Talk at Temple University (Philadelphia)

Relation between quantum toroidal algebras of $\mathfrak{sl}_n$ and affine Yangians of $\mathfrak{sl}_n$ for different n.

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§3 Quantum toroidal and affine Yangian alg-s

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1.1 Quantum loop algebras and Yangians.

$\mathfrak{g}$ - simple Lie algebra $\longleftrightarrow$ Cartan matrix A.

Given $\mathfrak{g} \rightsquigarrow$ one can consider two interesting Hopf algebras:

* $U_q(L\mathfrak{g})$ - quantum loop algebra
* $Y_h(\mathfrak{g})$ — Yangian

Here we can either treat q, h as complex numbers ($q \in \mathbb{C}^*$, $h \in \mathbb{C}$) or formal variables.
These algebras are deformations of $U(\mathfrak{g}[z,z^{-1}])$ and $U(\mathfrak{g}[w])$, respectively, as $q \longrightarrow 1$ or $h \longrightarrow 0$.

Generators

- The algebra $U_q(L\mathfrak{g})$ is generated by $\{e_{i,k}, f_{i,k}, h_{i,k}\}_{i \in I}^{k \in \mathbb{Z}}$
- The algebra $Y_h(\mathfrak{g})$ is generated by $\{x^{\pm}_{i,r}, \xi_{i,r}\}_{i \in I}^{r \in \mathbb{Z}_+}$

where I - the set of vertices of the Dynkin diagram associated with $\mathfrak{g}$.

Relations

We will write down relations on the next page, but what's important is that

- As $q \longrightarrow 1$, the aforementioned identification "$\lim\limits_{q \to 1} U_q(L\mathfrak{g})$" $\simeq U(\mathfrak{g}[z,z^{-1}])$ sends

$$e_{i,k} \longmapsto \tilde{e}_i \otimes z^k, \quad f_{i,k} \longmapsto \tilde{f}_i \otimes z^k, \quad h_{i,k} \longmapsto \tilde{h}_i \otimes z^k.$$ (Here $\hat{e}_i, \hat{f}_i, \hat{h}_i$ - rescaled usual generators, see below)

- As $h \longrightarrow 0$, the aforementioned identification "$\lim\limits_{h \to 0} Y_h(\mathfrak{g})$" $\simeq U(\mathfrak{g}[w])$ sends

$$x^+_{i,r} \longmapsto \tilde{e}_i \otimes w^r, \quad x^-_{i,r} \longmapsto \tilde{f}_i \otimes w^r, \quad \xi_{i,r} \longmapsto \tilde{h}_i \otimes w^r.$$ (—//—)

In other words, all the defining relations are appropriate deformations of the corresponding relations between the generators $\{e_i \otimes z^?, f_i \otimes z^?, h_i \otimes z^?\}$.

Notation: Set $d_i := \frac{(\alpha_i, \alpha_i)}{2}$, while we have $a_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \Rightarrow d_i a_{ij} = d_j a_{ji}$

Also set $q_i := q^{d_i}$, $[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}$, $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q \cdot [n-1]_q \cdot \ldots \cdot [1]_q}{[k]_q \cdot \ldots \cdot [1]_q \cdot [n-k]_q \cdot \ldots \cdot [1]_q}$

Recall that $\mathfrak{g}$ is generated by $\{e_i, f_i, h_i\}_{i \in I}$ with the defining rel-s:

$$\boxed{\begin{array}{l} [h_i, h_j] = 0, \ [h_i, e_j] = a_{ij} e_j, \ [h_i, f_j] = -a_{ij} f_j, \ [e_i, f_j] = \delta_{ij} h_i \\ \text{(Serre relation) For any } i \neq j \in I: \ (\mathrm{ad}(e_i))^{1-a_{ij}} e_j = 0 = (\mathrm{ad}(f_i))^{1-a_{ij}} f_j \end{array}}$$

Set $\hat{e}_i = e_i$, $\hat{f}_i = d_i f_i$, $\hat{h}_i = d_i h_i$ for example

[1.1] Quantum loop algebras and Yangians: Relations

Yangian Case

(Y1) $[\xi_{i,r}, \xi_{j,s}]=0$

(Y2.1) $[\xi_{i,0}, x_{j,s}^{\pm}] = \pm d_i a_{ij} x_{j,s}^{\pm}$

(Y2.2) $[\xi_{i,r+1}, x_{j,s}^{\pm}]-[\xi_{i,r}, x_{j,s+1}^{\pm}] = \pm \frac{d_i a_{ij} h}{2} (\xi_{i,r} x_{j,s}^{\pm} + x_{j,s}^{\pm} \xi_{i,r})$

(Y3) $[x_{i,r+1}^{\pm}, x_{j,s}^{\pm}]-[x_{i,r}^{\pm}, x_{j,s+1}^{\pm}] = \pm \frac{d_i a_{ij} h}{2} (x_{i,r}^{\pm} x_{j,s}^{\pm} + x_{j,s}^{\pm} x_{i,r}^{\pm})$

(Y4) $[x_{i,r}^{+}, x_{j,s}^{-}] = \delta_{ij} \cdot \xi_{i,r+s}$

(Y5) (Serre relation) For any $i\neq j\in I$ and any $r_1,\dots,r_{1-a_{ij}}, s\in \mathbb{Z}_+$:

$$\mathrm{Sym}_{\mathfrak{S}_{1-a_{ij}}} [x_{i,r_1}^{\pm}, [x_{i,r_2}^{\pm},\dots,[x_{i,r_{1-a_{ij}}}^{\pm}, x_{j,s}^{\pm}]\dots]]=0$$

Quantum Loop Algebra Case

(Q1) $[h_{i,k}, h_{j,\ell}]=0$

(Q2.1) $[h_{i,0}, e_{j,k}]=a_{ij}\cdot e_{j,k}$, $[h_{i,0}, f_{j,k}]=-a_{ij}\cdot f_{j,k}$

(Q2.2) $[h_{i,r}, e_{j,k}] = \frac{[ra_{ij}]_{q_i}}{r}\cdot e_{j,k+r}$, $[h_{i,r}, f_{j,k}] = -\frac{[ra_{ij}]_{q_i}}{r}\cdot f_{j,r+k}$ $(r\neq 0)$

(Q3) $e_{i,k+1} e_{j,\ell} - q_i^{a_{ij}} e_{j,\ell}\cdot e_{i,k+1} = q_i^{a_{ij}} e_{i,k} e_{j,\ell+1} - e_{j,\ell+1}\cdot e_{i,k}$

$f_{i,k+1} f_{j,\ell} - q_i^{-a_{ij}} f_{j,\ell}\cdot f_{i,k+1} = q_i^{-a_{ij}} f_{i,k} f_{j,\ell+1} - f_{j,\ell+1}\cdot f_{i,k}$

(Q4) $[e_{i,k}, f_{j,\ell}] = \delta_{ij}\cdot \frac{\psi^+_{i,k+\ell}-\psi^-_{i,k+\ell}}{q_i-q_i^{-1}}$

(Q5) (Serre relation) For any $i\neq j\in I$ and any $k_1,\dots,k_m,\ell\in\mathbb{Z}$ (where $m:=1-a_{ij}$):

$$\mathrm{Sym}_{\mathfrak{S}_m}\Big\{\sum_{s=0}^{m}(-1)^s \begin{bmatrix} m\\ s\end{bmatrix}_{q_i} e_{i,k_1}\cdots e_{i,k_s} e_{j,\ell} e_{i,k_{s+1}}\cdots e_{i,k_m}\Big\}=0$$

$$\mathrm{Sym}_{\mathfrak{S}_m}\Big\{\sum_{s=0}^{m}(-1)^s \begin{bmatrix} m\\ s\end{bmatrix}_{q_i} f_{i,k_1}\cdots f_{i,k_s} f_{j,\ell} f_{i,k_{s+1}}\cdots f_{i,k_m}\Big\}=0$$

__Here__: $\boxed{\psi_i^{\pm}(z)=\sum_{r\geq 0}\psi^{\pm}_{i,\pm r} z^{\mp r} = \exp\left(\pm\frac{hd_i}{2}h_{i,0}\right)\cdot \exp\left(\pm(q_i-q_i^{-1})\sum_{s\geq 1} h_{i,\pm s} z^{\mp s}\right)}$

__Rmk__: The defining relations (Q2.1, Q2.2, Q3) can be rewritten nicely via generating f-s:

$$\boxed{\begin{array}{l}(z-q_i^{a_{ij}}w)e_i(z)e_j(w) = (q_i^{a_{ij}}z-w)e_j(w)e_i(z), \quad (q_i^{a_{ij}}z-w)f_i(z)f_j(w)=(z-q_i^{a_{ij}}w)f_j(w)f_i(z)\\ (z-q_i^{a_{ij}}w)\psi_i^{\pm}(z)e_j(w)=(q_i^{a_{ij}}z-w)e_j(w)\psi_i^{\pm}(z), \quad (q_i^{a_{ij}}z-w)\psi_i^{\pm}(z)f_j(w)=(z-q_i^{a_{ij}}w)f_j(w)\psi_i^{\pm}(z)\\ \mathrm{Sym}_{\mathfrak{S}_m}[e_i(z_1),\dots,[e_i(z_m),e_j(w)]_{q_i}\dots]_{q_i}=0, \quad \mathrm{Sym}_{\mathfrak{S}_m}[f_i(z_1),\dots,[f_i(z_m),f_j(w)]_{q_i}\dots]_{q_i}=0\end{array}}$$

where $e_i(z)=\sum_{k=-\infty}^{+\infty} e_{i,k}z^{-k}$, $f_i(z)=\sum_{k=-\infty}^{+\infty} f_{i,k}z^{-k}$.

1.2 Drinfeld's degeneration

The relation between the quantum loop algebra $U_q(L\mathfrak{g})$ and $Y_\hbar(\mathfrak{g})$ has been stated in [Drinfeld, Quantum gps, Proceedings of ICM, 1986], while the written proof appeared more than 20 years later in [Guay - Ma, 2010].

To state the result, we consider the formal versions of our alg-s:

- $Y_\hbar(\mathfrak{g})$ - algebra over $\mathbb{C}[[\hbar]]$ with the same collections of generators and relations
- $U_\hbar(L\mathfrak{g})$ - algebra over $\mathbb{C}[[\hbar]]$ with the same generators and relations (for $q = \exp(\frac{\hbar}{2})$).

We also define $\mathcal{J} \subset U_\hbar(L\mathfrak{g})$ to be the kernel of the composition

$$\boxed{U_\hbar(L\mathfrak{g}) \xrightarrow{\hbar \to 0} U(\mathfrak{g}[z,z^{-1}]) \xrightarrow{z \mapsto 1} U(\mathfrak{g})}$$

(we use the fact that $U_\hbar(L\mathfrak{g})/(\hbar) \simeq U(\mathfrak{g}[z,z^{-1}])$ as mentioned before)

Consider the associated descending filtration on $U_\hbar(L\mathfrak{g})$ given by powers of $\mathcal{J}$ and set $\boxed{gr_{\mathcal{J}}(U_\hbar(L\mathfrak{g})) := \bigoplus_{n\geq 0} \mathcal{J}^n/\mathcal{J}^{n+1}}$ to be its associated graded.

Theorem: Let $\{d_i^\pm\}_{i\in I} \subset \mathbb{C}^*$ be such that $d_i^+ d_i^- = d_i$.

Then there exists a unique isomorphism of graded algebras

$$\boxed{Y_\hbar(\mathfrak{g}) \xrightarrow{\sim} gr_{\mathcal{J}}(U_\hbar(L\mathfrak{g}))}$$

such that

$$\xi_{i,0} \longmapsto d_i h_{i,0} \in U_\hbar(L\mathfrak{g})/\mathcal{J}$$

$$x_{i,0}^+ \longmapsto d_i^+ e_{i,0} \in U_\hbar(L\mathfrak{g})/\mathcal{J}$$

$$x_{i,0}^- \longmapsto d_i^- f_{i,0} \in U_\hbar(L\mathfrak{g})/\mathcal{J}$$

$$x_{i,1}^+ \longmapsto d_i^+ (e_{i,1} - e_{i,0}) \in \mathcal{J}/\mathcal{J}^2$$

$$x_{i,1}^- \longmapsto d_i^- (f_{i,1} - f_{i,0}) \in \mathcal{J}/\mathcal{J}^2.$$

Rmk: Here $Y_\hbar(\mathfrak{g})$ is $\mathbb{Z}_+$-graded via $\deg(h_{i,r}) = \deg(x_{i,r}^\pm) = r$, $\deg(\hbar) = 1$.

1.3 Finite-dimensional simple modules

Yangian Case

Given a $Y_\hbar(\mathfrak{g})$-representation V, we say that $v \in V$ is a highest weight vector if

(a) $x_{i,r}^+(v) = 0 \quad \forall i \in I, r \in \mathbb{Z}_+$.

(b) $\xi_{i,r}(v) = \gamma_{i,r} \cdot v \quad \forall i \in I, r \in \mathbb{Z}_+; \ \gamma_{i,r} \in \mathbb{C}$.

The collection $\gamma = \{\gamma_{i,r}\}_{i \in I}^{r \in \mathbb{Z}_+}$ is called a weight of v.

We say that V is a highest weight module if $V = Y_\hbar(\mathfrak{g})(v)$, v – high. w. vector

Given any such γ, we define M_γ as the quotient of $Y_\hbar(\mathfrak{g})$ by the left ideal generated by $\{x_{i,r}^+, \xi_{i,r} - \gamma_{i,r}\}$. By standard arguments, M_γ has a unique irreducible quotient L_γ.

Theorem [Drinfeld]:

(a) Every simple finite-dimensional $Y_\hbar(\mathfrak{g})$-repr. V is a highest weight repr. (i.e. $V \simeq L_\gamma$).

(b) The irreducible $Y_\hbar(\mathfrak{g})$-highest weight repr. L_γ is finite dimensional ~~iff~~

$$\boxed{1 + \sum_{r \geq 0} \gamma_{i,r} z^{-r-1} = \frac{P_i(z+d_i)}{P_i(z)}}$$

for some polynomials $P_i \in \mathbb{C}[z]$ (called Drinfeld polynomials)

Quantum Loop Algebra Case

Given an $U_\hbar(L\mathfrak{g})$-representation V, we say that $v \in V$ is a highest weight vector if

(a) $e_{i,k}(v) = 0 \quad \forall i \in I, k \in \mathbb{Z}$

(b) $\psi_{i,k}^\pm(v) = \gamma_{i,k}^\pm \cdot v \quad \forall i \in I, k \in \mathbb{Z}, \ \gamma_{i,k}^\pm \in \mathbb{C}$.

As in the Yangian Case above, we call the collection $\gamma = \{\gamma_{i,k}^\pm\}$ – a weight of v.

We also define M_γ and L_γ as above.

Theorem [Chari – Pressley]

(a) Every simple finite-dimensional $U_\hbar(L\mathfrak{g})$-repr. V is a highest weight repr. (i.e. $V \simeq L_\gamma$)

(b) The irreducible $U_\hbar(L\mathfrak{g})$-highest weight repr. L_γ is finite dimensional if

$$\boxed{\sum_{r \geq 0} \gamma_{i,\pm r}^\pm \cdot z^{\mp r} = \left(q_i^{\deg(P_i)} \cdot \frac{P_i(q_i^{-2} z)}{P_i(z)}\right)^\pm}$$

for some polynomials $P_i(z) \in 1 + z\,\mathbb{C}[z]$ (called Drinfeld polynomials).

1.4. Representations via Nakajima Quiver Varieties.

For a simply laced $\mathfrak{g}$ (i.e. all $d_i = 1$), Nakajima introduced certain alg. varieties $Z(w)$ (with $w \in \mathbb{Z}_+^I$) endowed with an action of $\prod_{i\in I} GL_{w_i} \times \mathbb{C}^\times =: G$.

<u>Theorem</u> [Nakajima, Varagnolo]:

(a) There exists a natural algebra homomorphism

$$\boxed{\Psi_U : U_\hbar(L\mathfrak{g}) \longrightarrow K^G(Z(w))}$$

(b) There exists a natural algebra homomorphism

$$\boxed{\Psi_Y : Y_\hbar(\mathfrak{g}) \longrightarrow H^G(Z(w))}$$

An alternative way to view this is to consider the Nakajima quiver varieties $\underline{\mathcal{M}(v,w) \quad \text{with } v, w \in (\mathbb{Z}_+)^I}$

We define

$$\boxed{H(w) := \bigoplus_v H^G(\mathcal{M}(v,w)) \quad , \quad K(w) := \bigoplus_v K^G(\mathcal{M}(v,w))}$$

<u>Rmk</u>: (i) One can consider $\mathbb{T}$-equivariance instead of G-equiv, where $\mathbb{T} \subset G$ - max. torus.

(ii) In what follows we will need to localize both spaces $\rightsquigarrow H(w)_{loc}$, $K(w)_{loc}$

<u>Theorem</u> [Nakajima]:

(a) There is a natural action $\boxed{U_\hbar(L\mathfrak{g}) \curvearrowright K(w)_{loc} \quad \forall w \in (\mathbb{Z}_+)^I}$

Moreover, if we grade $K(w)_{loc}$ by $(\mathbb{Z}_+)^I$ just by assigning v, then $\deg(e_{i,k}) = (0, \ldots, \underset{i}{-1}, \ldots, 0)$, $\deg(f_{i,k}) = (0, \ldots, \underset{i}{1}, \ldots, 0)$, $\deg(h_{i,k}) = \bar{0}$

(b) There is a natural action $\boxed{Y_\hbar(\mathfrak{g}) \curvearrowright H(w)_{loc} \quad \forall w \in \mathbb{Z}_+^I}$

Moreover, if we grade $H(w)_{loc}$ by $(\mathbb{Z}_+)^I$ just by assigning v, then

$$\deg(x_{i,r}^\pm) = (0, \ldots, 0, \underset{i}{\mp 1}, 0, \ldots, 0) \quad , \quad \deg(\xi_{i,k}) = \bar{0} = (0, \ldots, 0)$$

2.1 Main Result of Gautam and Toledano Laredo

In Part 1, we saw that the theories of $Y_\hbar(\mathfrak{g})$ and $U_\hbar(L\mathfrak{g})$ have common features:

- Drinfeld's degeneration result
- Classification of irreducible finite-dimensional representations.
- Geometric realization of representations.

Motivated by this, the authors of [GTL] found a more deep connection between these two algebras.

<u>Theorem 1</u> [GTL]: There exists an explicit algebra homomorphism

$$\boxed{\Phi : U_\hbar(L\mathfrak{g}) \longrightarrow \widehat{Y_\hbar(\mathfrak{g})}}$$

given on the generators by explicit formulas:

$$\boxed{\begin{aligned}
h_{i,0} &\longmapsto d_i^{-1}\cdot \xi_{i,0}\\
h_{i,r} &\longmapsto \frac{\hbar}{q_i-q_i^{-1}}\sum_{m\geq 0} t_{i,m}\frac{r^m}{m!} = \frac{B_i(r)}{q_i-q_i^{-1}}\\
e_{i,k} &\longmapsto e^{k\sigma_i^+}\sum_{m\geq 0} g_{i,m}\, x_{i,m}^+ = e^{k\sigma_i^+} g_i(\sigma_i^+)\, x_{i,0}^+\\
f_{i,k} &\longmapsto e^{k\sigma_i^-}\sum_{m\geq 0} g_{i,m}\, x_{i,m}^- = e^{k\sigma_i^-} g_i(\sigma_i^-)\, x_{i,0}^-
\end{aligned}}$$

where we use the following notation:

- $\hbar\sum_{m\geq 0} t_{i,m}u^{-m-1} = \log\left(1+\hbar\sum_{m\geq 0}\xi_{i,m}u^{-m-1}\right)$
- $B_i(w) = \hbar\sum_{m\geq 0} t_{i,m}\frac{w^m}{m!}$ – inverse Borel transform of $t_i(u)$ from previous line.
- $\sigma_i^\pm : Y_\hbar(\mathfrak{b}^\pm) \longrightarrow Y_\hbar(\mathfrak{b}^\pm)$ given by $\xi_{j,r}\longmapsto \xi_{j,r}$, $x_{j,r}^\pm \longmapsto x_{j,r+\delta_{ij}}^\pm$
- $\sum_{m\geq 0} g_{i,m}v^m = \left(\frac{\hbar}{q_i-q_i^{-1}}\right)^{1/2}\cdot \exp\left(\frac{\gamma_i(v)}{2}\right)$, where $\gamma_i(v) := -B_i(-\partial_v)\,\partial_v \log\left(\frac{v}{e^{v/2}-e^{-v/2}}\right)$

  Here $g_i(v), \gamma_i(v) \in \widehat{Y_\hbar^0(\mathfrak{g})}[\![v]\!]$
- Finally, let us remind that $\widehat{Y_\hbar(\mathfrak{g})}$ and $\widehat{Y_\hbar^0(\mathfrak{g})}$ stay for the completions of the Yangian $Y_\hbar(\mathfrak{g})$ and its "Cartan subalgebra" $Y_\hbar^0(\mathfrak{g})$ with respect to natural $\mathbb{Z}_+$-grading.

2.2 Properties of Φ

Let us now discuss the key properties of the homomorphism Φ from previous page.

(1) Φ restricts to a homomorphism $U_\hbar(L\mathfrak{sl}_2^i) \to \widehat{Y_\hbar(\mathfrak{sl}_2^i)}$ $\forall i \in I$.

(2) Φ restricts to a homomorphism $U_\hbar(L\mathfrak{b}^\pm) \longrightarrow \widehat{Y_\hbar(\mathfrak{b}^\pm)}$

(3) Classical limit of Φ

Factoring both source and target of Φ by $(\hbar)$, we get its classical limit

$$\boxed{\bar{\Phi}: U(\mathfrak{g}[z,z^{-1}]) \longrightarrow \widehat{U(\mathfrak{g}[w])}}$$

It is easy to see that it is induced by

$$\mathfrak{g}[z,z^{-1}] \longrightarrow \mathfrak{g}[[w]] \qquad \text{with} \quad X\otimes z^k \longmapsto X\otimes e^{kw}$$

(4) Relation to the Drinfeld's degeneration

Considering the associated graded of both source & target of Φ, we get

$$\boxed{gr(\Phi): gr_{\mathcal{J}}(U_\hbar(L\mathfrak{g})) \longrightarrow gr.\left(\widehat{Y_\hbar(\mathfrak{g})}\right) = Y_\hbar(\mathfrak{g})}$$

Then it's easy to see that $gr(\Phi)$ is the inverse of the Drinfeld's degeneration isomorphism (with $d_i^\pm = d_i^{1/2}$).

(5) Relating roots of Drinfeld polynomials

The homomorphism Φ restricts to a homomorphism $U_\hbar(L\mathfrak{h}) \longrightarrow \widehat{Y_\hbar(\mathfrak{h})}$ which induces the exponentiation of roots on Drinfeld polynomials.

Let us define the completion $\widehat{U_\hbar(L\mathfrak{g})}$ by:

$$\boxed{\widehat{U_\hbar(L\mathfrak{g})} := \varprojlim U_\hbar(L\mathfrak{g})/\mathcal{J}^r}$$

Theorem 2 [GTL] (a) The homomorphism Φ from Thm 1 extends to a homomorphism $\widehat{\Phi}: \widehat{U_\hbar(L\mathfrak{g})} \longrightarrow \widehat{Y_\hbar(\mathfrak{g})}$

(b) $\widehat{\Phi}$ is an isomorphism.

► Follows from the fact that $U_\hbar(L\mathfrak{g}), Y_\hbar(\mathfrak{g})$ - flat $\mathbb{C}[[\hbar]]$-deformations of $U(\mathfrak{g}[z,z^{-1}]), U(\mathfrak{g}[w])$ and the observation that the classical limit $\bar{\widehat{\Phi}}$ is induced by an isom.

$$\varprojlim \mathfrak{g}[z,z^{-1}]/(z-1)^r \xrightarrow{\sim} \varprojlim \mathfrak{g}[w]/(w)^r \quad , \text{see (3) above}$$

■

2.3 Ideas of the proof from [GTL].

Step #1 : Determine images of $h_{i,k}$ (pretty simple guess)

Step #2 : As we want Φ to satisfy (1) & (2) from previous page, we expect

$$\Phi: e_{i,0} \longmapsto \sum_{m\geq 0} g_{i,m}^{+} x_{i,m}^{+} \qquad \text{for some } g_{i,m}^{\pm} \in \widehat{Y_\hbar^0(\mathfrak{g})}$$
$$f_{i,0} \longmapsto \sum_{m\geq 0} g_{i,m}^{-} x_{i,m}^{-}$$

Step #3 : Using the defining relation (Q2.2) from the def-n of $U_q(L\mathfrak{g})$, get:

$$\Phi: e_{i,k} \longmapsto e^{k\sigma_i^+} g_i^+(\sigma_i^+)\, x_{i,0}^{+}$$
$$f_{i,k} \longmapsto e^{k\sigma_i^-} g_i^-(\sigma_i^-)\, x_{i,0}^{-}$$

Rewrite the defining rel-s (Q3, Q4) as certain equalities on $\{g_i^\pm(v)\}$.
The key non-trivial thing is in finding an appropriate collection of $\{g_i^\pm(v)\}$.

<u>Main Idea</u>: Use requirement / property (5) from the previous page to replace one of the aforementioned equalities on $\{g_i^\pm(v)\}$ by a more rigorous, but simpler relation.

Once this is done, the authors immediately guess $g_i^\pm(v)$ and show that other equalities also hold... Straightforward computations!

<u>Finally</u>, one needs to show that Φ also preserves the Serre rel-n.
There is no straightforward proof of this known so far.
Instead, in [GTL] authors use the following reasoning:

* Φ(LHS of Serre − RHS of Serre) acts trivially on any fin. dim. $\widehat{Y_\hbar(\mathfrak{g})}$-repr-n.
* The intersection of kernels $\bigcap_\rho \mathrm{Ker}(\rho)$ over all finite-dimensional graded $Y_\hbar(\mathfrak{g})$-modules is <u>Zero</u>!

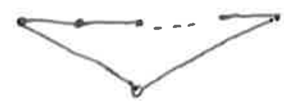
Combining these two observations, one gets compatibility of Φ with Serre rel-s almost for free!

3.1 Quantum toroidal and affine Yangian algebras

Recall that the way we introduced the algebras $U_q(L\mathfrak{g})$ and $Y_h(\mathfrak{g})$ via generators and relations really depended on the Cartan matrix A, not the intrinsic structure of the Lie algebra $\mathfrak{g}$.

As such, the next interesting case to consider is when $\mathfrak{g}$ - affine KM algebra, whose Dynkin diagram is obtained from an associated Dynkin diagram of simple Lie algebra $\mathfrak{g}^\circ$ by adding 1 vertex.

Today: We will be interested in $A_{n-1}^{(1)}$-case.



$n \geq 3$

Motivation: This is the only affine case, when we have a cycle in Dynkin diagram. As a result, in Nakajima's construction we get an action of the 2-dim torus $\mathbb{C}^* \times \mathbb{C}^*$ besides for $\prod GL_{w_i}$-action.

Therefore: One should expect to have 2 parameters in play for the quantum loop algebra and Yangian of $A_{n-1}^{(1)}$-type.

Historically:

- The quantum toroidal algebras of $\mathfrak{sl}_n$ (= quantum loop of $A_{n-1}^{(1)}$-type) first appeared in the work of [Ginzburg-Kapranov-Vasserot '95]
- The affine Yangian of $\mathfrak{sl}_n$ (= Yangian of $A_{n-1}^{(1)}$-type) was considered first by Guay around 2005.
- A similar class of algebras, called the quantum toroidal of $\mathfrak{gl}_1$ and the affine Yangian of $\mathfrak{gl}_1$ became of interest in the recent years (see Maulik-Okounkov, Schiffmann-Vasserot, Feigin-Tsymbaliuk, Negut, ...).
- In somewhere between $\mathfrak{gl}_1$ and $\mathfrak{sl}_n (n\geq 3)$ cases, we have the $\mathfrak{sl}_2$-setting: quantum toroidal and affine Yangian of $\mathfrak{sl}_2$.

Uniform Notation: * $\mathcal{U}^{(n)}_{q_1,q_2,q_3}$ $(n \in \mathbb{N},\ q_1q_2q_3=1)$ – q. toroidal of $\mathfrak{sl}_n$ $(n>1)$ or $\mathfrak{gl}_1$ $(n=1)$

* $\mathcal{Y}^{(n)}_{h_1,h_2,h_3}$ $(n\in\mathbb{N},\ h_1+h_2+h_3=0)$ – a. Yangian of $\mathfrak{sl}_n$ $(n>1)$ or $\mathfrak{gl}_1$ $(n=1)$.

3.2 Motivation

*** Geometry**

As already mentioned before, the q. toroidal and a. Yangian algebras have natural geometric actions. Moreover, the initial motivation of Nakajima to introduce the quiver varieties came from his studies of moduli spaces of instantons on ALE spaces with Kronheimer. In the latter situation, the corresponding quivers are of affine type. Therefore, q. toroidal and a. Yangian algebras have particular importance from this viewpoint.

gl_1-case: the corresponding quiver Q is the Jordan quiver 

$\Rightarrow$ double of this quiver is 

Then the Nakajima quiver variety $M(r,n)$ is defined as:

$$\left\{ \overset{B_1, B_2}{\mathbb{C}^n} \underset{j}{\overset{i}{\rightleftarrows}} \mathbb{C}^r \mid [B_1, B_2] + ij = 0 \right\}^s / GL_n \qquad g.(B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1})$$

Geometrically: $\boxed{M(r,n) = \left\{ (E, \Phi) \,\middle|\, \begin{array}{l} E\text{- torsion free sheaf on } \mathbb{P}^2 \text{ of rank } r,\ c_2(E) = n \\ \text{loc. free in nbhd of } \ell_\infty = (0:*:*);\ \Phi: E|_{\ell_\infty} \xrightarrow{\sim} \mathcal{O}_{\ell_\infty}^{\oplus r} \end{array} \right\} / \text{iso.}}$

According to [SV, FT], we have:

$$\boxed{\mathcal{U}^{(1)}_{q_1,q_2,q_3} \curvearrowright \bigoplus_n K^{(\mathbb{C}^*)^r \times \mathbb{C}^* \times \mathbb{C}^*}(M(r,n))_{loc}, \qquad \mathcal{Y}^{(1)}_{h_1,h_2,h_3} \curvearrowright \bigoplus_n H^{(\mathbb{C}^*)^r \times \mathbb{C}^* \times \mathbb{C}^*}(M(r,n))_{loc.}}$$

(q_1, q_3 – natural characters of $\mathbb{C}^* \times \mathbb{C}^*$) $\qquad$ (h_1, h_3 – natural basis of Lie $(\mathbb{C}^* \times \mathbb{C}^*)$)

*** Physical expectation**

In the recent paper of Belavin – Bershtein – Tarnopolsky ([BBT]), a 4d AGT rel-n on the ALE space $X_n = \widehat{\mathbb{C}^2/\mathbb{Z}_n}$ was studied. Main tool of [BBT]: the limit of the K-theoretic (= 5d) AGT relation on $\mathbb{C}^2$, where $q_1 \to \omega_n = \sqrt[n]{1}$, $q_2 \to 1$.

Since $\mathcal{U}^{(1)}_{q_1,q_2,q_3}$ acts on the K-theory of moduli spaces of torsion free sheaves/$\mathbb{C}^2$
$\mathcal{Y}^{(n)}_{h_1,h_2,h_3}$ acts on the cohomology of moduli spaces of torsion free sheaves/X_n,
it was conjectured that the "limit" of $\mathcal{U}^{(1)}_{q_1,q_2,q_3}$ as $q_1 \to \omega_n$, $q_2 \to 1$ should be related to $\mathcal{Y}^{(n)}_{h_1,h_2,h_3}$.

This was the key motivation for our main result: update of [GTL].

3.3 Main Result

- Let $Y^{(n),'}_{\hbar_1,\hbar_2}$ be the formal version of $\mathcal{Y}^{(n)}_{h_1,h_2,h_3}$ with $h_1 = \frac{\hbar_1}{n}$, $h_2 = \frac{\hbar_2}{n}$, $h_3 = -h_1 - h_2$.

In other words, $\mathcal{Y}^{(n),'}_{\hbar_1,\hbar_2}$ is the associative $\mathbb{C}[[\hbar_1,\hbar_2]]$-algebra generated by $\{x^\pm_{i,r}, \xi_{i,r}\}^{r\in\mathbb{Z}_+}_{i\in\mathbb{Z}/n\mathbb{Z}}$ and with the defining rel-s similar to those of $Y_\hbar(\widehat{\mathfrak{sl}}_n)$

- Let $\mathcal{U}^{(n),\omega,'}_{\hbar_1,\hbar_2}$ be the formal version of $\mathcal{U}^{(n),'}_{q_1,q_2,q_3}$ with

$$q_1 = \omega \cdot \exp\left(\frac{\hbar_1}{n}\right),\quad q_2 = \exp\left(\frac{\hbar_2}{n}\right),\quad q_3 = \omega^{-1}\cdot \exp\left(\frac{-\hbar_1-\hbar_2}{n}\right).$$

<u>Here</u>: ω - root of unity.

- The algebra $\mathcal{Y}^{(n),'}_{\hbar_1,\hbar_2}$ is naturally graded with

$$\deg(x^\pm_{i,r}) = \deg(\xi_{i,r}) = r,\quad \deg(\hbar_1) = \deg(\hbar_2) = 1.$$

We use $\widehat{\mathcal{Y}}^{(n),'}_{\hbar_1,\hbar_2}$ to denote the completion of $\mathcal{Y}^{(n),'}_{\hbar_1,\hbar_2}$ w.r.t. this grading

- Fix $m,n\in\mathbb{N}$ and ω_{mn} - mn^{th} root of unity

<u>Main Theorem</u>: There exists a $\mathbb{C}[[\hbar_1,\hbar_2]]$-algebra homomorphism

$$\boxed{\Phi^{\omega_{mn}}_{m,n} : U^{(m),\omega_{mn},'}_{\hbar_1,\hbar_2} \longrightarrow \widehat{\mathcal{Y}}^{(mn),'}_{\hbar_1,\hbar_2}}$$

given on the generators by explicit formulas:

$$\boxed{\begin{array}{ll}
h_{i,0} \longmapsto \sum\limits_{\substack{i'\in\mathbb{Z}/mn\mathbb{Z}\\ i'\equiv_m i}} \xi_{i',0} & \\
h_{i,l} \longmapsto \dfrac{n}{q-q^{-1}} \sum\limits_{\substack{i'\in\mathbb{Z}/mn\mathbb{Z}\\ i'\equiv_m i}} \omega_{mn}^{-li'}\, B_{i'}(ln) & q = \sqrt{q_2} = \exp\left(\frac{\hbar_2}{2m}\right)\\
e_{i,k} \longmapsto \sum\limits_{\substack{i'\in\mathbb{Z}/mn\mathbb{Z}\\ i'\equiv_m i}} \omega_{mn}^{-ki'}\, e^{kn\sigma^+_{i'}} g_{i'}(\sigma^+_{i'})\, x^+_{i',0} & \\
f_{i,k} \longmapsto \sum\limits_{\substack{i'\in\mathbb{Z}/mn\mathbb{Z}\\ i'\equiv_m i}} \omega_{mn}^{-ki'}\, e^{kn\sigma^-_{i'}} g_{i'}(\sigma^-_{i'})\, x^-_{i',0} &
\end{array}}$$

where we use the same notation $\sigma^\pm_{i'}$, $B_{i'}(w)$, though $g_{i'}(v)\in \widehat{\mathcal{Y}}^{(mn),0,'}_{\hbar_1,\hbar_2}[[v]]$ is given by a more complicated f-la which we omit at the moment.

<u>Rmk</u>: When $m=1$ and $\omega_{mn}=1$, this morally coincides with [GTL].

3.4 Classical limits

- Let $\mathcal{D}'_\hbar$ be an associative $\mathbb{C}[[\hbar]]$-algebra generated by $\langle \partial, x^{\pm1}\rangle$ subject to the relation $\boxed{\partial\cdot x = x\cdot(\partial+\hbar)}$ (called algebra of $\hbar$-differential operators on $\mathbb{C}^*$).

  Let $\mathcal{D}^{(n),\prime}_\hbar := M_n \otimes \mathcal{D}'_\hbar$, where $M_n = \mathrm{Mat}_{n\times n}(\mathbb{C})$. We will need a 1-dim central extension $\bar{\mathcal{D}}^{(n),\prime}_\hbar = \mathcal{D}^{(n),\prime}_\hbar \oplus \mathbb{C}[[\hbar]]\cdot c_{\mathcal{D}}$ (as a Lie alg.) via the 2-cocycle from [BKLY] (2^{nd} cohomology are 1-dim $\Rightarrow$ essentially 1 such extension).

- Let $\ddot{\mathcal{D}}'_q$ be an associative $\mathbb{C}[[\hbar]]$-alg. generated by $\langle D^{\pm1}, Z^{\pm1}\rangle$ subject to the relation $DZ = qZD$, wher $q \in \mathbb{C}[[\hbar]]^*$ - fixed in the beginning. This algebra is often called the algebra of q-difference operators on $\mathbb{C}^*$

  Let $\ddot{\mathcal{D}}^{(n),\prime}_q := M_n \otimes \ddot{\mathcal{D}}'_q$ and $\bar{\ddot{\mathcal{D}}}^{(n),\prime}_q = \ddot{\mathcal{D}}^{(n),\prime}_q \oplus \mathbb{C}[[\hbar]]\cdot c_{\ddot{\mathcal{D}}}$ be a non-trivial 1-dim extension (the 2^{nd} cohomology are 2-dim $\Rightarrow$ need to specify the choice of 2-cocycle).

- The relation of these algebras to q. toroidal & a. Yangian is established by the following result:

Prop: (a) There exists a homomorphism of $\mathbb{C}[[\hbar_1]]$-algebras

$$\boxed{\theta^{(m)}: \mathcal{U}^{(m),\omega,\prime}_{\hbar_1,\hbar_2}/(\hbar_2) \longrightarrow U(\bar{\ddot{\mathcal{D}}}^{(m),\prime}_{q_1^m})}$$

(b) There exists a homom. of $\mathbb{C}[[\hbar_1]]$-algebras

$$\boxed{\vartheta^{(n)}: \mathcal{Y}^{(n),\prime}_{\hbar_1,\hbar_2}/(\hbar_2) \longrightarrow U(\bar{\mathcal{D}}^{(n),\prime}_{\hbar_1})}$$

(c) The images are $U(\bar{\ddot{\mathcal{D}}}^{(m),0,\prime}_{q_1^m})$, $U(\bar{\mathcal{D}}^{(n),0,\prime}_{\hbar_1})$ - universal enveloping of certain Lie subalgebras (which are almost $\mathbb{C}[[\hbar_1]]$-lattices).

Thm: The homomorphisms $\theta^{(m)}$ and $\vartheta^{(n)}$ are injective.

- Therefore, factoring by $(\hbar_2)$, we get $\boxed{\bar{\Phi}^{\omega_{mn}}_{m,n}: U(\bar{\ddot{\mathcal{D}}}^{(m),0,\prime}_{q_1^m}) \longrightarrow \widehat{U(\bar{\mathcal{D}}^{(mn),0,\prime}_{\hbar_1})}}$

  This one is induced by $\Xi^{\omega_n}_{m,n}: \bar{\ddot{\mathcal{D}}}^{(m),\prime}_{\omega_n\cdot e^{\hbar_1}} \longrightarrow \widehat{\mathcal{D}}^{(mn),\prime}_{\hbar_1}$ (here $\omega_n = \omega_{mn}^m = \sqrt[n]{1}$).

  In turn, $\Xi^{\omega_n}_{m,n}$ is induced by $\otimes M_m$ from $\Xi^{\omega_n}_{1,n}: \ddot{\mathcal{D}}'_{\omega_n\cdot e^{\hbar_1}} \longrightarrow \widehat{\mathcal{D}}^{(n),\prime}_{\hbar_1}$ given by

$$\boxed{Z \longmapsto \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ x & & & 0 \end{pmatrix}, \quad D \longmapsto \begin{pmatrix} q^{n-1}e^{\hbar\partial} & & & \\ & q^{n-2}e^{\hbar\partial} & & \\ & & \ddots & \\ & & & q^0\cdot e^{\hbar\partial} \end{pmatrix}, \quad q = \omega_n\cdot e^{\hbar_1}}$$

3.5 Sketch of the proof

We can prove by straightforward computations that the assignment $\Phi^{\omega_{mn}}_{m,n}$ prescribing to the generators $\{e_{i,k}, f_{i,k}, h_{i,k}\}^{k\in\mathbb{Z}}_{i\in\mathbb{Z}/m\mathbb{Z}}$ certain elements of $\widehat{\mathcal{Y}}^{(mn),'}_{\hbar_1,\hbar_2}$ is compatible with all defining rel-s, except for Serre rel-s.

Unfortunately: we can't use the trick of [GTL] to deduce automatically compatibility with Serre rel-s

(Recall that their argument used $U_\hbar(L\mathfrak{sl}_2^i) \subset U_\hbar(L\mathfrak{g})$, which don't have Serre rel.)

Instead: We use their representations and check compatibility on the level of faithful repr-s.

Step 1: We show that the action of $\mathcal{U}^{(m),\omega_{mn},'}_{\hbar_1,\hbar_2}$ on the formal version of $\bigoplus_w K(w)_{loc}$ is faithful, while the action of $\mathcal{Y}^{(mn),'}_{\hbar_1,\hbar_2}$ on the formal version of $\bigoplus_w H(w)_{loc}$ is also faithful.

Step 2: For every w, the spaces $K(w)_{loc}$ & $H(w)_{loc}$ have natural bases parametrized by tuples of "colored" partitions: $\{\bar{\lambda}\}$.

We construct a diagonal map

$$\boxed{K(w)_{loc} \xrightarrow{\mathcal{I}_w} H(w)_{loc} \qquad [\bar{\lambda}] \longmapsto c_{\bar{\lambda}}\cdot[\bar{\lambda}]}$$

which is compatible with $\Phi^{\omega_{mn}}_{m,n}$ in the following sense:

$\forall\, v \in K(w)_{loc}$, $X \in \{e_{i,k}, f_{i,k}, h_{i,k}\}$: $\boxed{\mathcal{I}_w(X(v)) = \Phi^{\omega_{mn}}_{m,n}(X)(\mathcal{I}_w(v))}$

These 2 steps immediately imply that $\Phi^{\omega_{mn}}_{m,n}$ is compatible with all the defining rel-s of the alg. $\mathcal{U}^{(m),\omega_{mn},'}_{\hbar_1,\hbar_2} \Rightarrow \Phi^{\omega_{mn}}_{m,n}$ induces a required homomorphism.

Rem: When $m=1$, $\omega_{mn}=1 \Rightarrow \mathcal{I}_w$ - Chern character map

In general, it is a composition of the Chern character map and the restriction to the invariant (w.r.t. bigger cyclic group) locus.