_{13} \prec \cdots \prec \gamma_{1N} \prec \gamma_{23} \prec \cdots \prec \gamma_{2N} \prec \cdots \prec \gamma_{N-2,N-1} \prec \gamma_{N-2,N} \prec \gamma_{N-1,N}\,. \tag{3.5}$$

Theorem 3.6. *(a) The sets of ordered monomials*

$$\left\{ \overleftarrow{\prod_{\gamma\in\Phi^+}} e_\gamma^{m_\gamma} \,\middle|\, {}^{m_\gamma\in\mathbb{Z}_{\geq 0}}_{m_\gamma\leq 1 \text{ if } \gamma\in\Phi_{\bar 1}} \right\} \quad \text{and} \quad \left\{ \overleftarrow{\prod_{\gamma\in\Phi^+}} f_\gamma^{m_\gamma} \,\middle|\, {}^{m_\gamma\in\mathbb{Z}_{\geq 0}}_{m_\gamma\leq 1 \text{ if } \gamma\in\Phi_{\bar 1}} \right\}$$

form bases for $U_q^>(\mathfrak{gl}(V))$ and $U_q^<(\mathfrak{gl}(V))$, respectively. Here, the arrow $\leftarrow$ indicates that the factors in the product are ordered such that the roots γ increase from right to left with respect to the convex order (3.5).
(b) The skew-pairing (2.41) is orthogonal with respect to these bases.

Using the dual PBW bases above, one can express the universal R-matrix $\mathfrak{R}$ in the completion of the tensor square of $U_q(\mathfrak{gl}(V))$:

$$\mathfrak{R} = \mathfrak{R}_u\mathfrak{R}_s \in U_q(\mathfrak{gl}(V))\hat{\otimes} U_q(\mathfrak{gl}(V))\,,$$

where (cf. [9, Theorem 5.18])

$$\mathfrak{R}_s = q^{-\sum_{i=1}^N (-1)^{\bar i} H_i\otimes H_i}\,, \qquad \mathfrak{R}_u = \overleftarrow{\prod_{\gamma\in\Phi^+}} \left(\sum_{k\leq 1 \text{ if } \gamma\in\Phi_{\bar 1}}^{k\geq 0} \frac{e_\gamma^k\otimes f_\gamma^k}{(f_\gamma^k, e_\gamma^k)_{\mathrm{DJ}}} \right). \tag{3.7}$$

We now evaluate $\mathfrak{R}$ on the *first fundamental representation* $\varrho\colon U_q(\mathfrak{gl}(V)) \to \mathrm{End}(V_q)$ (see [9, Proposition A.1]). Here, $V_q \coloneqq \mathbb{C}(q)\otimes_{\mathbb{C}} V$ is the $\mathbb{C}(q)$-superspace obtained by extension of scalars, and $\mathrm{End}(V_q)$ denotes the $\mathbb{C}(q)$-superalgebra of all $\mathbb{C}(q)$-linear maps. The representation ϱ is defined via:

$$\varrho(e_i) = \mathsf{e}_i\,, \qquad \varrho(f_i) = \mathsf{f}_i\,, \qquad \varrho(q^{\pm H_k}) = q^{\pm \mathsf{H}_k}\,,$$

for $1\leq i<N$ and $1\leq k\leq N$, where $\mathsf{e}_i, \mathsf{f}_i, \mathsf{H}_k$ are the Chevalley-type generators of $\mathfrak{gl}(V)$ from (2.9, 2.12). Henceforth, for a diagonal matrix $D = \mathrm{diag}(d_1,\ldots,d_N)$, the notation q^D signifies:

$$q^D \coloneqq \sum_{1\leq i\leq N} q^{d_i} E_{ii} \quad \in \quad \mathrm{End}(V_q)\,. \tag{3.8}$$

The evaluation of the semisimple part $\varrho^{\otimes 2}(\mathfrak{R}_s)$ follows from a direct computation (interpreting $q = e^\hbar$ in the $\hbar$-adic setting), while the evaluation of the unipotent part $\varrho^{\otimes 2}(\mathfrak{R}_u)$ is established in [9, Proposition A.15]:

$$R_s \coloneqq \varrho^{\otimes 2}(\mathfrak{R}_s) = \sum_{1\leq i,j\leq N} q^{-(\varepsilon_i,\varepsilon_j)} E_{ii}\otimes E_{jj}\,, \qquad R_u \coloneqq \varrho^{\otimes 2}(\mathfrak{R}_u) = \mathrm{I} - (q-q^{-1})\sum_{i<j} (-1)^{\bar j} E_{ij}\otimes E_{ji}\,.$$

Combining these, we obtain the explicit formula for $R \coloneqq \varrho^{\otimes 2}(\mathfrak{R})$ and its inverse R^{-1} in $\mathrm{End}(V_q)^{\otimes 2}$ as given in [9, Remark A.5]:

$$\begin{aligned} R &= \sum_{1\leq i,j\leq N} q^{-(\varepsilon_i,\varepsilon_j)} E_{ii}\otimes E_{jj} - (q-q^{-1})\sum_{i<j}(-1)^{\bar j} E_{ij}\otimes E_{ji}\,, \\ R^{-1} &= \sum_{1\leq i,j\leq N} q^{(\varepsilon_i,\varepsilon_j)} E_{ii}\otimes E_{jj} + (q-q^{-1})\sum_{i<j}(-1)^{\bar j} E_{ij}\otimes E_{ji}\,. \end{aligned} \tag{3.9}$$

Remark 3.10. We note that $\mathfrak{R}$ (as well as its evaluation R) satisfies the *Yang–Baxter equation* (cf. Proposition A.31):

$$\mathfrak{R}_{(12)}\mathfrak{R}_{(13)}\mathfrak{R}_{(23)} = \mathfrak{R}_{(23)}\mathfrak{R}_{(13)}\mathfrak{R}_{(12)} \quad \in \quad U_q(\mathfrak{gl}(V))^{\hat{\otimes} 3}\,, \tag{3.11}$$

and the following comultiplication properties:

$$(\Delta\otimes \mathrm{Id})\mathfrak{R} = \mathfrak{R}_{(13)}\mathfrak{R}_{(23)}\,, \qquad (\mathrm{Id}\otimes\Delta)\mathfrak{R} = \mathfrak{R}_{(13)}\mathfrak{R}_{(12)}\,, \tag{3.12}$$

where the indices indicate the tensor factors on which $\mathfrak{R}$ acts (e.g., $\mathfrak{R}_{(12)} = \mathfrak{R}\otimes 1 \in U_q(\mathfrak{gl}(V))^{\hat{\otimes} 3}$, with analogous definitions for $\mathfrak{R}_{(13)}$ and $\mathfrak{R}_{(23)}$). This leg-numbering notation will be formalized in the next subsection. The multiplication in $U_q(\mathfrak{gl}(V))^{\hat{\otimes} 3}$ follows the standard sign convention for the tensor product of superalgebras, cf. (2.4). Moreover, the operator

$$\hat{R} \coloneqq \tau_{VV}\circ R \in \mathrm{End}(V_q)^{\otimes 2}$$

is a $U_q(\mathfrak{gl}(V))$-module homomorphism from $V_q^{\otimes 2}$ to itself, and satisfies the *braid relation*:

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23} \quad \in \quad \mathrm{End}(V_q)^{\otimes 3}\,,$$

where the indices follow the same leg-numbering conventions.

3.2. **Leg-numbering notation.**

Let $\mathcal{A}$ be a $\mathbb{C}(q)$-superalgebra (in many cases, $\mathcal{A}$ will be the free superalgebra $\mathcal{T}$ defined in (3.15) below, or one of its subquotients). We will frequently encounter superalgebras of the form $\mathcal{A}^{\otimes r} \otimes \mathrm{End}(V_q)^{\otimes s}$ $(r, s \geq 0)$ and morphisms between them. To keep equations involving these tensor products concise, we devote this separate subsection to formalizing the leg-numbering notation.

For each $1 \leq a \leq r'$ and $1 \leq b \leq s'$, there are canonical embeddings of $\mathcal{A}$ and $\mathrm{End}(V_q)$ as the a-th and b-th tensor arguments of $\mathcal{A}^{\otimes r'} \otimes \mathrm{End}(V_q)^{\otimes s'}$ respectively:

$$\iota_{(a)} \colon \mathcal{A} \hookrightarrow \mathcal{A}^{\otimes r'} \otimes \mathrm{End}(V_q)^{\otimes s'}, \qquad x \mapsto \big(1^{\otimes (a-1)} \otimes x \otimes 1^{\otimes (r'-a)}\big) \otimes \mathrm{I}^{\otimes s'},$$
$$\iota_b \colon \mathrm{End}(V_q) \hookrightarrow \mathcal{A}^{\otimes r'} \otimes \mathrm{End}(V_q)^{\otimes s'}, \qquad Y \mapsto 1^{\otimes r'} \otimes \big(\mathrm{I}^{\otimes (b-1)} \otimes Y \otimes \mathrm{I}^{\otimes (s'-b)}\big).$$

By the universal property of tensor products, for any $r \leq r'$, $s \leq s'$, and strictly increasing sequences of indices $1 \leq a_1 < \cdots < a_r \leq r'$ and $1 \leq b_1 < \cdots < b_s \leq s'$, one can construct a canonical ordered embedding:

$$\iota_{(a_1,\ldots,a_r),b_1,\ldots,b_s} \colon \mathcal{A}^{\otimes r} \otimes \mathrm{End}(V_q)^{\otimes s} \hookrightarrow \mathcal{A}^{\otimes r'} \otimes \mathrm{End}(V_q)^{\otimes s'}.$$

We uniquely extend this construction to arbitrary sequences of distinct indices $(a_1, \ldots, a_r)$ and $(b_1, \ldots, b_s)$. Let $(a'_1, \ldots, a'_r)$ and $(b'_1, \ldots, b'_s)$ be the strictly increasing rearrangements of these sequences. There exist unique permutations $\sigma_r \in S_r$ and $\sigma_s \in S_s$ such that $a_i = a'_{\sigma_r(i)}$ and $b_j = b'_{\sigma_s(j)}$. Letting $\sigma \in S_{r+s}$ be the image of $(\sigma_r, \sigma_s) \in S_r \times S_s$ under the canonical embedding $S_r \times S_s \to S_{r+s}$, we define the permuted embedding by precomposing the canonical ordered embedding with the natural isomorphism τ^σ from (2.2):

$$\iota_{(a_1,\ldots,a_r),b_1,\ldots,b_s} \coloneqq \iota_{(a'_1,\ldots,a'_r),b'_1,\ldots,b'_s} \circ \tau^\sigma.$$

For any sequences of distinct indices $(a_1, \ldots, a_r)$, $(b_1, \ldots, b_s)$, the image of an element $M \in \mathcal{A}^{\otimes r} \otimes \mathrm{End}(V_q)^{\otimes s}$ under the above permuted embedding is denoted by:

$$M_{(a_1,\ldots,a_r),b_1,\ldots,b_s} \coloneqq \iota_{(a_1,\ldots,a_r),b_1,\ldots,b_s}(M) \quad \in \quad \mathcal{A}^{\otimes r'} \otimes \mathrm{End}(V_q)^{\otimes s'}.$$

In practice, the integers r' and s' will be clear from the context. When $0 \leq r \leq 1$ or $0 \leq s \leq 1$ and no confusion can arise, the corresponding subscript is omitted. We note that this convention is compatible with our earlier notation in (3.11); for instance, the elements R_{12}, R_{13}, R_{23} are precisely the images of $R \in \mathrm{End}(V_q)^{\otimes 2}$ under the respective embeddings $\iota_{12}, \iota_{13}, \iota_{23}$ into $\mathrm{End}(V_q)^{\otimes 3}$.

Analogously, one can generalize this convention to morphisms $\phi \colon \mathcal{A}^{\otimes k} \to \mathcal{A}^{\otimes k'}$ and $\psi \colon \mathrm{End}(V_q)^{\otimes l} \to \mathrm{End}(V_q)^{\otimes l'}$. For any integers r and s large enough to contain the consecutive index blocks $(a, \ldots, a+k-1)$ and $(b, \ldots, b+l-1)$ respectively, we define:

$$\phi_{(a,\ldots,a+k-1)} \coloneqq \big(\mathrm{Id}^{\otimes (a-1)} \otimes \phi \otimes \mathrm{Id}^{\otimes (r-a-k+1)}\big) \otimes \mathrm{Id}^{\otimes s} \colon \mathcal{A}^{\otimes r} \otimes \mathrm{End}(V_q)^{\otimes s} \to \mathcal{A}^{\otimes (r-k+k')} \otimes \mathrm{End}(V_q)^{\otimes s},$$
$$\psi_{b,\ldots,b+l-1} \coloneqq \mathrm{Id}^{\otimes r} \otimes \big(\mathrm{Id}^{\otimes (b-1)} \otimes \psi \otimes \mathrm{Id}^{\otimes (s-b-l+1)}\big) \colon \mathcal{A}^{\otimes r} \otimes \mathrm{End}(V_q)^{\otimes s} \to \mathcal{A}^{\otimes r} \otimes \mathrm{End}(V_q)^{\otimes (s-l+l')}.$$

Again, these subscripts will frequently be omitted when the context is clear and no confusion can arise.

3.3. **The superalgebra $U(R)$.**

Following the construction in [7], we now define the RLL-realization $U(R)$ of the general linear quantum supergroup, corresponding to the matrix R:

Definition 3.13. The $\mathbb{C}(q)$-superalgebra $U(R)$ is generated by the elements $\{l^+_{ij}, l^-_{ji}\}_{1 \leq i \leq j \leq N}$ subject to the relations (3.16)–(3.18) below. The $\mathbb{Z}_2$-grading is compatible via (2.13) with the Q-grading determined by

$$\deg(l^\pm_{ij}) = \varepsilon_i - \varepsilon_j. \tag{3.14}$$

To rigorously formulate the defining relations, we first consider the free $\mathbb{C}(q)$-superalgebra

$$\mathcal{T} \coloneqq \mathbb{C}(q)\big\langle l^+_{ij}, l^-_{ji} \,\big|\, 1 \leq i \leq j \leq N \big\rangle, \tag{3.15}$$

equipped with the same Q-grading and $\mathbb{Z}_2$-grading as in (3.14). By adopting the convention that $l^+_{ij} = 0 = l^-_{ji}$ for $i > j$, we now organize these generators into formal matrices

$$L^\pm \coloneqq \sum_{1 \leq i,j \leq N} l^\pm_{ij} \otimes E_{ij} \quad \in \quad \mathcal{T} \otimes \mathrm{End}(V_q).$$

Following notations of Subsection 3.2, the algebra $U(R)$ is the quotient of $\mathcal{T}$ by the following relations:

$$l^\pm_{ii} l^\mp_{ii} = 1 \qquad \text{for } 1 \leq i \leq N, \tag{3.16}$$

together with all coefficients in $\mathcal{T}$ of the following matrix equations evaluated in the superalgebra $\mathcal{T}\otimes \mathrm{End}(V_q)^{\otimes 2}$:

$$R_{12}L_1^{\pm}L_2^{\pm} = L_2^{\pm}L_1^{\pm}R_{12}\,, \tag{3.17}$$

$$R_{12}L_1^{+}L_2^{-} = L_2^{-}L_1^{+}R_{12}\,. \tag{3.18}$$

Remark 3.19. (a) As the relations (3.16)–(3.18) are homogeneous, (3.14) indeed defines a Q-grading on $U(R)$.
(b) As L^+ and L^- are upper and lower triangular matrices, respectively, and their diagonal entries are invertible by (3.16), they are invertible matrices in the superalgebra $U(R)\otimes \mathrm{End}(V_q)$.

Furthermore, $U(R)$ admits the structure of a Hopf superalgebra. To construct the latter, we shall first introduce the bialgebra structure on the above free superalgebra $\mathcal{T}$. Let W be the $\mathbb{C}(q)$-superspace with basis $\{l^+_{ij}, l^-_{ji}\}_{1\le i\le j\le N}$, equipped with the Q-grading and the compatible $\mathbb{Z}_2$-grading given by (3.14). Then the free superalgebra $\mathcal{T}$ is isomorphic to the tensor superalgebra $T(W)$ of W over the base field $\mathbb{C}(q)$.

We endow W with a supercoalgebra structure via the following matrix equations on the generators:

$$\Delta(L^{\pm}) = L^{\pm}_{(1)}L^{\pm}_{(2)} \in W^{\otimes 2}\otimes \mathrm{End}(V_q)\,, \qquad \epsilon(L^{\pm}) = \mathrm{I}\in \mathrm{End}(V_q)\,, \tag{3.20}$$

where we use the above leg-numbering notation for the superspace W. Component-wise these yield

$$\Delta(l^{\pm}_{ij}) = \sum_{1\le k\le N} (-1)^{(\bar{i}+\bar{k})(\bar{k}+\bar{j})}\, l^{\pm}_{ik}\otimes l^{\pm}_{kj}\,, \qquad \epsilon(l^{\pm}_{ij}) = \delta_{ij} \qquad \text{for all} \quad 1\le i,j\le N\,. \tag{3.21}$$

Since the supercoalgebra structure morphisms (3.21) preserve the Q-grading, W is a Q-graded supercoalgebra. By the universal property of $\mathcal{T}=T(W)$, the morphisms

$$W \xrightarrow{\Delta} W\otimes W \hookrightarrow \mathcal{T}\otimes\mathcal{T}\,, \qquad W \xrightarrow{\epsilon} \mathbb{C}(q)$$

can be lifted uniquely to superalgebra morphisms

$$\Delta\colon \mathcal{T}\to\mathcal{T}\otimes\mathcal{T}\,, \qquad \epsilon\colon \mathcal{T}\to \mathbb{C}(q)\,,$$

endowing $\mathcal{T}$ with a Q-graded superbialgebra structure (the verification of coassociativity for Δ and the compatibility of the two maps is straightforward).

We claim that this superbialgebra structure on $\mathcal{T}$ descends to a superbialgebra structure (in fact, a Hopf superalgebra structure) on its quotient $U(R)$. To facilitate this verification, we record a few identities using the leg-numbering notation from Subsection 3.2. Let μ denote the multiplication morphism of $\mathcal{A}$, which is either $\mathcal{T}$ or one of its subquotients. We then have:

$$\mu(L^{\nu}_{(1),1}L^{\eta}_{(2),2}) = L^{\nu}_1 L^{\eta}_2 \in \mathcal{A}\otimes \mathrm{End}(V_q)^{\otimes 2}\,, \tag{3.22}$$

where $\nu,\eta\in\{\pm\}$. Since the braiding $\tau\colon A\otimes B\to B\otimes A$ itself is a superalgebra morphism for any superalgebras A and B, we also obtain:

$$\begin{aligned} \tau_{(12)}(L^{\nu}_{(1),i}L^{\eta}_{(2),j}) &= \tau_{(12)}(L^{\nu}_{(1),i})\tau_{(12)}(L^{\eta}_{(2),j}) = L^{\nu}_{(2),i}L^{\eta}_{(1),j}\,,\\ \tau_{12}(L^{\nu}_{(i),1}L^{\eta}_{(j),2}) &= \tau_{12}(L^{\nu}_{(i),1})\tau_{12}(L^{\eta}_{(j),2}) = L^{\nu}_{(i),2}L^{\eta}_{(j),1}\,, \end{aligned} \tag{3.23}$$

where i and j are not necessarily distinct. Furthermore, since $L^{\pm}\in\mathcal{A}\otimes\mathrm{End}(V_q)$ have $\mathbb{Z}_2$-degree $\bar{\mathbf{0}}$, we obtain the following commutativity relation between matrices acting on disjoint tensor arguments:

$$L^{\nu}_{(i),a}L^{\eta}_{(j),b} = L^{\eta}_{(j),b}L^{\nu}_{(i),a} \qquad \text{for} \quad a\ne b \text{ and } i\ne j\,. \tag{3.24}$$

The verification of the compatibility of (3.20) with the relations (3.17) and (3.18) is now straightforward. Let $\mathcal{I}$ be the two-sided ideal of $\mathcal{T}$ generated by the defining relations (3.16)–(3.18). For any $(\nu,\eta)\in\{(+,+),(-,-),(+,-)\}$, we have:

$$\Delta(R_{12}L^{\nu}_1L^{\eta}_2) = R_{12}\big(L^{\nu}_{(1),1}L^{\nu}_{(2),1}\big)\big(L^{\eta}_{(1),2}L^{\eta}_{(2),2}\big) \overset{(3.24)}{=} \big(R_{12}L^{\nu}_{(1),1}L^{\eta}_{(1),2}\big)\big(L^{\nu}_{(2),1}L^{\eta}_{(2),2}\big)\,,$$

$$\Delta(L^{\eta}_2L^{\nu}_1R_{12}) = \big(L^{\eta}_{(1),2}L^{\eta}_{(2),2}\big)\big(L^{\nu}_{(1),1}L^{\nu}_{(2),1}\big)R_{12} \overset{(3.24)}{=} \big(L^{\eta}_{(1),2}L^{\nu}_{(1),1}\big)\big(L^{\eta}_{(2),2}L^{\nu}_{(2),1}R_{12}\big)\,.$$

Subtracting these expansions and evoking the relations (3.17) and (3.18) yields:

$$\Delta(R_{12}L^{\nu}_1L^{\eta}_2 - L^{\eta}_2L^{\nu}_1R_{12}) \in (\mathcal{I}\otimes\mathcal{T}+\mathcal{T}\otimes\mathcal{I})\otimes \mathrm{End}(V_q)^{\otimes 2}\,. \tag{3.25}$$

Along with the straightforward evaluation of (3.21) on (3.16), this confirms that $\mathcal{I}$ is a superbialgebra ideal, and that Δ indeed preserves the defining relations of $U(R)$. The verification for the counit ϵ proceeds analogously.

We note that in contrast to $\mathcal{T}$, the superalgebra $U(R)$ actually admits the structure of a Hopf superalgebra, where the antipode is uniquely determined by the following formula on the generators:

$$S(L^{\pm}) = (L^{\pm})^{-1} \in U(R) \otimes \mathrm{End}(V_q)\,. \tag{3.26}$$

Remark 3.27. Recall the notion of the *supertranspose*. For any superalgebra $\mathcal{A}$ and matrix $X = \sum_{i,j=1}^{N} x_{ij} \otimes E_{ij} \in \mathcal{A} \otimes \mathrm{End}(V_q)$, its supertranspose is defined as

$$X^{\mathrm{st}} := \sum_{1 \le i,j \le N} (-1)^{\bar{j}(\bar{i}+\bar{j})} x_{ij} \otimes E_{ji}\,. \tag{3.28}$$

Suppose $X = \sum_{i,j=1}^{N} x_{ij} \otimes E_{ij}$ and $Y = \sum_{i,j=1}^{N} y_{ij} \otimes E_{ij}$ are elements in $\mathcal{A} \otimes \mathrm{End}(V_q)$ that are homogeneous with respect to the total $\mathbb{Z}_2$-grading, and whose entries satisfy the following supercommutativity condition:

$$[x_{ik}, y_{kj}] = 0 \qquad \text{for all} \qquad 1 \le i,j,k \le N\,, \tag{3.29}$$

where $[-,-]$ denotes the supercommutator (2.5). Under this condition, the following identity holds:

$$(XY)^{\mathrm{st}} = (-1)^{|X||Y|} Y^{\mathrm{st}} X^{\mathrm{st}}\,. \tag{3.30}$$

Following the notation of Subsection 3.2, for any element $X \in \mathcal{A} \otimes \mathrm{End}(V_q)^{\otimes s}$ and $1 \le i \le s$, we let X^{st_i} denote the *partial supertranspose* applied to the i-th factor of $\mathrm{End}(V_q)$. We note that these partial supertranspose operations acting on different factors mutually commute ($(X^{\mathrm{st}_i})^{\mathrm{st}_j} = (X^{\mathrm{st}_j})^{\mathrm{st}_i}$ for $i \neq j$) and satisfy (cf. (2.2)):

$$\tau^{\sigma}(X^{\mathrm{st}_i}) = (\tau^{\sigma}(X))^{\mathrm{st}_{\sigma(i)}} \qquad \text{for any} \quad \sigma \in S_s\,. \tag{3.31}$$

The *full supertranspose*, obtained by applying the partial supertranspose to all s matrix factors, is denoted by $X^{\mathrm{st}} := X^{\mathrm{st}_1 \cdots \mathrm{st}_s}$. By (3.31) and the mutual commutativity of partial supertransposes, we have:

$$\tau^{\sigma}(X^{\mathrm{st}}) = (\tau^{\sigma}(X))^{\mathrm{st}}\,. \tag{3.32}$$

Moreover, for homogeneous elements $X, Y \in \mathcal{A} \otimes \mathrm{End}(V_q)^{\otimes s}$ with respect to the total $\mathbb{Z}_2$-grading, whose entries satisfy the supercommutativity condition (3.29), the full supertranspose obeys the exact same identity (3.30). Furthermore, for any elements $X, Y \in \mathcal{A} \otimes \mathrm{End}(V_q)$ (not necessarily satisfying (3.29)), we have in $\mathcal{A} \otimes \mathrm{End}(V_q)^{\otimes s}$:

$$(X_i Y_j)^{\mathrm{st}} = X_i^{\mathrm{st}} Y_j^{\mathrm{st}} \qquad \forall\, 1 \le i \neq j \le s\,. \tag{3.33}$$

Now consider the defining relations (3.17) and (3.18):

$$R_{12} L_1^{\nu} L_2^{\eta} = L_2^{\eta} L_1^{\nu} R_{12}\,,$$

where $(\nu, \eta) \in \{(+,+), (-,-), (+,-)\}$. Since R_{12} has coefficients in the base field, its entries supercommute with those of $L_1^{\nu} L_2^{\eta}, L_2^{\eta} L_1^{\nu}$. Thus, applying the identity (3.30) to the full supertranspose of this relation yields:

$$(L_1^{\nu} L_2^{\eta})^{\mathrm{st}} (R_{12})^{\mathrm{st}} = (R_{12})^{\mathrm{st}} (L_2^{\eta} L_1^{\nu})^{\mathrm{st}}\,.$$

Furthermore, direct computation using the explicit formula (3.9) yields[1]

$$(R_{12})^{\mathrm{st}} = \sum_{1 \le i,j \le N} q^{-(\varepsilon_i, \varepsilon_j)} E_{ii} \otimes E_{jj} - (q - q^{-1}) \sum_{i<j} (-1)^{\bar{i}} E_{ji} \otimes E_{ij} = \Theta_1 \Theta_2 \cdot R_{21} \cdot (\Theta_1 \Theta_2)^{-1}\,, \tag{3.34}$$

where $\Theta = \sum_{i=1}^{N} q^{(\varepsilon_i, \varepsilon_i)/2} E_{ii}$, and by (3.33) we obtain:

$$(L_1^{\nu})^{\mathrm{st}} (L_2^{\eta})^{\mathrm{st}} \cdot \Theta_1 \Theta_2 \cdot R_{21} \cdot (\Theta_1 \Theta_2)^{-1} = \Theta_1 \Theta_2 \cdot R_{21} \cdot (\Theta_1 \Theta_2)^{-1} \cdot (L_2^{\eta})^{\mathrm{st}} (L_1^{\nu})^{\mathrm{st}}\,.$$

Applying the permutation operator τ_{12} to both sides gives (cf. (3.23), (3.32)):

$$(L_2^{\nu})^{\mathrm{st}} (L_1^{\eta})^{\mathrm{st}} \cdot \Theta_1 \Theta_2 \cdot R_{12} \cdot (\Theta_1 \Theta_2)^{-1} = \Theta_1 \Theta_2 \cdot R_{12} \cdot (\Theta_1 \Theta_2)^{-1} \cdot (L_1^{\eta})^{\mathrm{st}} (L_2^{\nu})^{\mathrm{st}}\,,$$

and conjugating further by $(\Theta_1 \Theta_2)^{-1}$ gives:

$$(\Theta_1 \Theta_2)^{-1} \cdot (L_2^{\nu})^{\mathrm{st}} (L_1^{\eta})^{\mathrm{st}} \cdot \Theta_1 \Theta_2 \cdot R_{12} = R_{12} \cdot (\Theta_1 \Theta_2)^{-1} \cdot (L_1^{\eta})^{\mathrm{st}} (L_2^{\nu})^{\mathrm{st}} \cdot \Theta_1 \Theta_2\,.$$

As $L_i^{\pm}$ and Θ_j commute for $i \neq j$, and $(L^+)^{\mathrm{st}}$ and $(L^-)^{\mathrm{st}}$ are lower and upper triangular, respectively, we thus obtain a superalgebra isomorphism

$$\omega_R \colon U(R) \xrightarrow{\sim} U(R) \qquad \text{given by} \qquad L^{\pm} \mapsto \Theta^{-1} (L^{\mp})^{\mathrm{st}} \Theta\,. \tag{3.35}$$

Similarly to Remark 2.37, it gives rise to the Hopf superalgebra isomorphism $\omega_R \colon U(R) \xrightarrow{\sim} U(R)^{\mathrm{cop}}$ since

$$\Delta((L^{\pm})^{\mathrm{st}}) = (\Delta(L^{\pm}))^{\mathrm{st}} \overset{(3.30)}{=} (L^{\pm}_{(2)})^{\mathrm{st}} (L^{\pm}_{(1)})^{\mathrm{st}}\,.$$

[1]While one simply has $(R_{12})^{\mathrm{st}} = R_{21}$ for the general linear type, we introduce the conjugation by Θ to provide a uniform treatment that will also apply to orthosymplectic types below, where it is absolutely necessary.

It is immediate that ω_R restricts to superalgebra isomorphisms between the subbialgebras $U^+(R)$ and $U^-(R)$ introduced in the next subsection, cf. Remark 2.51.

3.4. Generalized double construction of $U(R)$.

Let $U^+(R)$ and $U^-(R)$ be the superalgebras generated respectively by $\{l^+_{ij}\}_{1\leq i\leq j\leq N}$ and $\{l^-_{ji}\}_{1\leq i\leq j\leq N}$, equipped with the Q-grading (3.14) and the compatible $\mathbb{Z}_2$-grading, subject to the respective relations (3.17). The same formulas (3.20) endow both $U^\pm(R)$ with superbialgebra structures. Similarly to the case of $U_q(\mathfrak{gl}(V))$, the superalgebra $U(R)$ can be realized as a quotient of the double of the superbialgebra pair $(U^+(R), U^-(R))$.

Proposition 3.36. *(a) There exists a unique skew-pairing (cf. (A.21))*

$$(\cdot,\cdot)_R\colon U^+(R)\times U^-(R)\to \mathbb{C}(q) \tag{3.37}$$

satisfying the following property on the generators:

$$\sum_{1\leq i,j,k,l\leq N} (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})}(l^+_{ij}, l^-_{kl})_R\, E_{ij}\otimes E_{kl} = R\,. \tag{3.38}$$

(b) The superalgebra $U(R)$ is isomorphic to the quotient of the generalized double $\mathcal{D}_R := \mathcal{D}(U^+(R), U^-(R))$ modulo the relations $l^\pm_{ii}l^\mp_{ii}=1$ for all $1\leq i\leq N$.

Remark 3.39. Using the notation $\sigma := (\cdot,\cdot)_R$, the formula (3.38) can be written concisely as a matrix relation:

$$\sigma(L^+_{(1),1}L^-_{(2),2}) = R\,, \tag{3.40}$$

where we employ the leg-numbering notation defined in Subsection 3.2. We will frequently use this below.

Remark 3.41. We also note that (3.38) is compatible with the convention $l^+_{ij}=0=l^-_{ji}$ for $i>j$, which is evident from the explicit formula for R presented in (3.9).

Proof of Proposition 3.36. (a) Recall the supercoalgebra W defined via (3.20). We note that the subspaces

$$W^+ := \mathrm{Span}\{l^+_{ij}\mid 1\leq i\leq j\leq N\}\,,\qquad W^- := \mathrm{Span}\{l^-_{ji}\mid 1\leq i\leq j\leq N\}\,,$$

are subcoalgebras of W. Define a bilinear pairing $\sigma\colon W^+\times W^-\to\mathbb{C}(q)$ via the matrix equations (3.40). We note that this pairing is of Q-degree zero (cf. (2.43)), and equivalent to a morphism $W^+\to (W^-)^*$ (or $W^-\to (W^+)^*$) in sVect. Since the dual space $(W^-)^*$ of a supercoalgebra W^- has a canonical superalgebra structure, the universal property of the tensor superalgebra $T(W^+)$ yields a superalgebra morphism

$$T(W^+)\to (W^-)^*\,. \tag{3.42}$$

We shall now reverse the roles of the two spaces above. Since $T(W^+)$ is locally finite with respect to the Q-grading, its graded dual $T(W^+)^\vee$ is also a superbialgebra. Taking the graded dual of the superalgebra morphism (3.42), we obtain a supercoalgebra morphism $W^-\to T(W^+)^\vee$. Because $T(W^+)^\vee$ and its opposite superalgebra $(T(W^+)^\vee)^{\mathrm{op}}$ share the same supercoalgebra structure, we may view the above map as a supercoalgebra morphism taking values in $(T(W^+)^\vee)^{\mathrm{op}}$. Invoking the universal property of the tensor superalgebra on W^- then yields a unique extension to a superbialgebra morphism

$$T(W^-)\to (T(W^+)^\vee)^{\mathrm{op}}\,,$$

which is exactly equivalent to a skew-pairing (cf. Remarks A.7, A.22)

$$\sigma\colon T(W^+)\times T(W^-)\to\mathbb{C}(q)\,. \tag{3.43}$$

Now we prove that (3.43) factors through the defining relations (3.17) on both tensor factors, hence inducing a skew-pairing on $U^+(R)\times U^-(R)$. We note that the last two identities of the skew-pairing (A.21) evaluated on the generators can be written concisely in leg-numbering notation as follows:

$$\begin{gathered}\sigma(L^+_{(1),1}L^+_{(1),2}L^-_{(2),3}) = (\sigma\otimes\sigma)(L^+_{(1),1}L^-_{(2),3}L^+_{(3),2}L^-_{(4),3})\,,\\ \sigma(L^+_{(1),1}L^-_{(2),2}L^-_{(2),3}) = (\sigma\otimes\sigma)(L^+_{(1),1}L^-_{(2),2}L^+_{(3),1}L^-_{(4),3})\,,\end{gathered} \tag{3.44}$$

where $\sigma \otimes \sigma \colon \mathcal{T}^{\otimes 4} \to \mathbb{C}(q)$ is considered to have codomain $\mathbb{C}(q)$ after the identification $\mathbb{C}(q)^{\otimes 2} \simeq \mathbb{C}(q)$. For example, the first formula of (3.44) follows from:

$$\begin{aligned}
&\sigma\Big(\mu_{(12)}(L^+_{(1),1}L^+_{(2),2}L^-_{(3),3})\Big) = ((\sigma\otimes\sigma)\circ\tau_{(23)})\Big(\Delta_{(3)}(L^+_{(1),1}L^+_{(2),2}L^-_{(3),3})\Big)\\
\overset{(3.22)}{\Rightarrow}\quad &\sigma(L^+_{(1),1}L^+_{(1),2}L^-_{(2),3}) = ((\sigma\otimes\sigma)\circ\tau_{(23)})(L^+_{(1),1}L^+_{(2),2}L^-_{(3),3}L^-_{(4),3})\\
\overset{(3.23)}{\Rightarrow}\quad &\sigma(L^+_{(1),1}L^+_{(1),2}L^-_{(2),3}) = (\sigma\otimes\sigma)(L^+_{(1),1}L^+_{(3),2}L^-_{(2),3}L^-_{(4),3})\\
\overset{(3.24)}{\Rightarrow}\quad &\sigma(L^+_{(1),1}L^+_{(1),2}L^-_{(2),3}) = (\sigma\otimes\sigma)(L^+_{(1),1}L^-_{(2),3}L^+_{(3),2}L^-_{(4),3})\,.
\end{aligned}$$

Using these properties, we obtain:

$$\begin{aligned}
\sigma\big(R_{12}L^+_{(1),1}L^+_{(1),2}L^-_{(2),3}\big) &= R_{12}\,\sigma\big(L^+_{(1),1}L^+_{(1),2}L^-_{(2),3}\big)\\
&= R_{12}\,(\sigma\otimes\sigma)(L^+_{(1),1}L^-_{(2),3}L^+_{(3),2}L^-_{(4),3}) = R_{12}R_{13}R_{23}\,,
\end{aligned}$$

and analogously:

$$\begin{aligned}
\sigma\big(L^+_{(1),2}L^+_{(1),1}R_{12}L^-_{(2),3}\big) &= \sigma\big(L^+_{(1),2}L^+_{(1),1}L^-_{(2),3}\big)R_{12} \overset{(3.23)}{=} \tau_{12}\Big(\sigma\big(L^+_{(1),1}L^+_{(1),2}L^-_{(2),3}\big)\Big)R_{12}\\
&= \tau_{12}\Big((\sigma\otimes\sigma)(L^+_{(1),1}L^-_{(2),3}L^+_{(3),2}L^-_{(4),3})\Big)R_{12} = \tau_{12}(R_{13}R_{23})R_{12} \overset{(3.23)}{=} R_{23}R_{13}R_{12}\,.
\end{aligned}$$

Comparing these results, we note that the equality follows directly from the Yang–Baxter equation (3.11). This implies that the expression $R_{12}L_1^+L_2^+ - L_2^+L_1^+R_{12}$ pairs trivially with every generator of $T(W^-)$. A computation completely analogous to that of (3.25) shows that the two-sided ideal of $T(W^+)$ generated by the positive sign case of relation (3.17) is also a superbialgebra ideal. Consequently, $R_{12}L_1^+L_2^+ - L_2^+L_1^+R_{12}$ pairs trivially with every element of $T(W^-)$, and therefore lies in the left kernel of σ. The verification for the right kernel proceeds analogously, completing the proof of part (a).

(b) The proof is completely parallel to that of Proposition 2.40(b). We note again that the double $\mathcal{D}_R$ is the quotient of the free product superalgebra $U^+(R) * U^-(R)$ modulo the cross-relations (2.48). Following the notation in Proposition A.24, let $\mathcal{I}$ be the two-sided ideal generated by all elements $u_{x,y}$ for $\mathbb{Z}_2$-homogeneous $x \in U^+(R)$ and $y \in U^-(R)$. Since $U^+(R)$ and $U^-(R)$ are respectively graded by Q^+ and $-Q^+$, the explicit comultiplication formula (3.21), together with Proposition A.25 (see Remark A.26), implies that $\mathcal{I}$ is generated by the elements $u_{x,y}$ as x and y range over the generators of $U^+(R)$ and $U^-(R)$, respectively. Applying (2.48) to pairs of generators (l^+_{ij}, l^-_{kl}) and organizing them into a matrix yields:

$$\begin{aligned}
&\sum_{i,j,k,l} u_{l^+_{ij},l^-_{kl}} \otimes E_{ij}\otimes E_{kl}\\
&\quad= \sum_{i,j,k,l}\sum_{a,b}(-1)^{(\bar b+\bar l)(\bar i+\bar j)}\cdot(-1)^{(\bar i+\bar a)(\bar a+\bar j)}(-1)^{(\bar k+\bar b)(\bar b+\bar l)}\\
&\qquad\qquad\cdot\Big\{(-1)^{(\bar i+\bar a)(\bar k+\bar b)}(l^+_{ia},l^-_{kb})_R\cdot l^+_{aj}l^-_{bl} - (l^+_{aj},l^-_{bl})_R\cdot l^-_{kb}l^+_{ia}\Big\}\otimes E_{ij}\otimes E_{kl}\\
&\quad\overset{(\star)}{=} \sum_{i,j,k,l}\sum_{a,b}(-1)^{(\bar a+\bar j)(\bar k+\bar l)}\cdot\Big\{(-1)^{(\bar i+\bar a)(\bar k+\bar b)}(l^+_{ia},l^-_{kb})_R\cdot l^+_{aj}l^-_{bl} - (l^+_{aj},l^-_{bl})_R\cdot l^-_{kb}l^+_{ia}\Big\}\otimes E_{ij}\otimes E_{kl}\\
&\quad= R_{12}L_1^+L_2^- - L_2^-L_1^+R_{12}\,,
\end{aligned}$$

where the simplification of signs $(\star)$ is due to Remark A.9:

$$\begin{aligned}
&(-1)^{(\bar b+\bar l)(\bar i+\bar j)}(-1)^{(\bar i+\bar a)(\bar a+\bar j)}(-1)^{(\bar k+\bar b)(\bar b+\bar l)}(-1)^{(\bar i+\bar a)(\bar k+\bar b)}\cdot(l^+_{ia},l^-_{kb})_R\\
&= (-1)^{(\bar b+\bar l)(\bar i+\bar j)}(-1)^{(\bar k+\bar b)(\bar a+\bar j)}(-1)^{(\bar i+\bar a)(\bar b+\bar l)}(-1)^{(\bar i+\bar a)(\bar k+\bar b)}\cdot(l^+_{ia},l^-_{kb})_R\\
&= (-1)^{(\bar k+\bar l)(\bar a+\bar j)}(-1)^{(\bar i+\bar a)(\bar k+\bar b)}\cdot(l^+_{ia},l^-_{kb})_R
\end{aligned}$$

and

$$\begin{aligned}
&(-1)^{(\bar b+\bar l)(\bar i+\bar j)}(-1)^{(\bar i+\bar a)(\bar a+\bar j)}(-1)^{(\bar k+\bar b)(\bar b+\bar l)}\cdot(l^+_{aj},l^-_{bl})_R\\
&= (-1)^{(\bar b+\bar l)(\bar i+\bar j)}(-1)^{(\bar i+\bar a)(\bar b+\bar l)}(-1)^{(\bar k+\bar b)(\bar a+\bar j)}\cdot(l^+_{aj},l^-_{bl})_R = (-1)^{(\bar k+\bar l)(\bar a+\bar j)}\cdot(l^+_{aj},l^-_{bl})_R\,.
\end{aligned}$$

Thus, the vanishing of all $u_{l^+_{ij},l^-_{kl}}$ is precisely equivalent to relation (3.18).

Finally, by taking the quotient of the double $\mathcal{D}_R$ by the two-sided ideal

$$\mathcal{J} \coloneqq \big\langle l^\pm_{ii}l^\mp_{ii} - 1 \,\big|\, 1 \le i \le N\big\rangle\,,$$

we obtain the isomorphism $\mathcal{D}_R/\mathcal{J} \simeq U(R)$. □

Remark 3.45. Let $U^{\geq}(R)$ and $U^{\leq}(R)$ be the *Borel subalgebras* of $U(R)$ generated by $\{l^+_{ij}\}_{1\leq i\leq j\leq N} \cup \{l^-_{kk}\}_{1\leq k\leq N}$ and $\{l^-_{ji}\}_{1\leq i\leq j\leq N} \cup \{l^+_{kk}\}_{1\leq k\leq N}$, respectively. From the double construction above, one can check that the defining relations of $U^{\geq}(R)$ consist of the internal relations of $U^+(R)$ (the "+"-sign of (3.17)) and the cross-relations between $\{l^+_{ij}\}_{1\leq i\leq j\leq N}$ and $\{l^-_{kk}\}_{1\leq k\leq N}$ (encompassing the diagonal relations (3.16)). An analogous statement holds for $U^{\leq}(R)$. Consequently, it can be shown that the skew-pairing (3.37) extends uniquely to a skew-pairing $U^{\geq}(R)\times U^{\leq}(R)\to \mathbb{C}(q)$, denoted by the same symbol. Furthermore, $U^{\geq}(R)$ and $U^{\leq}(R)$ are Hopf subalgebras of $U(R)$, as they are closed under the antipode (3.26). As in Remark 2.50, the generalized double construction for $U(R)$ can alternatively be carried out using these Hopf subalgebras instead of the subbialgebras $U^+(R)$ and $U^-(R)$. Also the isomorphism ω_R of (3.35) restricts to a superalgebra isomorphism between $U^{\geq}(R)$ and $U^{\leq}(R)$.

Remark 3.46. As in Remark 2.49, one may alternatively construct the generalized double of $U(R)$ by taking the subbialgebras in the opposite order. This utilizes the transposed inverse of the original pairing $\sigma = (\cdot,\cdot)_R$:

$$\widetilde{\sigma} = (-,-)_{\widetilde{R}}\colon U^-(R)\times U^+(R)\to \mathbb{C}(q)\,.$$

By direct computation using the convolution product, one can verify that its evaluation on the generators is explicitly given by the matrix formula (cf. (3.40)):

$$\widetilde{\sigma}(L^-_{(1),1}L^+_{(2),2}) = R^{-1}_{21}\,, \tag{3.47}$$

using the notation of Subsection 3.2. Furthermore, as in Remark 3.45, this pairing uniquely extends to a skew-pairing $(-,-)_{\widetilde{R}}\colon U^{\leq}(R)\times U^{\geq}(R)\to\mathbb{C}(q)$ between the Borel subalgebras.

3.5. The twisted superalgebra $\widetilde{U}(R)$.

Although the defining relations (3.17) and (3.18) are concise in matrix form, extracting their component-wise formulas can be computationally complicated. This difficulty arises from the Koszul signs generated when swapping elements of the coefficient superalgebra $\mathcal{A}$ with the auxiliary matrix algebra $\mathrm{End}(V_q)$ during the multiplication of formal matrices like $L^{\pm}_{(a),b}$. To circumvent this issue, we introduce a *twisted superalgebra*.

Let G be an abelian group, written additively, and let $\mathcal{C} := \mathrm{Vect}^G_{\mathbb{K}}$ be the category of G-graded $\mathbb{K}$-vector spaces with degree-preserving linear maps, endowed with the standard tensor product of underlying $\mathbb{K}$-vector spaces. Consider a 2-cochain (that is, an arbitrary function) $\zeta\colon G\times G\to\mathbb{K}^\times$ on G. The identity functor F on $\mathcal{C}$, together with the identity morphism $\mathbb{K}\xrightarrow{\mathrm{Id}} F(\mathbb{K})$ and the natural isomorphism

$$J^\zeta_{X,Y}\colon F(X)\otimes F(Y) \xrightarrow{\sim} F(X\otimes Y)\,, \qquad x\otimes y\mapsto \zeta(|x|_G,|y|_G)\, x\otimes y\,,$$

defined for any G-homogeneous elements $x\in X, y\in Y$ of degrees $|x|_G,|y|_G\in G$, form a monoidal endofunctor on $\mathcal{C}$ if and only if ζ is a *normalized* 2*-cocycle*:

$$\zeta(g',g'')\zeta(g,g'+g'')\zeta(g,g')^{-1}\zeta(g+g',g'')^{-1} = 1\,, \qquad \zeta(g,0)=1=\zeta(0,g) \qquad \forall\, g,g',g''\in G\,. \tag{3.48}$$

Let A be an algebra object in $\mathcal{C}$ (i.e., a G-graded algebra), and let μ_A denote its multiplication morphism. Using the above isomorphism J^ζ, one can define a *twisted algebra* A^ζ whose multiplication is given by:

$$\mu^\zeta_A := \mu_A\circ J^\zeta_{A,A}\colon a\otimes a'\mapsto \zeta(|a|_G,|a'|_G)\, aa'\,. \tag{3.49}$$

Furthermore, if $\phi\colon A\to B$ is an algebra morphism, functoriality ensures that the map $\phi = F(\phi)$ itself provides an algebra morphism $\phi\colon A^\zeta\to B^\zeta$ between the twisted algebras.

Now let us consider braided structures on $\mathcal{C}$. For any 2-cochain $\beta\colon G\times G\to\mathbb{K}^\times$, one can define a natural isomorphism

$$c^\beta_{X,Y}\colon X\otimes Y\xrightarrow{\sim} Y\otimes X\,, \qquad x\otimes y\mapsto \beta(|x|_G,|y|_G)\, y\otimes x\,,$$

for any G-homogeneous elements $x\in X, y\in Y$ of degrees $|x|_G,|y|_G\in G$. This natural isomorphism equips $\mathcal{C}$ with a braided monoidal structure if and only if β is a *bicharacter*, meaning it is additive in both arguments:

$$\beta(g+g',h)=\beta(g,h)\beta(g',h)\,, \qquad \beta(g,h+h')=\beta(g,h)\beta(g,h') \qquad \forall\, g,g',h,h'\in G\,.$$

Equivalently, viewing G as a $\mathbb{Z}$-module, a bicharacter is simply a $\mathbb{Z}$-module homomorphism $G\otimes_{\mathbb{Z}} G\to\mathbb{K}^\times$.

For any algebra objects A and B in $\mathcal{C}$, the braiding c^β endows the tensor product $A\otimes B$ with an algebra structure, denoted $A\otimes^\beta B$, with multiplication given on G-homogeneous elements by (cf. (2.4)):

$$(a\otimes b)*(a'\otimes b') := \beta(|b|_G,|a'|_G)\, aa'\otimes bb'\,.$$

Remark 3.50. The category sVect is recovered in this framework by setting $G=\mathbb{Z}_2$ and $\beta(g,h)=(-1)^{gh}$.

We shall now apply these constructions for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{K} = \mathbb{C}(q)$. When working with tensor products of the form $\mathcal{A}^{\otimes r} \otimes \mathrm{End}(V_q)^{\otimes s}$, we view the coefficient superalgebra $\mathcal{A}$ and the matrix algebra $\mathrm{End}(V_q)$ as objects in $\mathcal{C}$ by assigning their standard $\mathbb{Z}_2$-degrees to the first and second components of G, respectively. Specifically, homogeneous elements in $\mathcal{A}$ have degrees in $\mathbb{Z}_2 \times \{0\}$, while those in $\mathrm{End}(V_q)$ have degrees in $\{0\} \times \mathbb{Z}_2$. We endow $\mathcal{C}$ with two different braidings given by bicharacters.

- **The standard braiding:** Consider the bicharacter

$$\beta(g,h) := (-1)^{(g_1+g_2)(h_1+h_2)} \in \{\pm 1\},$$

defined for any elements $g = (g_1, g_2)$ and $h = (h_1, h_2)$ in $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. For any objects X, Y in $\mathcal{C}$, the standard braiding $c := c^\beta$ evaluates on G-homogeneous elements $x \in X$ and $y \in Y$ as:

$$c_{X,Y}(x \otimes y) := \beta(|x|_G, |y|_G)\, y \otimes x = (-1)^{(|x|_1+|x|_2)(|y|_1+|y|_2)} y \otimes x\,,$$

where their G-degrees are denoted by $|x|_G = (|x|_1, |x|_2)$ and $|y|_G = (|y|_1, |y|_2)$, respectively. Under this braiding, multiplying two formal matrices yields:

$$L_1^\nu L_2^\eta = \sum_{1\le i,j,k,l\le N} (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})} l_{ij}^\nu l_{kl}^\eta \otimes E_{ij} \otimes E_{kl} \quad \in \quad \mathcal{A} \otimes^\beta \mathrm{End}(V_q)^{\otimes 2}\,.$$

Here, the Koszul sign appears because the matrix unit E_{ij} "passes through" the generator l_{kl}^η. We note that this braiding is compatible with the standard symmetric monoidal structure of sVect: the group homomorphism

$$G \to \mathbb{Z}_2\,, \qquad (a,b) \mapsto a+b$$

induces a braided monoidal functor from $(\mathcal{C}, c)$ to sVect. Therefore, computing the matrix relations (3.17) and (3.18) in $(\mathcal{C}, c)$ yields the exact same component-wise formulas as computing them in sVect.

- **The twisted braiding:** Consider the alternative bicharacter

$$\beta'(g,h) := (-1)^{g_1h_1+g_2h_2}\,,$$

defined for any $g = (g_1, g_2), h = (h_1, h_2) \in G$. For any objects X, Y in $\mathcal{C}$, this yields the twisted braiding $c' := c^{\beta'}$ given by:

$$c'_{X,Y}(x \otimes y) := \beta'(|x|_G, |y|_G)\, y \otimes x = (-1)^{|x|_1|y|_1+|x|_2|y|_2} y \otimes x\,.$$

Under this twisted braiding, no signs appear when swapping an object concentrated in degree $\mathbb{Z}_2 \times \{0\}$ with an object concentrated in degree $\{0\} \times \mathbb{Z}_2$. Consequently, formal matrix multiplication simplifies as follows:

$$L_1^\nu L_2^\eta = \sum_{1\le i,j,k,l\le N} l_{ij}^\nu l_{kl}^\eta \otimes E_{ij} \otimes E_{kl} \quad \in \quad \mathcal{A} \otimes^{\beta'} \mathrm{End}(V_q)^{\otimes 2}\,.$$

Now consider the superalgebra $\widetilde{U}(R)$, which has the exact same generators and diagonal relations (3.16) as $U(R)$, and whose remaining defining relations take the identical matrix form as (3.17) and (3.18) but are evaluated in the superalgebra $\mathcal{T} \otimes^{\beta'} \mathrm{End}(V_q)^{\otimes 2}$ instead of $\mathcal{T} \otimes^{\beta} \mathrm{End}(V_q)^{\otimes 2}$.

Proposition 3.51. *Let $\zeta\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{C}(q)^\times$ be the 2-cochain on $\mathbb{Z}_2$ given by $\zeta(\bar{\mathbf{i}}, \bar{\mathbf{j}}) := (-1)^{\bar{\mathbf{i}}\bar{\mathbf{j}}}$. Then, the assignment $l_{ij}^\pm \mapsto l_{ij}^\pm$ for all i,j gives rise to a superalgebra isomorphism $\widetilde{U}(R) \xrightarrow{\sim} U(R)^\zeta$.*

Proof. We first note that it is straightforward to verify that ζ is a normalized 2-cocycle, i.e. satisfies (3.48). To avoid confusion, let $*$ and $*'$ denote the multiplications in $\mathcal{T}^{\otimes r} \otimes^\beta \mathrm{End}(V_q)^{\otimes 2}$ and $\mathcal{T}^{\otimes r} \otimes^{\beta'} \mathrm{End}(V_q)^{\otimes 2}$ for any $r \ge 1$, respectively. We also note that since $R_{12} \in \mathrm{End}(V_q)^{\otimes 2}$ has G-degree 0, multiplication by R_{12} does not depend on the choice of $*$ or $*'$.

Consider the RLL defining relations (3.17, 3.18) of $U(R)$ written as $R_{12}(L_1^\nu * L_2^\eta) = (L_2^\eta * L_1^\nu)R_{12}$ with $(\nu, \eta) \in \{(+,+), (-,-), (+,-)\}$. Using the multiplication morphism $\mu_\mathcal{T}$ of $\mathcal{T}$, this can be written as follows:

$$R_{12}\, \mu_\mathcal{T}(L_{(1),1}^\nu * L_{(2),2}^\eta) = \mu_\mathcal{T}(L_{(2),2}^\eta * L_{(1),1}^\nu) R_{12}\,.$$

Since $\mu_\mathcal{T}^\zeta = \mu_\mathcal{T} \circ J_{\mathcal{T},\mathcal{T}}^\zeta$, this is equivalent to:

$$R_{12}\, \mu_\mathcal{T}^\zeta\big((J_{\mathcal{T},\mathcal{T}}^\zeta)^{-1}(L_{(1),1}^\nu * L_{(2),2}^\eta)\big) = \mu_\mathcal{T}^\zeta\big((J_{\mathcal{T},\mathcal{T}}^\zeta)^{-1}(L_{(2),2}^\eta * L_{(1),1}^\nu)\big) R_{12}\,. \tag{3.52}$$

Moreover, direct computation yields (cf. (3.24)):

$$(J_{\mathcal{T},\mathcal{T}}^\zeta)^{-1}(L_{(1),1}^\nu * L_{(2),2}^\eta) = \sum_{1\le i,j,k,l\le N} (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})} (J_{\mathcal{T},\mathcal{T}}^\zeta)^{-1}(l_{ij}^\nu \otimes l_{kl}^\eta) \otimes E_{ij} \otimes E_{kl} = L_{(1),1}^\nu *' L_{(2),2}^\eta\,,$$

$$(J_{\mathcal{T},\mathcal{T}}^\zeta)^{-1}(L_{(2),2}^\eta * L_{(1),1}^\nu) = \sum_{1\le i,j,k,l\le N} (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})} (J_{\mathcal{T},\mathcal{T}}^\zeta)^{-1}(l_{ij}^\nu \otimes l_{kl}^\eta) \otimes E_{ij} \otimes E_{kl} = L_{(2),2}^\eta *' L_{(1),1}^\nu\,.$$

Substituting these back into (3.52), we obtain:

$$R_{12}\,\mu^{\zeta}_{\mathcal{T}}(L^{\nu}_{(1),1} *' L^{\eta}_{(2),2}) = \mu^{\zeta}_{\mathcal{T}}(L^{\eta}_{(2),2} *' L^{\nu}_{(1),1})R_{12}\,.$$

This implies that the defining relations of $U(R)^{\zeta}$ coincide exactly with those of $\widetilde{U}(R)$ (where the compatibility of the diagonal relations (3.16) is trivial due to $\zeta(|l^{\pm}_{ii}|,|l^{\mp}_{ii}|)=1$), which completes the proof. □

Remark 3.53. The above isomorphism $\widetilde{U}(R) \xrightarrow{\sim} U(R)^{\zeta}$ implies that the original superalgebra $U(R)$ can be recovered from $\widetilde{U}(R)$ via an *untwisting* procedure (i.e. twisting by ζ^{-1}), that is, $U(R) \xrightarrow{\sim} \widetilde{U}(R)^{\zeta^{-1}}$.

3.6. Gauss decomposition and twisted identities: general linear case.

In this Subsection, we shall assume that all algebraic manipulations occur within the twisted superalgebra $\widetilde{U}(R)$ or the tensor product $\widetilde{U}(R)^{\otimes r} \otimes^{\beta'} \mathrm{End}(V_q)^{\otimes s}$ (for $r,s\geq 1$), rather than in their untwisted counterparts.

Any element in $\widetilde{U}(R)\otimes^{\beta'}\mathrm{End}(V_q)$ can be expressed in matrix form as follows:

$$\sum_{1\leq i,j\leq N} a_{ij}\otimes E_{ij} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \quad \in \quad \widetilde{U}(R)\otimes^{\beta'}\mathrm{End}(V_q)\,. \tag{3.54}$$

Since the Koszul signs no longer arise when swapping elements of $\widetilde{U}(R)$ and $\mathrm{End}(V_q)$, the multiplication in $\widetilde{U}(R)\otimes^{\beta'}\mathrm{End}(V_q)$ coincides exactly with the standard matrix multiplication:

$$\left(\sum_{1\leq i,j\leq N} a_{ij}\otimes E_{ij}\right)\left(\sum_{1\leq i,j\leq N} b_{ij}\otimes E_{ij}\right) = \sum_{1\leq i,j\leq N}\left(\sum_{1\leq k\leq N} a_{ik}b_{kj}\right)\otimes E_{ij}\,. \tag{3.55}$$

We will frequently utilize this matrix form. In particular, L^+ and L^- are genuinely upper and lower triangular matrices, respectively. Using the diagonal identity (3.16), we obtain the following *Gauss decompositions*:

$$\begin{gathered} L^+ = \begin{bmatrix} l^+_{11} & 0 & \cdots & 0 \\ 0 & l^+_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & l^+_{NN} \end{bmatrix} \begin{bmatrix} 1 & (l^+_{11})^{-1}l^+_{12} & \cdots & (l^+_{11})^{-1}l^+_{1N} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & (l^+_{N-1,N-1})^{-1}l^+_{N-1,N} \\ 0 & \cdots & 0 & 1 \end{bmatrix}, \\ L^- = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ l^-_{21}(l^-_{11})^{-1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ l^-_{N1}(l^-_{11})^{-1} & \cdots & l^-_{N,N-1}(l^-_{N-1,N-1})^{-1} & 1 \end{bmatrix} \begin{bmatrix} l^-_{11} & 0 & \cdots & 0 \\ 0 & l^-_{22} & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & l^-_{NN} \end{bmatrix}. \end{gathered} \tag{3.56}$$

For convenience, we define the elements $\{\mathbf{e}_{ij},\mathbf{f}_{ji}\}_{1\leq i<j\leq N}$ in $\widetilde{U}(R)$ via the following formulas (cf. (2.10)):

$$\begin{aligned} (l^+_{ii})^{-1}l^+_{ij} &= (-1)^{\overline{i}\,\overline{j}}q^{-(\varepsilon_i,\varepsilon_i)}q^{-(\rho,\varepsilon_i-\varepsilon_j)}(q-q^{-1})\cdot\mathbf{e}_{ij}\,, \\ l^-_{ji}(l^-_{ii})^{-1} &= -(-1)^{\overline{i}+\cdots+\overline{j-1}}(-1)^{\overline{i}\,\overline{j}}q^{(\varepsilon_i,\varepsilon_i)}q^{(\rho,\varepsilon_i-\varepsilon_j)}(q-q^{-1})\cdot\mathbf{f}_{ji}\,. \end{aligned} \tag{3.57}$$

We also introduce $\{q^{\pm\mathbf{H}_k}\}_{k=1}^N$ in $\widetilde{U}(R)$ via $q^{\pm\mathbf{H}_k}=l^{\pm}_{kk}$ and define $q^{\pm\mathbf{H}_{ij}}:=q^{\pm\mathbf{H}_i}(q^{\pm\mathbf{H}_j})^{-1}$ for $1\leq i<j\leq N$.

We now exhibit a list of identities satisfied by the above elements of $\widetilde{U}(R)$. These identities are extracted by comparing the appropriate matrix components of the defining relations (3.17) and (3.18) of $\widetilde{U}(R)$. We illustrate this methodology with a few selected computations. To prevent ambiguity, we use $[-,-]^{\zeta}$ and $[\![-,-]\!]^{\zeta}$ to denote the supercommutator and the q-supercommutator of elements in $\widetilde{U}(R)$, cf. (2.5, 2.25).

1. Commutativity of Cartan generators: $[q^{\mathbf{H}_i},q^{\mathbf{H}_j}]^{\zeta}=0$.

We compute the $(E_{ii}\otimes E_{jj})$-component of (3.17) for the "+"-sign case:

- **LHS:** $(q^{-(\varepsilon_i,\varepsilon_j)}E_{ii}\otimes E_{jj})(l^+_{ii}l^+_{jj}E_{ii}\otimes E_{jj})$;
- **RHS:** $(l^+_{jj}l^+_{ii}E_{ii}\otimes E_{jj})(q^{-(\varepsilon_i,\varepsilon_j)}E_{ii}\otimes E_{jj})$.

Equating both sides yields $l^+_{ii}l^+_{jj}=l^+_{jj}l^+_{ii}$, which gives $q^{\mathbf{H}_i}q^{\mathbf{H}_j}=q^{\mathbf{H}_j}q^{\mathbf{H}_i}$.

2. Inverses: $q^{\pm\mathbf{H}_i}q^{\mp\mathbf{H}_i}=1$.

This follows directly from the diagonal identity (3.16).

3. Cartan action on root vectors: $q^{\mathbf{H}_a}\mathbf{e}_{ij}q^{-\mathbf{H}_a}=q^{(\varepsilon_a,\varepsilon_i-\varepsilon_j)}\mathbf{e}_{ij}$.

We compute the $(E_{aa}\otimes E_{ij})$-component of (3.17) for the "+"-sign case (evoking that L^+ is upper triangular):

- **LHS:** $(q^{-(\varepsilon_a,\varepsilon_i)}E_{aa}\otimes E_{ii})(l^+_{aa}l^+_{ij}E_{aa}\otimes E_{ij})$;
- **RHS:** $(l^+_{ij}l^+_{aa}E_{aa}\otimes E_{ij})(q^{-(\varepsilon_a,\varepsilon_j)}E_{aa}\otimes E_{jj})$.

Equating both sides yields $q^{-(\varepsilon_a,\varepsilon_i)} l^+_{aa} l^+_{ij} = q^{-(\varepsilon_a,\varepsilon_j)} l^+_{ij} l^+_{aa}$, which simplifies to $l^+_{aa} l^+_{ij} (l^+_{aa})^{-1} = q^{(\varepsilon_a,\varepsilon_i-\varepsilon_j)} l^+_{ij}$. This establishes the desired identity $q^{\mathbf{H}_a} \mathbf{e}_{ij} q^{-\mathbf{H}_a} = q^{(\varepsilon_a,\varepsilon_i-\varepsilon_j)} \mathbf{e}_{ij}$.

To exhibit the rest of aforementioned identities, we summarize the relevant matrix components and the resulting twisted identities in Table 1 (the identities verified above are also included for ease of reference):

Matrix Component	Conditions	Identity	
$E_{ii} \otimes E_{jj}$	$1 \le i,j \le N$	$[q^{\mathbf{H}_i}, q^{\mathbf{H}_j}]^\zeta = 0$	(3.58)
	$1 \le i \le N$	$q^{\pm\mathbf{H}_i} q^{\mp\mathbf{H}_i} = 1$	(3.59)
$E_{aa} \otimes E_{ij}$	$1 \le i < j \le N,\ 1 \le a \le N$	$q^{\mathbf{H}_a} \mathbf{e}_{ij} q^{-\mathbf{H}_a} = q^{(\varepsilon_a,\varepsilon_i-\varepsilon_j)} \mathbf{e}_{ij}$	(3.60)
$E_{ij} \otimes E_{j,j+1}$	$1 \le i < j < N$	$\mathbf{e}_{i,j+1} = (-1)^{(\bar{i}+\bar{j})(\bar{j}+\overline{j+1})} [\![\mathbf{e}_{ij}, \mathbf{e}_{j,j+1}]\!]^\zeta$	(3.61)
$E_{i,i+1} \otimes E_{j,j+1}$	$1 \le i < j < N,\ j-i > 1$	$[\![\mathbf{e}_{i,i+1}, \mathbf{e}_{j,j+1}]\!]^\zeta = 0$	(3.62)
$E_{i,i+1} \otimes E_{i,i+1}$	$1 \le i < N,\ \lvert\mathbf{e}_{i,i+1}\rvert = \bar{1}$	$[\![\mathbf{e}_{i,i+1}, \mathbf{e}_{i,i+1}]\!]^\zeta = 0$	(3.63)
$E_{i,i+1} \otimes E_{i,i+2}$	$1 \le i \le N-2$	$[\![\mathbf{e}_{i,i+1}, [\![\mathbf{e}_{i,i+1}, \mathbf{e}_{i+1,i+2}]\!]^\zeta]\!]^\zeta = 0$	(3.64)
$E_{i,i+1} \otimes E_{i-1,i+1}$	$2 \le i \le N-1$	$[\![[\![\mathbf{e}_{i-1,i}, \mathbf{e}_{i,i+1}]\!]^\zeta, \mathbf{e}_{i,i+1}]\!]^\zeta = 0$	(3.65)
$E_{i,i+1} \otimes E_{i-1,i+2}$	$2 \le i \le N-2$	$[\![[\![[\![\mathbf{e}_{i-1,i}, \mathbf{e}_{i,i+1}]\!]^\zeta, \mathbf{e}_{i+1,i+2}]\!]^\zeta, \mathbf{e}_{i,i+1}]\!]^\zeta = 0$	(3.66)
$E_{ij} \otimes E_{ij}$	$1 \le i < j \le N,\ \lvert\mathbf{e}_{ij}\rvert = \bar{1}$	$[\![\mathbf{e}_{ij}, \mathbf{e}_{ij}]\!]^\zeta = 0$	(3.67)

TABLE 1. A-type, identities in $\widetilde{U}(R)$ via $RL^+_1 L^+_2 = L^+_2 L^+_1 R$ and diagonal relations

Remark 3.68. While (3.62) is a special case of (3.67), it is recorded separately as it corresponds to the Serre relations. We also note that (3.67) implies $\mathbf{e}^2_{ij} = 0$ for $|\mathbf{e}_{ij}| = \bar{1}$.

Remark 3.69. The relations (3.58)–(3.61) are applied repeatedly to derive the higher-order relations in Table 1. For instance, the $E_{i,i+1} \otimes E_{i,i+2}$ component of (3.17) yields $[\![\mathbf{e}_{i,i+1}, \mathbf{e}_{i,i+2}]\!]^\zeta = 0$, which directly recovers (3.64) upon substituting the iterative formula (3.61) and removing the overall sign.

Remark 3.70. Similar relations for $\mathbf{f}_{ij}$'s can be obtained by applying Remark 3.27 to Table 1. We note that while $(-1)^{(\bar{i}+\overline{i+1})(\bar{j}+\overline{j+1})} [\mathbf{e}_{i,i+1}, \mathbf{f}_{j+1,j}]^\zeta = \delta_{ij} \frac{q^{\mathbf{H}_{i,i+1}} - q^{-\mathbf{H}_{i,i+1}}}{q-q^{-1}}$ can be derived alike, it will not be needed.

3.7. **Homomorphism theorem: general linear case.**

The main result of this section is the identification of the Hopf superalgebras $U_q(\mathfrak{gl}(V))^{\mathcal{F}}$ and $U(R)$:

Theorem 3.71. *(a) The assignment*

$$e_i \mapsto \mathbf{e}_{i,i+1}\,, \qquad f_i \mapsto \mathbf{f}_{i+1,i}\,, \qquad q^{\pm H_i} \mapsto q^{\pm\mathbf{H}_i} \tag{3.72}$$

gives rise to a Q-graded Hopf superalgebra isomorphism $\xi\colon U_q(\mathfrak{gl}(V))^{\mathcal{F}} \xrightarrow{\sim} U(R)$.
(b) The images of the quantum root vectors from Subsection 3.1 under ξ are given by

$$\xi(e_{\gamma_{ij}}) = \mathbf{e}_{ij}\,, \qquad \xi(f_{\gamma_{ij}}) = \mathbf{f}_{ji}\,.$$

(c) The isomorphism ξ is compatible with the pairings of Remarks 2.55, 3.45:

$$(\xi(x), \xi(y))_R = (x,y)^{\mathcal{F}}_{\mathrm{DJ}} \qquad \text{for all} \quad x \in U^{\geq}_q(\mathfrak{gl}(V))^{\mathcal{F}},\ y \in U^{\leq}_q(\mathfrak{gl}(V))^{\mathcal{F}}\,.$$

Remark 3.73. Since the elements $l^{\pm}_{ii}$ are even, the defining formulas (3.57) for $\mathbf{e}_{ij}$ and $\mathbf{f}_{ji}$ still hold when the multiplication in the left-hand sides is replaced by that of $U(R)$, instead the one of $\widetilde{U}(R)$.

Remark 3.74. For the later use in the proof of Theorem 3.71, we record here how the superalgebra isomorphism ω_R of (3.35) acts on the elements $\mathbf{e}_{ij}$, $\mathbf{f}_{ji}$ of (3.57), and $q^{\pm\mathbf{H}_k}$. We have $\omega_R(q^{\pm\mathbf{H}_k}) = q^{\mp\mathbf{H}_k}$ and

$$\omega_R(\mathbf{e}_{ij}) = -(-1)^{\bar{i}+\cdots+\overline{j-1}} (-1)^{\bar{i}(\bar{i}+\bar{j})} q^{(\varepsilon_i,\varepsilon_i)/2} q^{(\varepsilon_j,\varepsilon_j)/2} q^{2(\rho,\varepsilon_i-\varepsilon_j)} \mathbf{f}_{ji}\,,$$
$$\omega_R(\mathbf{f}_{ji}) = -(-1)^{\bar{i}+\cdots+\overline{j-1}} (-1)^{\bar{j}(\bar{i}+\bar{j})} q^{-(\varepsilon_i,\varepsilon_i)/2} q^{-(\varepsilon_j,\varepsilon_j)/2} q^{-2(\rho,\varepsilon_i-\varepsilon_j)} \mathbf{e}_{ij}\,.$$

In particular, we have (cf. (2.11)):

$$\omega_R(\mathbf{e}_{i,i+1}) = -(-1)^{\bar{i}\,\overline{i+1}} q^{3(\rho,\alpha_i)} \mathbf{f}_{i+1,i}\,,$$
$$\omega_R(\mathbf{f}_{i+1,i}) = -(-1)^{\bar{i}\,\overline{i+1}} (-1)^{\bar{i}+\overline{i+1}} q^{-3(\rho,\alpha_i)} \mathbf{e}_{i,i+1}\,.$$

Proof of Theorem 3.71. We first prove that the assignment (3.72) gives rise to a morphism of Hopf superalgebras $U_q^{\geq}(\mathfrak{gl}(V))^{\mathcal{F}} \to U^{\geq}(R)$. By untwisting the relations in Table 1 via (cf. (3.49))

$$[\![x,y]\!]^{\zeta} = (-1)^{|x||y|}[\![x,y]\!]\,, \tag{3.75}$$

we obtain the following:

- equations (3.58)–(3.60) imply the Chevalley-type relations (2.33)–(2.34) for $U_q^{\geq}(\mathfrak{gl}(V))^{\mathcal{F}}$;
- equations (3.62)–(3.66) imply the Serre relations (2.26)–(2.28) for $U_q^{\geq}(\mathfrak{gl}(V))^{\mathcal{F}}$.

This proves that ξ gives a superalgebra morphism $U_q^{\geq}(\mathfrak{gl}(V))^{\mathcal{F}} \to U^{\geq}(R)$. Moreover, the following computations show that ξ also intertwines the comultiplications:

$$\Delta(q^{\mathbf{H}_k}) = \Delta(l^+_{kk}) = l^+_{kk}\otimes l^+_{kk} = q^{\mathbf{H}_k}\otimes q^{\mathbf{H}_k}\,,$$

$$\Delta(\mathbf{e}_{i,i+1}) = (-1)^{\overline{i}\,\overline{i+1}} q^{(\varepsilon_i,\varepsilon_i)} q^{(\rho,\varepsilon_i-\varepsilon_{i+1})}(q-q^{-1})^{-1}\cdot \Delta(l^-_{ii})\Delta(l^+_{i,i+1})$$

$$= (-1)^{\overline{i}\,\overline{i+1}} q^{(\varepsilon_i,\varepsilon_i)} q^{(\rho,\varepsilon_i-\varepsilon_{i+1})}(q-q^{-1})^{-1}\cdot (1\otimes l^-_{ii}l^+_{i,i+1} + l^-_{ii}l^+_{i,i+1}\otimes l^-_{ii}l^+_{i+1,i+1}) = 1\otimes \mathbf{e}_{i,i+1} + \mathbf{e}_{i,i+1}\otimes q^{-\mathbf{H}_{i,i+1}}\,.$$

Furthermore, untwisting (3.61) via (3.75) gives $\mathbf{e}_{i,j+1} = [\![\mathbf{e}_{ij},\mathbf{e}_{j,j+1}]\!]$, which is compatible with (3.4), thus establishing part (b) for the positive Borel subalgebras. We note that this iterative formula also proves the surjectivity of $\xi\colon U_q^{\geq}(\mathfrak{gl}(V))^{\mathcal{F}} \to U^{\geq}(R)$.

To derive the analogous results for the negative Borel subalgebras, we utilize the isomorphisms from Remarks 2.37 and 3.27. Specifically, using Remark 3.74, one can check that the composed morphism

$$U_q^{\leq}(\mathfrak{gl}(V))^{\mathcal{F}} \xrightarrow{\omega_{\mathrm{DJ}}} (U_q^{\geq}(\mathfrak{gl}(V))^{\mathcal{F}})^{\mathrm{cop}} \xrightarrow{\xi} (U^{\geq}(R))^{\mathrm{cop}} \xrightarrow{\omega_R^{-1}} U^{\leq}(R)$$

coincides with ξ on the generators. This proves that the assignment (3.72) gives rise to a well-defined Hopf superalgebra morphism $U_q^{\leq}(\mathfrak{gl}(V))^{\mathcal{F}} \to U^{\leq}(R)$. Moreover, this morphism is automatically surjective since both ω_{DJ} and ω_R restrict to isomorphisms between the respective positive and negative Borel subalgebras (cf. Remarks 2.51 and 3.45). Furthermore, we obtain an analogous iterative formula $\mathbf{f}_{j+1,i} = -(-1)^{(\overline{i}+\overline{j})(\overline{j}+\overline{j+1})}q^{(\varepsilon_j,\varepsilon_j)}[\![\mathbf{f}_{ji},\mathbf{f}_{j+1,j}]\!]$, which is also compatible with (3.4). This completes the proof of part (b) for the negative Borel subalgebras.

Let us now compare the pairings. Since both pairings are of Q-degree zero, cf. (2.43), we get

$$(\mathbf{e}_{i,i+1},q^{\pm\mathbf{H}_k})_R = 0 = (e_i,q^{\pm H_k})^{\mathcal{F}}_{\widetilde{\mathrm{DJ}}}\,,\qquad (q^{\pm\mathbf{H}_k},\mathbf{f}_{i+1,i})_R = 0 = (q^{\pm H_k},f_i)^{\mathcal{F}}_{\widetilde{\mathrm{DJ}}}\,.$$

Moreover, we have

$$(q^{\mathbf{H}_i},q^{-\mathbf{H}_j})_R = (l^+_{ii},l^-_{jj})_R = q^{-(\varepsilon_i,\varepsilon_j)} = (q^{H_i},q^{-H_j})^{\mathcal{F}}_{\widetilde{\mathrm{DJ}}}\,.$$

Finally

$$\begin{aligned}
(l^-_{ii}l^+_{i,i+1}\,,\,l^-_{j+1,j}l^+_{jj})_R &= (l^-_{ii}\otimes l^+_{i,i+1}\,,\,\Delta(l^-_{j+1,j})\Delta(l^+_{jj}))_R \overset{(2.43)}{=} (l^-_{ii}\otimes l^+_{i,i+1}\,,\,l^-_{j+1,j+1}l^+_{jj}\otimes l^-_{j+1,j}l^+_{jj})_R\\
&= (l^-_{ii}\,,\,l^-_{j+1,j+1}l^+_{jj})_R\,(l^+_{i,i+1}\,,\,l^-_{j+1,j}l^+_{jj})_R = q^{-(\varepsilon_i,\alpha_j)}\,(l^+_{i,i+1}\,,\,l^-_{j+1,j}l^+_{jj})_R\\
&\overset{(2.43)}{=} q^{-(\varepsilon_i,\alpha_j)}\,(l^+_{i,i+1}\otimes l^+_{ii}\,,\,l^-_{j+1,j}\otimes l^+_{jj})_R = q^{(\varepsilon_i,\varepsilon_{j+1})}\,(l^+_{i,i+1}\,,\,l^-_{j+1,j})_R\\
&= -\delta_{ij}q^{(\varepsilon_i,\varepsilon_{j+1})}(-1)^{\overline{i}+\overline{i+1}}\cdot(-1)^{\overline{i+1}}(q-q^{-1}) = -\delta_{ij}(-1)^{\overline{i}}(q-q^{-1})
\end{aligned}$$

which implies

$$(\mathbf{e}_{i,i+1},\mathbf{f}_{j+1,j})_R = \delta_{ij}(q-q^{-1})^{-1} = (e_i,f_j)^{\mathcal{F}}_{\widetilde{\mathrm{DJ}}}\,.$$

This establishes the compatibility of the pairings on the generators, which completes the proof of part (c).

Thus, ξ defines a surjective Hopf superalgebra morphism between the corresponding doubles:

$$\xi\colon \mathcal{D}^{\mathcal{F}}_{\mathrm{DJ}} = U_q^{\geq}(\mathfrak{gl}(V))^{\mathcal{F}}\otimes U_q^{\leq}(\mathfrak{gl}(V))^{\mathcal{F}} \longrightarrow U^{\geq}(R)\otimes U^{\leq}(R) = \mathcal{D}_R\,,$$

see Remark 2.55. Since the pairing $(-,-)^{\mathcal{F}}_{\widetilde{\mathrm{DJ}}}$ is non-degenerate, $\xi\colon \mathcal{D}^{\mathcal{F}}_{\mathrm{DJ}} \to \mathcal{D}_R$ must be injective on each Borel subalgebra, and so is an isomorphism. Finally, the identification of the Cartan subalgebras yields an isomorphism $\xi\colon U_q(\mathfrak{gl}(V))^{\mathcal{F}} \xrightarrow{\sim} U(R)$, thus completing the proof of part (a). □

3.8. **Inverse morphism: general linear case.**

Following [5], we shall now construct a morphism in the opposite direction (which does require to work with $\hbar$-adic completion, as emphasized in Remark 3.3).

Proposition 3.76. *The assignment*

$$L^+ \mapsto (\mathrm{Id}\otimes\varrho)(\mathfrak{R}_{(21)})\,,\qquad L^- \mapsto (\mathrm{Id}\otimes\varrho)(\mathfrak{R}^{-1})\,, \tag{3.77}$$

gives rise to a Q-graded Hopf superalgebra morphism $\phi^{\mathrm{DF}}\colon U(R)^{\mathrm{cop}} \to U_q(\mathfrak{gl}(V))$.

Proof. We first verify that the assignment (3.77) defines a superalgebra morphism. The explicit formula (3.7) (together with an $\hbar$-adic computation, see Remark 3.3) ensures that $\phi^{\mathrm{DF}}(L^+)$ and $\phi^{\mathrm{DF}}(L^-)$ are indeed upper and lower triangular matrices, respectively. Furthermore, since the diagonal components of $(\mathrm{Id}\otimes\varrho)((\mathfrak{R}_u)_{(21)})$ and $(\mathrm{Id}\otimes\varrho)(\mathfrak{R}_u^{-1})$ reduce to the identity matrix, the diagonal components of $(\mathrm{Id}\otimes\varrho)(\mathfrak{R}_{(21)})$ and $(\mathrm{Id}\otimes\varrho)(\mathfrak{R}^{-1})$ simplify respectively to $(\mathrm{Id}\otimes\varrho)((\mathfrak{R}_s)_{(21)}) = (\mathrm{Id}\otimes\varrho)(\mathfrak{R}_s)$ and $(\mathrm{Id}\otimes\varrho)(\mathfrak{R}_s^{-1})$, verifying the diagonal relations (3.16).

From the Yang–Baxter equation (3.11), we deduce the following equalities:

$$\mathfrak{R}_{(23)}\mathfrak{R}_{(21)}\mathfrak{R}_{(31)} = \mathfrak{R}_{(31)}\mathfrak{R}_{(21)}\mathfrak{R}_{(23)}\,, \qquad \mathfrak{R}_{(23)}\mathfrak{R}^{-1}_{(12)}\mathfrak{R}^{-1}_{(13)} = \mathfrak{R}^{-1}_{(13)}\mathfrak{R}^{-1}_{(12)}\mathfrak{R}_{(23)}\,, \tag{3.78}$$

with the former obtained by applying $\tau_{(12)}\tau_{(23)}$ to (3.11). Applying $\mathrm{Id}\otimes\varrho^{\otimes 2}$ to (3.78) verifies (3.17):

$$R_{12}\phi^{\mathrm{DF}}(L^\pm)_1\phi^{\mathrm{DF}}(L^\pm)_2 = \phi^{\mathrm{DF}}(L^\pm)_2\phi^{\mathrm{DF}}(L^\pm)_1 R_{12}\,.$$

We note that because $\mathrm{Id}\otimes\varrho^{\otimes 2}$ maps the second and third tensor factors of $U_q(\mathfrak{gl}(V))^{\hat\otimes 3}$ to the last two factors in $U(R)\otimes \mathrm{End}(V_q)^{\otimes 2}$, the element $\mathfrak{R}_{(23)}$ is sent to R_{12}.

Likewise, applying $\tau_{(12)}$ to (3.11) and multiplying by $\mathfrak{R}^{-1}_{(13)}$, we get $\mathfrak{R}_{(23)}\mathfrak{R}_{(21)}\mathfrak{R}^{-1}_{(13)} = \mathfrak{R}^{-1}_{(13)}\mathfrak{R}_{(21)}\mathfrak{R}_{(23)}$. Applying $\mathrm{Id}\otimes\varrho^{\otimes 2}$ to the latter equality then verifies (3.18):

$$R_{12}\phi^{\mathrm{DF}}(L^+)_1\phi^{\mathrm{DF}}(L^-)_2 = \phi^{\mathrm{DF}}(L^-)_2\phi^{\mathrm{DF}}(L^+)_1 R_{12}\,.$$

To verify that ϕ^{DF} also intertwines the respective comultiplications, we note that (3.12) implies

$$(\Delta\otimes\mathrm{Id})\mathfrak{R}_{(21)} = \mathfrak{R}_{(32)}\mathfrak{R}_{(31)}\,, \qquad (\Delta\otimes\mathrm{Id})\mathfrak{R}^{-1} = \mathfrak{R}^{-1}_{(23)}\mathfrak{R}^{-1}_{(13)}\,.$$

Applying $\mathrm{Id}^{\otimes 2}\otimes\varrho$ to this yields $(\Delta\otimes\mathrm{Id})\phi^{\mathrm{DF}}(L^\pm) = \phi^{\mathrm{DF}}(L^\pm)_{(2)}\phi^{\mathrm{DF}}(L^\pm)_{(1)}$. This completes the proof. □

To understand the relation between ϕ^{DF} and ξ, we start with the following explicit computation.

Lemma 3.79. *The images of the generators $\{l^\pm_{ii}\}_{i=1}^N$ and $\{l^+_{i,i+1}, l^-_{i+1,i}\}_{i=1}^N$ under ϕ^{DF} are as follows:*

$$\phi^{\mathrm{DF}}:\quad l^\pm_{ii}\mapsto q^{\mp H_i}\,, \qquad l^+_{i,i+1}\mapsto -(q-q^{-1})f_i q^{-H_{i+1}}\,, \qquad l^-_{i+1,i}\mapsto (-1)^{\overline{i+1}}(q-q^{-1})q^{H_{i+1}}e_i\,.$$

Proof. We begin by evaluating the semisimple part. A direct $\hbar$-adic computation yields:

$$(\mathrm{Id}\otimes\varrho)(\mathfrak{R}_s) = \sum_{1\le i\le N} q^{-H_i}\otimes E_{ii}\,,$$

so that $\phi^{\mathrm{DF}}(l^\pm_{ii}) = q^{\mp H_i}$. Next, we evaluate the unipotent part. We first determine the $E_{i,i+1}$-component of

$$(\mathrm{Id}\otimes\varrho)((\mathfrak{R}_u)_{(21)}) = \prod_{\gamma\in\Phi^+}^{\leftarrow}\left(\sum_{k\le 1 \text{ if } \gamma\in\Phi_{\bar 1}}^{k\ge 0}(-1)^{|e_\gamma^k|}\frac{f_\gamma^k\otimes\varrho(e_\gamma^k)}{(f_\gamma^k, e_\gamma^k)_{\mathrm{DJ}}}\right).$$

Since ϱ preserves the Q-degree and $\deg(E_{i,i+1}) = \alpha_i$ is a simple root, the only term contributing to this component is

$$(-1)^{|e_{\alpha_i}|}\frac{f_{\alpha_i}\otimes\varrho(e_{\alpha_i})}{(f_{\alpha_i}, e_{\alpha_i})_{\mathrm{DJ}}} = -(q-q^{-1})f_i\otimes E_{i,i+1}\,,$$

which implies that $\phi^{\mathrm{DF}}(l^+_{i,i+1}) = -(q-q^{-1})f_i q^{-H_{i+1}}$. Likewise, let us compute the $E_{i+1,i}$-component of

$$(\mathrm{Id}\otimes\varrho)(\mathfrak{R}_u^{-1}) = \prod_{\gamma\in\Phi^+}^{\rightarrow}\left(\sum_{k\le 1 \text{ if } \gamma\in\Phi_{\bar 1}}^{k\ge 0}\frac{e_\gamma^k\otimes\varrho(f_\gamma^k)}{(f_\gamma^k, e_\gamma^k)_{\mathrm{DJ}}}\right)^{-1} = \prod_{\gamma\in\Phi^+}^{\rightarrow}\sum_{r\ge 0}\left(-\sum_{k\le 1 \text{ if } \gamma\in\Phi_{\bar 1}}^{k\ge 1}\frac{e_\gamma^k\otimes\varrho(f_\gamma^k)}{(f_\gamma^k, e_\gamma^k)_{\mathrm{DJ}}}\right)^r,$$

where we use the fact that $\sum_{k\le 1 \text{ if } \gamma\in\Phi_{\bar 1}}^{k\ge 1}\frac{e_\gamma^k\otimes\varrho(f_\gamma^k)}{(f_\gamma^k, e_\gamma^k)_{\mathrm{DJ}}}$ is nilpotent. As above, the only contributing term is thus

$$-\frac{e_{\alpha_i}\otimes\varrho(f_{\alpha_i})}{(f_{\alpha_i}, e_{\alpha_i})_{\mathrm{DJ}}} = (-1)^{\bar i+\overline{i+1}}(q-q^{-1})e_i\otimes(-1)^{\bar i}E_{i+1,i} = (-1)^{\overline{i+1}}(q-q^{-1})e_i\otimes E_{i+1,i}\,,$$

which implies that $\phi^{\mathrm{DF}}(l^-_{i+1,i}) = (-1)^{\overline{i+1}}(q-q^{-1})q^{H_{i+1}}e_i$. This completes the proof. □

We shall now exhibit ϕ^{DF} as the inverse of ξ, up to canonical superalgebra automorphisms.

Proposition 3.80. *The composition*

$$U_q(\mathfrak{gl}(V)) \xrightarrow[(2.53)]{\phi_{\mathcal{F}}^{-1}} U_q(\mathfrak{gl}(V))^{\mathcal{F}} \xrightarrow[(3.72)]{\xi} U(R) \xrightarrow[(3.35)]{\omega_R^{-1}} U(R)^{\mathrm{cop}} \xrightarrow[(3.77)]{\phi^{\mathrm{DF}}} U_q(\mathfrak{gl}(V))$$

is the identity morphism.

Proof. This is verified by direct computation (cf. (2.11)):

$$q^{\pm H_k} \xmapsto{\phi_{\mathcal{F}}^{-1}} q^{\pm H_k} \overset{\xi}{\mapsto} q^{\pm \mathbf{H}_k} \xmapsto{\omega_R^{-1}} q^{\mp \mathbf{H}_k} \xmapsto{\phi^{\mathrm{DF}}} q^{\pm H_k}\,,$$

$$e_i \xmapsto{\phi_{\mathcal{F}}^{-1}} e_i q^{h_i} \overset{\xi}{\mapsto} \mathbf{e}_{i,i+1} q^{\mathbf{H}_{i,i+1}} \xmapsto{\omega_R^{-1}} -(-1)^{\overline{i}\,\overline{i+1}}(-1)^{\overline{i}+\overline{i+1}} q^{3(\rho,\alpha_i)} \mathbf{f}_{i+1,i} q^{-\mathbf{H}_{i,i+1}} \xmapsto{\phi^{\mathrm{DF}}} e_i\,,$$

$$f_i \xmapsto{\phi_{\mathcal{F}}^{-1}} q^{-h_i} f_i \overset{\xi}{\mapsto} q^{-\mathbf{H}_{i,i+1}} \mathbf{f}_{i+1,i} \xmapsto{\omega_R^{-1}} -(-1)^{\overline{i}\,\overline{i+1}} q^{-3(\rho,\alpha_i)} q^{\mathbf{H}_{i,i+1}} \mathbf{e}_{i,i+1} \xmapsto{\phi^{\mathrm{DF}}} f_i\,. \qquad \square$$

3.9. **Factorization of the reduced canonical element: general linear case.**

In this subsection, we derive the canonical element $\mathfrak{R}^R$ of $U(R)$ corresponding to the double construction and establish the factorization of the associated reduced R-matrix. Although this recovers the main result of [24], we present the complete argument here, as it will require minimal modifications to extend present results to the orthosymplectic setup in Subsection 5.6. To align with the order of the Borel subalgebras utilized in (2.41), we employ the transposed inverse pairing $(-,-)_{\widetilde{R}}$ introduced in Remark 3.46 (see also Proposition A.28 and Corollary A.35). The non-degeneracy of this pairing follows directly from that of $(-,-)_R$ via Theorem 3.71(c), together with (A.10) and (A.27).

To construct the canonical element with respect to this pairing, we first establish bases for the relevant subalgebras. Let $U^>(R)$, $U^<(R)$, and $U^0(R)$ be the subalgebras of $U(R)$ generated by $\{\mathbf{e}_{ij}\}_{1\le i<j\le N}$, $\{\mathbf{f}_{ji}\}_{1\le i<j\le N}$, and $\{q^{\pm\mathbf{H}_k}\}_{k=1}^N$, respectively. By Theorem 3.71 and Remark 2.39, we obtain the following isomorphisms of underlying superspaces given by the multiplication morphisms:

$$U^0(R)\otimes U^>(R) \xrightarrow{\sim} U^{\ge}(R)\,, \qquad U^0(R)\otimes U^<(R) \xrightarrow{\sim} U^{\le}(R)\,. \tag{3.81}$$

It is clear that the set $\big\{q^{\sum_{k=1}^N m_k \mathbf{H}_k} \,\big|\, m_1,\ldots,m_N\in\mathbb{Z}\big\}$ forms a basis for $U^0(R)$. For notational convenience, given any positive root $\gamma=\gamma_{ij}=\varepsilon_i-\varepsilon_j\in\Phi^+$, we shall use the notation $\mathbf{e}_\gamma := \mathbf{e}_{ij}$, $\mathbf{f}_\gamma := \mathbf{f}_{ji}$, and $q^{\pm\mathbf{H}_\gamma} := q^{\pm\mathbf{H}_{ij}}$. To establish the bases for $U^>(R)$ and $U^<(R)$, we need the following technical result:

Lemma 3.82. *For any non-negative integers m_γ, n_γ with $m_\gamma, n_\gamma \le 1$ if $\gamma\in\Phi_{\bar{1}}$, we have*

$$\left(\overleftarrow{\prod_{\gamma\in\Phi^+}} \mathbf{f}_\gamma^{m_\gamma},\ \overleftarrow{\prod_{\gamma\in\Phi^+}} \mathbf{e}_\gamma^{n_\gamma}\right)_{\widetilde{R}} = (-1)^{\chi(\mathbf{m})} \prod_{\gamma\in\Phi^+} \delta_{m_\gamma,n_\gamma} \mathsf{C}_{\gamma,m_\gamma} (\mathbf{f}_\gamma,\mathbf{e}_\gamma)_{\widetilde{R}}^{m_\gamma}\,, \tag{3.83}$$

where the scalar $\mathsf{C}_{\gamma,p}$ is defined by

$$\mathsf{C}_{\gamma,p} := \prod_{k=1}^{p} \frac{1-\big((-1)^{|\mathbf{e}_\gamma|} q^{(\gamma,\gamma)}\big)^k}{1-(-1)^{|\mathbf{e}_\gamma|} q^{(\gamma,\gamma)}}\,,$$

and for the sequence $\mathbf{m}=(m_\gamma)_{\gamma\in\Phi^+}$ we define $\chi(\mathbf{m})$ via

$$\chi(\mathbf{m}) := \sum_{\gamma\prec\gamma'} m_\gamma m_{\gamma'} |\mathbf{e}_\gamma||\mathbf{e}_{\gamma'}| + \sum_{\gamma\in\Phi^+} \binom{m_\gamma}{2} |\mathbf{e}_\gamma|\,.$$

Proof. The proof proceeds by induction on $\sum_{\gamma\in\Phi^+}(m_\gamma+n_\gamma)$. The base cases, when this sum is 0 or 1, are obvious due to degree reasons. For any $\lambda\in Q$, let $U(R)_\lambda, U^{\ge}(R)_\lambda, U^{\le}(R)_\lambda$ denote the corresponding degree λ components. By (3.21), the coproducts of the quantum root vectors satisfy the following inclusions:

$$\begin{aligned}
\Delta(\mathbf{e}_\gamma) &\in \mathbf{e}_\gamma\otimes q^{-\mathbf{H}_\gamma} + 1\otimes \mathbf{e}_\gamma + \sum_{\gamma'\prec\gamma\prec\gamma''} U^{\ge}(R)_{\gamma'}\otimes U^{\ge}(R)_{\gamma''}\,,\\
\Delta(\mathbf{f}_\gamma) &\in q^{\mathbf{H}_\gamma}\otimes \mathbf{f}_\gamma + \mathbf{f}_\gamma\otimes 1 + \sum_{\gamma'\succ\gamma\succ\gamma''} U^{\le}(R)_{-\gamma'}\otimes U^{\le}(R)_{-\gamma''}\,,
\end{aligned} \tag{3.84}$$

where $\prec$ denotes the convex order (3.5), and thus $\gamma',\gamma''\in\Phi^+$ satisfy $\gamma'+\gamma''=\gamma$. Let $\mathbf{E}=\overleftarrow{\prod}_{\gamma\in\Phi^+}\mathbf{e}_\gamma^{n_\gamma}$ and $\mathbf{F}=\overleftarrow{\prod}_{\gamma\in\Phi^+}\mathbf{f}_\gamma^{m_\gamma}$. Assume the sequences (m_γ) and (n_γ) are not both identically zero, and let $\alpha\in\Phi^+$ be the minimal root with respect to the convex order $\prec$ such that $m_\alpha\ne 0$ or $n_\alpha\ne 0$.

First assume that $m_\alpha>0$. We factor $\mathbf{F}=\mathbf{F}'\mathbf{f}_\alpha$ with $\mathbf{F}' := \big(\overleftarrow{\prod}_{\gamma\succ\alpha}\mathbf{f}_\gamma^{m_\gamma}\big)\mathbf{f}_\alpha^{m_\alpha-1}$. Then we have

$$(\mathbf{F},\mathbf{E})_{\widetilde{R}} = (\mathbf{F}'\otimes\mathbf{f}_\alpha, \Delta(\mathbf{E}))_{\widetilde{R}} = \sum_{(\mathbf{E})} (-1)^{|\mathbf{f}_\alpha||\mathbf{E}_1|} (\mathbf{F}',\mathbf{E}_1)_{\widetilde{R}} (\mathbf{f}_\alpha,\mathbf{E}_2)_{\widetilde{R}}\,.$$

Because $n_\gamma=0$ for all $\gamma\prec\alpha$, the product $\mathbf{E}$ consists of root vectors $\mathbf{e}_\gamma$ with $\gamma\succeq\alpha$, and furthermore the second tensor factor of each $\Delta(\mathbf{e}_\gamma)$ is a sum of homogeneous elements of degree 0 or $\succeq\alpha$ by (3.84). Since the skew-pairing is of Q-degree zero and $\deg(\mathbf{f}_\alpha)=-\alpha$, every nonzero contribution requires $\mathbf{E}_2$ to have degree exactly α. Because all contributing terms have degree 0 or $\succeq\alpha$, convexity of the order (3.5) implies that the only way to form a total degree of α is to select the $(1\otimes\mathbf{e}_\alpha)$ term from the comultiplication of exactly one $\mathbf{e}_\alpha$ factor, and the $(\mathbf{e}_\gamma \otimes q^{-\mathbf{H}_\gamma})$ term from all other factors in $\mathbf{E}$. Thus, the only contributing part of $\Delta(\mathbf{E})$ is:

$$\overleftarrow{\prod_{\gamma\succ\alpha}}(\mathbf{e}_\gamma \otimes q^{-\mathbf{H}_\gamma}) \cdot (\mathbf{e}_\alpha \otimes q^{-\mathbf{H}_\alpha} + 1 \otimes \mathbf{e}_\alpha)^{n_\alpha} .$$

Together with the following computation:

$$\begin{aligned}\sum_{i=1}^{n_\alpha}(\mathbf{e}_\alpha \otimes q^{-\mathbf{H}_\alpha})^{i-1}(1\otimes \mathbf{e}_\alpha)(\mathbf{e}_\alpha \otimes q^{-\mathbf{H}_\alpha})^{n_\alpha-i} &= \sum_{i=1}^{n_\alpha}\Big((-1)^{|\mathbf{e}_\alpha|}q^{(\alpha,\alpha)}\Big)^{n_\alpha-i}(\mathbf{e}_\alpha \otimes q^{-\mathbf{H}_\alpha})^{n_\alpha-1}(1\otimes \mathbf{e}_\alpha)\\ &= \frac{1-\big((-1)^{|\mathbf{e}_\alpha|}q^{(\alpha,\alpha)}\big)^{n_\alpha}}{1-(-1)^{|\mathbf{e}_\alpha|}q^{(\alpha,\alpha)}}(\mathbf{e}_\alpha \otimes q^{-\mathbf{H}_\alpha})^{n_\alpha-1}(1\otimes \mathbf{e}_\alpha)\,,\end{aligned}$$

this yields:

$$\begin{aligned}(\mathbf{F},\mathbf{E})_{\widetilde{R}} &= (-1)^{|\mathbf{f}_\alpha||\mathbf{F}'|}\frac{1-\big((-1)^{|\mathbf{e}_\alpha|}q^{(\alpha,\alpha)}\big)^{n_\alpha}}{1-(-1)^{|\mathbf{e}_\alpha|}q^{(\alpha,\alpha)}}(\mathbf{F}',\mathbf{E}')_{\widetilde{R}}\left(\mathbf{f}_\alpha, q^{-\sum_{\gamma\succ\alpha}n_\gamma \mathbf{H}_\gamma}q^{-(n_\alpha-1)\mathbf{H}_\alpha}\mathbf{e}_\alpha\right)_{\widetilde{R}}\\ &= (-1)^{|\mathbf{f}_\alpha||\mathbf{F}'|}\frac{1-\big((-1)^{|\mathbf{e}_\alpha|}q^{(\alpha,\alpha)}\big)^{n_\alpha}}{1-(-1)^{|\mathbf{e}_\alpha|}q^{(\alpha,\alpha)}}(\mathbf{F}',\mathbf{E}')_{\widetilde{R}}(\mathbf{f}_\alpha,\mathbf{e}_\alpha)_{\widetilde{R}}\,,\end{aligned}$$

where $\mathbf{E}' := \big(\overleftarrow{\prod}_{\gamma\succ\alpha}\mathbf{e}_\gamma^{n_\gamma}\big)\mathbf{e}_\alpha^{n_\alpha-1}$, and we use that $(\mathbf{f}_\alpha,\mathbf{K}\mathbf{e}_\alpha)_{\widetilde{R}} = (\mathbf{f}_\alpha,\mathbf{e}_\alpha)_{\widetilde{R}}$ for any $\mathbf{K}\in U^0(R)$ by (3.84). Applying the induction hypothesis to evaluate $(\mathbf{F}',\mathbf{E}')_{\widetilde{R}}$ then yields the desired factorization for $(\mathbf{F},\mathbf{E})_{\widetilde{R}}$.

The case when $m_\alpha = 0$ but $n_\alpha > 0$ proceeds analogously by reversing the roles of $\mathbf{e}$ and $\mathbf{f}$. □

From the lemma above, we obtain the following PBW-type basis of $U^>(R)$ and $U^<(R)$:

Corollary 3.85. *The sets of ordered monomials*

$$\left\{\overleftarrow{\prod_{\gamma\in\Phi^+}}\mathbf{e}_\gamma^{m_\gamma}\,\middle|\, {\scriptstyle m_\gamma\in\mathbb{Z}_{\geq 0} \atop m_\gamma\leq 1 \text{ if } \gamma\in\Phi_{\bar 1}}\right\} \qquad \textit{and} \qquad \left\{\overleftarrow{\prod_{\gamma\in\Phi^+}}\mathbf{f}_\gamma^{m_\gamma}\,\middle|\, {\scriptstyle m_\gamma\in\mathbb{Z}_{\geq 0} \atop m_\gamma\leq 1 \text{ if } \gamma\in\Phi_{\bar 1}}\right\} \tag{3.86}$$

form bases for $U^>(R)$ and $U^<(R)$, respectively.

Proof. Since Lemma 3.82 establishes that these monomials are orthogonal with respect to the skew-pairing $(-,-)_{\widetilde{R}}$, their linear independence immediately follows from the non-degeneracy of this pairing.

Furthermore, Theorem 3.71 ensures that each degree λ component of $U^>(R)$ and $U^<(R)$ is finite-dimensional and shares the exact same dimension as its counterpart in $U_q^>(\mathfrak{gl}(V))$ and $U_q^<(\mathfrak{gl}(V))$. Since the degrees of the respective generators match ($\deg(\mathbf{e}_\gamma)=\gamma=\deg(e_\gamma)$ and $\deg(\mathbf{f}_\gamma)=-\gamma=\deg(f_\gamma)$), the number of monomials of degree λ in (3.86) precisely equals this dimension, due to Theorem 3.6. Thus, (3.86) indeed form bases. □

Consider the following subspaces (cf. (2.44)):

$$U^{\geq}(R)_{\deg>0} := \bigoplus_{\lambda>0}U^{\geq}(R)_\lambda\,, \qquad U^{\leq}(R)_{\deg<0} := \bigoplus_{\lambda<0}U^{\leq}(R)_\lambda\,.$$

For any Q-homogeneous elements $\mathbf{e}\in U^>(R)$ and $\mathbf{f}\in U^<(R)$, formula (3.84) implies (cf. (2.45)):

$$\Delta(\mathbf{e})\in 1\otimes\mathbf{e}+U^{\geq}(R)_{\deg>0}\otimes U^{\geq}(R)\,, \qquad \Delta(\mathbf{f})\in \mathbf{f}\otimes 1+U^{\leq}(R)\otimes U^{\leq}(R)_{\deg<0}\,. \tag{3.87}$$

Thus, for any Q-homogeneous $\mathbf{e}\in U^>(R), \mathbf{f}\in U^<(R)$ and group-like elements $\mathbf{k},\mathbf{k}'\in U^0(R)$, we obtain:

$$(\mathbf{k}'\mathbf{f},\mathbf{k}\mathbf{e})_{\widetilde{R}} = (\mathbf{k}'\otimes\mathbf{f},\Delta(\mathbf{k})\Delta(\mathbf{e}))_{\widetilde{R}} = (\mathbf{k}',\mathbf{k})_{\widetilde{R}}(\mathbf{f},\mathbf{k}\mathbf{e})_{\widetilde{R}} = (\mathbf{k}',\mathbf{k})_{\widetilde{R}}(\Delta^{\mathrm{op}}(\mathbf{f}),\mathbf{k}\otimes\mathbf{e})_{\widetilde{R}} = (\mathbf{k}',\mathbf{k})_{\widetilde{R}}(\mathbf{f},\mathbf{e})_{\widetilde{R}}\,. \tag{3.88}$$

Bilinearity ensures that (3.88) holds for all $\mathbf{k},\mathbf{k}'\in U^0(R)$. Together with the triangular decomposition (3.81), this allows us to factor the canonical element $\mathfrak{R}^R$ of the double $\mathcal{D}(U^{\leq}(R),U^{\geq}(R))$ (cf. Subsection 3.1)

$$\mathfrak{R}^R = \mathfrak{R}^R_s\mathfrak{R}^R_u$$

into the canonical elements associated with the restrictions of $(-,-)_{\widetilde{R}}$ to $U^0(R)\times U^0(R)$ and $U^<(R)\times U^>(R)$. Using the PBW-type bases (3.86) and their orthogonality from Lemma 3.82, we can explicitly evaluate $\mathfrak{R}^R_u$.

Proposition 3.89. *$\mathfrak{R}^R_u$ factorizes as*

$$\mathfrak{R}^R_u = \overleftarrow{\prod_{\gamma\in\Phi^+}}\left(\sum_{k\leq 1 \text{ if } \gamma\in\Phi_{\bar 1}}^{k\geq 0}\frac{\mathbf{e}_\gamma^k\otimes\mathbf{f}_\gamma^k}{(\mathbf{f}_\gamma^k,\mathbf{e}_\gamma^k)_{\widetilde{R}}}\right)\,.$$

Proof. Let us first record the following special case of (3.83):

$$(\mathbf{f}_\gamma^{m_\gamma},\mathbf{e}_\gamma^{n_\gamma})_{\widetilde{R}} = \delta_{m_\gamma,n_\gamma}(-1)^{\binom{m_\gamma}{2}|\mathbf{e}_\gamma|}\mathsf{C}_{\gamma,m_\gamma}(\mathbf{f}_\gamma,\mathbf{e}_\gamma)_{\widetilde{R}}^{m_\gamma}\,.$$

This allows us to rewrite the general pairing (3.83) as:

$$\left(\overleftarrow{\prod_{\gamma\in\Phi^+}}\mathbf{f}_\gamma^{m_\gamma},\ \overleftarrow{\prod_{\gamma\in\Phi^+}}\mathbf{e}_\gamma^{n_\gamma}\right)_{\widetilde{R}} = (-1)^{\sum_{\gamma\prec\gamma'} m_\gamma m_{\gamma'}|\mathbf{e}_\gamma||\mathbf{e}_{\gamma'}|}\prod_{\gamma\in\Phi^+}\delta_{m_\gamma,n_\gamma}(\mathbf{f}_\gamma^{m_\gamma},\mathbf{e}_\gamma^{m_\gamma})_{\widetilde{R}}\,.$$

Combining this with the straightforward tensor product rearrangement

$$\overleftarrow{\prod_{\gamma\in\Phi^+}}\mathbf{e}_\gamma^{m_\gamma}\otimes\overleftarrow{\prod_{\gamma\in\Phi^+}}\mathbf{f}_\gamma^{m_\gamma} = (-1)^{\sum_{\gamma\prec\gamma'} m_\gamma m_{\gamma'}|\mathbf{e}_\gamma||\mathbf{e}_{\gamma'}|}\overleftarrow{\prod_{\gamma\in\Phi^+}}\mathbf{e}_\gamma^{m_\gamma}\otimes\mathbf{f}_\gamma^{m_\gamma}$$

and the equalities $\mathbf{e}_\gamma^2=0=\mathbf{f}_\gamma^2$ for odd γ (see Remark 3.68), we obtain the claimed factorization of $\mathfrak{R}_u^R$:

$$\begin{aligned}\mathfrak{R}_u^R &= \sum_{m_\gamma\le 1 \text{ if } \gamma\in\Phi_{\bar 1}}^{m_\gamma\ge 0}\left(\overleftarrow{\prod_{\gamma\in\Phi^+}}\mathbf{e}_\gamma^{m_\gamma}\otimes\overleftarrow{\prod_{\gamma\in\Phi^+}}\mathbf{f}_\gamma^{m_\gamma}\right)\Big/\left(\overleftarrow{\prod_{\gamma\in\Phi^+}}\mathbf{f}_\gamma^{m_\gamma},\ \overleftarrow{\prod_{\gamma\in\Phi^+}}\mathbf{e}_\gamma^{m_\gamma}\right)_{\widetilde{R}}\\ &= \sum_{m_\gamma\le 1 \text{ if } \gamma\in\Phi_{\bar 1}}^{m_\gamma\ge 0}\ \overleftarrow{\prod_{\gamma\in\Phi^+}}\frac{\mathbf{e}_\gamma^{m_\gamma}\otimes\mathbf{f}_\gamma^{m_\gamma}}{(\mathbf{f}_\gamma^{m_\gamma},\mathbf{e}_\gamma^{m_\gamma})_{\widetilde{R}}} = \overleftarrow{\prod_{\gamma\in\Phi^+}}\left(\sum_{k\le 1 \text{ if } \gamma\in\Phi_{\bar 1}}^{k\ge 0}\frac{\mathbf{e}_\gamma^{k}\otimes\mathbf{f}_\gamma^{k}}{(\mathbf{f}_\gamma^{k},\mathbf{e}_\gamma^{k})_{\widetilde{R}}}\right).\end{aligned}$$

□

4. Orthosymplectic Lie superalgebras and quantum supergroups

4.1. **Orthosymplectic Lie superalgebras.**

As in Subsection 2.3, we fix a superspace V of dimension (m,n), and its homogeneous basis $\{v_1,\ldots,v_N\}$, where $N=m+n$. We assume that n is even, and define $s=\lfloor\frac{N}{2}\rfloor$. Consider the involution $'$ on the index set $\mathbb{I}=\{1,2,\ldots,N\}$ defined by:

$$i\mapsto i' := N+1-i\,.$$

We shall assume that $\bar i=\bar{i'}$ for all $i\in\mathbb{I}$. In particular, v_{s+1} is necessarily even when N is odd. We also choose a sequence $\vartheta_V:=(\vartheta_1,\vartheta_2,\ldots,\vartheta_N)\in\{\pm1\}^N$ such that $\vartheta_i=1$ if $\bar i=\bar{0}$ and $\vartheta_i=-\vartheta_{i'}$ if $\bar i=\bar 1$ (we do not assume $\vartheta_i=1$ for $i\le s$). From the conditions above, we immediately have:

$$\vartheta_i^2=1 \qquad \text{and} \qquad \vartheta_i\vartheta_{i'}=(-1)^{\bar i} \qquad \text{for all} \quad i\in\mathbb{I}\,.$$

Consider a bilinear form $B_G\colon V\times V\to\mathbb{C}$ defined by the anti-diagonal matrix $G=\sum_{i=1}^N\vartheta_i E_{i'i}$. The *orthosymplectic Lie superalgebra* $\mathfrak{osp}(V)$ is defined as the Lie subalgebra of $\mathfrak{gl}(V)$ consisting of all matrices X that preserve the bilinear form B_G, meaning (see notation (3.28))

$$X^{\mathrm{st}}G+GX=0\,.$$

Explicitly, $\mathfrak{osp}(V)$ is spanned by the elements

$$X_{ij}=E_{ij}-(-1)^{\bar i(\bar i+\bar j)}\vartheta_i\vartheta_j E_{j'i'}\,.$$

Since $X_{j'i'}=-(-1)^{\bar i(\bar i+\bar j)}\vartheta_i\vartheta_j\cdot X_{ij}$, the following elements form a basis for $\mathfrak{osp}(V)$:

$$\big\{X_{ij}\,\big|\,i+j\le N\big\}\cup\big\{X_{ii'}\,\big|\,\bar i=\bar 1\,,\ 1\le i\le s\big\}\,.$$

We also introduce a closely related object, the *extended orthosymplectic Lie superalgebra*[2]

$$\mathfrak{gosp}(V):=\mathfrak{osp}(V)\oplus\mathbb{C}\cdot\mathrm{I}\subset\mathfrak{gl}(V)\,,$$

where I denotes the identity matrix. We note that both $\mathfrak{osp}(V)$ and $\mathfrak{gosp}(V)$ inherit the $\mathbb{Z}_2$-grading from $\mathfrak{gl}(V)$. Since $[\mathrm{I},X]=0$ for all $X\in\mathfrak{osp}(V)$, the orthosymplectic Lie superalgebra $\mathfrak{osp}(V)$ forms a codimension 1 ideal of $\mathfrak{gosp}(V)$. To avoid confusion, the central element I will frequently be denoted by C.

Similarly to $\mathfrak{gl}(V)$, we choose the Borel subalgebras of $\mathfrak{osp}(V)$ and $\mathfrak{gosp}(V)$ consisting of all upper triangular matrices. The Cartan subalgebra $\mathfrak{h}_{\mathfrak{osp}}$ of $\mathfrak{osp}(V)$ consists of all diagonal matrices and has basis $\{X_{ii}\}_{i=1}^s$, while the Cartan subalgebra $\mathfrak{h}_{\mathfrak{gosp}}$ of $\mathfrak{gosp}(V)$ is the direct sum $\mathfrak{h}_{\mathfrak{osp}}\oplus\mathbb{C}\cdot C$. Recall the linear functionals $\{\varepsilon_i\}_{i=1}^N$ on $\mathfrak{gl}(V)$, and let $\tilde\varepsilon_i:=\varepsilon_i|_{\mathfrak{h}_{\mathfrak{gosp}}}$ be their restrictions to the Cartan subalgebra $\mathfrak{h}_{\mathfrak{gosp}}$ of $\mathfrak{gosp}(V)$. Then $\tilde\varepsilon_i+\tilde\varepsilon_{i'}=\tilde\varepsilon_j+\tilde\varepsilon_{j'}$ for all $1\le i,j\le N$. Thus $\tilde\varepsilon_C:=(\tilde\varepsilon_i+\tilde\varepsilon_{i'})/2$ is well-defined independently from the choice of i, and $\tilde\varepsilon_C=\tilde\varepsilon_{s+1}$ for odd N. The set $\{\tilde\varepsilon_i-\tilde\varepsilon_C\}_{i=1}^s\cup\{\tilde\varepsilon_C\}$ forms a basis of $\mathfrak{h}^*_{\mathfrak{gosp}}$ dual to

[2]Our choice of notation $\mathfrak{gosp}$ follows that of [20].

the basis $\{X_{ii}\}_{i=1}^s \cup \{C\}$ of $\mathfrak{h}_{\mathfrak{gosp}}$. Moreover, as $\tilde{\varepsilon}_C|_{\mathfrak{h}_{\mathfrak{osp}}} = 0$, we note that $\varepsilon_i|_{\mathfrak{h}_{\mathfrak{osp}}} = (\tilde{\varepsilon}_i - \tilde{\varepsilon}_C)|_{\mathfrak{h}_{\mathfrak{osp}}}$ for all $1 \le i \le N$, and therefore:

$$\varepsilon_i|_{\mathfrak{h}_{\mathfrak{osp}}} = -\varepsilon_{i'}|_{\mathfrak{h}_{\mathfrak{osp}}} \quad \text{for all } i\,, \qquad \varepsilon_{s+1}|_{\mathfrak{h}_{\mathfrak{osp}}} = 0 \quad \text{for odd } N\,.$$

This implies that $\{\varepsilon_i|_{\mathfrak{h}_{\mathfrak{osp}}}\}_{i=1}^s$ forms a basis of $\mathfrak{h}_{\mathfrak{osp}}^*$ dual to the basis $\{X_{ii}\}_{i=1}^s$ of $\mathfrak{h}_{\mathfrak{osp}}$. The functionals $\varepsilon_i|_{\mathfrak{h}_{\mathfrak{osp}}} \in \mathfrak{h}_{\mathfrak{osp}}^*$ and their extension to $\mathfrak{h}_{\mathfrak{gosp}}^*$ by zero on C will be simply denoted by ε_i, so that

$$\varepsilon_i = \tilde{\varepsilon}_i - \tilde{\varepsilon}_C \in \mathfrak{h}_{\mathfrak{gosp}}^* \qquad \forall\, 1 \le i \le N. \tag{4.1}$$

In complete parallel with the $\mathfrak{gl}(V)$ case, a direct computation shows that $[X_{ii}, X_{ab}] = (\varepsilon_a - \varepsilon_b)(X_{ii})X_{ab}$, and thus X_{ab} is a root vector corresponding to the root $\varepsilon_a - \varepsilon_b = \tilde{\varepsilon}_a - \tilde{\varepsilon}_b$. This yields the root space decomposition $\mathfrak{osp}(V) = \mathfrak{h}_{\mathfrak{osp}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{osp}(V)_\alpha$ with the root system given by

$$\Phi = \big\{\varepsilon_a - \varepsilon_b \,\big|\, a + b \le N\,,\ a \ne b\big\} \cup \big\{2\varepsilon_a \,\big|\, \overline{a} = \bar{\mathbf{1}}\big\}\,,$$

and the corresponding polarization:

$$\Phi^+ = \big\{\varepsilon_a - \varepsilon_b \,\big|\, a < b < a'\big\} \cup \big\{2\varepsilon_a \,\big|\, \overline{a} = \bar{\mathbf{1}}\,,\ a \le s\big\}\,,$$
$$\Phi^- = \big\{\varepsilon_a - \varepsilon_b \,\big|\, b < a < b'\big\} \cup \big\{2\varepsilon_a \,\big|\, \overline{a} = \bar{\mathbf{1}}\,,\ a' \le s\big\}\,.$$

We use $\Phi_{\bar{\mathbf{0}}}, \Phi_{\bar{\mathbf{1}}}$ to denote the sets of even, odd roots. Lastly, we consider the following *reduced* root system:

$$\bar{\Phi} = \big\{\gamma \in \Phi \,\big|\, \tfrac{1}{2}\gamma \notin \Phi\big\}\,, \qquad \bar{\Phi}_{\bar{\mathbf{0}}} = \bar{\Phi} \cap \Phi_{\bar{\mathbf{0}}}\,, \qquad \bar{\Phi}_{\bar{\mathbf{1}}} = \bar{\Phi} \cap \Phi_{\bar{\mathbf{1}}}\,. \tag{4.2}$$

We also note that $\mathfrak{gosp}(V)$ shares the same root system Φ and admits the corresponding root space decomposition

$$\mathfrak{gosp}(V) = \mathfrak{h}_{\mathfrak{gosp}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{gosp}(V)_\alpha \qquad \text{with} \quad \mathfrak{gosp}(V)_\alpha = \mathfrak{osp}(V)_\alpha \quad \forall\, \alpha \in \Phi\,.$$

4.2. **Chevalley–Serre-type presentation.**

With respect to above polarization, the simple roots and corresponding root vectors in $\mathfrak{osp}(V)$ are as follows:

- **Case 1 (*B*-type):** m is odd.

$$\begin{aligned} &\alpha_i = \varepsilon_i - \varepsilon_{i+1}\,, \quad \mathsf{e}_i = X_{i,i+1}\,, \quad \mathsf{f}_i = (-1)^{\bar{i}} X_{i+1,i} \qquad \text{for} \ \ 1 \le i \le s\,, \\ &\mathsf{h}_i = (-1)^{\bar{i}} X_{ii} - (-1)^{\overline{i+1}} X_{i+1,i+1} \qquad \text{for} \ \ 1 \le i \le s\,. \end{aligned} \tag{4.3}$$

- **Case 2 (*C*-type):** m is even and $\bar{s} = \bar{\mathbf{1}}$.

$$\begin{aligned} &\alpha_i = \begin{cases} \varepsilon_i - \varepsilon_{i+1} & \text{if} \ \ 1 \le i < s \\ 2\varepsilon_s & \text{if} \ \ i = s \end{cases}, \qquad \mathsf{e}_i = \begin{cases} X_{i,i+1} & \text{if} \ \ 1 \le i < s \\ E_{ss'} & \text{if} \ \ i = s \end{cases}, \\ &\mathsf{f}_i = \begin{cases} (-1)^{\bar{i}} X_{i+1,i} & \text{if} \ \ 1 \le i < s \\ -2E_{s's} & \text{if} \ \ i = s \end{cases}, \qquad \mathsf{h}_i = \begin{cases} (-1)^{\bar{i}} X_{ii} - (-1)^{\overline{i+1}} X_{i+1,i+1} & \text{if} \ \ 1 \le i < s \\ -2X_{ss} & \text{if} \ \ i = s \end{cases}. \end{aligned} \tag{4.4}$$

- **Case 3 (*D*-type):** m is even and $\bar{s} = \bar{\mathbf{0}}$.

$$\begin{aligned} &\alpha_i = \begin{cases} \varepsilon_i - \varepsilon_{i+1} & \text{if} \ \ 1 \le i < s \\ \varepsilon_{s-1} + \varepsilon_s & \text{if} \ \ i = s \end{cases}, \qquad \mathsf{e}_i = \begin{cases} X_{i,i+1} & \text{if} \ \ 1 \le i < s \\ X_{s-1,s'} & \text{if} \ \ i = s \end{cases}, \\ &\mathsf{f}_i = \begin{cases} (-1)^{\bar{i}} X_{i+1,i} & \text{if} \ \ 1 \le i < s \\ (-1)^{\overline{s-1}} X_{s',s-1} & \text{if} \ \ i = s \end{cases}, \qquad \mathsf{h}_i = \begin{cases} (-1)^{\bar{i}} X_{ii} - (-1)^{\overline{i+1}} X_{i+1,i+1} & \text{if} \ \ 1 \le i < s \\ (-1)^{\overline{s-1}} X_{s-1,s-1} + (-1)^{\bar{s}} X_{ss} & \text{if} \ \ i = s \end{cases}. \end{aligned} \tag{4.5}$$

While $\{\mathsf{h}_i\}_{i=1}^s$ is a basis for $\mathfrak{h}_{\mathfrak{osp}}$, it is more often convenient to employ an alternative basis. To this end, we consider the following elements:

$$\mathsf{H}_i := (-1)^{\bar{i}} X_{ii} = (-1)^{\bar{i}} (E_{ii} - E_{i'i'}) \qquad \text{for} \ \ 1 \le i \le N\,.$$

Specifically, $\mathsf{H}_i = -\mathsf{H}_{i'}$ for all $1 \le i \le N$, and $\mathsf{H}_{s+1} = 0$ when N is odd. The set $\{\mathsf{H}_i\}_{i=1}^s$ also forms a basis for $\mathfrak{h}_{\mathfrak{osp}}$, and in terms of this alternative basis, the elements h_i from (4.3)–(4.5) take the following form:

$$\mathsf{h}_i = \begin{cases} \mathsf{H}_i - \mathsf{H}_{i+1} & \text{if} \ \ 1 \le i < s \ \ (\text{in all } BCD\text{-types})\,, \\ \mathsf{H}_s & \text{if} \ \ i = s \ \ (B\text{-type})\,, \\ 2\mathsf{H}_s & \text{if} \ \ i = s \ \ (C\text{-type})\,, \\ \mathsf{H}_{s-1} + \mathsf{H}_s & \text{if} \ \ i = s \ \ (D\text{-type})\,. \end{cases} \tag{4.6}$$

Furthermore, the lattices $P^\vee := \mathbb{Z}C \oplus \bigoplus_{i=1}^s \mathbb{Z}\mathsf{H}_i$ and $Q^\vee := \bigoplus_{i=1}^s \mathbb{Z}\mathsf{h}_i$ constitute the coweight and coroot lattices of $\mathfrak{gosp}(V)$, respectively. We also define the weight lattice

$$P := \sum_{i=1}^N \mathbb{Z}\tilde{\varepsilon}_i = \begin{cases} \mathbb{Z}\tilde{\varepsilon}_C \oplus \bigoplus_{i=1}^s \mathbb{Z}\tilde{\varepsilon}_i & (B\text{-type}), \\ \mathbb{Z}(2\tilde{\varepsilon}_C) \oplus \bigoplus_{i=1}^s \mathbb{Z}\tilde{\varepsilon}_i & (CD\text{-types}), \end{cases}$$

the root lattice $Q := \bigoplus_{i=1}^s \mathbb{Z}\alpha_i \subseteq P$, and $Q^+ := \bigoplus_{i=1}^s \mathbb{Z}_{\geq 0}\alpha_i$, the latter giving rise to the partial order on P. One can verify that $\mathfrak{gosp}(V)$ is graded by Q (and hence also by P) via

$$\deg(\mathsf{e}_i) = \alpha_i\,, \qquad \deg(\mathsf{f}_i) = -\alpha_i\,, \qquad \deg(\mathsf{h}_i) = 0\,, \qquad \deg(\mathsf{H}_i) = 0\,, \qquad \deg(C) = 0\,, \tag{4.7}$$

which restricts to a Q-grading on $\mathfrak{osp}(V)$, and is compatible with the $\mathbb{Z}_2$-grading via the group homomorphism

$$P \to \mathbb{Z}_2\,, \qquad \tilde{\varepsilon}_i \mapsto \bar{i} \quad \text{for all} \quad 1 \leq i \leq N\,, \tag{4.8}$$

cf. (2.13). Specifically, this implies $\tilde{\varepsilon}_C \mapsto \bar{\mathbf{0}}$ for B-type, and $2\tilde{\varepsilon}_C \mapsto \bar{\mathbf{0}}$ for CD-types.

We define a non-degenerate bilinear form $(\cdot,\cdot)\colon \mathfrak{gosp}(V) \times \mathfrak{gosp}(V) \to \mathbb{C}$ by

$$(X,Y) = \tfrac{1}{2}\mathrm{sTr}(XY)\,, \qquad (C,X) = 0 = (X,C)\,, \qquad (C,C) = 1\,, \tag{4.9}$$

for all $X, Y \in \mathfrak{osp}(V)$. As in the $\mathfrak{gl}(V)$ case, its restriction to $\mathfrak{h}_{\mathfrak{gosp}}$ is non-degenerate. Through the identification $\mathfrak{h}^*_{\mathfrak{gosp}} \simeq \mathfrak{h}_{\mathfrak{gosp}}$ via $\varepsilon_i \leftrightarrow \mathsf{H}_i$ and $\tilde{\varepsilon}_C \leftrightarrow C$, this induces a bilinear form $(-,-)\colon \mathfrak{h}^*_{\mathfrak{gosp}} \times \mathfrak{h}^*_{\mathfrak{gosp}} \to \mathbb{C}$ satisfying (for all $1 \leq i,j \leq s$)

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}(-1)^{\bar{i}}\,, \qquad (\tilde{\varepsilon}_C, \varepsilon_i) = 0 = (\varepsilon_i, \tilde{\varepsilon}_C)\,, \qquad (\tilde{\varepsilon}_C, \tilde{\varepsilon}_C) = 1\,. \tag{4.10}$$

We note that the factor $1/2$ in (4.9) is used to ensure the first formula above akin to (2.15).

We define the *symmetrized Cartan matrix* by $a_{ij} = (\alpha_i, \alpha_j)$ for $1 \leq i,j \leq s$. The elements $\{\mathsf{e}_i, \mathsf{f}_i, \mathsf{h}_i\}_{i=1}^s$ generate $\mathfrak{osp}(V)$ and satisfy the exact same Chevalley-type relations (2.16) as in the $\mathfrak{gl}(V)$ case. By [25, Main Theorem], this set of generators and relations provides a complete presentation of $\mathfrak{osp}(V)$ when augmented with the appropriate Serre relations (which we omit here).

Remark 4.11. We remark that our choice of generators corresponds to a rescaled version of those used in [23, 25], as we work with the symmetrized Cartan matrix rather than the non-symmetrized one.

Remark 4.12. As in Remark 2.18, instead of using Cartan generators h_i or H_i, one can use an additive subgroup $\Gamma \subseteq \mathfrak{h}_{\mathfrak{gosp}}$ containing $\{\mathsf{h}_i\}_{i=1}^s$ to provide a uniform Chevalley-Serre type presentation for both $\mathfrak{osp}(V)$ and $\mathfrak{gosp}(V)$. Consider the Lie superalgebra $\mathfrak{g}(\Gamma)$ over $\mathbb{C}$ generated by $\{\mathsf{e}_i, \mathsf{f}_i\}_{i=1}^s$ and Γ, subject to the Serre relations together with the Chevalley-type relations as in Remark 2.18, but for all $\mathsf{H}, \mathsf{H}' \in \Gamma$ and $1 \leq i,j \leq s$. It is clear that $\mathfrak{g}(Q^\vee) = \mathfrak{osp}(V)$ and $\mathfrak{g}(\Gamma_{\mathfrak{gosp}}) = \mathfrak{gosp}(V)$, where

$$\Gamma_{\mathfrak{gosp}} = \begin{cases} \mathbb{Z}C \oplus \bigoplus_{i=1}^s \mathbb{Z}\mathsf{H}_i & (B\text{-type}), \\ \mathbb{Z}(2C) \oplus \bigoplus_{i=1}^s \mathbb{Z}(\mathsf{H}_i + C) & (CD\text{-types}), \end{cases}$$

is the image of P inside $P^\vee$ under the identification $\mathfrak{h}_{\mathfrak{gosp}} \simeq \mathfrak{h}^*_{\mathfrak{gosp}}$ induced by the above non-degenerate form. One can uniformly choose a different basis of the above lattice $\Gamma_{\mathfrak{gosp}} = \bigoplus_{i=1}^{s+1} \mathbb{Z}\tilde{\mathsf{H}}_i$ via

$$\tilde{\mathsf{H}}_i := \mathsf{H}_i + C \qquad \text{for} \quad 1 \leq i \leq N\,. \tag{4.13}$$

4.3. Drinfeld–Jimbo-type orthosymplectic quantum supergroups.

Similarly to the general linear case in Subsection 2.4, we define a class of quantum supergroups $U_q(\Gamma)$ parameterized by an additive subgroup $\Gamma \subseteq P^\vee$ containing the coroot lattice $Q^\vee$, see Remark 4.12.

Remark 4.14. As in Remark 2.19, we shall denote the elements $\mathsf{h}_i, \mathsf{H}_i, \tilde{\mathsf{H}}_i \in \mathfrak{h}_{\mathfrak{gosp}}$ by $h_i, H_i, \tilde{H}_i$, respectively, in the quantum supergroup setting in order to distinguish them from their classical counterparts.

Definition 4.15. For a fixed additive subgroup $\Gamma \subseteq P^\vee$ containing $Q^\vee$, the quantum supergroup $U_q(\Gamma)$ is the $\mathbb{C}(q)$-superalgebra generated by $\{e_i, f_i\}_{i=1}^s$ and the formal symbols $\{q^{\pm H} \mid H \in \Gamma\}$. It is equipped with the Q-grading (cf. (4.7)):

$$\deg(e_i) = \alpha_i\,, \qquad \deg(f_i) = -\alpha_i\,, \qquad \deg(q^{\pm H}) = 0\,, \tag{4.16}$$

and the compatible $\mathbb{Z}_2$-grading, see (4.8). These generators are subject to the Serre relations specified later in this subsection, alongside with the Chevalley-type relations (2.22)–(2.24) as in the $\mathfrak{gl}(V)$ case, but for all $H, H' \in \Gamma$ and $1 \leq i,j \leq s$, where $\alpha_j(H)$ denote the values of roots α_j on the Cartan elements $H \in P^\vee$.

To express the Serre relations, we recall the q-supercommutator $[\![-,-]\!]$ from (2.25). First of all, we impose (2.26) for all $1 \leq i,j \leq s$ as well as (2.27, 2.28) provided all featured indices are strictly less than s. The remaining Serre relations involving the index s are specified by the following type-dependent list:

- ***B*-type:**

$$\begin{aligned}
&[\![e_{s-1},[\![e_{s-1},e_s]\!]]\!]=0\,,\quad [\![f_{s-1},[\![f_{s-1},f_s]\!]]\!]=0 \qquad \text{if}\ \ \alpha_{s-1}\in\Phi_{\bar{0}}\,,\\
&[\![[\![[\![e_{s-1},e_s]\!],e_s]\!],e_s]\!]=0\,,\quad [\![[\![[\![f_{s-1},f_s]\!],f_s]\!],f_s]\!]=0 \qquad \text{if}\ \ \alpha_s\in\Phi_{\bar{0}}\,,\\
&[\![[\![[\![e_{s-2},e_{s-1}]\!],e_s]\!],e_{s-1}]\!]=0\,,\quad [\![[\![[\![f_{s-2},f_{s-1}]\!],f_s]\!],f_{s-1}]\!]=0 \qquad \text{if}\ \ \alpha_{s-1}\in\Phi_{\bar{1}}\,.
\end{aligned} \tag{4.17}$$

- ***C*-type:**

$$\begin{aligned}
&[\![e_{s-1},[\![e_{s-1},[\![e_{s-1},e_s]\!]]\!]]\!]=0\,,\quad [\![f_{s-1},[\![f_{s-1},[\![f_{s-1},f_s]\!]]\!]]\!]=0 \qquad \text{if}\ \ \alpha_{s-1}\in\Phi_{\bar{0}}\,,\\
&[\![[\![e_{s-1},e_s]\!],e_s]\!]=0\,,\quad [\![[\![f_{s-1},f_s]\!],f_s]\!]=0\,,\\
&\begin{cases} [\![[\![[\![[\![e_{s-2},e_{s-1}]\!],e_s]\!],[\![e_{s-2},e_{s-1}]\!]]\!],e_{s-1}]\!]=0\\ [\![[\![[\![[\![f_{s-2},f_{s-1}]\!],f_s]\!],[\![f_{s-2},f_{s-1}]\!]]\!],f_{s-1}]\!]=0\end{cases} \qquad \text{if}\ \ \alpha_{s-2},\alpha_{s-1}\in\Phi_{\bar{1}}\,,\\
&\begin{cases} [\![[\![[\![[\![[\![[\![e_{s-3},e_{s-2}]\!],e_{s-1}]\!],e_s]\!],e_{s-1}]\!],e_{s-2}]\!],e_{s-1}]\!]=0\\ [\![[\![[\![[\![[\![[\![f_{s-3},f_{s-2}]\!],f_{s-1}]\!],f_s]\!],f_{s-1}]\!],f_{s-2}]\!],f_{s-1}]\!]=0\end{cases} \qquad \text{if}\ \ \alpha_{s-2}\in\Phi_{\bar{0}},\ \alpha_{s-1}\in\Phi_{\bar{1}}\,.
\end{aligned} \tag{4.18}$$

- ***D*-type:**

$$\begin{aligned}
&[\![e_{s-2},[\![e_{s-2},e_s]\!]]\!]=0\,,\quad [\![f_{s-2},[\![f_{s-2},f_s]\!]]\!]=0 \qquad \text{if}\ \ \alpha_{s-2}\in\Phi_{\bar{0}}\,,\\
&[\![[\![e_{s-2},e_s]\!],e_s]\!]=0\,,\quad [\![[\![f_{s-2},f_s]\!],f_s]\!]=0 \qquad \text{if}\ \ \alpha_s\in\Phi_{\bar{0}}\,,\\
&[\![[\![[\![e_{s-3},e_{s-2}]\!],e_s]\!],e_{s-2}]\!]=0\,,\quad [\![[\![[\![f_{s-3},f_{s-2}]\!],f_s]\!],f_{s-2}]\!]=0 \qquad \text{if}\ \ \alpha_{s-2}\in\Phi_{\bar{1}}\,,\\
&[\![[\![e_{s-2},e_{s-1}]\!],e_s]\!]=[\![[\![e_{s-2},e_s]\!],e_{s-1}]\!]\,,\quad [\![[\![f_{s-2},f_{s-1}]\!],f_s]\!]=[\![[\![f_{s-2},f_s]\!],f_{s-1}]\!] \qquad \text{if}\ \ \alpha_s\in\Phi_{\bar{1}}\,.
\end{aligned} \tag{4.19}$$

The Hopf superalgebra structure on $U_q(\Gamma)$ is naturally defined by the exactly same formulas (2.30)–(2.31).

Definition 4.20. (a) The quantum supergroup $U_q(\Gamma_{\mathfrak{gosp}})$ is called the *extended Drinfeld–Jimbo-type orthosymplectic quantum supergroup* and is denoted by $U_q(\mathfrak{gosp}(V))$. Using the basis $\{\tilde{\mathsf{H}}_i\}_{i=1}^{s+1}$ of $\Gamma_{\mathfrak{gosp}}$, see (4.13), it admits an equivalent presentation with finitely many generators. Specifically, it is the $\mathbb{C}(q)$-superalgebra generated by $\{e_i,f_i\}_{i=1}^s\cup\{q^{\pm\tilde{H}_i}\}_{i=1}^{s+1}$, equipped with the same Q-grading as in (4.16) and compatible $\mathbb{Z}_2$-grading. These generators are subject to the Serre relations specified above, alongside with the Chevalley-type relations (2.22)–(2.24) evaluated on the chosen basis of $\Gamma_{\mathfrak{gosp}}$. Here, the elements $q^{\pm C}=q^{\pm\tilde{H}_{s+1}}$ for B-type (resp. $q^{\pm 2C}=q^{\pm\tilde{H}_s}q^{\pm\tilde{H}_{s+1}}$ for CD-types) are central, and the elements $q^{\pm h_i}$ in (2.24) are described through the chosen Cartan generators $\{q^{\pm\tilde{H}_k}\}_{k=1}^{s+1}$ akin to (4.6) via:

$$q^{\pm h_i}=\begin{cases} q^{\pm\tilde{H}_i}q^{\mp\tilde{H}_{i+1}} & \text{if}\ \ 1\le i<s\ \ (\text{in all } BCD\text{-types})\,,\\ q^{\pm\tilde{H}_s}q^{\mp\tilde{H}_{s+1}} & \text{if}\ \ i=s\ \ (BC\text{-types})\,,\\ q^{\pm\tilde{H}_{s-1}}q^{\mp\tilde{H}_{s+1}} & \text{if}\ \ i=s\ \ (D\text{-type})\,.\end{cases}$$

(b) Analogously, the *Drinfeld–Jimbo-type orthosymplectic quantum supergroup* is defined as $U_q(\mathfrak{osp}(V)):=U_q(Q^\vee)$. Equivalently, it is the $\mathbb{C}(q)$-superalgebra generated by $\{e_i,f_i\}_{i=1}^s$ and the Cartan generators $\{q^{\pm h_i}\}_{i=1}^s$, subject to the same Q-grading and $\mathbb{Z}_2$-grading. These generators satisfy the Serre relations specified above, alongside with the following explicit Chevalley-type relations (for all $1\le i,j\le s$):

$$[q^{h_i},q^{h_j}]=0\,,\ \ q^{\pm h_i}q^{\mp h_i}=1\,,\ \ q^{h_i}e_jq^{-h_i}=q^{(\alpha_i,\alpha_j)}e_j\,,\ \ q^{h_i}f_jq^{-h_i}=q^{-(\alpha_i,\alpha_j)}f_j\,,\ \ [e_i,f_j]=\delta_{ij}\frac{q^{h_i}-q^{-h_i}}{q-q^{-1}}\,.$$

By definition, there is a canonical Hopf superalgebra morphism from $U_q(\mathfrak{osp}(V))$ to $U_q(\mathfrak{gosp}(V))$. This morphism can be shown to be an embedding via the triangular decomposition (4.23) discussed below.

Remark 4.21. Our choice of the denominator $q-q^{-1}$ in the Chevalley-type relations follows the conventions of [23], and may therefore differ from some other literature by a scalar rescaling of the f_j's.

Remark 4.22. Analogously to Remark 2.37, the assignment $\omega_{\mathrm{DJ}}(q^{\pm\tilde{H}_k})=q^{\mp\tilde{H}_k}$ for $1\le k\le s+1$, together with the following action on the root vectors:

$$\omega_{\mathrm{DJ}}(e_i)=\begin{cases}\frac{1}{q^5(1+q^2)}f_s & \text{for } C\text{-type and } i=s\,,\\ -(-1)^{\overline{i}\,\overline{i+1}}q^{3(\rho,\alpha_i)}f_i & \text{otherwise}\,,\end{cases}$$

$$\omega_{\mathrm{DJ}}(f_i)=\begin{cases} q^5(1+q^2)e_s & \text{for } C\text{-type and } i=s\,,\\ -(-1)^{\overline{i}+\overline{i+1}}(-1)^{\overline{i}\,\overline{i+1}}q^{-3(\rho,\alpha_i)}e_i & \text{otherwise}\,,\end{cases}$$

gives rise to a Hopf superalgebra isomorphism $\omega_{\mathrm{DJ}}\colon U_q(\mathfrak{gosp}(V))\xrightarrow{\sim}U_q(\mathfrak{gosp}(V))^{\mathrm{cop}}$, where ρ is the *Weyl vector* of Φ given by the formula (2.10) and satisfying property (2.11) for simple roots α_i of (4.3)–(4.5).

4.4. **Drinfeld double and Drinfeld twist of $U_q(\mathfrak{gosp}(V))$.**

In complete analogy with the $\mathfrak{gl}(V)$ case (cf. (2.38)), $U_q(\mathfrak{gosp}(V))$ admits a *triangular decomposition*

$$U_q^<(\mathfrak{gosp}(V)) \otimes U_q^0(\mathfrak{gosp}(V)) \otimes U_q^>(\mathfrak{gosp}(V)) \xrightarrow{\sim} U_q(\mathfrak{gosp}(V)), \tag{4.23}$$

where the factors are the subalgebras generated by $\{f_i\}_{i=1}^s$, $\{q^{\pm H} \mid H \in \Gamma_{\mathfrak{gosp}}\}$, and $\{e_i\}_{i=1}^s$, respectively. This gives rise to the corresponding positive and negative Borel Hopf subalgebras $U_q^{\geq}(\mathfrak{gosp}(V))$ and $U_q^{\leq}(\mathfrak{gosp}(V))$, along with the following isomorphisms of underlying superspaces:

$$U_q^>(\mathfrak{gosp}(V)) \otimes U_q^0(\mathfrak{gosp}(V)) \xrightarrow{\sim} U_q^{\geq}(\mathfrak{gosp}(V)), \qquad U_q^<(\mathfrak{gosp}(V)) \otimes U_q^0(\mathfrak{gosp}(V)) \xrightarrow{\sim} U_q^{\leq}(\mathfrak{gosp}(V)). \tag{4.24}$$

The following result establishes the Drinfeld double realization of $U_q(\mathfrak{gosp}(V))$. This is a direct analogue of Proposition 2.40 (we omit the proof which is identical).

Proposition 4.25. *(a) There exists a unique skew-pairing (cf. (A.21))*

$$(\cdot,\cdot)_{\mathrm{DJ}} \colon U_q^{\leq}(\mathfrak{gosp}(V)) \times U_q^{\geq}(\mathfrak{gosp}(V)) \to \mathbb{C}(q) \tag{4.26}$$

satisfying the following properties on the generators for $1 \leq i,j \leq s$ and $H, H' \in \Gamma_{\mathfrak{gosp}}$:

$$(f_i, q^H)_{\mathrm{DJ}} = 0\,, \quad (q^H, e_i)_{\mathrm{DJ}} = 0\,, \quad (f_i, e_j)_{\mathrm{DJ}} = \frac{\delta_{ij}(-1)^{|f_i||e_j|}}{q^{-1}-q}\,, \quad (q^H, q^{H'})_{\mathrm{DJ}} = q^{-(H,H')}\,,$$

where the pairing in the exponent is evaluated via (4.9). Moreover, this pairing is non-degenerate, as is its restriction to the Cartan subalgebras $U_q^0(\mathfrak{gosp}(V)) \times U_q^0(\mathfrak{gosp}(V))$.

(b) The superalgebra $U_q(\mathfrak{gosp}(V))$ is isomorphic to the quotient of the Drinfeld double $\mathcal{D}_{\mathrm{DJ}}$ formed by the pair of Hopf subalgebras $U_q^{\leq}(\mathfrak{gosp}(V))$ and $U_q^{\geq}(\mathfrak{gosp}(V))$ with respect to (4.26), obtained by identifying the Cartan subalgebras $U_q^0(\mathfrak{gosp}(V))$ from both factors.

Remark 4.27. Similarly to Remark 2.49, one may alternatively construct the Drinfeld double by taking the Borel subalgebras in the opposite order. This utilizes the transposed inverse $(-,-)_{\widetilde{\mathrm{DJ}}} \colon U_q^{\geq}(\mathfrak{gosp}(V)) \times U_q^{\leq}(\mathfrak{gosp}(V)) \to \mathbb{C}(q)$, which is also non-degenerate and evaluates on the generators as follows:

$$(e_i, q^H)_{\widetilde{\mathrm{DJ}}} = 0\,, \quad (q^H, f_i)_{\widetilde{\mathrm{DJ}}} = 0\,, \quad (e_i, f_j)_{\widetilde{\mathrm{DJ}}} = \frac{\delta_{ij}}{q-q^{-1}}\,, \quad (q^H, q^{H'})_{\widetilde{\mathrm{DJ}}} = q^{(H,H')}\,.$$

We now introduce a specific Drinfeld twist of $U_q(\mathfrak{gosp}(V))$ using the element $\mathcal{F} = \mathfrak{R}_s$, the semisimple part of the universal R-matrix, see Subsection 5.1. This yields the exact same formulas for the twisted comultiplication $\Delta^{\mathcal{F}}$ and antipode $S^{\mathcal{F}}$ on the generators e_i, f_i, q^H as in the $\mathfrak{gl}(V)$ case from Subsection 2.6:

$$\begin{aligned}
&\Delta^{\mathcal{F}}(e_i) = 1 \otimes e_i + e_i \otimes q^{-h_i}\,, \quad \Delta^{\mathcal{F}}(f_i) = q^{h_i} \otimes f_i + f_i \otimes 1\,, \quad \Delta^{\mathcal{F}}(q^H) = q^H \otimes q^H\,,\\
&S^{\mathcal{F}}(e_i) = -e_i q^{h_i}\,, \quad S^{\mathcal{F}}(f_i) = -q^{-h_i} f_i\,, \quad S^{\mathcal{F}}(q^H) = q^{-H}\,.
\end{aligned}$$

The following structural properties are completely parallel to Proposition 2.52:

Proposition 4.28. *(a) The assignment $\phi_{\mathcal{F}}(e_i) = e_i q^{-h_i}$, $\phi_{\mathcal{F}}(f_i) = q^{h_i} f_i$, $\phi_{\mathcal{F}}(q^H) = q^H$ uniquely extends to a Q-graded Hopf superalgebra isomorphism $\phi_{\mathcal{F}} \colon U_q(\mathfrak{gosp}(V))^{\mathcal{F}} \xrightarrow{\sim} U_q(\mathfrak{gosp}(V))$.*

(b) The isomorphism $\phi_{\mathcal{F}}$ preserves the Borel subalgebras $U_q^{\geq}(\mathfrak{gosp}(V))$ and $U_q^{\leq}(\mathfrak{gosp}(V))$.

(c) The pullback skew-pairing $(x,y)^{\mathcal{F}}_{\mathrm{DJ}} := (\phi_{\mathcal{F}}(x), \phi_{\mathcal{F}}(y))_{\mathrm{DJ}}$ evaluates identically to the skew-pairing $(-,-)_{\mathrm{DJ}}$ on the generators. Consequently, $U_q(\mathfrak{gosp}(V))^{\mathcal{F}}$ can also be realized as the Drinfeld double of $U_q^{\leq}(\mathfrak{gosp}(V))^{\mathcal{F}}$ and $U_q^{\geq}(\mathfrak{gosp}(V))^{\mathcal{F}}$ with respect to $(-,-)^{\mathcal{F}}_{\mathrm{DJ}}$ (and similarly for $(-,-)^{\mathcal{F}}_{\widetilde{\mathrm{DJ}}}$ with the opposite order, see Remark 4.27).

5. RLL-realization of extended orthosymplectic quantum supergroups

5.1. **The universal R-matrix.**

Following the combinatorial approach developed in [9] (based on [4]), one defines the corresponding *quantum root vectors* $\{e_\gamma, f_\gamma\}_{\gamma \in \bar{\Phi}^+}$ recursively as in [9, (5.20)], using iterated q-supercommutators (2.25) and combinatorics of dominant Lyndon words. As in the $\mathfrak{gl}(V)$ case, let $\gamma_{ij} := \varepsilon_i - \varepsilon_j$. Since $\gamma_{ij} = \gamma_{j'i'}$, the reduced positive root system $\bar{\Phi}^+$ of (4.2) is uniquely parameterized by a subset of $\{(i,j) \mid i < j \leq i'\}$ via

$$\bar{\Phi}^+ = \begin{cases} \{\gamma_{ij} \,|\, i < j < i'\} & (B\text{-type})\,,\\ \{\gamma_{ij} \,|\, i < j < i'\} \cup \{\gamma_{ii'} \,|\, \bar{i} = \bar{1}\} & (CD\text{-types})\,. \end{cases} \tag{5.1}$$

The set L^+ of dominant Lyndon words depends on the order of simple roots, and we choose it as in [9]:

$$\alpha_1 \prec \alpha_2 \prec \cdots \prec \alpha_s\,. \tag{5.2}$$

Via the aforementioned bijection $\mathrm{l}\colon \mathsf{L}^+ \xrightarrow{\sim} \bar{\Phi}^+$, the lexicographic order on L^+ endows $\bar{\Phi}^+$ with a convex order $\prec$, which for (5.2) is simply induced by the lexicographic order on the pairs $\{(i,j) \mid i<j\leq i'\}$, cf. (3.5). For simple roots α_i, we set $e_{\alpha_i}=e_i$ and $f_{\alpha_i}=f_i$. For non-simple roots $\gamma\in\bar{\Phi}^+$, the root vectors are defined inductively via (cf. (3.4)):

$$e_\gamma = [\![e_\alpha, e_\beta]\!]\,, \qquad f_\gamma = -(-1)^{|f_\alpha||f_\beta|}q^{-(\alpha,\beta)}[\![f_\alpha, f_\beta]\!]\,,$$

where the pair of roots (α,β) corresponds to the so-called costandard factorization $\ell=\ell_1\ell_2$ of the dominant Lyndon word $\ell=\mathrm{l}^{-1}(\gamma)$ into $\ell_1=\mathrm{l}^{-1}(\alpha)$ and $\ell_2=\mathrm{l}^{-1}(\beta)$. The costandard factorization for the order (5.2) is generically given by the uniform rule $\gamma_{ij}\mapsto(\gamma_{i,j-1},\gamma_{j-1,j})$ for $i<j\leq i'$, which applies exhaustively for B-type, while the type-dependent exceptions for CD-types are as follows:

- ***C*-type:** $\gamma_{ii'}\mapsto(\gamma_{is},\gamma_{is'})$ for $i<s$;
- ***D*-type:** $\gamma_{is'}\mapsto(\gamma_{i,s-1},\gamma_{s-1,s'})$ for $i<s-1$, and $\gamma_{ii'}\mapsto(\gamma_{is},\gamma_{is'})$ for $i<s$.

In complete parallel with the $\mathfrak{gl}(V)$ case (cf. Theorem 3.6), these quantum root vectors provide PBW-type bases for the strictly positive and negative subalgebras $U_q^>(\mathfrak{gosp}(V))$ and $U_q^<(\mathfrak{gosp}(V))$, which are orthogonal with respect to the skew-pairing $(-,-)_{\mathrm{DJ}}$ from (4.26), see [9, Theorem 5.16].

Theorem 5.3. *(a) The sets of ordered monomials*

$$\left\{ \overleftarrow{\prod_{\gamma\in\bar{\Phi}^+}} e_\gamma^{m_\gamma} \,\middle|\, {\substack{m_\gamma\in\mathbb{Z}_{\geq 0} \\ m_\gamma\leq 1 \text{ if } \gamma\in\bar{\Phi}_{\bar 1} \text{ isotropic}}} \right\} \quad \text{and} \quad \left\{ \overleftarrow{\prod_{\gamma\in\bar{\Phi}^+}} f_\gamma^{m_\gamma} \,\middle|\, {\substack{m_\gamma\in\mathbb{Z}_{\geq 0} \\ m_\gamma\leq 1 \text{ if } \gamma\in\bar{\Phi}_{\bar 1} \text{ isotropic}}} \right\}$$

form bases for $U_q^>(\mathfrak{gosp}(V))$ and $U_q^<(\mathfrak{gosp}(V))$, respectively. Here, the arrow $\leftarrow$ indicates that the factors are ordered such that the roots γ increase from right to left with respect to the convex order $\prec$ on $\bar{\Phi}^+$.
(b) These dual bases are orthogonal with respect to the skew-pairing $(-,-)_{\mathrm{DJ}}$.

Using these dual PBW bases, the universal R-matrix $\mathfrak{R}$ in the completion of the tensor square of $U_q(\mathfrak{gosp}(V))$ (cf. Remark 3.3) factors exactly as it does in the $\mathfrak{gl}(V)$ case:

$$\mathfrak{R}=\mathfrak{R}_u\mathfrak{R}_s\,,\quad \mathfrak{R}_s=q^{-C\otimes C}q^{-\sum_{i=1}^s(-1)^{\bar i}H_i\otimes H_i}\,,\quad \mathfrak{R}_u=\overleftarrow{\prod_{\gamma\in\bar{\Phi}^+}}\left(\sum_{k\leq 1 \text{ if } \gamma\in\bar{\Phi}_{\bar 1} \text{ isotropic}}^{k\geq 0}\frac{e_\gamma^k\otimes f_\gamma^k}{(f_\gamma^k,e_\gamma^k)_{\mathrm{DJ}}}\right),$$

cf. [9, Theorem 5.18]. We now evaluate $\mathfrak{R}$ on the *first fundamental representation* $\varrho\colon U_q(\mathfrak{gosp}(V))\to\mathrm{End}(V_q)$ (see [9, Proposition 3.1]), where $V_q:=\mathbb{C}(q)\otimes_{\mathbb{C}}V$ as before. The representation ϱ is defined on the generators by $\varrho(e_i)=\mathsf{e}_i$ and $\varrho(f_i)=\kappa_i\mathsf{f}_i$ for $1\leq i\leq s$, where

$$\kappa_i=\begin{cases}\frac{q+q^{-1}}{2} & \text{for } C\text{-type and } i=s\,,\\ 1 & \text{otherwise}\,,\end{cases} \tag{5.4}$$

while on the Cartan generators it is given by $\varrho(q^{\pm\tilde H_i})=q^{\pm(\mathsf{H}_i+\mathrm{I})}$ for $1\leq i\leq s+1$, where I denotes the identity matrix and the exponentials of diagonal matrices are evaluated as in (3.8).

Again, the evaluation of the semisimple part $\varrho^{\otimes 2}(\mathfrak{R}_s)$ follows from a direct $\hbar$-adic computation (cf. (4.1) and (4.10)), while that of the unipotent part $\varrho^{\otimes 2}(\mathfrak{R}_u)$ is derived in [9, Propositions 5.23, 5.28]:

$$R_s:=\varrho^{\otimes 2}(\mathfrak{R}_s)=q^{-(\tilde\varepsilon_C,\tilde\varepsilon_C)}\sum_{1\leq i,j\leq N}q^{-(\varepsilon_i,\varepsilon_j)}E_{ii}\otimes E_{jj}=\sum_{1\leq i,j\leq N}q^{-(\tilde\varepsilon_i,\tilde\varepsilon_j)}E_{ii}\otimes E_{jj}\,,$$

$$\begin{aligned}R_u&:=\varrho^{\otimes 2}(\mathfrak{R}_u)\\ &=\mathrm{I}-(q-q^{-1})\sum_{i<j}(-1)^{\bar j}E_{ij}\otimes\Big(q^{(\varepsilon_i,\varepsilon_j)}E_{ji}-(-1)^{\bar j(\bar i+\bar j)}\vartheta_i\vartheta_j q^{(\rho,\varepsilon_i-\varepsilon_j)}q^{-(\varepsilon_i,\varepsilon_i)/2}q^{-(\varepsilon_j,\varepsilon_j)/2}E_{i'j'}\Big)\,,\end{aligned}$$

where ρ is the Weyl vector. Combining these we get explicit formulas for $R:=\varrho^{\otimes 2}(\mathfrak{R})$ and its inverse R^{-1} in $\mathrm{End}(V_q)^{\otimes 2}$, cf. [9, Remark 4.7]:

$$\begin{aligned}R&=q^{-(\tilde\varepsilon_C,\tilde\varepsilon_C)}\sum_{1\leq i,j\leq N}q^{-(\varepsilon_i,\varepsilon_j)}E_{ii}\otimes E_{jj}\\ &\quad-(q-q^{-1})q^{-(\tilde\varepsilon_C,\tilde\varepsilon_C)}\sum_{i<j}(-1)^{\bar j}E_{ij}\otimes\Big(E_{ji}-(-1)^{\bar j(\bar i+\bar j)}\vartheta_i\vartheta_j q^{(\rho,\varepsilon_i-\varepsilon_j)}q^{-(\varepsilon_i,\varepsilon_i)/2}q^{(\varepsilon_j,\varepsilon_j)/2}E_{i'j'}\Big)\,,\\ R^{-1}&=q^{(\tilde\varepsilon_C,\tilde\varepsilon_C)}\sum_{1\leq i,j\leq N}q^{(\varepsilon_i,\varepsilon_j)}E_{ii}\otimes E_{jj}\\ &\quad+(q-q^{-1})q^{(\tilde\varepsilon_C,\tilde\varepsilon_C)}\sum_{i<j}(-1)^{\bar j}E_{ij}\otimes\Big(E_{ji}-(-1)^{\bar j(\bar i+\bar j)}\vartheta_i\vartheta_j q^{-(\rho,\varepsilon_i-\varepsilon_j)}q^{-(\varepsilon_i,\varepsilon_i)/2}q^{(\varepsilon_j,\varepsilon_j)/2}E_{i'j'}\Big)\,.\end{aligned} \tag{5.5}$$

Remark 5.6. We note that $\mathfrak{R}$ (as well as its evaluation R) satisfies the exact same Yang–Baxter equation, comultiplication properties, and braid relations as those detailed in Remark 3.10 for the $\mathfrak{gl}(V)$ case.

5.2. The superalgebra $U(R)$.

Following the general linear case, we define the RLL-realization $U(R)$ of the extended orthosymplectic quantum supergroup using the corresponding R-matrix (5.5). The overarching algebraic structure is defined in exactly the same way as in the $\mathfrak{gl}(V)$ setting, with the only difference being the specific choice of R.

Definition 5.7. The $\mathbb{C}(q)$-superalgebra $U(R)$ is generated by the elements $\{l^+_{ij}, l^-_{ji}\}_{1\le i\le j\le N}$. Adopting the convention $l^+_{ij}=0=l^-_{ji}$ for $i>j$, we organize these generators into formal matrices $L^\pm := \sum_{i,j=1}^N l^\pm_{ij}\otimes E_{ij}$. The superalgebra $U(R)$ is defined as the quotient of the free superalgebra $\mathcal{T}:=\mathbb{C}(q)\langle l^+_{ij}, l^-_{ji}\,|\,1\le i\le j\le N\rangle$ by the diagonal relations (3.16), together with all coefficients in $\mathcal{T}$ of the matrix equations (3.17) and (3.18) evaluated in $\mathcal{T}\otimes \mathrm{End}(V_q)^{\otimes 2}$. The $\mathbb{Z}_2$-grading is compatible via (4.8) with the Q-grading determined by

$$\deg(l^\pm_{ij}) = \varepsilon_i - \varepsilon_j\,. \tag{5.8}$$

As the defining relations are homogeneous, we note that $U(R)$ is indeed a Q-graded superalgebra. Furthermore, the exact same formulas (3.20) and (3.26) endow $U(R)$ with the structure of a Hopf superalgebra.

Remark 5.9. As follows from the explicit formula (5.5), we have $(R_{12})^{\mathrm{st}} = \Theta_1\Theta_2\cdot R_{21}\cdot(\Theta_1\Theta_2)^{-1}$ where $\Theta=\sum_{i=1}^N q^{(\varepsilon_i,\varepsilon_i)/2}E_{ii}$, similarly to (3.34). Thus, arguing exactly as in Remark 3.27, we note that applying the full supertranspose to the defining relations of $U(R)$ yields an isomorphism of Hopf superalgebras

$$\omega_R\colon U(R)\xrightarrow{\sim} U(R)^{\mathrm{cop}} \qquad \text{given by} \qquad L^\pm \mapsto \Theta^{-1}(L^\mp)^{\mathrm{st}}\Theta\,. \tag{5.10}$$

This ω_R induces a superalgebra isomorphism between $U^+(R)$ and $U^-(R)$ introduced below.

Let $U^+(R)$ and $U^-(R)$ be the superalgebras generated respectively by $\{l^+_{ij}\}_{1\le i\le j\le N}$ and $\{l^-_{ji}\}_{1\le i\le j\le N}$, equipped with the Q-grading (5.8) and the compatible $\mathbb{Z}_2$-grading, subject to the respective single-sign relations (3.17). The same formulas (3.20) endow both $U^\pm(R)$ with superbialgebra structures. As in the general linear setting, the superalgebra $U(R)$ can be realized as a quotient of the generalized double of the superbialgebra pair $(U^+(R),U^-(R))$.

Proposition 5.11. *(a) There exists a unique skew-pairing*

$$(\cdot,\cdot)_R\colon U^+(R)\times U^-(R)\to\mathbb{C}(q) \tag{5.12}$$

satisfying the following property on the generators:

$$\sum_{1\le i,j,k,l\le N} (-1)^{(\bar{i}+\bar{j})(\bar{k}+\bar{l})}(l^+_{ij}, l^-_{kl})_R\, E_{ij}\otimes E_{kl} = R\,. \tag{5.13}$$

(b) The superalgebra $U(R)$ is isomorphic to the quotient of the generalized double $\mathcal{D}_R := \mathcal{D}(U^+(R),U^-(R))$ modulo the relations $l^\pm_{ii}l^\mp_{ii}=1$ for all $1\le i\le N$.

We omit the proof of Proposition 5.11, as it relies entirely on the structural compatibilities governed by the Yang–Baxter equation (cf. Remark 5.6) and is thus completely analogous to that of Proposition 3.36. We note that

$$(l^+_{ij}, l^-_{kl})_R\neq 0 \iff (i,k)=(j,l) \ \text{ or } \ (i,k)=(l,j) \ \text{ or } \ k=i' \,\&\, l=j',$$

which in view of (5.8) implies that the skew-pairing (5.12) is of Q-degree zero, and consequently of $\mathbb{Z}_2$-degree zero. The latter is crucial for $(\star)$ used in the proof of Proposition 3.36.

Remark 5.14. As in Remark 3.41, the evaluation (5.13) is compatible with the convention $l^+_{ij}=0=l^-_{ji}$ for $i>j$, as evident from the explicit formula (5.5).

Remark 5.15. Following Remark 3.45, let $U^{\ge}(R)$ and $U^{\le}(R)$ denote the Hopf subalgebras of $U(R)$ generated by $\{l^+_{ij}\}_{1\le i\le j\le N}\cup\{l^-_{kk}\}_{1\le k\le N}$ and $\{l^-_{ji}\}_{1\le i\le j\le N}\cup\{l^+_{kk}\}_{1\le k\le N}$, respectively. The skew-pairing (5.12) extends uniquely to a skew-pairing $U^{\ge}(R)\times U^{\le}(R)\to\mathbb{C}(q)$. Thus, the generalized double construction of $U(R)$ can equivalently be carried out using these Hopf subalgebras. Furthermore, the supertranspose isomorphism ω_R from (5.10) restricts to a superalgebra isomorphism between $U^{\ge}(R)$ and $U^{\le}(R)$.

Remark 5.16. As in Remark 3.46, the generalized double construction for $U(R)$ can equivalently be carried out by taking the subbialgebras in the opposite order. This utilizes the transposed inverse of $(-,-)_R$:

$$\widetilde{\sigma} = (-,-)_{\widetilde{R}}\colon U^-(R)\times U^+(R)\to\mathbb{C}(q)\,.$$

By direct computation, its evaluation on the generators is given by the matrix formula (3.47). As in Remark 5.15, this pairing uniquely extends to a skew-pairing $(-,-)_{\widetilde{R}}\colon U^{\le}(R)\times U^{\ge}(R)\to\mathbb{C}(q)$.

Finally, the twisting procedure used to bypass the emergence of Koszul signs during formal matrix multiplication carries over verbatim, yielding the analogue of Proposition 3.51:

Proposition 5.17. *Let $\zeta\colon \mathbb{Z}_2\times\mathbb{Z}_2\to\mathbb{C}(q)^\times$ be the normalized 2-cocycle given by $\zeta(\bar{\mathbf{i}},\bar{\mathbf{j}}) := (-1)^{\bar{\mathbf{i}}\bar{\mathbf{j}}}$. Then, the assignment $l_{ij}^{\pm}\mapsto l_{ij}^{\pm}$ for all $1\le i\le j\le N$ gives rise to a superalgebra isomorphism $\widetilde{U}(R)\xrightarrow{\sim}U(R)^\zeta$.*

5.3. **Gauss decomposition and twisted identities: orthosymplectic case.**

As in Subsection 3.6, we shall assume that all algebraic manipulations occur within the twisted superalgebra $\widetilde{U}(R)$ or the tensor product $\widetilde{U}(R)^{\otimes r}\otimes^{\beta'}\mathrm{End}(V_q)^{\otimes s}$ (for $r,s\ge 1$), rather than in their untwisted counterparts. Again, the elements in $\widetilde{U}(R)^{\otimes r}\otimes^{\beta'}\mathrm{End}(V_q)^{\otimes s}$ will be considered as formal matrices (3.54), following the standard matrix multiplication (3.55). Consequently, L^+ and L^- are genuinely upper and lower triangular matrices, respectively, and using (3.16) we obtain the exact same *Gauss decompositions* (3.56).

Similarly to the general linear setup, we define the elements $\{\mathbf{e}_{ij},\mathbf{f}_{ji}\}_{1\le i<j\le N}$ in $\widetilde{U}(R)$ via the exact same formulas (3.57). We also introduce $q^{\pm\tilde{\mathbf{H}}_k}:=l_{kk}^{\pm}$ in $\widetilde{U}(R)$ for $1\le k\le N$ and define $q^{\pm\tilde{\mathbf{H}}_{ij}}:=q^{\pm\tilde{\mathbf{H}}_i}(q^{\pm\tilde{\mathbf{H}}_j})^{-1}$ for all $1\le i<j\le N$.

By evaluating the specific matrix components of the defining relations (3.17) and (3.18), we deduce a set of identities governing the elements $\mathbf{e}_{ij}$, $\mathbf{f}_{ji}$, and $q^{\pm\tilde{\mathbf{H}}_k}$ within $\widetilde{U}(R)$. As before, $[-,-]^\zeta$ and $[\![-,-]\!]^\zeta$ denote the supercommutator and the q-supercommutator (cf. (2.5) and (2.25)) of elements in $\widetilde{U}(R)$.

We start with type-independent identities, the first three of which are derived exactly as in Subsection 3.6:

Matrix Component	Conditions	Identity	
$E_{ii}\otimes E_{jj}$	$1\le i,j\le N$	$[q^{\tilde{\mathbf{H}}_i},q^{\tilde{\mathbf{H}}_j}]^\zeta=0$	(5.18)
	$1\le i\le N$	$q^{\pm\tilde{\mathbf{H}}_i}q^{\mp\tilde{\mathbf{H}}_i}=1$	(5.19)
$E_{aa}\otimes E_{ij}$	$1\le i<j\le N,\ 1\le a\le N$	$q^{\tilde{\mathbf{H}}_a}\mathbf{e}_{ij}q^{-\tilde{\mathbf{H}}_a}=q^{(\varepsilon_a,\varepsilon_i-\varepsilon_j)}\mathbf{e}_{ij}$	(5.20)
$E_{ij}\otimes E_{ij}$	$1\le i<j\le N,\ \varepsilon_i-\varepsilon_j$ odd isotropic	$[\![\mathbf{e}_{ij},\mathbf{e}_{ij}]\!]^\zeta=0$	(5.21)

TABLE 2. Type-independent identities in $\widetilde{U}(R)$

We now proceed to list the type-dependent identities:

Matrix Component	Conditions	Identity	
$E_{ij}\otimes E_{i'j'}$	$1\le i<j\le s+1$	$q^{\tilde{\mathbf{H}}_i}q^{\tilde{\mathbf{H}}_{i'}}=q^{\tilde{\mathbf{H}}_j}q^{\tilde{\mathbf{H}}_{j'}}$	(5.22)
$E_{i,i+1}\otimes E_{i'i'}$	$1\le i\le s$	$\mathbf{e}_{i,i+1}=-(-1)^{\bar{i}(\bar{i}+\overline{i+1})}\vartheta_i\vartheta_{i+1}\mathbf{e}_{(i+1)'i'}$	(5.23)
$E_{ij}\otimes E_{j,j+1}$	$i<j<(i+1)'$	$\mathbf{e}_{i,j+1}=(-1)^{(\bar{i}+\bar{j})(\bar{j}+\overline{j+1})}[\![\mathbf{e}_{ij},\mathbf{e}_{j,j+1}]\!]^\zeta$	(5.24)
$E_{i,(i+1)'}\otimes E_{(i+1)'i'}$	$1\le i\le s$	$\mathbf{e}_{ii'}=(-1)^{\bar{i}+\overline{i+1}}\mathbf{e}_{i,(i+1)'}\mathbf{e}_{(i+1)'i'}$ $\quad -q^{-(\varepsilon_i,\varepsilon_i)}q^{-(\varepsilon_{i+1},\varepsilon_{i+1})}\mathbf{e}_{(i+1)'i'}\mathbf{e}_{i,(i+1)'}$	(5.25)
$E_{(j+1)'j'}\otimes E_{j'i'}$	$i<j<(i+1)'$	$\mathbf{e}_{(j+1)'i'}=(-1)^{(\bar{i}+\bar{j})(\bar{j}+\overline{j+1})}[\![\mathbf{e}_{(j+1)'j'},\mathbf{e}_{j'i'}]\!]^\zeta$	(5.26)
$E_{i,i+1}\otimes E_{i+1,i'}$	$1\le i\le s$	$\mathbf{e}_{ii'}=(-1)^{\bar{i}+\overline{i+1}}\mathbf{e}_{i,i+1}\mathbf{e}_{i+1,i'}$ $\quad -q^{-(\varepsilon_i,\varepsilon_i)}q^{-(\varepsilon_{i+1},\varepsilon_{i+1})}\mathbf{e}_{i+1,i'}\mathbf{e}_{i,i+1}$	(5.27)
$E_{i,i+1}\otimes E_{j,j+1}$	$1\le i<j\le s,\ j-i>1$	$[\![\mathbf{e}_{i,i+1},\mathbf{e}_{j,j+1}]\!]^\zeta=0$	(5.28)
$E_{i,i+1}\otimes E_{i,i+1}$	$1\le i<s,\ \lvert\mathbf{e}_{i,i+1}\rvert=\bar{1}$	$[\![\mathbf{e}_{i,i+1},\mathbf{e}_{i,i+1}]\!]^\zeta=0$	(5.29)
$E_{i,i+1}\otimes E_{i,i+2}$	$1\le i<s$	$[\![\mathbf{e}_{i,i+1},[\![\mathbf{e}_{i,i+1},\mathbf{e}_{i+1,i+2}]\!]^\zeta]\!]^\zeta=0$	(5.30)
$E_{i,i+1}\otimes E_{i-1,i+1}$	$1<i<s$	$[\![[\![\mathbf{e}_{i-1,i},\mathbf{e}_{i,i+1}]\!]^\zeta,\mathbf{e}_{i,i+1}]\!]^\zeta=0$	(5.31)
$E_{s,s+1}\otimes E_{s-1,s'}$		$[\![[\![[\![\mathbf{e}_{s-1,s},\mathbf{e}_{s,s+1}]\!]^\zeta,\mathbf{e}_{s,s+1}]\!]^\zeta,\mathbf{e}_{s,s+1}]\!]^\zeta=0$	(5.32)
$E_{i,i+1}\otimes E_{i-1,i+2}$	$1<i<s$	$[\![[\![[\![\mathbf{e}_{i-1,i},\mathbf{e}_{i,i+1}]\!]^\zeta,\mathbf{e}_{i+1,i+2}]\!]^\zeta,\mathbf{e}_{i,i+1}]\!]^\zeta=0$	(5.33)

TABLE 3. B-type, identities in $\widetilde{U}(R)$ derived from $RL_1^+L_2^+=L_2^+L_1^+R$

Matrix Component	Conditions	Identity	
$E_{ij} \otimes E_{i'j'}$	$1 \le i < j \le s$	$q^{\tilde{\mathbf{H}}_i} q^{\tilde{\mathbf{H}}_{i'}} = q^{\tilde{\mathbf{H}}_j} q^{\tilde{\mathbf{H}}_{j'}}$	(5.34)
$E_{i,i+1} \otimes E_{i'i'}$	$1 \le i \le s-1$	$\mathbf{e}_{i,i+1} = -(-1)^{\bar{i}(\bar{i}+\overline{i+1})} \vartheta_i \vartheta_{i+1} \mathbf{e}_{(i+1)'i'}$	(5.35)
$E_{ij} \otimes E_{j,j+1}$	$i < j < (i+1)',\ j \neq s$	$\mathbf{e}_{i,j+1} = (-1)^{(\bar{i}+\bar{j})(\bar{j}+\overline{j+1})} [\![\mathbf{e}_{ij}, \mathbf{e}_{j,j+1}]\!]^{\zeta}$	(5.36)
$E_{ss'} \otimes E_{is}$	$1 \le i \le s-1$	$(1+q^2)\mathbf{e}_{is'} = [\![\mathbf{e}_{is}, \mathbf{e}_{ss'}]\!]^{\zeta}$	(5.37)
$E_{i,(i+1)'} \otimes E_{(i+1)'i'}$	$1 \le i \le s-1$	$\begin{aligned} \mathbf{e}_{ii'} = (-1)^{\overline{i+\overline{i+1}}} \mathbf{e}_{i,(i+1)'} \mathbf{e}_{(i+1)'i'} \\ - q^{-(\varepsilon_i,\varepsilon_i)} q^{-(\varepsilon_{i+1},\varepsilon_{i+1})} \mathbf{e}_{(i+1)'i'} \mathbf{e}_{i,(i+1)'} \end{aligned}$	(5.38)
$E_{(j+1)'j'} \otimes E_{j'i'}$	$i < j < (i+1)',\ j \neq s$	$\mathbf{e}_{(j+1)'i'} = (-1)^{(\bar{i}+\bar{j})(\bar{j}+\overline{j+1})} [\![\mathbf{e}_{(j+1)'j'}, \mathbf{e}_{j'i'}]\!]^{\zeta}$	(5.39)
$E_{ss'} \otimes E_{s'i'}$	$1 \le i \le s-1$	$(1+q^2)\mathbf{e}_{si'} = [\![\mathbf{e}_{ss'}, \mathbf{e}_{s'i'}]\!]^{\zeta}$	(5.40)
$E_{i,i+1} \otimes E_{i+1,i'}$	$1 \le i \le s-1$	$\begin{aligned} \mathbf{e}_{ii'} = (-1)^{\overline{i+\overline{i+1}}} \mathbf{e}_{i,i+1} \mathbf{e}_{i+1,i'} \\ - q^{-(\varepsilon_i,\varepsilon_i)} q^{-(\varepsilon_{i+1},\varepsilon_{i+1})} \mathbf{e}_{i+1,i'} \mathbf{e}_{i,i+1} \end{aligned}$	(5.41)
$E_{i,i+1} \otimes E_{j,j+1}$	$1 \le i < j \le s,\ j-i > 1$	$[\![\mathbf{e}_{i,i+1}, \mathbf{e}_{j,j+1}]\!]^{\zeta} = 0$	(5.42)
$E_{i,i+1} \otimes E_{i,i+1}$	$1 \le i < s,\ \lvert\mathbf{e}_{i,i+1}\rvert = \bar{\mathbf{1}}$	$[\![\mathbf{e}_{i,i+1}, \mathbf{e}_{i,i+1}]\!]^{\zeta} = 0$	(5.43)
$E_{i,i+1} \otimes E_{i,i+2}$	$1 \le i \le s-2$	$[\![\mathbf{e}_{i,i+1}, [\![\mathbf{e}_{i,i+1}, \mathbf{e}_{i+1,i+2}]\!]^{\zeta}]\!]^{\zeta} = 0$	(5.44)
$E_{i,i+1} \otimes E_{i-1,i+1}$	$1 < i \le s$	$[\![[\![\mathbf{e}_{i-1,i}, \mathbf{e}_{i,i+1}]\!]^{\zeta}, \mathbf{e}_{i,i+1}]\!]^{\zeta} = 0$	(5.45)
$E_{i,i+1} \otimes E_{i-1,i+2}$	$1 < i < s-1$	$[\![[\![[\![\mathbf{e}_{i-1,i}, \mathbf{e}_{i,i+1}]\!]^{\zeta}, \mathbf{e}_{i+1,i+2}]\!]^{\zeta}, \mathbf{e}_{i,i+1}]\!]^{\zeta} = 0$	(5.46)
$E_{s-1,s} \otimes E_{s-3,(s-2)'}$		$\begin{aligned} [\![[\![[\![[\![[\![[\![\mathbf{e}_{s-3,s-2}, \mathbf{e}_{s-2,s-1}]\!]^{\zeta}, \mathbf{e}_{s-1,s}]\!]^{\zeta}, \\ \mathbf{e}_{ss'}]\!]^{\zeta}, \mathbf{e}_{s-1,s}]\!]^{\zeta}, \mathbf{e}_{s-2,s-1}]\!]^{\zeta}, \mathbf{e}_{s-1,s}]\!]^{\zeta} = 0 \end{aligned}$	(5.47)

TABLE 4. C-type, identities in $\widetilde{U}(R)$ derived from $RL_1^+L_2^+ = L_2^+L_1^+R$

Matrix Component	Conditions	Identity	
$E_{ij} \otimes E_{i'j'}$	$1 \le i < j \le s$	$q^{\tilde{\mathbf{H}}_i} q^{\tilde{\mathbf{H}}_{i'}} = q^{\tilde{\mathbf{H}}_j} q^{\tilde{\mathbf{H}}_{j'}}$	(5.48)
$E_{i,i+1} \otimes E_{i'i'}$	$1 \le i \le s$	$\mathbf{e}_{i,i+1} = -(-1)^{\bar{i}(\bar{i}+\overline{i+1})} \vartheta_i \vartheta_{i+1} \mathbf{e}_{(i+1)'i'}$	(5.49)
$E_{s-1,s'} \otimes E_{(s-1)'(s-1)'}$		$\mathbf{e}_{s-1,s'} = -(-1)^{\overline{s-1}(\overline{s-1}+\bar{s})} \vartheta_{s-1} \vartheta_{s'} \mathbf{e}_{s,(s-1)'}$	(5.50)
$E_{ij} \otimes E_{j,j+1}$	$i < j < (i+1)',\ j \neq s$	$\mathbf{e}_{i,j+1} = (-1)^{(\bar{i}+\bar{j})(\bar{j}+\overline{j+1})} [\![\mathbf{e}_{ij}, \mathbf{e}_{j,j+1}]\!]^{\zeta}$	(5.51)
$E_{i,s-1} \otimes E_{s-1,s'}$	$1 \le i \le s-2$	$\mathbf{e}_{is'} = (-1)^{(\bar{i}+\overline{s-1})(\overline{s-1}+\bar{s})} [\![\mathbf{e}_{i,s-1}, \mathbf{e}_{s-1,s'}]\!]^{\zeta}$	(5.52)
$E_{is} \otimes E_{s,(s-1)'}$	$1 \le i \le s-2$	$\mathbf{e}_{i,(s-1)'} = (-1)^{(\bar{i}+\bar{s})(\overline{s-1}+\bar{s})} [\![\mathbf{e}_{is}, \mathbf{e}_{s,(s-1)'}]\!]^{\zeta}$	(5.53)
$E_{i,(i+1)'} \otimes E_{(i+1)'i'}$	$1 \le i \le s-1$	$\begin{aligned} \mathbf{e}_{ii'} = (-1)^{\overline{i+\overline{i+1}}} \mathbf{e}_{i,(i+1)'} \mathbf{e}_{(i+1)'i'} \\ - q^{-(\varepsilon_i,\varepsilon_i)} q^{-(\varepsilon_{i+1},\varepsilon_{i+1})} \mathbf{e}_{(i+1)'i'} \mathbf{e}_{i,(i+1)'} \end{aligned}$	(5.54)
$E_{(j+1)'j'} \otimes E_{j'i'}$	$i < j < (i+1)',\ j \neq s$	$\mathbf{e}_{(j+1)'i'} = (-1)^{(\bar{i}+\bar{j})(\bar{j}+\overline{j+1})} [\![\mathbf{e}_{(j+1)'j'}, \mathbf{e}_{j'i'}]\!]^{\zeta}$	(5.55)
$E_{s,(s-1)'} \otimes E_{(s-1)'i'}$	$1 \le i \le s-2$	$\mathbf{e}_{si'} = (-1)^{(\bar{i}+\overline{s-1})(\overline{s-1}+\bar{s})} [\![\mathbf{e}_{s,(s-1)'}, \mathbf{e}_{(s-1)'i'}]\!]^{\zeta}$	(5.56)
$E_{s-1,s'} \otimes E_{s'i'}$	$1 \le i \le s-2$	$\mathbf{e}_{s-1,i'} = (-1)^{(\bar{i}+\bar{s})(\overline{s-1}+\bar{s})} [\![\mathbf{e}_{s-1,s'}, \mathbf{e}_{s'i'}]\!]^{\zeta}$	(5.57)
$E_{i,i+1} \otimes E_{i+1,i'}$	$1 \le i \le s-1$	$\begin{aligned} \mathbf{e}_{ii'} = (-1)^{\overline{i+\overline{i+1}}} \mathbf{e}_{i,i+1} \mathbf{e}_{i+1,i'} \\ - q^{-(\varepsilon_i,\varepsilon_i)} q^{-(\varepsilon_{i+1},\varepsilon_{i+1})} \mathbf{e}_{i+1,i'} \mathbf{e}_{i,i+1} \end{aligned}$	(5.58)
$E_{i,i+1} \otimes E_{j,j+1}$	$1 \le i < j < s,\ j-i > 1$	$[\![\mathbf{e}_{i,i+1}, \mathbf{e}_{j,j+1}]\!]^{\zeta} = 0$	(5.59)
$E_{i,i+1} \otimes E_{s-1,s'}$	$1 \le i \le s-3$	$[\![\mathbf{e}_{i,i+1}, \mathbf{e}_{s-1,s'}]\!]^{\zeta} = 0$	(5.60)
$E_{i,i+1} \otimes E_{i,i+1}$	$1 \le i < s,\ \lvert\mathbf{e}_{i,i+1}\rvert = \bar{\mathbf{1}}$	$[\![\mathbf{e}_{i,i+1}, \mathbf{e}_{i,i+1}]\!]^{\zeta} = 0$	(5.61)
$E_{s-1,s'} \otimes E_{s-1,s'}$	$\lvert\mathbf{e}_{s-1,s'}\rvert = \bar{\mathbf{1}}$	$[\![\mathbf{e}_{s-1,s'}, \mathbf{e}_{s-1,s'}]\!]^{\zeta} = 0$	(5.62)

Matrix Component	Conditions	Identity	
$E_{i,i+1} \otimes E_{i,i+2}$	$1 \le i \le s-2$	$[\![\mathbf{e}_{i,i+1}, [\![\mathbf{e}_{i,i+1}, \mathbf{e}_{i+1,i+2}]\!]^{\zeta}]\!]^{\zeta} = 0$	(5.63)
$E_{s-2,s-1} \otimes E_{s-2,s'}$		$[\![\mathbf{e}_{s-2,s-1}, [\![\mathbf{e}_{s-2,s-1}, \mathbf{e}_{s-1,s'}]\!]^{\zeta}]\!]^{\zeta} = 0$	(5.64)
$E_{i,i+1} \otimes E_{i-1,i+1}$	$1 < i < s$	$[\![[\![\mathbf{e}_{i-1,i}, \mathbf{e}_{i,i+1}]\!]^{\zeta}, \mathbf{e}_{i,i+1}]\!]^{\zeta} = 0$	(5.65)
$E_{s-1,s'} \otimes E_{s-2,s'}$		$[\![[\![\mathbf{e}_{s-2,s-1}, \mathbf{e}_{s-1,s'}]\!]^{\zeta}, \mathbf{e}_{s-1,s'}]\!]^{\zeta} = 0$	(5.66)
$E_{i,i+1} \otimes E_{i-1,i+2}$	$1 < i < s-1$	$[\![[\![[\![\mathbf{e}_{i-1,i}, \mathbf{e}_{i,i+1}]\!]^{\zeta}, \mathbf{e}_{i+1,i+2}]\!]^{\zeta}, \mathbf{e}_{i,i+1}]\!]^{\zeta} = 0$	(5.67)
$E_{s-2,s-1} \otimes E_{s-3,s'}$		$[\![[\![[\![\mathbf{e}_{s-3,s-2}, \mathbf{e}_{s-2,s-1}]\!]^{\zeta}, \mathbf{e}_{s-1,s'}]\!]^{\zeta}, \mathbf{e}_{s-2,s-1}]\!]^{\zeta} = 0$	(5.68)

TABLE 5. D-type, identities in $\widetilde{U}(R)$ derived from $RL_1^+L_2^+ = L_2^+L_1^+R$

Remark 5.69. As in Remark 3.68, while (5.29, 5.43, 5.61–5.62) are special cases of (5.21), they are recorded separately as they correspond to the Serre relations. Also (5.21) implies $\mathbf{e}_{ij}^2 = 0$ for odd isotropic $\varepsilon_i - \varepsilon_j$.

Remark 5.70. For each type, both the type-independent identities (Table 2) and the type-dependent identities listed earlier for that same type are applied iteratively to establish the subsequent relations, cf. Remark 3.69.

Finally, we record the following exceptional identities for C- and D-types:

Type	Conditions	Identity	
C-type		$[\![\mathbf{e}_{s-1,s}, [\![\mathbf{e}_{s-1,s}, [\![\mathbf{e}_{s-1,s}, \mathbf{e}_{ss'}]\!]^{\zeta}]\!]^{\zeta}]\!]^{\zeta} = 0$	(5.71)
	$\lvert\mathbf{e}_{s-2,s-1}\rvert, \lvert\mathbf{e}_{s-1,s}\rvert = \bar{\mathbf{1}}$	$[\![[\![[\![[\![\mathbf{e}_{s-2,s-1}, \mathbf{e}_{s-1,s}]\!]^{\zeta}, \mathbf{e}_{ss'}]\!]^{\zeta}, [\![\mathbf{e}_{s-2,s-1}, \mathbf{e}_{s-1,s}]\!]^{\zeta}]\!]^{\zeta}, \mathbf{e}_{s-1,s}]\!]^{\zeta} = 0$	(5.72)
D-type	$\lvert\mathbf{e}_{s-1,s}\rvert = \bar{\mathbf{0}}$	$[\![\mathbf{e}_{s-1,s}, \mathbf{e}_{s-1,s'}]\!]^{\zeta} = 0$	(5.73)
		$[\![[\![\mathbf{e}_{s-2,s-1}, \mathbf{e}_{s-1,s}]\!]^{\zeta}, \mathbf{e}_{s-1,s'}]\!]^{\zeta} = [\![[\![\mathbf{e}_{s-2,s-1}, \mathbf{e}_{s-1,s'}]\!]^{\zeta}, \mathbf{e}_{s-1,s}]\!]^{\zeta}$	(5.74)

TABLE 6. Exceptional identities in $\widetilde{U}(R)$

The proofs of the exceptional identities are as follows:

- Since relation (5.71) obviously holds when $\lvert\mathbf{e}_{s-1,s}\rvert = \bar{\mathbf{1}}$, due to $\mathbf{e}_{s-1,s}^2 = 0$ by (5.43), we shall assume now that $\lvert\mathbf{e}_{s-1,s}\rvert = \bar{\mathbf{0}}$. Then identity (5.71) reduces via (5.37) to $[\![\mathbf{e}_{s-1,s}, [\![\mathbf{e}_{s-1,s}, \mathbf{e}_{s-1,s'}]\!]^{\zeta}]\!]^{\zeta} = 0$, expanding to
$$\mathbf{e}_{s-1,s}^2\mathbf{e}_{s-1,s'} - (1+q^{-2})\mathbf{e}_{s-1,s}\mathbf{e}_{s-1,s'}\mathbf{e}_{s-1,s} + q^{-2}\mathbf{e}_{s-1,s'}\mathbf{e}_{s-1,s}^2 = 0\,. \tag{5.75}$$
Evaluating the $E_{s',(s-1)'} \otimes E_{s-1,(s-1)'}$-component of $RL_1^+L_2^+ = L_2^+L_1^+R$ (together with (5.20) and (5.35) for $i = s-1$) yields
$$[\mathbf{e}_{s-1,s}, \mathbf{e}_{s-1,(s-1)'}]^{\zeta} = 0\,, \tag{5.76}$$
while (5.38) for $i = s-1$ reads $\mathbf{e}_{s-1,(s-1)'} = \mathbf{e}_{s-1,s'}\mathbf{e}_{s',(s-1)'} - q^2\mathbf{e}_{s',(s-1)'}\mathbf{e}_{s-1,s'}$. Substituting the latter into (5.76) and applying (5.35) for $i = s-1$ yields a scalar multiple of (5.75), and identity (5.71) follows.
- Using the iterative formulas (5.36) and (5.37), identity (5.72) reduces to $[\![[\![\mathbf{e}_{s-2,s'}, \mathbf{e}_{s-2,s}]\!]^{\zeta}, \mathbf{e}_{s-1,s}]\!]^{\zeta} = 0$, which by (5.35) is equivalent to $[\![[\![\mathbf{e}_{s-2,s'}, \mathbf{e}_{s-2,s}]\!]^{\zeta}, \mathbf{e}_{s',(s-1)'}]\!]^{\zeta} = 0$. This is further equivalent to $[\![[\![\mathbf{e}_{s-2,s}, \mathbf{e}_{s-2,s'}]\!]^{\zeta}, \mathbf{e}_{s',(s-1)'}]\!]^{\zeta} = 0$ as the condition $\lvert\mathbf{e}_{s-2,s-1}\rvert = \lvert\mathbf{e}_{s-1,s}\rvert = \bar{\mathbf{1}}$ implies $\lvert\mathbf{e}_{s-2,s}\rvert = \lvert\mathbf{e}_{s-2,s'}\rvert = \bar{\mathbf{0}}$. We note the following algebraic equality:
$$[\![[\![\mathbf{e}_{s-2,s}, \mathbf{e}_{s-2,s'}]\!]^{\zeta}, \mathbf{e}_{s',(s-1)'}]\!]^{\zeta} = [\![\mathbf{e}_{s-2,s}, [\![\mathbf{e}_{s-2,s'}, \mathbf{e}_{s',(s-1)'}]\!]^{\zeta}]\!]^{\zeta} - [\![\mathbf{e}_{s-2,s'}, [\![\mathbf{e}_{s-2,s}, \mathbf{e}_{s',(s-1)'}]\!]^{\zeta}]\!]^{\zeta}\,. \tag{5.77}$$
The second term in (5.77) vanishes because $[\![\mathbf{e}_{s-2,s}, \mathbf{e}_{s-1,s}]\!]^{\zeta} = 0$ by (5.45) together with (5.35) and (5.36). The first term reduces via (5.36) to a scalar multiple of $[\![\mathbf{e}_{s-2,s}, \mathbf{e}_{s-2,(s-1)'}]\!]^{\zeta}$, which can be shown to vanish by evaluating the $E_{s-2,s} \otimes E_{s-2,(s-1)'}$-component of $RL_1^+L_2^+ = L_2^+L_1^+R$ (together with (5.20)). This completes the verification of (5.72).
- Relation (5.73) follows by comparing the iterative formulas (5.54) and (5.58) at $i = s-1$, alongside with formulas (5.49) and (5.50) for $i = s-1$.

- Applying formulas (5.50) and (5.49) at $i=s-1$ reduces relation (5.74) to
$$[\![[\![\mathbf{e}_{s-2,s-1},\mathbf{e}_{s-1,s}]\!]^{\zeta},\mathbf{e}_{s,(s-1)'}]\!]^{\zeta} = [\![[\![\mathbf{e}_{s-2,s-1},\mathbf{e}_{s-1,s'}]\!]^{\zeta},\mathbf{e}_{s',(s-1)'}]\!]^{\zeta}\,.$$
Formulas (5.51)–(5.53) imply that both sides equal $(-1)^{\overline{s-1}+\overline{s}}\mathbf{e}_{s-2,(s-1)'}$, thus completing the proof.

5.4. **Homomorphism theorem: orthosymplectic case.**

The key result of this section is the identification between the Hopf superalgebras $U_q(\mathfrak{gosp}(V))^{\mathcal{F}}$ and $U(R)$:

Theorem 5.78. *(a) The assignment*

$$q^{\pm\tilde{H}_i}\mapsto q^{\pm\tilde{\mathbf{H}}_i}\qquad \textit{for}\ \ 1\leq i\leq s+1\,,$$
$$e_i\mapsto\begin{cases}\mathbf{e}_{i,i+1} & \textit{if}\ \ 1\leq i<s\ \ \textit{(in all BCD-types)}\,,\\ \mathbf{e}_{s,s+1} & \textit{if}\ \ i=s\ \ \textit{(B-type)}\,,\\ \mathbf{e}_{ss'}/(1+q^2) & \textit{if}\ \ i=s\ \ \textit{(C-type)}\,,\\ \mathbf{e}_{s-1,s'} & \textit{if}\ \ i=s\ \ \textit{(D-type)}\,,\end{cases}\tag{5.79}$$
$$f_i\mapsto\begin{cases}\mathbf{f}_{i+1,i} & \textit{if}\ \ 1\leq i<s\ \ \textit{(in all BCD-types)}\,,\\ \mathbf{f}_{s+1,s} & \textit{if}\ \ i=s\ \ \textit{(B-type)}\,,\\ q\mathbf{f}_{s's} & \textit{if}\ \ i=s\ \ \textit{(C-type)}\,,\\ \mathbf{f}_{s',s-1} & \textit{if}\ \ i=s\ \ \textit{(D-type)}\,.\end{cases}$$

gives rise to a Q-graded Hopf superalgebra isomorphism $\xi\colon U_q(\mathfrak{gosp}(V))^{\mathcal{F}}\xrightarrow{\sim}U(R)$.

(b) Under the isomorphism ξ, the images of the quantum root vectors from Subsection 5.1 are given by:

- ***B-type***

  If $\gamma=\varepsilon_i-\varepsilon_j$ with $1\leq i<j\leq s+1$, then
$$\xi(e_\gamma)=\mathbf{e}_{ij}\,,\qquad \xi(f_\gamma)=\mathbf{f}_{ji}\,.$$
  If $\gamma=\varepsilon_i+\varepsilon_j$ with $1\leq i<j\leq s$, then
$$\xi(e_\gamma)=\vartheta_j\vartheta_{s+1}\cdot\prod_{k=j}^{s}\Big(-(-1)^{\overline{k}(\overline{k}+\overline{k+1})}\Big)\cdot\mathbf{e}_{ij'}\,,\qquad \xi(f_\gamma)=\vartheta_j\vartheta_{s+1}\cdot\prod_{k=j}^{s}\Big(-(-1)^{\overline{k}(\overline{k}+\overline{k+1})}\Big)\cdot\mathbf{f}_{j'i}\,.$$
- ***C-type***

  If $\gamma=\varepsilon_i-\varepsilon_j$ with $1\leq i<j\leq s$, then
$$\xi(e_\gamma)=\mathbf{e}_{ij}\,,\qquad \xi(f_\gamma)=\mathbf{f}_{ji}\,.$$
  If $\gamma=\varepsilon_i+\varepsilon_j$ with $1\leq i<j\leq s$, then
$$\xi(e_\gamma)=-\vartheta_j\vartheta_s\cdot\prod_{k=j}^{s}\Big(-(-1)^{\overline{k}(\overline{k}+\overline{k+1})}\Big)\cdot\mathbf{e}_{ij'}\,,\qquad \xi(f_\gamma)=-(q+q^{-1})\vartheta_j\vartheta_s\cdot\prod_{k=j}^{s}\Big(-(-1)^{\overline{k}(\overline{k}+\overline{k+1})}\Big)\cdot\mathbf{f}_{j'i}\,.$$
- ***D-type***

  If $\gamma=\varepsilon_i-\varepsilon_j$ with $1\leq i<j\leq s$, then
$$\xi(e_\gamma)=\mathbf{e}_{ij}\,,\qquad \xi(f_\gamma)=\mathbf{f}_{ji}\,.$$
  If $\gamma=\varepsilon_i+\varepsilon_j$ with $1\leq i<j\leq s$, then
$$\xi(e_\gamma)=-\vartheta_j\vartheta_s\cdot\prod_{k=j}^{s}\Big(-(-1)^{\overline{k}(\overline{k}+\overline{k+1})}\Big)\cdot\mathbf{e}_{ij'}\,,\qquad \xi(f_\gamma)=-\vartheta_j\vartheta_s\cdot\prod_{k=j}^{s}\Big(-(-1)^{\overline{k}(\overline{k}+\overline{k+1})}\Big)\cdot\mathbf{f}_{j'i}\,.$$

(c) The isomorphism ξ is compatible with the pairings introduced in Proposition 4.28(c) and Remark 5.15:
$$(\xi(x),\xi(y))_R=(x,y)^{\mathcal{F}}_{\mathrm{DJ}}\qquad \textit{for all}\quad x\in U^{\geq}_q(\mathfrak{gosp}(V))^{\mathcal{F}},\ y\in U^{\leq}_q(\mathfrak{gosp}(V))^{\mathcal{F}}\,.$$

Proof. The proof is completely analogous to that of Theorem 3.71. We first establish that the assignment (5.79) defines a surjective Hopf superalgebra morphism $\xi\colon U^{\geq}_q(\mathfrak{gosp}(V))^{\mathcal{F}}\to U^{\geq}(R)$. By untwisting the relations in Tables 2–6 via (3.75), we deduce the following:

- The Chevalley-type relations for $U^{\geq}_q(\mathfrak{gosp}(V))^{\mathcal{F}}$ are recovered directly from identities (5.18)–(5.20);
- The standard Serre relations (2.26)–(2.28) (for the appropriate indices) and the higher-order Serre relations (4.17)–(4.19) for each respective type follow from identities (5.28)–(5.33) for B-type; (5.42)–(5.47) along with (5.71, 5.72) for C-type; and (5.59)–(5.68) along with (5.73, 5.74) for D-type;

- The surjectivity of ξ restricted to the Cartan subalgebra follows from identities (5.22), (5.34), and (5.48) for the respective types. The fact that all remaining generators $\mathbf{e}_{ij}$ $(i<j)$ lie in the image is then guaranteed by the iterative formulas: (5.23)–(5.27) for B-type, (5.35)–(5.41) for C-type, and (5.49)–(5.58) for D-type. We also emphasize that for D-type, setting $i=s$ in (5.49), one actually gets

$$\mathbf{e}_{ss'}=0\,; \tag{5.80}$$

- In conjunction with (3.4), these iterative formulas simultaneously establish part (b);
- The fact that ξ intertwines the respective comultiplications follows by the exact same computation as in the $\mathfrak{gl}(V)$ setting. The only case deserving a separate treatment is the generator $\mathbf{e}_{s-1,s'}$ for D-type, in which case one uses $l^+_{ss'}=0$, due to (5.80), to derive $\Delta(l^+_{s-1,s'})=l^+_{s-1,s-1}\otimes l^+_{s-1,s'}+l^+_{s-1,s'}\otimes l^+_{s's'}$.

This confirms that ξ is a well-defined, surjective Hopf superalgebra morphism of the positive Borel subalgebras. The corresponding morphism for the negative Borel subalgebras $U_q^{\leq}(\mathfrak{osp}(V))^{\mathcal{F}}\to U^{\leq}(R)$ follows by utilizing the superalgebra isomorphisms ω_{DJ} and ω_R from Remarks 4.22 and 5.9, precisely as executed in the $\mathfrak{gl}(V)$ case.

The compatibility of the pairings stated in part (c) is verified by a direct evaluation on the generators. Since both pairings are of Q-degree zero (cf. (2.43)), we have:

$$(\mathbf{e}_{ij},q^{\pm\tilde{\mathbf{H}}_k})_R=0=(q^{\pm\tilde{\mathbf{H}}_k},\mathbf{f}_{ji})_R\,,$$

for all $1\leq i<j\leq N$, and

$$(q^{\tilde{\mathbf{H}}_i},q^{-\tilde{\mathbf{H}}_j})_R=(l^+_{ii},l^-_{jj})_R\overset{(5.5)}{=}q^{-(\tilde{\varepsilon}_C,\tilde{\varepsilon}_C)}q^{-(\varepsilon_i,\varepsilon_j)}=q^{-(\tilde{\varepsilon}_i,\tilde{\varepsilon}_j)}=(q^{\tilde{H}_i},q^{-\tilde{H}_j})^{\mathcal{F}}_{\widetilde{\mathrm{DJ}}}\,.$$

Furthermore, for any $1\leq i<j\leq N$ and $1\leq k<l\leq N$, expanding the pairing yields:

$$\begin{aligned}(l^-_{ii}l^+_{ij}\,,\,l^-_{lk}l^+_{kk})_R&=(l^-_{ii}\otimes l^+_{ij}\,,\,\Delta(l^-_{lk})\Delta(l^+_{kk}))_R\overset{(2.43)}{=}(l^-_{ii}\otimes l^+_{ij}\,,\,l^-_{ll}l^+_{kk}\otimes l^-_{lk}l^+_{kk})_R\\&=(l^-_{ii}\,,\,l^-_{ll}l^+_{kk})_R\,(l^+_{ij}\,,\,l^-_{lk}l^+_{kk})_R=q^{-(\tilde{\varepsilon}_i,\tilde{\varepsilon}_k-\tilde{\varepsilon}_l)}\,(l^+_{ij}\,,\,l^-_{lk}l^+_{kk})_R\\&\overset{(2.43)}{=}q^{-(\tilde{\varepsilon}_i,\tilde{\varepsilon}_k-\tilde{\varepsilon}_l)}\,(l^+_{ij}\otimes l^+_{ii}\,,\,l^-_{lk}\otimes l^+_{kk})_R=q^{(\tilde{\varepsilon}_i,\tilde{\varepsilon}_l)}\,(l^+_{ij}\,,\,l^-_{lk})_R\,,\end{aligned} \tag{5.81}$$

which vanishes unless $\deg(l^+_{ij})=-\deg(l^-_{lk})$. Therefore $(\xi(e_i),\xi(f_j))_R=0$ for $i\neq j$ due to degree reasons, and it remains to evaluate the pairing for $i=j$. Thus, the proof of part (c) is concluded by the following explicit type-dependent evaluations for all $1\leq i\leq s$ using (5.81), cf. (4.10):

- *$1\leq i<s$ in BCD-types*

  $$q^{(\tilde{\varepsilon}_i,\tilde{\varepsilon}_{i+1})}(l^+_{i,i+1}\,,\,l^-_{i+1,i})_R=-(-1)^{\overline{i+1}}(q-q^{-1})q^{-(\tilde{\varepsilon}_C,\tilde{\varepsilon}_C)}\cdot(-1)^{\bar{i}+\overline{i+1}}q^{(\tilde{\varepsilon}_i,\tilde{\varepsilon}_{i+1})}=-(-1)^{\bar{i}}(q-q^{-1})\,,$$

  and therefore

  $$(\xi(e_i),\xi(f_i))_R=(\mathbf{e}_{i,i+1},\mathbf{f}_{i+1,i})_R=-(-1)^{\bar{i}}(q-q^{-1})^{-2}(l^-_{ii}l^+_{i,i+1}\,,\,l^-_{i+1,i}l^+_{ii})_R=(q-q^{-1})^{-1}\,.$$

- *$i=s$ in B-type*

  $$q^{(\tilde{\varepsilon}_s,\tilde{\varepsilon}_{s+1})}(l^+_{s,s+1}\,,\,l^-_{s+1,s})_R=-(-1)^{\overline{s+1}}(q-q^{-1})q^{-(\tilde{\varepsilon}_C,\tilde{\varepsilon}_C)}\cdot(-1)^{\bar{s}+\overline{s+1}}q^{(\tilde{\varepsilon}_s,\tilde{\varepsilon}_{s+1})}=-(-1)^{\bar{s}}(q-q^{-1})\,,$$

  and thus

  $$(\xi(e_s),\xi(f_s))_R=(\mathbf{e}_{s,s+1},\mathbf{f}_{s+1,s})_R=-(-1)^{\bar{s}}(q-q^{-1})^{-2}(l^-_{ss}l^+_{s,s+1}\,,\,l^-_{s+1,s}l^+_{ss})_R=(q-q^{-1})^{-1}\,.$$

- *$i=s$ in C-type*

  $$\begin{aligned}q^{(\tilde{\varepsilon}_s,\tilde{\varepsilon}_{s'})}(l^+_{ss'}\,,\,l^-_{s's})_R&=-(q-q^{-1})q^{-(\tilde{\varepsilon}_C,\tilde{\varepsilon}_C)}\Big((-1)^{\bar{s}}-q^{(\rho,2\varepsilon_s)}\Big)\cdot q^{(\tilde{\varepsilon}_s,\tilde{\varepsilon}_{s'})}\\&=-(q-q^{-1})q^{-(\varepsilon_s,\varepsilon_s)}\Big((-1)^{\bar{s}}-q^{2(\varepsilon_s,\varepsilon_s)}\Big)=(q-q^{-1})(q+q^{-1})\,,\end{aligned}$$

  and hence

$$(\xi(e_s),\xi(f_s))_R=(q+q^{-1})^{-1}(\mathbf{e}_{ss'},\mathbf{f}_{s's})_R=-(-1)^{\bar{s}}(q+q^{-1})^{-1}(q-q^{-1})^{-2}(l^-_{ss}l^+_{ss'}\,,\,l^-_{s's}l^+_{ss})_R=(q-q^{-1})^{-1}\,.$$

- *$i=s$ in D-type*

  $$q^{(\tilde{\varepsilon}_{s-1},\tilde{\varepsilon}_{s'})}(l^+_{s-1,s'}\,,\,l^-_{s',s-1})_R=-(-1)^{\bar{s}}(q-q^{-1})q^{-(\tilde{\varepsilon}_C,\tilde{\varepsilon}_C)}\cdot(-1)^{\overline{s-1}+\bar{s}}q^{(\tilde{\varepsilon}_{s-1},\tilde{\varepsilon}_{s'})}=-(-1)^{\overline{s-1}}(q-q^{-1})\,,$$

  and therefore

  $$\begin{aligned}(\xi(e_s),\xi(f_s))_R&=(\mathbf{e}_{s-1,s+1},\mathbf{f}_{s+1,s-1})_R\\&=-(-1)^{\overline{s-1}+\bar{s}}(q-q^{-1})^{-2}(l^-_{s-1,s-1}l^+_{s-1,s'}\,,\,l^-_{s',s-1}l^+_{s-1,s-1})_R=(q-q^{-1})^{-1}\,.\end{aligned}$$

We conclude that in all cases, the right-hand side evaluates precisely to $(q-q^{-1})^{-1} = (e_i, f_i)^{\mathcal{F}}_{\widetilde{\mathrm{DJ}}}$.

As in $\mathfrak{gl}(V)$ case, ξ naturally extends to a surjective Hopf superalgebra morphism between the doubles

$$\xi\colon \mathcal{D}^{\mathcal{F}}_{\mathrm{DJ}} = U^{\geq}_q(\mathfrak{osp}(V))^{\mathcal{F}} \otimes U^{\leq}_q(\mathfrak{osp}(V))^{\mathcal{F}} \longrightarrow U^{\geq}(R) \otimes U^{\leq}(R) = \mathcal{D}_R\,.$$

The non-degeneracy of the pairing $(-,-)^{\mathcal{F}}_{\widetilde{\mathrm{DJ}}}$ guarantees that $\xi\colon \mathcal{D}^{\mathcal{F}}_{\mathrm{DJ}} \to \mathcal{D}_R$ is injective on each Borel subalgebra, thereby establishing an isomorphism of doubles. Finally, the identification of the Cartan subalgebras yields the desired isomorphism $\xi\colon U_q(\mathfrak{osp}(V))^{\mathcal{F}} \xrightarrow{\sim} U(R)$, completing the proof of (a). □

5.5. **Inverse morphism: orthosymplectic case.**

Following the approach in the general linear setting (see Subsection 3.8), we shall now construct a similar morphism for the orthosymplectic quantum supergroups. The proof of the following proposition is completely analogous to that of Proposition 3.76:

Proposition 5.82. *The assignment*

$$L^+ \mapsto (\mathrm{Id}\otimes\varrho)(\mathfrak{R}_{(21)})\,, \qquad L^- \mapsto (\mathrm{Id}\otimes\varrho)(\mathfrak{R}^{-1})\,, \tag{5.83}$$

gives rise to a Q-graded Hopf superalgebra morphism $\phi^{\mathrm{DF}}\colon U(R)^{\mathrm{cop}} \to U_q(\mathfrak{osp}(V))$.

We now determine the explicit images of the generators under the morphism ϕ^{DF}, cf. Lemma 3.79:

Lemma 5.84. *The images of the diagonal generators $\{l^{\pm}_{ii}\}_{i=1}^N$ under ϕ^{DF} are given by $\phi^{\mathrm{DF}}(l^{\pm}_{ii}) = q^{\mp\tilde{H}_i}$. For the generators l^+_{ij}, l^-_{ji} associated with a simple root $\gamma_{ij} = \varepsilon_i - \varepsilon_j$, their images under ϕ^{DF} are as follows:*

$$l^+_{ij} \mapsto -(q-q^{-1}) f_{\gamma_{ij}} q^{-\tilde{H}_j}\,,$$
$$l^-_{ji} \mapsto \begin{cases} -(q^2-q^{-2}) q^{\tilde{H}_{s'}} e_s & \textit{if } \ \gamma_{ij} = \alpha_s \textit{ for } C\textit{-type}\,, \\ (-1)^{\bar{j}}(q-q^{-1}) q^{\tilde{H}_j} e_{\gamma_{ij}} & \textit{otherwise}\,. \end{cases}$$

Proof. As in the general linear case, we first evaluate the semisimple component. An explicit $\hbar$-adic computation yields:

$$(\mathrm{Id}\otimes\varrho)(\mathfrak{R}_s) = \sum_{1\leq i\leq N} q^{-\tilde{H}_i} \otimes E_{ii}\,,$$

which establishes $\phi^{\mathrm{DF}}(l^{\pm}_{ii}) = q^{\mp\tilde{H}_i}$. Proceeding to the unipotent part, for the simple root $\gamma_{ij} = \varepsilon_i - \varepsilon_j$, we need to determine the E_{ij}-component of

$$(\mathrm{Id}\otimes\varrho)((\mathfrak{R}_u)_{(21)}) = \prod^{\leftarrow}_{\gamma\in\bar{\Phi}^+} \left(\sum^{k\geq 0}_{k\leq 1 \text{ if } \gamma\in\bar{\Phi}_{\bar{1}} \text{ isotropic}} (-1)^{|e^k_\gamma|} \frac{f^k_\gamma \otimes \varrho(e^k_\gamma)}{(f^k_\gamma, e^k_\gamma)_{\mathrm{DJ}}} \right).$$

Because the representation ϱ preserves the Q-degree, the only contribution to this component arises from

$$(-1)^{|e_{\gamma_{ij}}|} \frac{f_{\gamma_{ij}} \otimes \varrho(e_{\gamma_{ij}})}{(f_{\gamma_{ij}}, e_{\gamma_{ij}})_{\mathrm{DJ}}} = -(q-q^{-1}) f_{\gamma_{ij}} \otimes \varrho(e_{\gamma_{ij}})\,.$$

As the coefficient of E_{ij} in each $\varrho(e_{\gamma_{ij}})$ is 1 by (4.3)–(4.5), this implies that $\phi^{\mathrm{DF}}(l^+_{ij}) = -(q-q^{-1}) f_{\gamma_{ij}} q^{-\tilde{H}_j}$.

Let us now determine the E_{ji}-component of

$$(\mathrm{Id}\otimes\varrho)(\mathfrak{R}_u^{-1}) = \prod^{\rightarrow}_{\gamma\in\bar{\Phi}^+} \left(\sum^{k\geq 0}_{k\leq 1 \text{ if } \gamma\in\bar{\Phi}_{\bar{1}} \text{ isotropic}} \frac{e^k_\gamma \otimes \varrho(f^k_\gamma)}{(f^k_\gamma, e^k_\gamma)_{\mathrm{DJ}}} \right)^{-1} = \prod^{\rightarrow}_{\gamma\in\bar{\Phi}^+} \sum_{r\geq 0} \left(- \sum^{k\geq 1}_{k\leq 1 \text{ if } \gamma\in\bar{\Phi}_{\bar{1}} \text{ isotropic}} \frac{e^k_\gamma \otimes \varrho(f^k_\gamma)}{(f^k_\gamma, e^k_\gamma)_{\mathrm{DJ}}} \right)^r$$

where we utilize the nilpotency of $\sum^{k\geq 1}_{k\leq 1 \text{ if } \gamma\in\bar{\Phi}_{\bar{1}} \text{ isotropic}} \frac{e^k_\gamma\otimes\varrho(f^k_\gamma)}{(f^k_\gamma, e^k_\gamma)_{\mathrm{DJ}}}$. The sole contributing term is:

$$-\frac{e_{\gamma_{ij}} \otimes \varrho(f_{\gamma_{ij}})}{(f_{\gamma_{ij}}, e_{\gamma_{ij}})_{\mathrm{DJ}}} = (-1)^{\bar{i}+\bar{j}}(q-q^{-1}) e_{\gamma_{ij}} \otimes \varrho(f_{\gamma_{ij}})\,.$$

Within $\varrho(f_{\gamma_{ij}})$, the coefficient of the matrix unit E_{ji} evaluates to $-(q+q^{-1})$ when $\gamma_{ij} = \alpha_s$ in C-type, and to $(-1)^{\bar{i}}$ in all other cases, see (5.4) combined with (4.3)–(4.5). Therefore:

$$\phi^{\mathrm{DF}}(l^-_{ji}) = \begin{cases} -(q^2-q^{-2}) q^{\tilde{H}_{s'}} e_s & \text{if } \ \gamma_{ij} = \alpha_s \text{ for } C\text{-type}\,, \\ (-1)^{\bar{j}}(q-q^{-1}) q^{\tilde{H}_j} e_{\gamma_{ij}} & \text{otherwise}\,. \end{cases}$$

This completes the proof. □

We conclude with the following analogue of Proposition 3.80:

Proposition 5.85. *The composition*

$$U_q(\mathfrak{gosp}(V)) \xrightarrow{\phi_{\mathcal{F}}^{-1}} U_q(\mathfrak{gosp}(V))^{\mathcal{F}} \xrightarrow[(5.79)]{\xi} U(R) \xrightarrow[(5.10)]{\omega_R^{-1}} U(R)^{\mathrm{cop}} \xrightarrow[(5.83)]{\phi^{\mathrm{DF}}} U_q(\mathfrak{gosp}(V))$$

(see Proposition 4.28 for $\phi_{\mathcal{F}}$) is the identity morphism.

Proof. This is verified by direct computation. For the Cartan generators, we have:

$$q^{\pm \tilde{H}_i} \xmapsto{\phi_{\mathcal{F}}^{-1}} q^{\pm \tilde{H}_i} \xmapsto{\xi} q^{\pm \tilde{\mathbf{H}}_i} \xmapsto{\omega_R^{-1}} q^{\mp \tilde{\mathbf{H}}_i} \xmapsto{\phi^{\mathrm{DF}}} q^{\pm \tilde{H}_i}\,,$$

while the rest is based on the following type-dependent computations:

- $1 \leq i < s$ *in BCD-types*

$$e_i \xmapsto{\phi_{\mathcal{F}}^{-1}} e_i q^{h_i} \xmapsto{\xi} \mathbf{e}_{i,i+1} q^{\tilde{\mathbf{H}}_{i,i+1}} \xmapsto{\omega_R^{-1}} -(-1)^{\bar{i}\,\overline{i+1}} (-1)^{\bar{i}+\overline{i+1}} q^{3(\rho,\alpha_i)} \mathbf{f}_{i+1,i} q^{-\tilde{\mathbf{H}}_{i,i+1}} \xmapsto{\phi^{\mathrm{DF}}} e_i\,,$$
$$f_i \xmapsto{\phi_{\mathcal{F}}^{-1}} q^{-h_i} f_i \xmapsto{\xi} q^{-\tilde{\mathbf{H}}_{i,i+1}} \mathbf{f}_{i+1,i} \xmapsto{\omega_R^{-1}} -(-1)^{\bar{i}\,\overline{i+1}} q^{-3(\rho,\alpha_i)} q^{\tilde{\mathbf{H}}_{i,i+1}} \mathbf{e}_{i,i+1} \xmapsto{\phi^{\mathrm{DF}}} f_i\,.$$

- $i = s$ *in B-type*

$$e_s \xmapsto{\phi_{\mathcal{F}}^{-1}} e_s q^{h_s} \xmapsto{\xi} \mathbf{e}_{s,s+1} q^{\tilde{\mathbf{H}}_{s,s+1}} \xmapsto{\omega_R^{-1}} -(-1)^{\bar{s}\,\overline{s+1}} (-1)^{\bar{s}+\overline{s+1}} q^{3(\rho,\alpha_s)} \mathbf{f}_{s+1,s} q^{-\tilde{\mathbf{H}}_{s,s+1}} \xmapsto{\phi^{\mathrm{DF}}} e_s\,,$$
$$f_s \xmapsto{\phi_{\mathcal{F}}^{-1}} q^{-h_s} f_s \xmapsto{\xi} q^{-\tilde{\mathbf{H}}_{s,s+1}} \mathbf{f}_{s+1,s} \xmapsto{\omega_R^{-1}} -(-1)^{\bar{s}\,\overline{s+1}} q^{-3(\rho,\alpha_s)} q^{\tilde{\mathbf{H}}_{s,s+1}} \mathbf{e}_{s,s+1} \xmapsto{\phi^{\mathrm{DF}}} f_s\,.$$

- $i = s$ *in C-type*

$$e_s \xmapsto{\phi_{\mathcal{F}}^{-1}} e_s q^{h_s} \xmapsto{\xi} \frac{1}{1+q^2} \mathbf{e}_{ss'} q^{\tilde{\mathbf{H}}_{s,s'}} \xmapsto{\omega_R^{-1}} \frac{1}{q^4(1+q^2)} \mathbf{f}_{s's} q^{-\tilde{\mathbf{H}}_{s,s'}} \xmapsto{\phi^{\mathrm{DF}}} e_s\,,$$
$$f_s \xmapsto{\phi_{\mathcal{F}}^{-1}} q^{-h_s} f_s \xmapsto{\xi} q \cdot q^{-\tilde{\mathbf{H}}_{s,s'}} \mathbf{f}_{s's} \xmapsto{\omega_R^{-1}} q^5 \cdot q^{\tilde{\mathbf{H}}_{s,s'}} \mathbf{e}_{ss'} \xmapsto{\phi^{\mathrm{DF}}} f_s\,.$$

- $i = s$ *in D-type*

$$e_s \xmapsto{\phi_{\mathcal{F}}^{-1}} e_s q^{h_s} \xmapsto{\xi} \mathbf{e}_{s-1,s'} q^{\tilde{\mathbf{H}}_{s-1,s'}} \xmapsto{\omega_R^{-1}} -(-1)^{\overline{s-1}\,\bar{s}} (-1)^{\overline{s-1}+\bar{s}} q^{3(\rho,\alpha_s)} \mathbf{f}_{s',s-1} q^{-\tilde{\mathbf{H}}_{s-1,s'}} \xmapsto{\phi^{\mathrm{DF}}} e_s\,,$$
$$f_s \xmapsto{\phi_{\mathcal{F}}^{-1}} q^{-h_s} f_s \xmapsto{\xi} q^{-\tilde{\mathbf{H}}_{s-1,s'}} \mathbf{f}_{s',s-1} \xmapsto{\omega_R^{-1}} -(-1)^{\overline{s-1}\,\overline{s'}} q^{-3(\rho,\alpha_s)} q^{\tilde{\mathbf{H}}_{s-1,s'}} \mathbf{e}_{s-1,s'} \xmapsto{\phi^{\mathrm{DF}}} f_s\,.$$

□

5.6. **Factorization of the reduced canonical element: orthosymplectic case.**

We now derive the canonical element $\mathfrak{R}^R$ of $U(R)$ in the orthosymplectic setting and establish the factorization of the associated reduced R-matrix. The structural framework utilized in Subsection 3.9 adapts seamlessly to $U(R)$ for $\mathfrak{gosp}$. Since the underlying logic for the skew-pairing evaluations and the derivation of the PBW-type bases remains identical, we record the main structural results and the resulting factorization below, omitting the repetitive proofs.

As before, we employ the transposed inverse skew-pairing $(-,-)_{\widetilde{R}}$ to ensure compatibility with (4.26). Let $U^{>}(R)$, $U^{<}(R)$, and $U^0(R)$ be the subalgebras of $U(R)$ generated by $\{\mathbf{e}_{ij}\}_{1\leq i<j\leq N}$, $\{\mathbf{f}_{ji}\}_{1\leq i<j\leq N}$, and $\{q^{\pm\tilde{\mathbf{H}}_k}\}_{k=1}^N$, respectively. As in Subsection 3.9, for each $\gamma = \gamma_{ij} = \varepsilon_i - \varepsilon_j \in \bar{\Phi}^+$ with the choice of (i,j) as in (5.1), we set $\mathbf{e}_\gamma := \mathbf{e}_{ij}$, $\mathbf{f}_\gamma := \mathbf{f}_{ji}$, and $q^{\pm\tilde{\mathbf{H}}_\gamma} := q^{\pm\tilde{\mathbf{H}}_{ij}}$. We note that these listed elements indeed generate the subalgebras $U^{>}(R)$, $U^{<}(R)$, and $U^0(R)$ respectively, as follows from Theorem 5.78. Furthermore, the multiplication yields the triangular decomposition isomorphisms $U^0(R)\otimes U^{>}(R) \xrightarrow{\sim} U^{\geq}(R)$, $U^0(R)\otimes U^{<}(R) \xrightarrow{\sim} U^{\leq}(R)$ of superspaces.

The following counterpart of Lemma 3.82 establishes orthogonality of ordered products of root vectors:

Lemma 5.86. *For any sequences of non-negative integers $\mathbf{m} = (m_\gamma)_{\gamma\in\bar{\Phi}^+}$ and $\mathbf{n} = (n_\gamma)_{\gamma\in\bar{\Phi}^+}$ (restricted to $m_\gamma, n_\gamma \leq 1$ whenever $\gamma \in \Phi_{\bar{1}}$ is isotropic), we have*

$$\left(\overleftarrow{\prod_{\gamma\in\bar{\Phi}^+}} \mathbf{f}_\gamma^{m_\gamma},\ \overleftarrow{\prod_{\gamma\in\bar{\Phi}^+}} \mathbf{e}_\gamma^{n_\gamma} \right)_{\widetilde{R}} = (-1)^{\chi(\mathbf{m})} \prod_{\gamma\in\bar{\Phi}^+} \delta_{m_\gamma,n_\gamma} \mathsf{C}_{\gamma,m_\gamma} (\mathbf{f}_\gamma, \mathbf{e}_\gamma)_{\widetilde{R}}^{m_\gamma}\,,$$

where $\mathsf{C}_{\gamma,p}$ and $\chi(\mathbf{m})$ are defined exactly as in Lemma 3.82.

This orthogonality, together with the non-degeneracy of (4.26) and a dimensional comparison via the isomorphism ξ of (5.79), immediately yields the corresponding PBW-type bases for $U^{>}(R)$ and $U^{<}(R)$:

Corollary 5.87. *The sets of ordered monomials*

$$\left\{ \prod_{\gamma\in\bar{\Phi}^+}^{\leftarrow} \mathbf{e}_\gamma^{m_\gamma} \,\middle|\, {}^{m_\gamma\in\mathbb{Z}_{\geq 0}}_{m_\gamma\leq 1 \text{ if } \gamma\in\bar{\Phi}_{\bar{1}} \text{ isotropic}} \right\} \qquad \text{and} \qquad \left\{ \prod_{\gamma\in\bar{\Phi}^+}^{\leftarrow} \mathbf{f}_\gamma^{m_\gamma} \,\middle|\, {}^{m_\gamma\in\mathbb{Z}_{\geq 0}}_{m_\gamma\leq 1 \text{ if } \gamma\in\bar{\Phi}_{\bar{1}} \text{ isotropic}} \right\} \tag{5.88}$$

form bases for $U^>(R)$ and $U^<(R)$, respectively.

Similarly to the general linear case, we thus obtain the factorization of the canonical element

$$\mathfrak{R}^R = \mathfrak{R}^R_s \mathfrak{R}^R_u \,,$$

whereas $\mathfrak{R}^R_u$ can be evaluated using bases (5.88) whose pairing is computed in Lemma 5.86:

Proposition 5.89. *$\mathfrak{R}^R_u$ factorizes as*

$$\mathfrak{R}^R_u = \prod_{\gamma\in\bar{\Phi}^+}^{\leftarrow} \left(\sum_{k\leq 1 \text{ if } \gamma\in\bar{\Phi}_{\bar{1}} \text{ isotropic}}^{k\geq 0} \frac{\mathbf{e}_\gamma^k \otimes \mathbf{f}_\gamma^k}{(\mathbf{f}_\gamma^k, \mathbf{e}_\gamma^k)_{\widetilde{R}}} \right) .$$

Remark 5.90. We note that one can easily derive the multi-parameter version of the above factorization of the R-matrix within the RLL-realization, thus generalizing the super A-type results of [24], by utilizing the bicharacter twists exactly as Theorem B.7 is derived from Theorem B.6 in [19].

Appendix A. Generalized double construction in super setting

This Appendix is devoted to the generalization of the classical constructions from [18] to the setting of superbialgebras. Specifically, we systematically incorporate the $\mathbb{Z}_2$-grading and the corresponding super-sign conventions into the standard theory of matched pairs. Throughout this section, we assume for convenience that all elements of superbialgebras we consider are $\mathbb{Z}_2$-homogeneous.

A.1. **Matched pairs of superbialgebras.**

Let A and B be superbialgebras. Our goal is to endow the superspace $A\otimes B$ with a superbialgebra structure such that A and B can be identified as subbialgebras via the natural embeddings:

$$A \hookrightarrow A\otimes B\,, \qquad a\mapsto a\otimes 1\,,$$
$$B \hookrightarrow A\otimes B\,, \qquad b\mapsto 1\otimes b\,.$$

Requiring these embeddings to be superbialgebra morphisms (and assuming $A\otimes B$ is endowed with such a superalgebra structure) forces the following multiplication rules:

$$(a\otimes 1)(a'\otimes 1) = (aa'\otimes 1)\,, \qquad (1\otimes b)(1\otimes b') = (1\otimes bb')\,, \qquad (a\otimes 1)(1\otimes b) = (a\otimes b)\,. \tag{A.1}$$

Analogously, for the comultiplication:[3]

$$\Delta(a\otimes b) = \Delta(a\otimes 1)\Delta(1\otimes b) = \sum_{(a)(b)} (-1)^{|a_2||b_1|}(a_1\otimes 1)(1\otimes b_1)\otimes(a_2\otimes 1)(1\otimes b_2)$$
$$= \sum_{(a)(b)} (-1)^{|a_2||b_1|}(a_1\otimes b_1)\otimes(a_2\otimes b_2)\,,$$

and for the counit:

$$\epsilon(a\otimes b) = \epsilon(a\otimes 1)\epsilon(1\otimes b) = \epsilon(a)\epsilon(b)\,.$$

The only remaining nontrivial structure is the cross-multiplication $(1\otimes b)(a\otimes 1)$. To define this, we introduce morphisms (hence degree-preserving):

$$\triangleright\colon B\otimes A\to A\,, \qquad \triangleleft\colon B\otimes A\to B\,,$$

and define the multiplication as follows:

$$(1\otimes b)(a\otimes 1) = \sum_{(a)(b)} (-1)^{|b_2||a_1|}(b_1\triangleright a_1)\otimes(b_2\triangleleft a_2)\,. \tag{A.2}$$

[3]Throughout this section, unless stated otherwise, we omit parentheses in the subscripts and write the Sweedler notation as $\Delta(a)=\sum_{(a)} a_1\otimes a_2$ instead of $a_{(1)}\otimes a_{(2)}$. Additionally, to maintain brevity during iterated comultiplications, we suppress the summation symbols for subsequent components such as (a_i) or (b_i).

Proposition A.3. *Suppose that $\triangleright$ defines a left B-module supercoalgebra structure on A, and $\triangleleft$ defines a right A-module supercoalgebra structure on B. If these maps satisfy the following compatibility conditions:*

$$b \triangleright aa' = \sum_{(b)(a)} (-1)^{|b_2||a_1|}(b_1 \triangleright a_1)\cdot((b_2 \triangleleft a_2) \triangleright a')\,, \qquad b \triangleright 1 = \epsilon(b)\,,$$

$$bb' \triangleleft a = \sum_{(b')(a)} (-1)^{|b'_2||a_1|}(b \triangleleft (b'_1 \triangleright a_1))\cdot(b'_2 \triangleleft a_2)\,, \qquad 1 \triangleleft a = \epsilon(a)\,,$$

$$\sum_{(a)(b)} (-1)^{|b_2||a_1|}(b_1 \triangleleft a_1)\otimes(b_2 \triangleright a_2)$$
$$= \sum_{(a)(b)} (-1)^{|b_2||a_1|}(-1)^{(|a_1|+|b_1|)(|a_2|+|b_2|)}(b_2 \triangleleft a_2)\otimes(b_1 \triangleright a_1)\,,$$

*then the pair (A, B) is called a **matched pair** of superbialgebras. In this case, the preceding formulas (A.1)–(A.2) endow $A\otimes B$ with a well-defined superbialgebra structure, called the **double cross product** of A and B and denoted by $A \bowtie B$. Moreover, if A and B are Hopf superalgebras with respective antipodes S_A and S_B, then $A \bowtie B$ is also a Hopf superalgebra with antipode $S_A \otimes S_B$.*

Proof. By the definition of module supercoalgebra structures and module actions, the following identities hold:

$$\Delta(b \triangleright a) = \sum_{(a)(b)} (-1)^{|b_2||a_1|}(b_1 \triangleright a_1)\otimes(b_2 \triangleright a_2)\,, \qquad \epsilon(b \triangleright a) = \epsilon(b)\epsilon(a)\,,$$

$$\Delta(b \triangleleft a) = \sum_{(a)(b)} (-1)^{|b_2||a_1|}(b_1 \triangleleft a_1)\otimes(b_2 \triangleleft a_2)\,, \qquad \epsilon(b \triangleleft a) = \epsilon(b)\epsilon(a)\,,$$

as well as:

$$b \triangleright (b' \triangleright a) = bb' \triangleright a\,, \qquad 1 \triangleright a = a\,,$$
$$(b \triangleleft a) \triangleleft a' = b \triangleleft aa'\,, \qquad b \triangleright 1 = b\,.$$

We now verify the superbialgebra axioms. Since many of the required conditions hold trivially by construction, we exhibit only the nontrivial cases.

- **Associativity 1.**

$$(1\otimes b)\{(1\otimes b')(a\otimes 1)\} = (1\otimes b)\sum_{(b')(a)} (-1)^{|b'_2||a_1|}(b'_1 \triangleright a_1)\otimes(b'_2 \triangleleft a_2)$$
$$= \sum_{(b)(b')(a)} (-1)^{|b'_2||a_1|}(-1)^{|b'_{1,2}||a_{1,1}|}(-1)^{|b_2|(|b'_{1,1}|+|a_{1,1}|)}\{b_1 \triangleright (b'_{1,1} \triangleright a_{1,1})\}\otimes\{b_2 \triangleleft (b'_{1,2} \triangleright a_{1,2})\}(b'_2 \triangleleft a_2)$$
$$= \sum_{(b)(b')(a)} (-1)^{|b'_3|(|a_1|+|a_2|)}(-1)^{|b'_2||a_1|}(-1)^{|b_2|(|b'_1|+|a_1|)}\{b_1 \triangleright (b'_1 \triangleright a_1)\}\otimes\{b_2 \triangleleft (b'_2 \triangleright a_2)\}(b'_3 \triangleleft a_3)$$
$$= \sum_{(b)(b')(a)} (-1)^{|b'_2||a_1|}(-1)^{|b_2|(|b'_1|+|a_1|)}\{b_1b'_1 \triangleright a_1\}\otimes\{b_2b'_2 \triangleleft a_2\}\,,$$

which coincides with

$$\{(1\otimes b)(1\otimes b')\}(a\otimes 1) = (1\otimes bb')(a\otimes 1) = \sum_{(b)(b')(a)} (-1)^{|b_2||b'_1|}(-1)^{(|b_2|+|b'_2|)|a_1|}(b_1b'_1 \triangleright a_1)\otimes(b_2b'_2 \triangleleft a_2)\,.$$

- **Associativity 2.**

$$\{(1\otimes b)(a\otimes 1)\}(a'\otimes 1) = \sum_{(b)(a)} (-1)^{|b_2||a_1|}\{(b_1 \triangleright a_1)\otimes(b_2 \triangleleft a_2)\}(a'\otimes 1)$$
$$= \sum_{(b)(a)(a')} (-1)^{|b_2||a_1|}(-1)^{|b_{2,2}||a_{2,1}|}(-1)^{(|b_{2,2}|+|a_{2,2}|)|a'_1|}\{(b_1 \triangleright a_1)((b_{2,1} \triangleleft a_{2,1}) \triangleright a'_1)\}\otimes\{(b_{2,2} \triangleleft a_{2,2}) \triangleleft a'_2\}$$
$$= \sum_{(b)(a)(a')} (-1)^{(|b_2|+|b_3|)|a_1|}(-1)^{|b_3||a_2|}(-1)^{(|b_3|+|a_3|)|a'_1|}\{(b_1 \triangleright a_1)((b_2 \triangleleft a_2) \triangleright a'_1)\}\otimes\{(b_3 \triangleleft a_3) \triangleleft a'_2\}$$
$$= \sum_{(b)(a)(a')} (-1)^{|b_2||a_1|}(-1)^{(|b_2|+|a_2|)|a'_1|}\{b_1 \triangleright a_1a'_1\}\otimes\{b_2 \triangleleft a_2a'_2\}\,,$$

which coincides with

$$(1\otimes b)\{(a\otimes 1)(a'\otimes 1)\} = (1\otimes b)(aa'\otimes 1) = \sum_{(b)(a)(a')} (-1)^{|a_2||a'_1|}(-1)^{|b_2|(|a_1|+|a'_1|)}(b_1 \triangleright a_1a'_1)\otimes(b_2 \triangleleft a_2a'_2)\,.$$

• **Unit 1.**

$$(1\otimes b)(1\otimes 1)=\sum_{(b)}(b_1\triangleright 1)\otimes(b_2\triangleleft 1)=\sum_{(b)}\epsilon(b_1)\otimes b_2=1\otimes b\,.$$

• **Unit 2.**

$$(1\otimes 1)(a\otimes 1)=\sum_{(a)}(1\triangleright a_1)\otimes(1\triangleleft a_2)=\sum_{(a)}a_1\otimes\epsilon(a_2)=a\otimes 1\,.$$

• **Comultiplication and multiplication.**

$$\begin{aligned}
\Delta\{(1\otimes b)(a\otimes 1)\} &= \sum_{(a)(b)}(-1)^{|b_2||a_1|}\Delta\{(b_1\triangleright a_1)\otimes(b_2\triangleleft a_2)\}\\
&= \sum_{(a)(b)}(-1)^{(|b_3|+|b_4|)(|a_1|+|a_2|)}(-1)^{|b_2||a_1|}(-1)^{|b_4||a_3|}(-1)^{(|b_2|+|a_2|)(|b_3|+|a_3|)}\\
&\qquad\cdot\{(b_1\triangleright a_1)\otimes(b_3\triangleleft a_3)\}\otimes\{(b_2\triangleright a_2)\otimes(b_4\triangleleft a_4)\}\\
&= \sum_{(a)(b)}(-1)^{|b_4||a_1|}(-1)^{(|b_2|+|b_3|)|a_1|}(-1)^{|b_4|(|a_2|+|a_3|)}(-1)^{|b_3||a_2|}(-1)^{(|b_2|+|a_2|)(|b_3|+|a_3|)}\\
&\qquad\cdot\{(b_1\triangleright a_1)\otimes(b_3\triangleleft a_3)\}\otimes\{(b_2\triangleright a_2)\otimes(b_4\triangleleft a_4)\}\\
&= \sum_{(a)(b)}(-1)^{|b_4||a_1|}(-1)^{(|b_2|+|b_3|)|a_1|}(-1)^{|b_4|(|a_2|+|a_3|)}(-1)^{|b_3||a_2|}\\
&\qquad\cdot\{(b_1\triangleright a_1)\otimes(b_2\triangleleft a_2)\}\otimes\{(b_3\triangleright a_3)\otimes(b_4\triangleleft a_4)\}\,,
\end{aligned}$$

which coincides with

$$\begin{aligned}
&\Delta(1\otimes b)\Delta(a\otimes 1)=\sum_{(a)(b)}(-1)^{|b_2||a_1|}(1\otimes b_1)(a_1\otimes 1)\otimes(1\otimes b_2)(a_2\otimes 1)\\
&=\sum_{(a)(b)}(-1)^{(|b_3|+|b_4|)(|a_1|+|a_2|)}(-1)^{|b_2||a_1|}(-1)^{|b_4||a_3|}\{(b_1\triangleright a_1)\otimes(b_2\triangleleft a_2)\}\otimes\{(b_3\triangleright a_3)\otimes(b_4\triangleleft a_4)\}\,.
\end{aligned}$$

• **Counit and multiplication.** Since ϵ vanishes on odd-degree elements:

$$\begin{aligned}
\epsilon\{(1\otimes b)(a\otimes 1)\} &= \sum_{(a)(b)}(-1)^{|b_2||a_1|}\epsilon\{(b_1\triangleright a_1)\otimes(b_2\triangleleft a_2)\}=\sum_{(a)(b)}(-1)^{|b_2||a_1|}\epsilon(b_1\triangleright a_1)\epsilon(b_2\triangleleft a_2)\\
&= \sum_{(a)(b)}(-1)^{|b_2||a_1|}\epsilon(b_1)\epsilon(a_1)\epsilon(b_2)\epsilon(a_2)=\sum_{(a)(b)}\epsilon(b_1)\epsilon(a_1)\epsilon(b_2)\epsilon(a_2)=\epsilon(b)\epsilon(a)\,,
\end{aligned}$$

which coincides with $\epsilon(1\otimes b)\epsilon(a\otimes 1)=\epsilon(b)\epsilon(a)$. □

A.2. **Convolution product.**

Let us briefly recall the *convolution product* that will be frequently used below. Let (A,μ_A,η_A) be a superalgebra, (C,Δ_C,ϵ_C) be a supercoalgebra, and consider the space of morphisms $\mathrm{Hom}_{\mathrm{sVect}}(C,A)$. For any two morphisms $\phi,\psi\colon C\to A$, their convolution $\phi\star\psi\colon C\to A$ is defined as the composition $\mu_A\circ(\phi\otimes\psi)\circ\Delta_C$. This equips $\mathrm{Hom}_{\mathrm{sVect}}(C,A)$ with a unital associative (non-super) algebra structure, with the identity element $\eta_A\circ\epsilon_C$. A morphism is called *convolution invertible* if it admits an inverse with respect to this product.

Remark A.4. Here, we restrict the convolution product to the space of morphisms $\mathrm{Hom}_{\mathrm{sVect}}(C,A)$, i.e. degree-preserving linear maps. Alternatively, one could use the internal $\mathrm{Hom}_{\mathbb{K}}(C,A)$ (cf. Subsection 2.2) consisting of all linear maps, in which case the convolution algebra naturally becomes a superalgebra. Unless stated otherwise, we shall work in the space of morphisms and thus frequently omit the subscript sVect.

The following functoriality properties of the convolution product are straightforward:

Lemma A.5. *Let A and A' be superalgebras, and let C and C' be supercoalgebras.*

(a) For any superalgebra morphism $\phi\colon A\to A'$, the postcomposition $\phi_*\colon \mathrm{Hom}(C,A)\to\mathrm{Hom}(C,A')$ is an algebra morphism (i.e. it preserves the convolution product).

(b) For any supercoalgebra morphism $\phi\colon C\to C'$, the precomposition $\phi^*\colon \mathrm{Hom}(C',A)\to\mathrm{Hom}(C,A)$ is an algebra morphism (i.e. it preserves the convolution product).

We also note the following general result:

Lemma A.6. *Let A be a superalgebra, C be a supercoalgebra, and consider the space* $\mathrm{Hom}(C, A)$ *endowed with the convolution product.*

(a) If C has a superbialgebra structure and $\phi\colon C \to A$ is a superalgebra morphism, then its convolution inverse ϕ' is a superalgebra morphism from C to A^{op} (or equivalently, from C^{op} to A).

(b) If A has a superbialgebra structure and $\phi\colon C \to A$ is a supercoalgebra morphism, then its convolution inverse ϕ' is a supercoalgebra morphism from C^{cop} to A (or equivalently, from C to A^{cop}).

Proof. (a) As the multiplication $\mu_C\colon C \otimes C \to C$ is a supercoalgebra morphism, Lemma A.5(b) implies that $\phi' \circ \mu_C\colon C\otimes C \to A$ is a convolution inverse of $\phi\circ\mu_C$. Furthermore, one can check that $g \coloneqq \mu_A \circ \tau_{AA} \circ (\phi' \otimes \phi')$ is also a convolution inverse of $f \coloneqq \phi \circ \mu_C = \mu_A \circ (\phi \otimes \phi)$ via the following computation:

$$
\begin{aligned}
(f \star g)(c \otimes c') &= \sum_{(c)(c')} (-1)^{|c_2||c_1'|} f(c_1 \otimes c_1') g(c_2 \otimes c_2') \\
&= \sum_{(c)(c')} (-1)^{|c_2||c'|} \phi(c_1)\phi(c_1')\phi'(c_2')\phi'(c_2) = \sum_{(c)} \epsilon_C(c')\phi(c_1)\phi'(c_2) = \epsilon_C(c)\epsilon_C(c') = \epsilon_{C\otimes C}(c \otimes c')\,.
\end{aligned}
$$

By the uniqueness of the convolution inverse, we get $\phi' \circ \mu_C = \mu_A \circ \tau_{AA} \circ (\phi' \otimes \phi')$, completing the proof.
(b) The proof is completely analogous to that of part (a). ☐

A.3. **Coadjoint actions.**

Let A, B be superbialgebras over a field $\mathbb{K}$, and consider a *dual pairing* between A and B, i.e. a bilinear map

$$
(\cdot,\cdot)\colon A \otimes B \to \mathbb{K}\,,
$$

satisfying:

$$
\begin{aligned}
(1, b) &= \epsilon(b)\,, & (a, 1) &= \epsilon(a)\,,\\
(aa', b) &= (a \otimes a', \Delta(b))\,, & (a, bb') &= (\Delta(a), b \otimes b')\,,
\end{aligned}
$$

subject to the usual sign convention:

$$
(a \otimes a', b \otimes b') \coloneqq (-1)^{|a'||b|}(a, b)(a', b')\,.
$$

Remark A.7. Recall that the dual space of a supercoalgebra admits a canonical superalgebra structure. Therefore, a bilinear map $(\cdot,\cdot)\colon A \otimes B \to \mathbb{K}$ is a dual pairing iff the following are superalgebra morphisms:

$$
\begin{aligned}
A &\to B^*\,, & a &\mapsto \big(b \mapsto (a, b)\big)\,,\\
B &\to A^*\,, & b &\mapsto \big(a \mapsto (-1)^{|a||b|}(a, b)\big)\,.
\end{aligned}
\tag{A.8}
$$

Alternatively, recall the *restricted dual* H° of a superbialgebra H, consisting of all linear functionals $f \in H^*$ such that their pullback $f \circ \mu_H$ along the multiplication map $\mu_H\colon H \otimes H \to H$ lies in the subspace $H^* \otimes H^* \subseteq (H \otimes H)^*$ (cf. [16, Section 1.2.8]). By construction, the restricted dual of a superbialgebra is again a superbialgebra. Thus, a bilinear map is a dual pairing if and only if the induced linear map $A \to B^*$ takes values in B° and defines a superbialgebra morphism $A \to B^\circ$ (or, analogously, if the induced map $B \to A^*$ defines a superbialgebra morphism $B \to A^\circ$). Since the morphism $A \to B^\circ$ canonically identifies with the morphisms

$$
A^{\mathrm{op}} \to (B^\circ)^{\mathrm{op}} = (B^{\mathrm{cop}})^\circ\,, \qquad A^{\mathrm{cop}} \to (B^\circ)^{\mathrm{cop}} = (B^{\mathrm{op}})^\circ\,, \qquad A^{\mathrm{op,cop}} \to (B^\circ)^{\mathrm{op,cop}} = (B^{\mathrm{op,cop}})^\circ\,,
$$

the initial pairing naturally induces dual pairings between the following superbialgebras:

$$
A^{\mathrm{op}} \otimes B^{\mathrm{cop}} \to \mathbb{K}\,, \qquad A^{\mathrm{cop}} \otimes B^{\mathrm{op}} \to \mathbb{K}\,, \qquad A^{\mathrm{op,cop}} \otimes B^{\mathrm{op,cop}} \to \mathbb{K}\,.
$$

The underlying morphism of superspaces (hence the explicit formula for the pairing) is identical in all cases.

Remark A.9. Since $(\cdot,\cdot)$ is a morphism between superspaces, it is degree-preserving by definition and thus $(a, b) = 0$ if $|a| + |b| \neq \bar{\mathbf{0}}$.

Since $A \otimes B$ admits a canonical supercoalgebra structure, the space of superspace morphisms from $A \otimes B$ to $\mathbb{K}$ can be endowed with the convolution product: the convolution $\phi \star \psi$ of morphisms ϕ and ψ is given by

$$
(\phi \star \psi)(a \otimes b) \coloneqq \big(\mu_{\mathbb{K}} \circ (\phi \otimes \psi) \circ \Delta\big)(a \otimes b) = \sum_{(a)(b)} (-1)^{|a_2||b_1|} \phi(a_1 \otimes b_1)\psi(a_2 \otimes b_2)\,.
$$

Moreover, the morphism $a\otimes b \mapsto \epsilon(a)\epsilon(b)$ is the identity element for the convolution product. A dual pairing is called *convolution invertible* if it admits a convolution inverse. Notably, any dual pairing $(-,-)\colon A\otimes B \to \mathbb{K}$ between Hopf superalgebras is convolution invertible, with its inverse given by:

$$
a \otimes b \mapsto (S(a), b) = (a, S(b))\,. \tag{A.10}
$$

Let $(-,-)\colon A\otimes B\to \mathbb{K}$ be a convolution invertible dual pairing between superbialgebras, and let $(-,-)'$ denote its convolution inverse. By definition, the following identity holds:

$$\sum_{(a)(b)} (-1)^{|a_2||b_1|}(a_1,b_1)(a_2,b_2)' = \epsilon(a)\epsilon(b) = \sum_{(a)(b)} (-1)^{|a_2||b_1|}(a_1,b_1)'(a_2,b_2)\,. \tag{A.11}$$

We also note the following:

Lemma A.12. *Let $\sigma\colon A\otimes B\to\mathbb{K}$ be a convolution invertible dual pairing. Its convolution inverse σ' is a dual pairing between A^{op} and B^{op} (equivalently, a dual pairing between A^{cop} and B^{cop} by Remark A.7).*

Proof. Consider the superalgebra structure on the dual space B^*, and let $\lambda\colon\ \mathrm{Hom}(A\otimes B,\mathbb{K}) \xrightarrow{\sim} \mathrm{Hom}(A,B^*)$ be the canonical isomorphism of vector spaces mapping ϕ to $\lambda(\phi)\colon a\mapsto (b\mapsto \phi(a\otimes b))$. We claim that λ intertwines the convolution products. Indeed, for any $\phi,\psi\in \mathrm{Hom}(A\otimes B,\mathbb{K})$, we have:

$$\begin{aligned}(\lambda(\phi)\star\lambda(\psi))(a)(b) &= \sum_{(a)}\big(\lambda(\phi)(a_1)\cdot\lambda(\psi)(a_2)\big)(b) = \sum_{(a)(b)}(-1)^{|a_2||b_1|}\lambda(\phi)(a_1)(b_1)\cdot\lambda(\psi)(a_2)(b_2)\\ &= \sum_{(a)(b)}(-1)^{|a_2||b_1|}\phi(a_1\otimes b_1)\cdot\psi(a_2\otimes b_2) = (\phi\star\psi)(a\otimes b) = \lambda(\phi\star\psi)(a)(b)\,.\end{aligned}$$

Since σ is a dual pairing, $\lambda(\sigma)\colon A\to B^*$ is a superalgebra morphism (see Remark A.7). By Lemma A.6(a), its convolution inverse $\lambda(\sigma')=\lambda(\sigma)'$ is a superalgebra morphism from A^{op} to B^*, which establishes the first condition of (A.8) for A^{op} and B^{op}.

A completely similar argument utilizing A^* and the canonical isomorphism $\mathrm{Hom}(A\otimes B,\mathbb{K})\xrightarrow{\sim}\mathrm{Hom}(B,A^*)$ completes the proof. □

A convolution invertible dual pairing canonically induces a double cross product algebra structure via the following coadjoint actions:

Definition A.13. The *right coadjoint action* $\triangleleft_{\mathrm{Coad}}\colon B\otimes A\to B$ of A on B is defined by:

$$b\triangleleft_{\mathrm{Coad}} a = \sum_{(a)(b)}(-1)^{|a|}(-1)^{|b_1||b_2|+|b_1||b_3|}(a_1,b_1)(a_2,b_3)'b_2\,.$$

Analogously, the *right coadjoint action* $\triangleleft_{\mathrm{Coad}}\colon A\otimes B\to A$ of B on A is defined by:

$$a\triangleleft_{\mathrm{Coad}} b = \sum_{(a)(b)}(-1)^{|a_1||a_2|+|a_1||a_3|}(a_1,b_1)(a_3,b_2)'a_2\,.$$

These actions are structurally motivated by duality with the *left adjoint actions* of Hopf algebras A and B on themselves, given by:

$$\begin{aligned}&\triangleright_{\mathrm{Ad}}\colon A\otimes A\to A\,, \qquad a\triangleright_{\mathrm{Ad}} a' = \sum_{(a)}(-1)^{|a_2||a'|}a_1a'S(a_2)\,,\\ &\triangleright_{\mathrm{Ad}}\colon B\otimes B\to B\,, \qquad b\triangleright_{\mathrm{Ad}} b' = \sum_{(b)}(-1)^{|b_2||b'|}b_1b'S(b_2)\,.\end{aligned}$$

Proposition A.14. *Let A and B be Hopf superalgebras. For any $a,a'\in A$ and $b,b'\in B$, the following holds:*

$$(a',b\triangleleft_{\mathrm{Coad}} a) \;=\; (-1)^{|a|(|a'|+|b|)}(a\triangleright_{\mathrm{Ad}} a',b) \qquad\text{and}\qquad (a\triangleleft_{\mathrm{Coad}} b,b') \;=\; (a,b\triangleright_{\mathrm{Ad}} b')\,.$$

Proof. These identities follow from direct computation:

$$\begin{aligned}(-1)^{|a|(|a'|+|b|)}(a\triangleright_{\mathrm{Ad}} a',b) &= \sum_{(a)}(-1)^{|a|(|a'|+|b|)}(-1)^{|a_2||a'|}(a_1a'S(a_2),b)\\ &= \sum_{(a)(b)}(-1)^{|a|(|a'|+|b|)}(-1)^{|a_2||a'|}(a_1\otimes a'\otimes S(a_2),b_1\otimes b_2\otimes b_3)\\ &= \sum_{(a)(b)}(-1)^{|a|(|a'|+|b|)}(-1)^{|a_2||a'|}(-1)^{|b_1||b_2|+|b_1||b_3|+|b_2||b_3|}(a_1,b_1)(a',b_2)(S(a_2),b_3)\\ &= \sum_{(a)(b)}(-1)^{|a|}(-1)^{|b_1||b_2|+|b_1||b_3|}(a_1,b_1)(a',b_2)(S(a_2),b_3)\,,\end{aligned}$$

and

$$
\begin{aligned}
(a, b \triangleright_{\mathrm{Ad}} b') &= \sum_{(b)} (-1)^{|b_2||b'|}(a, b_1 b' S(b_2)) = \sum_{(a)(b)} (-1)^{|b_2||b'|}(a_1 \otimes a_2 \otimes a_3, b_1 \otimes b' \otimes S(b_2)) \\
&= \sum_{(a)(b)} (-1)^{|b_2||b'|}(-1)^{|a_1||a_2|+|a_1||a_3|+|a_2||a_3|}(a_1, b_1)(a_2, b')(a_3, S(b_2)) \\
&= \sum_{(a)(b)} (-1)^{|a_1||a_2|+|a_1||a_3|}(a_1, b_1)(a_2, b')(a_3, S(b_2)) . \qquad \square
\end{aligned}
$$

Remark A.15. The right coadjoint action $\triangleleft_{\mathrm{Coad}} \colon A \otimes B \to A$ of B on A induces a *left coadjoint action* $\triangleright_{\mathrm{Coad}} \colon B^{\mathrm{op}} \otimes A \to A$ of B^{op} on A via:

$$
b \triangleright_{\mathrm{Coad}} a = (-1)^{|a||b|} a \triangleleft_{\mathrm{Coad}} b = \sum_{(a)(b)} (-1)^{|a||b|}(-1)^{|a_1||a_2|+|a_1||a_3|}(a_1, b_1)(a_3, b_2)' a_2 .
$$

Since the supercoalgebra structures of B and B^{op} are the same, we note that $\triangleleft_{\mathrm{Coad}} \colon B \otimes A \to B$ can be thought of as a right coadjoint action of A on B^{op} as well. The following proposition ensures that the coadjoint actions are compatible in the sense of a double cross product.

Proposition A.16. *The morphisms $\triangleleft_{\mathrm{Coad}} \colon B^{\mathrm{op}} \otimes A \to B^{\mathrm{op}}$ and $\triangleright_{\mathrm{Coad}} \colon B^{\mathrm{op}} \otimes A \to A$ satisfy the conditions in Proposition A.3.*

Proof. We first verify that $\triangleright_{\mathrm{Coad}}$ defines a left B^{op}-module structure on A:

$$
\begin{aligned}
& b \triangleright_{\mathrm{Coad}} (b' \triangleright_{\mathrm{Coad}} a) = b \triangleright_{\mathrm{Coad}} \sum_{(a)(b')} (-1)^{|a||b'|}(-1)^{|a_1|(|a_2|+|a_3|)}(a_1, b'_1)(a_3, b'_2)' a_2 \\
&= \sum_{(a)(b)(b')} (-1)^{|a||b'|}(-1)^{|a_1|(|a_2|+|a_3|+|a_4|+|a_5|)}(-1)^{(|a_2|+|a_3|+|a_4|)|b|}(-1)^{|a_2|(|a_3|+|a_4|)} \\
&\qquad \cdot (a_1, b'_1)(a_5, b'_2)'(a_2, b_1)(a_4, b_2)' a_3 \\
&= \sum_{(a)(b)(b')} (-1)^{|a||b'_2|}(-1)^{|b'_1|+|b|}(-1)^{|b_2|(|a_3|+|b_1|)} \cdot (-1)^{|b_1||b'_1|+|b_2||b'_2|}(a_1 \otimes a_2, b'_1 \otimes b_1)(a_4 \otimes a_5, b_2 \otimes b'_2)' a_3 \\
&= \sum_{(a)(b)(b')} (-1)^{|a||b'_2|}(-1)^{|b'_1|+|b|}(-1)^{|b_2|(|a_2|+|b_1|)} \cdot (-1)^{|b_1||b'|}(a_1, (b'b)_1)(a_3, (b'b)_2)' a_2 \\
&= (-1)^{|b||b'|} \cdot \sum_{(a)(b)(b')} (-1)^{|a|(|b|+|b'|)}(-1)^{|a_1|(|a_2|+|a_3|)}(a_1, (b'b)_1)(a_3, (b'b)_2)' a_2 = (-1)^{|b||b'|}(b'b) \triangleright_{\mathrm{Coad}} a ,
\end{aligned}
$$

as well as

$$
1 \triangleright_{\mathrm{Coad}} a = \sum_{(a)} (-1)^{|a_1|(|a_2|+|a_3|)}(a_1, 1)(a_3, 1)' a_2 = \sum_{(a)} \epsilon(a_1)\epsilon(a_3) a_2 = a .
$$

Furthermore, this action is compatible with the supercoalgebra structure of A:

$$
\begin{aligned}
& \sum_{(a)(b)} (-1)^{|b_2||a_1|}(b_1 \triangleright_{\mathrm{Coad}} a_1) \otimes (b_2 \triangleright_{\mathrm{Coad}} a_2) \\
&= \sum_{(a)(b)} (-1)^{|b_2||a_1|}(-1)^{|a_1||b_1|}(-1)^{|a_{1,1}|(|a_{1,2}|+|a_{1,3}|)}(-1)^{|a_2||b_2|}(-1)^{|a_{2,1}|(|a_{2,2}|+|a_{2,3}|)} \\
&\qquad \cdot (a_{1,1}, b_{1,1})(a_{1,3}, b_{1,2})'(a_{2,1}, b_{2,1})(a_{2,3}, b_{2,2})' a_{1,2} \otimes a_{2,2} \\
&= \sum_{(a)(b)} (-1)^{|a||b|}(-1)^{|a_{2,1}||a_{1,3}|}(-1)^{|a_{1,1}|(|a_{1,2}|+|a_{1,3}|+|a_{2,1}|)}(-1)^{(|a_{1,1}|+|a_{1,3}|+|a_{2,1}|)(|a_{2,2}|+|a_{2,3}|)} \\
&\qquad \cdot (a_{1,1}, b_{1,1})(a_{1,3}, b_{1,2})'(a_{2,1}, b_{2,1})(a_{2,3}, b_{2,2})' a_{1,2} \otimes a_{2,2} \\
&\overset{(A.11)}{=} \sum_{(a)(b)} (-1)^{|a||b|}(-1)^{|a_1||a_2|}(-1)^{|a_1|(|a_4|+|a_5|)} \cdot \epsilon(a_3)\epsilon(b_2)(a_1, b_1)(a_5, b_3)' a_2 \otimes a_4 \\
&= \sum_{(a)(b)} (-1)^{|a||b|}(-1)^{|a_1|(|a_2|+|a_3|+|a_4|)} \cdot (a_1, b_1)(a_4, b_2)' a_2 \otimes a_3 = \Delta(b \triangleright_{\mathrm{Coad}} a) ,
\end{aligned}
$$

as well as

$$\epsilon(b \triangleright_{\mathrm{Coad}} a) = \sum_{(a)(b)} (-1)^{|a||b|}(-1)^{|a_1|(|a_2|+|a_3|)}(a_1,b_1)(a_3,b_2)'\epsilon(a_2)$$
$$= \sum_{(a)(b)} (-1)^{|a||b|}(-1)^{|a_1||a_2|}(a_1,b_1)(a_2,b_2)' \overset{(A.11)}{=} \epsilon(a)\epsilon(b)\,.$$

Analogous computations confirm that $\triangleleft_{\mathrm{Coad}}$ defines a right A-module supercoalgebra structure on B^{op}. Since the verification of the remaining compatibility conditions from Proposition A.3 is purely computational, we explicitly exhibit only the relation governing the left action on a product:

$$\sum_{(a)(b)} (-1)^{|a_1||b_2|}(b_1 \triangleright_{\mathrm{Coad}} a_1)\cdot((b_2 \triangleleft_{\mathrm{Coad}} a_2)\triangleright_{\mathrm{Coad}} a')$$
$$= \sum_{(a)(b)} (-1)^{|a_1||b_2|}(-1)^{|a_1||b_1|}(-1)^{|a_{1,1}|(|a_{1,2}|+|a_{1,3}|)}(-1)^{|a_2|}(-1)^{|b_{2,1}|(|b_{2,2}|+|b_{2,3}|)}$$
$$\cdot\,(a_{1,1},b_{1,1})(a_{1,3},b_{1,2})'(a_{2,1},b_{2,1})(a_{2,2},b_{2,3})'a_{1,2}\cdot(b_{2,2}\triangleright_{\mathrm{Coad}} a')$$
$$= \sum_{(a)(a')(b)} (-1)^{(|a|-|a_{2,2}|)|b|}(-1)^{|a_{1,1}|(|a_{1,2}|+|a_{1,3}|+|a_{2,1}|)}(-1)^{|a_{2,2}|}(-1)^{|a_{1,3}||a_{2,1}|}(-1)^{|a'|(|b_{2,2}|+|b_{2,3}|)+|a'_1|(|a'_2|+|a'_3|)}$$
$$\cdot\,(a_{1,1},b_{1,1})(a_{1,3},b_{1,2})'(a_{2,1},b_{2,1})(a_{2,2},b_{2,4})'(a'_1,b_{2,2})(a'_3,b_{2,3})'a_{1,2}a'_2$$
$$\overset{(A.11)}{=} \sum_{(a)(a')(b)} (-1)^{(|a|-|a_4|)|b|}(-1)^{|a_1|(|a_2|+|a_3|)}(-1)^{|a_4|}(-1)^{|a'|(|b_3|+|b_4|)+|a'_1|(|a'_2|+|a'_3|)}$$
$$\cdot\,\epsilon(a_3)\epsilon(b_2)(a_1,b_1)(a_4,b_5)'(a'_1,b_3)(a'_3,b_4)'a_2a'_2$$
$$= \sum_{(a)(a')(b)} (-1)^{(|a|+|a'|)|b|}(-1)^{|a_3||a'_3|}(-1)^{(|a_1|+|a'_1|)(|a_2|+|a'_2|+|a_3|+|a'_3|)}(-1)^{|a_2||a'_1|+|a_3||a'_1|+|a_3||a'_2|}$$
$$\cdot\,(a_1\otimes a'_1, b_1\otimes b_2)(a'_3\otimes a_3, b_3\otimes b_4)'a_2a'_2$$
$$= \sum_{(a)(a')(b)} (-1)^{(|a|+|a'|)|b|}(-1)^{(|a_1|+|a'_1|)(|a_2|+|a'_2|+|a_3|+|a'_3|)}(-1)^{|a_2||a'_1|+|a_3||a'_1|+|a_3||a'_2|}\cdot(a_1a'_1,b_1)(a_3a'_3,b_2)'a_2a'_2$$
$$= b\triangleright_{\mathrm{Coad}}(aa')\,. \qquad \square$$

Combining the above result with Proposition A.3, one obtains the double cross product $A \bowtie B^{\mathrm{op}}$.

Proposition A.17. *The double cross product $A \bowtie B^{\mathrm{op}}$ corresponding to the convolution invertible dual pairing has the cross-multiplication formula:*

$$(1\otimes b)(a\otimes 1) = \sum_{(a)(b)} (-1)^{(|a|-|a_3|)(|b|-|b_1|)}(-1)^{|b_1|}(-1)^{|a_3|}(a_1,b_1)(a_3,b_3)'a_2\otimes b_2\,. \tag{A.18}$$

Proof. Direct computation yields the following:

$$(1\otimes b)(a\otimes 1) = \sum_{(a)(b)} (-1)^{|b_2||a_1|}(b_1\triangleright a_1)\otimes(b_2\triangleleft a_2)$$
$$= \sum_{(a)(b)} (-1)^{|b_2||a_1|}\left\{(-1)^{|a_1||b_1|}(-1)^{|a_{1,1}||a_{1,2}|+|a_{1,1}||a_{1,3}|}(a_{1,1},b_{1,1})(a_{1,3},b_{1,2})'a_{1,2}\right\}$$
$$\otimes\left\{(-1)^{|a_2|}(-1)^{|b_{2,1}||b_{2,2}|+|b_{2,1}||b_{2,3}|}(a_{2,1},b_{2,1})(a_{2,2},b_{2,3})'b_{2,2}\right\}.$$

Expanding and re-indexing the terms, we have:

$$(1\otimes b)(a\otimes 1) = \sum_{(a)(b)} (-1)^{(|b_3|+|b_4|+|b_5|)(|a_1|+|a_2|+|a_3|)}(-1)^{(|a_1|+|a_2|+|a_3|)(|b_1|+|b_2|)}(-1)^{|a_1||a_2|+|a_1||a_3|}$$
$$\cdot\,(-1)^{|a_4|+|a_5|}(-1)^{|b_3||b_4|+|b_3||b_5|}(a_1,b_1)(a_3,b_2)'(a_4,b_3)(a_5,b_5)'a_2\otimes b_4$$
$$\overset{(\sharp)}{=} \sum_{(a)(b)} (-1)^{(|a|-|a_5|)(|b|-|b_1|)}(-1)^{|b_1|}(-1)^{|a_5|}(-1)^{|b_2||a_4|}(a_1,b_1)(a_3,b_2)'(a_4,b_3)(a_5,b_5)'a_2\otimes b_4$$
$$\overset{(A.11)}{=} \sum_{(a)(b)} (-1)^{(|a|-|a_4|)(|b|-|b_1|)}(-1)^{|b_1|}(-1)^{|a_4|}(a_1,b_1)\epsilon(a_3)\epsilon(b_2)(a_4,b_4)'a_2\otimes b_3$$
$$= \sum_{(a)(b)} (-1)^{(|a|-|a_3|)(|b|-|b_1|)}(-1)^{|b_1|}(-1)^{|a_3|}(a_1,b_1)(a_3,b_3)'a_2\otimes b_2\,,$$

where the sign identity (♯) follows from:

$$
\begin{aligned}
&(-1)^{(|a_1|+|a_2|+|a_3|)(|b_1|+|b_2|+|b_3|+|b_4|+|b_5|)}(-1)^{|a_1|(|a_2|+|a_3|)}(-1)^{|a_4|+|a_5|}(-1)^{|b_3|(|b_4|+|b_5|)}(-1)^{|b_2||a_4|}\\
&= (-1)^{(|a_1|+|a_2|+|a_3|)|b|}(-1)^{|a_1|(|a_2|+|a_3|)}(-1)^{|a_4|+|a_5|}(-1)^{|b_3|(|b_4|+|b_5|)}(-1)^{|b_2||a_4|}\\
&= (-1)^{(|a_1|+|a_2|+|a_3|)(|b|-|b_1|)}(-1)^{|a_1||b_1|}(-1)^{|a_4|+|a_5|}(-1)^{|b_3|(|b_4|+|b_5|)}(-1)^{|b_2||a_4|}\\
&= (-1)^{(|a_1|+|a_2|+|a_3|)(|b|-|b_1|)}(-1)^{|b_1|}(-1)^{|a_5|}(-1)^{|b_3|(|b|-|b_1|)} = (-1)^{(|a|-|a_5|)(|b|-|b_1|)}(-1)^{|b_1|}(-1)^{|a_5|}\,. \qquad \square
\end{aligned}
$$

The next result provides an equivalent definition of $A \bowtie B^{\mathrm{op}}$ without using the convolution inverse.

Proposition A.19. *The cross-multiplication admits the alternative form:*

$$
\sum_{(a)(b)} (-1)^{|b_2||a|}(1\otimes b_1)(a_1\otimes 1)(a_2,b_2) = \sum_{(a)(b)} (-1)^{|b_2||a|}(-1)^{|a_1||b_1|}(a_1,b_1)a_2\otimes b_2\,. \tag{A.20}
$$

Proof. Substituting (A.18) into the left-hand side, we obtain:

$$
\begin{aligned}
&\sum_{(a)(b)} (-1)^{|b_2||a|}(1\otimes b_1)(a_1\otimes 1)(a_2,b_2)\\
&= \sum_{(a)(b)} (-1)^{|b_4||a|}(-1)^{(|b_2|+|b_3|)(|a_1|+|a_2|)}(-1)^{|b_1|+|a_3|}(a_1,b_1)(a_3,b_3)'a_2\otimes b_2(a_4,b_4)\\
&= \sum_{(a)(b)} (-1)^{(|a_3|+|a_4|)|a|}(-1)^{|b_2|(|a_1|+|a_2|)}(-1)^{|b_1|}(-1)^{|a_3||a_4|}(a_1,b_1)(a_3,b_3)'(a_4,b_4)a_2\otimes b_2\\
&= \sum_{(a)(b)} (-1)^{|a_3||a|}(-1)^{|b_2|(|a_1|+|a_2|)}(-1)^{|b_1|}(a_1,b_1)\epsilon(a_3)\epsilon(b_3)a_2\otimes b_2\\
&= \sum_{(a)(b)} (-1)^{|b_2||a|}(-1)^{|b_1|}(a_1,b_1)a_2\otimes b_2 = \sum_{(a)(b)} (-1)^{|b_2||a|}(-1)^{|a_1||b_1|}(a_1,b_1)a_2\otimes b_2\,.
\end{aligned}
$$

Conversely, (A.18) is recovered from (A.20) as follows:

$$
\begin{aligned}
&\sum_{(a)(b)} (-1)^{(|a|-|a_3|)(|b|-|b_1|)}(-1)^{|b_1|}(-1)^{|a_3|}(a_1,b_1)(a_3,b_3)'a_2\otimes b_2\\
&= \sum_{(a)(b)} (-1)^{(|a_1|+|a_2|)|b_2|}(-1)^{|a_1||b_1|}(-1)^{|a||b_3|}(a_1,b_1)(a_3,b_3)'a_2\otimes b_2\\
&\overset{(A.20)}{=} \sum_{(a)(b)} (-1)^{(|a_1|+|a_2|)|b_2|}(-1)^{|a||b_3|}(a_2,b_2)(a_3,b_3)'(1\otimes b_1)(a_1\otimes 1)\\
&= \sum_{(a)(b)} (-1)^{|a|(|b_2|+|b_3|)}(-1)^{|b_2||a_3|}(a_2,b_2)(a_3,b_3)'(1\otimes b_1)(a_1\otimes 1)\\
&= \sum_{(a)(b)} (-1)^{|a||b_2|}\epsilon(a_2)\epsilon(b_2)(1\otimes b_1)(a_1\otimes 1) = (1\otimes b)(a\otimes 1)\,. \qquad \square
\end{aligned}
$$

A.4. **Generalized doubles.**

In contrast to the dual pairing used in the previous subsection, we now consider a *skew-pairing* between A and B, defined as a bilinear map $(\cdot,\cdot)\colon A\otimes B\to\mathbb{K}$ satisfying:

$$
\begin{aligned}
(1,b) &= \epsilon(b)\,, & (a,1) &= \epsilon(a)\,,\\
(aa',b) &= (a\otimes a',\Delta(b))\,, & (a,bb') &= (\Delta^{\mathrm{op}}(a),b\otimes b')\,,
\end{aligned} \tag{A.21}
$$

subject to the standard super-sign convention $(a\otimes a', b\otimes b') := (-1)^{|a'||b|}(a,b)(a',b')$.

Remark A.22. The skew-pairing property implies the following:

$$
(a,bb') = (\Delta^{\mathrm{op}}(a),b\otimes b') = \sum_{(a)}(-1)^{|a_1||a_2|}(a_2\otimes a_1,b\otimes b') = \sum_{(a)}(a_2,b)(a_1,b') = (-1)^{|b'||b|}(\Delta(a),b'\otimes b)\,,
$$

which yields $(a, b' *_{\mathrm{op}} b) = (\Delta(a), b'\otimes b)$, where $*_{\mathrm{op}}$ denotes the opposite multiplication of B.

The remark above shows that a skew-pairing is equivalent to a dual pairing between A and B^{op}. Accordingly, we call a skew-pairing convolution invertible if the corresponding dual pairing $A\otimes B^{\mathrm{op}}\to\mathbb{K}$ is convolution invertible. Consequently, for any convolution invertible skew-pairing, the construction in Subsection A.3 yields a double cross product $A\bowtie B$, which we denote by $\mathcal{D}(A,B)$ and call the **generalized double** of A and B. Moreover, if the skew-pairing is non-degenerate, $\mathcal{D}(A,B)$ is referred to as the **Drinfeld**

double. Since the underlying coalgebra structures of B and B^{op} are identical, the cross-multiplication formulas (A.18) and (A.20) remain unchanged.

Remark A.23. Since a skew-pairing between A and B and the corresponding dual pairing between A and B^{op} coincide as morphisms of the underlying superspaces, we use the exact same symbol, e.g. $(\cdot,\cdot)$ or $\sigma(\cdot,\cdot)$, to denote both a skew-pairing and its corresponding dual pairing.

The following is clear from the construction:

Proposition A.24. *The generalized double $\mathcal{D}(A,B)$ is isomorphic to the quotient of the free product $A*B$ of the algebras A and B by the relations (A.18) (or equivalently (A.20)). Specifically, for each pair of homogeneous elements $a\in A$ and $b\in B$, define*

$$u_{a,b} \coloneqq \sum_{(a)(b)} (-1)^{|b_2||a|}\cdot\Big\{(-1)^{|a_1||b_1|}(a_1,b_1)\cdot a_2b_2-(a_2,b_2)\cdot b_1a_1\Big\}$$

and let $\mathcal{I}$ be the two-sided ideal of $A*B$ generated by these elements. Then $\mathcal{D}(A,B)\simeq(A*B)/\mathcal{I}$.

The multiplicative properties of these elements are described in the next result.

Proposition A.25. *For any $a,a'\in A$ and $b,b'\in B$, the following equalities hold:*

$$u_{aa',b}=\sum_{(a)(b)}(-1)^{|b_2||a|}(-1)^{|a_1|}(a_1,b_1)\cdot a_2u_{a',b_2}+\sum_{(a')(b)}(-1)^{|b_2|(|a|+|a'|)}(a'_2,b_2)\cdot u_{a,b_1}a'_1\,,$$
$$u_{a,bb'}=\sum_{(a)(b')}(-1)^{|b'_2||a|}(-1)^{|b'_1|}(a_1,b'_1)\cdot u_{a_2,b}b'_2+\sum_{(a)(b)}(-1)^{|b_2|(|a|+|b'|)}(a_2,b_2)\cdot b_1u_{a_1,b'}\,.$$

Proof. For any $a,a'\in A$ and $b\in B$, direct expansion of the first term of $u_{aa',b}$ yields:

$$\begin{aligned}
&\sum_{(a)(a')(b)}(-1)^{|b_2|(|a|+|a'|)}(-1)^{(|a_1|+|a'_1|)|b_1|}(-1)^{|a_2||a'_1|}(a_1a'_1,b_1)\cdot a_2a'_2b_2\\
&=\sum_{(a)(a')(b)}(-1)^{|b_3|(|a|+|a'|)}(-1)^{|a_1|+|a'_1|}(-1)^{|a||a'_1|}(a_1,b_1)(a'_1,b_2)\cdot a_2a'_2b_3\\
&=\sum_{(a)(a')(b)}(-1)^{(|b_2|+|b_3|)|a|}(-1)^{|a_1|}(a_1,b_1)a_2\cdot\Big\{(-1)^{|b_3||a'|}(-1)^{|a'_1||b_2|}(a'_1,b_2)a'_2b_3\Big\}\\
&=\sum_{(a)(b)}(-1)^{|b_2||a|}(-1)^{|a_1|}(a_1,b_1)\cdot a_2u_{a',b_2}\\
&\qquad+\sum_{(a)(a')(b)}(-1)^{(|b_2|+|b_3|)|a|}(-1)^{|a_1|}(a_1,b_1)a_2\cdot\Big\{(-1)^{|b_3||a'|}(a'_2,b_3)b_2a'_1\Big\}\,.
\end{aligned}$$

Similarly, for the second term, we have:

$$\begin{aligned}
&\sum_{(a)(a')(b)}(-1)^{|b_2|(|a|+|a'|)}(-1)^{|a_2||a'_1|}(a_2a'_2,b_2)\cdot b_1a_1a'_1\\
&=\sum_{(a)(a')(b)}(-1)^{(|b_2|+|b_3|)(|a|+|a'|)}(-1)^{|a_2||a'|}(a_2,b_2)(a'_2,b_3)\cdot b_1a_1a'_1\\
&=\sum_{(a)(a')(b)}(-1)^{|b_3|(|a|+|a'|)}\Big\{(-1)^{|b_2||a|}(a_2,b_2)b_1a_1\Big\}\cdot(a'_2,b_3)a'_1\\
&=-\sum_{(a')(b)}(-1)^{|b_2|(|a|+|a'|)}(a'_2,b_2)\cdot u_{a,b_1}a'_1\\
&\qquad+\sum_{(a)(a')(b)}(-1)^{|b_3|(|a|+|a'|)}\Big\{(-1)^{|b_2||a|}(-1)^{|a_1||b_1|}(a_1,b_1)a_2b_2\Big\}\cdot(a'_2,b_3)a'_1\,.
\end{aligned}$$

Subtracting these expressions recovers the identity for $u_{aa',b}$. The other identity follows from an analogous computation:

$$\begin{aligned}
&\sum_{(a)(b)(b')} (-1)^{(|b_2|+|b_2'|)|a|}(-1)^{|a_1|(|b_1|+|b_1'|)}(-1)^{|b_2||b_1'|}(a_1, b_1 b_1')\cdot a_2 b_2 b_2' \\
&= \sum_{(a)(b)(b')} (-1)^{(|b_2|+|b_2'|)|a|}(-1)^{|b_1|+|b_1'|}(-1)^{|b_2||b_1'|}(a_1, b_1')(a_2, b_1)\cdot a_3 b_2 b_2' \\
&= \sum_{(a)(b)(b')} (-1)^{|b_2'||a|}(-1)^{|b_1'|}\left\{(-1)^{|b_2|(|a_2|+|a_3|)}(-1)^{|a_2||b_1|}(a_2, b_1)a_3 b_2\right\}\cdot(a_1, b_1')b_2' \\
&= \sum_{(a)(b')} (-1)^{|b_2'||a|}(-1)^{|b_1'|}(a_1, b_1')\cdot u_{a_2,b} b_2' \\
&\qquad + \sum_{(a)(b)(b')} (-1)^{|b_2'||a|}(-1)^{|b_1'|}\left\{(-1)^{|b_2|(|a_2|+|a_3|)}(a_3, b_2)b_1 a_2\right\}\cdot(a_1, b_1')b_2',
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{(a)(b)(b')} (-1)^{(|b_2|+|b_2'|)|a|}(-1)^{|b_2||b_1'|}(a_2, b_2 b_2')\cdot b_1 b_1' a_1 \\
&= \sum_{(a)(b)(b')} (-1)^{(|b_2|+|b_2'|)|a|}(-1)^{|b_2||b_1'|}(a_2, b_2')(a_3, b_2)\cdot b_1 b_1' a_1 \\
&= \sum_{(a)(b)(b')} (-1)^{|b_2|(|a|+|b'|)}(a_3, b_2)b_1\cdot\left\{(-1)^{|b_2'|(|a_1|+|a_2|)}(a_2, b_2')b_1' a_1\right\} \\
&= -\sum_{(a)(b)} (-1)^{|b_2|(|a|+|b'|)}(a_2, b_2)\cdot b_1 u_{a_1,b'} \\
&\qquad + \sum_{(a)(b)(b')} (-1)^{|b_2|(|a|+|b'|)}(a_3, b_2)b_1\cdot\left\{(-1)^{|b_2'|(|a_1|+|a_2|)}(-1)^{|a_1||b_1'|}(a_1, b_1')a_2 b_2'\right\}. \qquad \square
\end{aligned}$$

Remark A.26. The result above implies that if A and B are generated by subsets X and Y having well-behaved comultiplications, then the ideal $\mathcal{I}$ is generated by $u_{x,y}$ restricted to the generators $x \in X$ and $y \in Y$, rather than all $u_{a,b}$. In all cases of interest to us, specifically $U_q(\mathfrak{g})$ and $U(R)$, this reduction holds, thus allowing us to define their doubles by imposing the cross-multiplication (A.18) solely on the algebra generators.

A.5. **The opposite double construction.**

Consider the generalized double $\mathcal{D}(A,B)$ of superbialgebras A and B associated with a convolution invertible skew-pairing. It is often useful to formulate the double in the opposite order, namely as a generalized double $\mathcal{D}(B,A)$ with respect to an appropriately induced skew-pairing. We shall now detail this canonical construction.

Let $\sigma\colon A\otimes B\to\mathbb{K}$ be a convolution invertible skew-pairing, viewed as a dual pairing between A and B^{op}. Lemma A.12 implies that its convolution inverse σ' is a dual pairing between A^{op} and B, hence, the composition

$$\widetilde{\sigma} \coloneqq \sigma'\circ\tau\colon B\otimes A^{\mathrm{op}}\to\mathbb{K} \tag{A.27}$$

defines a dual pairing between B and A^{op} and thus a skew-pairing between B and A. We shall refer to $\widetilde{\sigma}$ as the *transposed inverse* of σ. This induced pairing allows us to construct the double $\mathcal{D}(B,A)$. We also note that $\widetilde{\sigma}' = \sigma\circ\tau$ by Lemma A.5(b). To prevent any ambiguity, we shall explicitly specify the chosen pairing in our notation for the double, denoting the resulting algebras by $\mathcal{D}(A,B,\sigma)$ and $\mathcal{D}(B,A,\widetilde{\sigma})$.

Proposition A.28. *There exists an isomorphism of superbialgebras $\mathcal{D}(B,A,\widetilde{\sigma})\xrightarrow{\sim}\mathcal{D}(A,B,\sigma)$ which is identity on both subbialgebras A and B.*

Proof. Since both doubles are quotients of the isomorphic free products $A*B\simeq B*A$, it suffices to show that their respective cross-multiplication relations (A.18) generate the exact same two-sided ideal. For simplicity, we identify the superbialgebras A and B with their respective images inside the doubles, thereby omitting the tensor product symbols.

For any $a, b \in \mathcal{D}(A, B, \sigma)$, a direct computation yields (cf. Remark A.9):

$$\sum_{(a)(b)} (-1)^{(|b|-|b_3|)(|a|-|a_1|)}(-1)^{|a_1|}(-1)^{|b_3|} \cdot \widetilde{\sigma}(b_1, a_1)\widetilde{\sigma}'(b_3, a_3)b_2a_2$$

$$= \sum_{(a)(b)} (-1)^{(|b|-|b_5|)(|a|-|a_1|)}(-1)^{|a_1|}(-1)^{|b_5|}(-1)^{(|a_2|+|a_3|)(|b_3|+|b_4|)}(-1)^{|b_2|}(-1)^{|a_4|}$$
$$\cdot \widetilde{\sigma}(b_1, a_1)\widetilde{\sigma}'(b_5, a_5)\sigma(a_2, b_2)\sigma'(a_4, b_4)a_3b_3$$

$$= \sum_{(a)(b)} (-1)^{(|b|-|b_5|)(|a|-|a_1|)}(-1)^{(|a_2|+|a_3|)(|b_3|+|b_4|)}(-1)^{|b_2|+|a_4|} \cdot \sigma'(a_1, b_1)\sigma(a_5, b_5)\sigma(a_2, b_2)\sigma'(a_4, b_4)a_3b_3$$

$$\overset{(\dagger)}{=} \sum_{(a)(b)} (-1)^{(|b|-|b_4|-|b_5|)(|a|-|a_1|-|a_2|)}(-1)^{|a_2||b_1|+|a_3||b_3|+|b_4||a_5|} \cdot \sigma'(a_1, b_1)\sigma(a_2, b_2)\sigma'(a_4, b_4)\sigma(a_5, b_5)a_3b_3$$

$$= \sum_{(a)(b)} (-1)^{(|b|-|b_3|)(|a|-|a_1|)}(-1)^{|a_2||b_2|} \cdot \epsilon(a_1)\epsilon(b_1)\epsilon(a_3)\epsilon(b_3)a_2b_2 = ab\,,$$

where the sign identity (†) follows from Remark A.9:

$$(-1)^{(|b|-|b_5|)(|a|-|a_1|)}(-1)^{(|a_2|+|a_3|)(|b_3|+|b_4|)}(-1)^{|b_2|}(-1)^{|a_4|} \cdot \sigma(a_2, b_2)\sigma'(a_4, b_4)$$
$$= (-1)^{(|b|-|b_5|)(|a|-|a_1|-|a_2|)}(-1)^{|a_2||b_1|}(-1)^{|a_3|(|b_3|+|b_4|)}(-1)^{|a_4|} \cdot \sigma(a_2, b_2)\sigma'(a_4, b_4)$$
$$= (-1)^{(|b|-|b_4|-|b_5|)(|a|-|a_1|-|a_2|)}(-1)^{|a_2||b_1|}(-1)^{|a_3||b_3|}(-1)^{|b_4||a_5|} \cdot \sigma(a_2, b_2)\sigma'(a_4, b_4)\,.$$

This computation demonstrates that the cross-multiplication relation defining $\mathcal{D}(B, A, \widetilde{\sigma})$ is satisfied within $\mathcal{D}(A, B, \sigma)$. By a completely analogous calculation, the cross-multiplication relation defining $\mathcal{D}(A, B, \sigma)$ is satisfied within $\mathcal{D}(B, A, \widetilde{\sigma})$, establishing the algebra isomorphism. Finally, because the coproducts on both doubles are determined by the exact same formulas on the elements of A and B, their supercoalgebra structures automatically coincide. □

A.6. **Canonical element.**

Suppose the skew-pairing $(\cdot, \cdot)\colon A \otimes B \to \mathbb{K}$ is non-degenerate and that both A and B are finite-dimensional. The non-degeneracy implies that A and B must have the same dimension, and we may choose bases $\{a_1, \ldots, a_N\}$ and $\{b_1, \ldots, b_N\}$ of A and B, respectively, dual to each other with respect to the skew-pairing: $(a_i, b_j) = \delta_{ij}$.

Definition A.29. The *canonical element* of $\mathcal{D}(A, B)$ is defined as

$$\mathfrak{R} = \sum_{1\leq i\leq N} b_i \otimes a_i \in \mathcal{D}(A, B) \otimes \mathcal{D}(A, B)\,. \tag{A.30}$$

This construction is independent of the choice of dual bases and clearly satisfies $|\mathfrak{R}| = \bar{\mathbf{0}}$.

The basic properties of this $\mathfrak{R}$ are summarized in the following result.

Proposition A.31. *The canonical element $\mathfrak{R}$ satisfies the following properties:*

$$(\Delta \otimes \mathrm{Id})\mathfrak{R} = \mathfrak{R}_{(13)}\mathfrak{R}_{(23)}\,, \qquad (\mathrm{Id} \otimes \Delta)\mathfrak{R} = \mathfrak{R}_{(13)}\mathfrak{R}_{(12)}\,,$$
$$(\epsilon \otimes \mathrm{Id})\mathfrak{R} = 1 = (\mathrm{Id} \otimes \epsilon)\mathfrak{R}\,, \qquad \mathfrak{R}_{(12)}\mathfrak{R}_{(13)}\mathfrak{R}_{(23)} = \mathfrak{R}_{(23)}\mathfrak{R}_{(13)}\mathfrak{R}_{(12)}\,,$$
$$\Delta^{\mathrm{op}}(u)\mathfrak{R} = \mathfrak{R}\Delta(u) \quad \textit{for all} \quad u \in \mathcal{D}(A, B)\,,$$

where the notation of Subsection 3.2 is used.

Proof. We first note that for any $a \in A$ and $b \in B$, the following identities hold:

$$a = \sum_{1\leq i\leq N} (a, b_i)a_i\,, \qquad b = \sum_{1\leq i\leq N} (a_i, b)b_i\,. \tag{A.32}$$

We begin by proving $\Delta_{(2)}\mathfrak{R} = (\mathrm{Id} \otimes \Delta)\mathfrak{R} = \mathfrak{R}_{(13)}\mathfrak{R}_{(12)}$. Since the skew-pairing is non-degenerate, it suffices to show that the equality holds after applying $(a_k, -) \otimes \mathrm{Id} \otimes \mathrm{Id}$ to both sides for all $1 \leq k \leq N$:

$$\sum_{1\leq i\leq N} (a_k, b_i) \cdot \Delta(a_i) = \sum_{1\leq i,j\leq N} (a_k, b_ib_j) \cdot a_j \otimes a_i\,. \tag{A.33}$$

The left-hand side of (A.33) is simply $\Delta(a_k)$, while the right-hand side becomes:

$$\sum_{1\leq i,j\leq N} (a_k, b_ib_j) \cdot a_j \otimes a_i = \sum_{1\leq i,j\leq N} \sum_{(a_k)} \big((a_k)_1, b_j\big)\big((a_k)_2, b_i\big) \cdot a_j \otimes a_i \overset{(A.32)}{=} \sum_{(a_k)} (a_k)_1 \otimes (a_k)_2 = \Delta(a_k)\,,$$

thus establishing (A.33). Analogous computations prove $\Delta_{(1)}\mathfrak{R} = (\Delta \otimes \mathrm{Id})\mathfrak{R} = \mathfrak{R}_{(13)}\mathfrak{R}_{(23)}$.

The identity $(\epsilon \otimes \mathrm{Id})\mathfrak{R} = 1$ can be verified by the following direct computation:

$$(\epsilon \otimes \mathrm{Id})\mathfrak{R} = \sum_{1\leq i\leq N} \epsilon(b_i)a_i = \sum_{1\leq i\leq N} (1, b_i)a_i \overset{(A.32)}{=} 1\,,$$

and analogously for $(\mathrm{Id} \otimes \epsilon)\mathfrak{R} = 1$.

We now prove $\Delta^{\mathrm{op}}(u)\mathfrak{R} = \mathfrak{R}\Delta(u)$ for all $u \in \mathcal{D}(A, B)$. As $\{a_i, b_i\}_{i=1}^N$ generate $\mathcal{D}(A, B)$, it is sufficient to check the identity for $u = a_k$ and $u = b_k$. Setting $u = b_k$ and applying $(a_l, -) \otimes \mathrm{Id}$ to both sides yields:

$$\begin{aligned}\big((a_l, -) \otimes \mathrm{Id}\big)(\Delta^{\mathrm{op}}(b_k)\mathfrak{R}) &= \sum_{1\leq i\leq N}\sum_{(b_k)} (-1)^{|(b_k)_1||(b_k)_2|}(-1)^{|(b_k)_1||b_i|}\big(a_l, (b_k)_2 b_i\big) \cdot (b_k)_1 a_i \\ &= \sum_{1\leq i\leq N}\sum_{(a_l)(b_k)} (-1)^{|(b_k)_1||a_l|}\big((a_l)_1, b_i\big)\big((a_l)_2, (b_k)_2\big) \cdot (b_k)_1 a_i \\ &\overset{(A.32)}{=} (-1)^{|b_k||a_l|} \sum_{(a_l)(b_k)} (-1)^{|(b_k)_2||a_l|}\big((a_l)_2, (b_k)_2\big) \cdot (b_k)_1 (a_l)_1\end{aligned}$$

and

$$\begin{aligned}\big((a_l, -) \otimes \mathrm{Id}\big)(\mathfrak{R}\Delta(b_k)) &= \sum_{1\leq i\leq N}\sum_{(b_k)} (-1)^{|(b_k)_1||b_i|}\big(a_l, b_i(b_k)_1\big) \cdot a_i(b_k)_2 \\ &= \sum_{1\leq i\leq N}\sum_{(a_l)(b_k)} (-1)^{|(b_k)_1||(a_l)_2|}\big((a_l)_1, (b_k)_1\big)\big((a_l)_2, b_i\big) \cdot a_i(b_k)_2 \\ &\overset{(A.32)}{=} (-1)^{|b_k||a_l|} \sum_{(a_l)(b_k)} (-1)^{|(b_k)_2||a_l|}(-1)^{|(b_k)_1||(a_l)_1|}\big((a_l)_1, (b_k)_1\big) \cdot (a_l)_2(b_k)_2\,,\end{aligned}$$

which coincide by (A.20). The verification for $u = a_k$ is analogous, concluding the proof of $\Delta^{\mathrm{op}}(u)\mathfrak{R} = \mathfrak{R}\Delta(u)$. Lastly, the Yang–Baxter equation $\mathfrak{R}_{(12)}\mathfrak{R}_{(13)}\mathfrak{R}_{(23)} = \mathfrak{R}_{(23)}\mathfrak{R}_{(13)}\mathfrak{R}_{(12)}$ follows directly from above identities:

$$\begin{aligned}\mathfrak{R}_{(12)}\mathfrak{R}_{(13)}\mathfrak{R}_{(23)} &= \mathfrak{R}_{(12)}\Delta_{(1)}(\mathfrak{R}) = \Delta^{\mathrm{op}}_{(1)}(\mathfrak{R})\mathfrak{R}_{(12)} \\ &= (\tau_{(12)}\Delta_{(1)})(\mathfrak{R})\mathfrak{R}_{(12)} = \tau_{(12)}(\mathfrak{R}_{(13)}\mathfrak{R}_{(23)})\mathfrak{R}_{(12)} = \mathfrak{R}_{(23)}\mathfrak{R}_{(13)}\mathfrak{R}_{(12)}\,.\end{aligned}$$ □

Furthermore, the existence of antipodes yields explicit formulas for the inverse of the canonical element:

Proposition A.34. *If A is a Hopf superalgebra with an invertible antipode S (thus A^{op} and A^{cop} are also Hopf superalgebras with antipode S^{-1}), then the canonical element $\mathfrak{R}$ is invertible, with its inverse given by*

$$\mathfrak{R}^{-1} = \sum_{1\leq i\leq N} b_i \otimes S^{-1}(a_i)\,.$$

Likewise, if B is a Hopf superalgebra with antipode S, then $\mathfrak{R}$ is invertible, with its inverse given by:

$$\mathfrak{R}^{-1} = \sum_{1\leq i\leq N} S(b_i) \otimes a_i\,.$$

Proof. Assume that A is a Hopf superalgebra with an invertible antipode. For any $1 \leq k \leq N$, we have

$$\begin{aligned}\Big((a_k, -) \otimes \mathrm{Id}\Big)\Big(\mathfrak{R}\sum_{1\leq j\leq N} b_j \otimes S^{-1}(a_j)\Big) &= \sum_{1\leq i,j\leq N} (-1)^{|a_i||a_j|}(a_k, b_i b_j) \cdot a_i S^{-1}(a_j) \\ &= \sum_{1\leq i,j\leq N}\sum_{(a_k)} (-1)^{|(a_k)_1||(a_k)_2|}\big((a_k)_1, b_j\big)\big((a_k)_2, b_i\big) \cdot a_i S^{-1}(a_j) \\ &\overset{(A.32)}{=} \sum_{(a_k)} (-1)^{|(a_k)_1||(a_k)_2|} \cdot (a_k)_2 S^{-1}((a_k)_1) = \epsilon(a_k)\,,\end{aligned}$$

which implies that $\mathfrak{R}\big(\sum_{j=1}^N b_j \otimes S^{-1}(a_j)\big) = 1$. The verification of $\big(\sum_{j=1}^N b_j \otimes S^{-1}(a_j)\big)\mathfrak{R} = 1$ and the analogous statements for B proceed similarly, completing the proof. □

Assume the hypotheses of Proposition A.34 hold. For notational convenience, let us denote the skew-pairing by $\sigma\colon A \otimes B \to \mathbb{K}$ while using $\widetilde{\sigma}\colon B \otimes A \to \mathbb{K}$ for its transposed inverse. Evoking $\mathcal{D}(B, A, \widetilde{\sigma}) \simeq \mathcal{D}(A, B, \sigma)$ from Proposition A.28, let us now relate their canonical elements.

Corollary A.35. *Under the assumptions of Proposition A.34, the canonical element of $\mathcal{D}(B, A, \widetilde{\sigma})$ with respect to $\widetilde{\sigma}$ is $\mathfrak{R}^{-1}_{(21)}$.*

Proof. First, consider the case when A is a Hopf superalgebra with an invertible antipode S. The evaluation

$$\widetilde{\sigma}(b_j, S^{-1}(a_i)) \overset{(A.10),(A.27)}{=} (-1)^{|a_i||b_j|}\sigma(a_i, b_j) = (-1)^{|a_i||b_j|}\delta_{ij}$$

implies that $\{b_j\}_{j=1}^N$ and $\{(-1)^{|a_i||b_i|}S^{-1}(a_i)\}_{i=1}^N$ form dual bases with respect to $\widetilde{\sigma}$. The result then follows from Proposition A.34.

The case where B is a Hopf superalgebra is analogous. $\square$

Remark A.36. The results of this subsection naturally generalize to any infinite-dimensional setting where the summation (A.30) is well-defined, such as the $\hbar$-adic quantum supergroups (cf. Remark 3.3).

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