Ch 1.2: Solutions of Some Differential Equations

- Recall the free fall and owl/mice differential equations:

$$v' = 9.8 - 0.2v, \qquad p' = 0.5p - 450$$

- These equations have the general form $y' = ay - b$
- We can use methods of calculus to solve differential equations of this form.

- Classify first order DE:

- (1) Integrable equations

- (2) Separable equations

- (3) Linear equations

Integrable Equations

- (Examples)

(1) $y' = t^2 + 1$

(2) $y' = \sin(t)$

(3) $e^{2t} y' = 5$

(4) $y' = y - 2$

- Question: Can we use the same approach for the fourth equation?
- It is called a separable equation.

Separable Equations

- Question: Is there any way to transform the 4th equation into some equation close to integrable equation?

$$y' = y - 2$$

(a) an equilibrium solution: y(t) = 2

(b) We assume that y is not an equilibrium solution.

- (Examples) Initial Value Problem (IVP)

$$y' = 2y - 5, \qquad y(0) = y_0$$

(1) Find an equilibrium solution

(2) Find a general solution and the solution of the IVP.

Solution to General Equation

- To solve the general equation (: separable equation)

$$y' = ay - b$$

we use **methods of calculus**, as follows.

$$\frac{dy}{dt} = a\left(y - \frac{b}{a}\right) \;\Rightarrow\; \frac{dy/dt}{y - b/a} = a \;\Rightarrow\; \int \frac{dy}{y - b/a} = \int a\,dt$$

$$\Rightarrow\; \ln\left|y - b/a\right| = at + C \;\Rightarrow\; \left|y - b/a\right| = e^{at+C}$$

$$\Rightarrow\; y - b/a = \pm e^{at} e^{C} \;\Rightarrow\; y = b/a + ce^{at},\; c = \pm e^{C}$$

- Thus the general solution is

$$y = \frac{b}{a} + ce^{at},$$

where c is a constant.

Initial Value Problem

- Next, we solve the initial value problem

$$y' = ay - b, \qquad y(0) = y_0$$

- From previous slide, the solution to differential equation is

$$y = b/a + ce^{at}$$

- Using the initial condition to solve for c, we obtain

$$y(0) = y_0 = \frac{b}{a} + ce^0 \quad \Rightarrow \quad c = y_0 - \frac{b}{a}$$

and hence the solution to the initial value problem is

$$y = \frac{b}{a} + \left[y_0 - \frac{b}{a} \right] e^{at}$$

Equilibrium Solution

- To find the equilibrium solution, set $y' = 0$ & solve for y:

$$y' = ay - b \overset{set}{=} 0 \quad \Rightarrow \quad y(t) = \frac{b}{a}$$

- From the previous slide, our solution to the initial value problem is:

$$y = \frac{b}{a} + \left[y_0 - \frac{b}{a} \right] e^{at}$$

- Note the following solution behavior:
  - If $y_0 = b/a$, then y is constant, with $y(t) = b/a$
  - If $y_0 > b/a$ and $a > 0$, then y increases exponentially without bound
  - If $y_0 > b/a$ and $a < 0$, then y decays exponentially to b/a
  - If $y_0 < b/a$ and $a > 0$, then y decreases exponentially without bound
  - If $y_0 < b/a$ and $a < 0$, then y increases asymptotically to b/a

Mice and Owls (1 of 4)

- Consider a mouse population that reproduces at a rate proportional to the current population, with a rate constant equal to 0.5 mice/month (assuming no owls present).
- When owls are present, they eat the mice. Suppose that the owls eat 15 per day (average). Write a differential equation describing mouse population in the presence of owls. (Assume that there are 30 days in a month.)
- Solution:

$$\frac{dp}{dt} = 0.5p - 450$$

Example 1: Mice and Owls (2 of 4)

- To solve the differential equation

$$\frac{dp}{dt} = 0.5p - 450$$

we use methods of calculus, as follows.

$$\frac{dp}{dt} = 0.5(p-900) \Rightarrow \frac{dp/dt}{p-900} = 0.5 \Rightarrow \int \frac{dp}{p-900} = \int 0.5dt$$

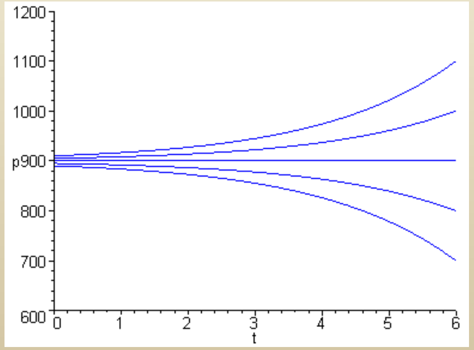
$$\Rightarrow \ln|p-900| = 0.5t + C \Rightarrow |p-900| = e^{0.5t+C}$$

$$\Rightarrow p - 900 = \pm e^{0.5t}e^{C} \Rightarrow p = 900 + ce^{0.5t}, \; c = \pm e^{C}$$

- Thus the solution is

$$p = 900 + ce^{0.5t}$$

where *c* is a constant.

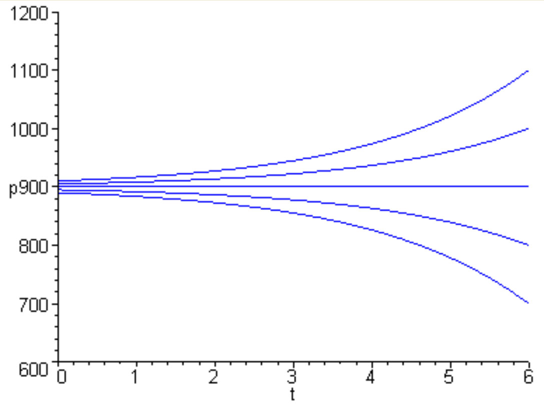


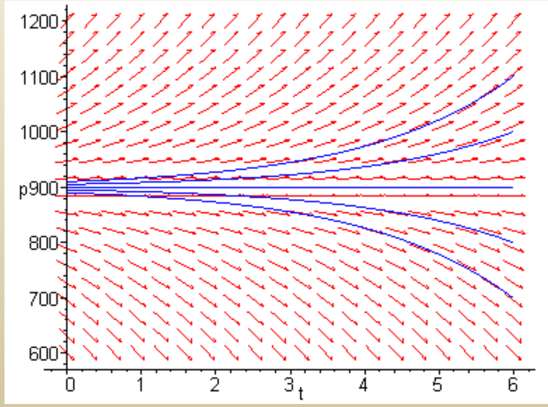
Example 1: Integral Curves (3 of 4)

- Thus we have infinitely many solutions to our equation,

$$p' = 0.5p - 450 \;\Rightarrow\; p = 900 + ce^{0.5t},$$

  since c is an arbitrary constant.
- Graphs of solutions (**integral curves**) for several values of c, and direction field for differential equation, are given below.
- Choosing $c = 0$, we obtain the equilibrium solution, while for $c \neq 0$, the solutions diverge from equilibrium solution.





Example 1: Initial Conditions (4 of 4)

- A differential equation often has infinitely many solutions. If a point on the solution curve is known, such as an initial condition, then this determines a unique solution.
- In the mice/owl differential equation, suppose we know that the mice population starts out at 850. Then $p(0) = 850$, and

$$p(t) = 900 + ce^{0.5t}$$

$$p(0) = 850 = 900 + ce^{0}$$

$$-50 = c$$

Solution :

$$p(t) = 900 - 50e^{0.5t}$$